1 Exercise 1.2.3

We want to show that

$$u(t,x,y) = e^{\imath(k_x x + k_y y - \omega t)} \tag{1}$$

satisfies the wave equation $u_{tt} = c^2(u_{xx} + u_{yy})$
We start by finding u_{tt}, u_{xx} and u_{yy}:

$$u_{tt} = \frac{\partial^2 u}{\partial t^2}(e^{\imath(k_x x + k_y y - \omega t)}) = -\omega^2 e^{\imath(k_x x + k_y y - \omega t)}$$
$$u_{xx} = \frac{\partial^2 u}{\partial x^2}(e^{\imath(k_x x + k_y y - \omega t)}) = -k_x^2 e^{\imath(k_x x + k_y y - \omega t)}$$
$$u_{yy} = \frac{\partial^2 u}{\partial y^2}(e^{\imath(k_x x + k_y y - \omega t)}) = -k_y^2 e^{\imath(k_x x + k_y y - \omega t)}$$

We can now substitute into the wave equation:

$$-\omega^2 u = -c^2(k_x^2 + k_y^2)u.$$

Since u never is equal to 0, we can cancel out u and we get:

$$\omega^2 = c^2(k_x^2 + k_y^2),$$

which is the dispersion relation, meaning that $u(t,x,y) = e^{\imath(k_x x + k_y y - \omega t)}$ satisfies the wave equation if and only if ω, k_x and k_y respect this relation.

2 Exercise 1.2.4

We now assume $m_x = m_y$ such that $k_x = k_y = k$. A discrete version of (1) is then given by

$$u_{i,j}^n = e^{\imath(kh(i+j) - \tilde{\omega} n \Delta t)}$$

where $\tilde{\omega}$ is a numerical dispersion coefficient, i.e., the numerical approximation of the exact ω. We want to insert this equation into the discretized equation

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2}\right)$$

and show that for CFL number $C = \frac{1}{\sqrt{2}}$, we get $\tilde{\omega} = \omega$ By inserting the discrete $u_{i,j}^n$ into the scheme, we get

$$u_{i,j}^{n+1} = e^{-\imath\tilde{\omega}\Delta t} u_{i,j}^n,$$
$$u_{i,j}^{n-1} = e^{\imath\tilde{\omega}\Delta t} u_{i,j}^n,$$
$$u_{i+1,j}^n = e^{\imath kh} u_{i,j}^n,$$
$$u_{i-1,j}^n = e^{-\imath kh} u_{i,j}^n,$$
$$u_{i,j+1}^n = e^{\imath kh} u_{i,j}^n,$$
$$u_{i,j-1}^n = e^{-\imath kh} u_{i,j}^n.$$

We will then use the identity $e^{\imath\theta} + e^{-\imath\theta} = 2\cos(\theta)$ to write the difference:

$$\frac{e^{-\imath\tilde{\omega}\Delta t}u_{i,j}^n - 2u_{i,j}^n + e^{\imath\tilde{\omega}\Delta t}u_{i,j}^n}{\Delta t^2} = c^2(\frac{e^{\imath kh}u_{i,j}^n - 2u_{i,j}^n + e^{-\imath kh}u_{i,j}^n}{h^2} + \frac{e^{\imath kh}u_{i,j}^n - 2u_{i,j}^n + e^{-\imath kh}u_{i,j}^n}{h^2}).$$

which yields

$$u_{i,j}^n \frac{e^{-\imath\tilde{\omega}\Delta t} + e^{\imath\tilde{\omega}\Delta t} - 2}{\Delta t^2} = 2c^2 u_{i,j}^n (\frac{e^{\imath kh} + e^{-\imath kh} - 2}{h^2}).$$

We then apply the identity. Since $u_{i,j}^n$ never is 0, we divide by $u_{i,j}^n$ on both sides, which yields:

$$\frac{2(\cos(\tilde{\omega}\Delta t) - 1)}{\Delta t^2} = 2c^2 \frac{(2\cos(kh) - 2)}{h^2}.$$

We now divide by a factor of 2 and multiply by a factor of Δt^2

$$\cos(\tilde{\omega}\Delta t) - 1 = (\frac{c\Delta t}{h})^2(2\cos(kh) - 2),$$

and we let $C = \frac{c\Delta t}{h}$ and factor out 2 in the RHS, which yields:

$$\cos(\tilde{\omega}\Delta t) - 1 = 2C^2(\cos(kh) - 1).$$

We can now choose $C = \frac{1}{\sqrt{2}} \Rightarrow C^2 = \frac{1}{2}$, which gives us:

$$\cos(\tilde{\omega}\Delta t) - 1 = \cos(kh) - 1,$$

which simplifies to

$$\cos(\tilde{\omega}\Delta t) = \cos(kh) \Rightarrow \tilde{\omega}\Delta t = kh$$

We know from the continuous case that

$$\omega = c\sqrt{k_x^2 + k_y^2}$$

and we know by assumption that $k_x = k_y = k$, which gives

$$\omega = c\sqrt{2k^2} = c\sqrt{2}k.$$

so we have

$$\omega\Delta t = c\sqrt{2}k\Delta t = \sqrt{2}(c\Delta t)k = \sqrt{2}C(kh),$$

and given that C $= \frac{1}{\sqrt{2}}$, we get

$$\omega\Delta t = kh.$$

Hence $\tilde{\omega} = \omega$