Python Exercise 3

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In this exercise we focus on elliptic and parabolic partial differential equations (PDEs). First, we ask you to solve the stationary heat conduction equation. Then, we ask you to solve the transient temperature field in a one-dimensional beam.

Contents

Elliptic partial differential equation

Here we are going to find the stationary solution of the temperature field in a square beam cross-section. In the top and left side of the cross-section the temperature is known (Dirichlet boundary condition), and on the right and bottom side the derivative is known (Neumann boundary condition), see Figure 1.

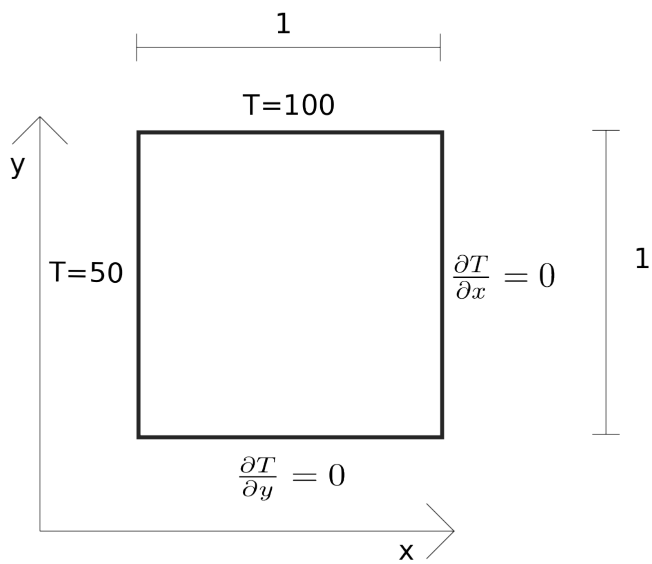

Figure 1: Beam cross-section.

If we assume isotropic heat conduction we can write the stationary heat conduction equation as:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1}$$

Problem 1: Stationary temperature field in beam cross-section

Create a python script that solves for the stationary temperature field in the beam cross-section.

Use the following details on discretization and treatment of boundary conditions:

- Discretize the problem using centered finite differences. Choose $\Delta x = \Delta y = \frac{1}{N}$.
- Use ghost nodes for the Neumann boundary conditions(centered differences)

Plot the solution for different values of N.

Hint 1.

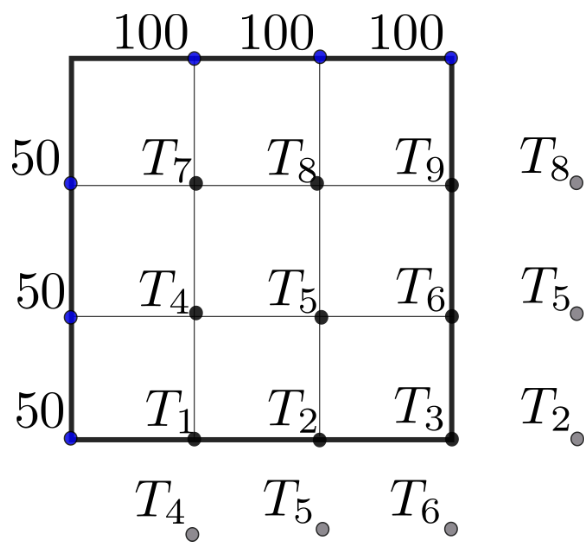

Figure 2: Sketch of discretization with $N = 3$

Figure 2 shows the discretized grid for $N = 3$. The nodes with unknown temperature are colored black ($T_1, \ldots T_9$), whereas ghost nodes are colored gray and nodes where the temperature is known are colored blue.

If the components of $\mathbf{T}$ are numbered by starting in the lower left part of the grid and the index increases in the x-direction first, the function **plotSurfaceNeumannDirichlet** could be used to plot the solution.

```
def plotSurfaceNeumannDirichlet(Temp, Ttop, Tleft, l, N, nxTicks=4, nyTicks=4):
    from mpl_toolkits.mplot3d import Axes3D
    from matplotlib import cm
    from matplotlib.ticker import LinearLocator, FormatStrFormatter

    """ Surface plot of the stationary temperature in quadratic beam cross-section
        .
    Note that the components of T has to be started in the
    lower left part of the grid with increasing indexes in the
    x-direction first.


    Args:
        Temp(array): the unknown temperatures, i.e. [T_1 .... T_(NxN)]
        Ttop(float): temperature at the top boundary
```

```
        Tleft(float): temperature at the left boundary
        l(float): height/width of the sides
        N(int): number of nodes with unknown temperature in x/y direction
        nxTicks(int): number of ticks on x-label (default=4)
        nyTicks(int): number of ticks on y-label (default=4)
    """
    x = np.linspace(0, l, N + 1)
    y = np.linspace(0, l, N + 1)

    X, Y = np.meshgrid(x, y)

    T = np.zeros_like(X)

    T[-1,:] = Ttop
    T[:,0] = Tleft
    k = 1
    for j in range(N):
        T[j,1:] = Temp[N*(k-1):N*k]
        k+=1

    fig = plt.figure()
    ax = fig.gca(projection='3d')
    surf = ax.plot_surface(X, Y, T, rstride=1, cstride=1, cmap=cm.coolwarm,
                           linewidth=0, antialiased=False)
    ax.set_zlim(0, Ttop)
    ax.set_xlabel('x')
    ax.set_ylabel('y')
    ax.set_zlabel('T [$^o$C]')

    xticks=np.linspace(0.0, l, nxTicks+1)
    ax.set_xticks(xticks)

    yticks=np.linspace(0.0, l, nyTicks+1)
    ax.set_yticks(yticks)
    plt.show()
```

Parabolic partial differential equation

Problem 2: Transient temperature field in one-dimensional beam

In this problem we shall solve a parabolic PDE. We are going to find the temperature field along a one-dimensional beam as function of time (transient temperature field). The temperature field is known in the initial state $t = 0$, and the temperature is known at the boundary.

This problem is governed by the one-dimensional transient heat conduction equation,

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{2}$$

where k is the heat conduction coefficient. In this exercise we set $k = 1$. The beam has length $L = 1$.

Boundary conditions: $T(0, t) = 100$ and $T(1, t) = 0$.

Initial conditions: $T(x, 0) = 0$

Find the temperature field along the beam as function of time. Discretize the length with a suitable number of nodes.

a) Discretize (2) with the FTCS scheme and write a script that solves the problem with given boundary conditions. Explore with different values for r (the numerical Fourier number), e.g. $r = 0.2$, $r = 0.5$ and $r = 1.0$, comment your findings. Plot the temperature field for $t = 0.001$, $t = 0.025$ and $t = 0.4$.

b) Same as **a)**, but now using the θ-scheme instead of the FTCS scheme. Use both $\theta = \frac{1}{2}$ (Crank) and $\theta = 1$ (Laasonen).