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TEP4156 Formula sheet

Velocity derivative

$$\frac{\partial u_i}{\partial x_j} = \dot{\varepsilon}_{ij} + \dot{\Omega}_{ij}$$

2D Shear angle rate

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

2D Rotation angle rate

$$\dot{\Omega}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

3D Rotation angle rate

$$(\dot{\vec{\Omega}})_k = \frac{1}{2}\epsilon_{ijk}\dot{\Omega}_{ij}$$

$$\dot{\vec{\Omega}} = \frac{1}{2}\vec{\nabla} \times \vec{u}$$

Mass conservation

$$\frac{D\rho}{Dt} + \vec{\nabla} \cdot (\rho\vec{u}) = 0$$

Momentum conservation

$$\rho\frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i$$

Energy conservation

$$\rho\frac{DE}{Dt} = \frac{\partial}{\partial x_j}(\sigma_{ij}u_i) + \rho f_i u_i - \frac{\partial q_k}{\partial x_k}$$

Total energy

$$E = e + \frac{1}{2}u_i u_i$$

Deformation

$$\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij}$$

Energy equation

$$\rho\frac{De}{Dt} + p\frac{\partial u_k}{\partial x_k} = \sigma'_{ij}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_k}{\partial x_k}$$

Enthalpy equation

$$\rho\frac{Dh}{Dt} - \frac{Dp}{Dt} = \sigma'_{ij}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_k}{\partial x_k}$$

Dissipation related. Positive for newtonian fluids:

$$\sigma'_{ij}\frac{\partial u_i}{\partial x_j} \geq 0$$

Fourier's law

$$q_k = -k\frac{\partial T}{\partial x_k}$$

Deformation law for newtonian fluids

$$\sigma'_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) + \mu_B\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

Creeping flow

$$\mathrm{Re} \to 0: \quad \vec{\nabla}p = \mu\vec{\nabla}^2\vec{u}$$

Boundary layer equation (LN s. 113)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

Steady heat equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}$$

Displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

Momentum thickness

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Stream function

$$\psi(x,y) = \sqrt{\frac{2}{m+1}\nu x U} f(\eta)$$

Similarity variable

$$\eta = y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}$$

Falkner Skan equation

$$f''' + ff'' + \frac{2m}{m+1}(1 - f'^2) = 0$$

Kármán integral relation

$$\frac{d\theta}{dx} + \left(2 + \frac{\delta^*}{\theta}\right)\frac{\theta}{U}\frac{dU}{dx} = \frac{C_f}{2}$$

Additional formulas

Fluid enthalpy

$h = e + \frac{p}{\rho}$

Bulk viscosity

$\mu_B = \lambda + \frac{2}{3}\mu$

Dynamic viscosity

$\nu = \frac{\mu}{\rho}$

Mechanical pressure

$\bar{P} = P - \mu_B \frac{\partial u_k}{\partial x_k}$

Navier-Stokes equation

$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\frac{1}{\rho}\vec{\nabla} P + \nu \vec{\nabla}^2 \vec{u} + \vec{g}$

Skin friction coefficient

$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$

Drag coefficient (flat plate)

$C_D = \frac{1}{L}\int_0^L C_f dx, \quad C_f = 2\frac{\partial \theta}{\partial x}$

$C_D = \frac{F_z}{\rho U^2 \pi a^2/2} = 24/Re$ (Creeping flow)

Shape factor

$H = \frac{\delta *}{\theta} > 1$

Laminar boundary layer

$\delta \ll L,$

$\mathrm{Re} \gg 1$

Stream function

$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$

Dissipation function:

$\Phi = \sigma_{ij}^{'} \frac{\partial u_i}{\partial x_j}$

Vorticity vector:

$\vec{w} = \nabla \times \vec{u}$

At incompressible limit:

$dh = c_p dT$

Fouriers law:

$\vec{q} = -k\vec{\nabla} T$

Notation

Material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u} \cdot \vec{\nabla})$$
$$\frac{D\vec{u}}{Dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$$

Levi-Civita symbol

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } ijk = 123 \text{ or cyclic permutation} \\ -1, & \text{if } ijk = 321 \text{ or cyclic permutation} \\ 0, & \text{otherwise} \end{cases}$$

Einstein summation

An index that appears twice is summed over

Example:

$$a_i b_i = \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 0, & \text{If } i \neq j \\ 1, & \text{If } i = j \end{cases}$$

Laplace operator

$\Delta = \nabla^2$

Notes

Strain rate tensor

Diagonal terms related to dilatation (extension) of fluis particle.

Off-diagonal terms related to shear-strain rates.

Dilatation rates ε_{ii} are $\dfrac{\partial u_i}{\partial x_i}$

Assuming incompressible flow of Newtonian fluid gives viscous part of stress tensor σ_{ij}:

$\sigma'_{ij} = 2\mu\epsilon_{ij} \implies \sigma_{ij} = -p\delta_{ij} + 2\mu\epsilon_{ij}$

Force required to pull sheet/plate with constant speed

Force balance, assuming only shear stress

Shear stress:

$\tau_{xy} = \mu(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}) \approx \mu\dfrac{\partial u}{\partial y}$

Last approximation justified by:

$\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \implies v = -\displaystyle\int \dfrac{\partial u}{\partial x} dy \sim \dfrac{U}{L}\delta$

$\implies \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \sim \dfrac{U}{\delta} + \dfrac{\frac{\delta}{L}U}{L}, \dfrac{\delta}{L} << 1$

Disturbances

We have a disturbance on the form:

$\hat{v}(x, y, t) = v(y)exp(i\alpha(x - ct))$

Disturbace is assumed small, periodic on the form of a wave in x-direction. Amplitude only dependant of y.

c: propagation speed (complex), α: wave number (real)

Differentiation:

$2u\dfrac{\partial u}{\partial x} = \dfrac{\partial (u)^2}{\partial x}$

$v\dfrac{\partial u}{\partial y} + u\dfrac{\partial v}{\partial y} = \dfrac{\partial (uv)}{\partial y}$

$\sigma'_{ij} = \sigma'_{ji}$

(At least for incompressible newtonian fluid, exam 2015)

Flow separation

$\tau_w|_{xs} = 0$

For Twaites this corresponds to:

$S(\lambda) = 0 \implies \lambda = -0.09$

Backflow

If flow between horizontal parallel plates with top plate

moving, back flow occurs when shear at lower plate vanishes is zero:

$\tau_w = 0$

BL thickness:

Where u/U = 0.99

$v \approx 0$ inside boundary layer.

Types of flow

Couette flow:

"Still + moving plate", fully developed, plates at $\pm h$

no-slip: $u(-h) = 0, u(h) = U_0$, no penetration: $v(\pm h) = 0$

same temperature: $T(h) = T_1, T(-h) = T_0$

Brinkman number:

$Br = \frac{\mu U_0^2}{k\Delta T} = dissipation/conduction$ **(LN s. 64)**

Flow in a pipe (Hagen-Poiseuille flow): (LN s. 67)

Stoke's 2nd problem: (LN s. 77)

Oscillating plate, parallel streamlines, separation ansatz

Stagnation point flow (Hiemenz flow): (LN s. 80)

Creeping flow: (LN s. 91)

$\nabla^2 P = 0$

$\nabla^2 \vec{\omega} = 0$

Stokes flow around a sphere (LN s. 93) $->$ Drag on sphere and sinking velocity of sphere in fluid

Boundary layer flow:

Flat plate (Blasius) flow: (LN s. 114)

$\psi = \sqrt{2\nu U x} f(\eta), \eta = y\sqrt{\frac{U}{2\nu x}}$

$u = Uf', v = \sqrt{\frac{2\nu U}{x}}(\eta f' - f)$

$f''' + f''f = 0, f'(\infty) = 1, f(0) = 0, f'(0) = 0$

Falkner-Skan flow: (LN s. 120)

Parallel flow "splitting" into two branches with angle θs between them.

$f''' + ff'' + \beta(1 - f'^2) = 0, \beta = \frac{\theta_s}{\pi/2} = \frac{2m}{m+1}, m = \frac{\theta_s}{\pi - \theta_s}$

$m = 0 \implies$ zero wedge angle = Blasius

$m = 1 \implies \pi/2$ wedge angle = Hiemenz

$\psi = \sqrt{\frac{2}{m+1}\nu U(x) x f(\eta)}, \eta = y\sqrt{\frac{m+1}{2}\frac{U(x)}{\nu x}}$

$\delta, \delta*, C_f, \theta$ relations LN s. 152

Boundary conditions:

$f(0) = f'(0) = 0, f'(\eta \to \infty) = 1$

Twaites method: (LN s. 154)

$\theta^2 = \frac{0.45\nu}{U(x)^6}\int_0^x U(x)^5 dx$

$\lambda = \frac{\theta^2 \frac{\partial U}{\partial x}}{\nu}, \delta^* = \theta H(\lambda), \tau_w = \frac{\mu U}{\theta} S(\lambda)$

$S(\lambda) = (\lambda + 0.09)^{0.62}$, flow separation at $\tau_w = 0 \implies S = 0 \implies \lambda = -0.09$

$H(\lambda) = 2 + 4.14z - 83.5z^2 + 854z^3 - 3337z^4 + 4576z^5, z = (0.25 - \lambda)$ (check this relation before use...)

Free shear flow: (LN s. 164)

Parallel streams

Boundary conditions:

$u_1(0) = u_2(0), \mu_1 \dfrac{\partial u_1}{\partial y}|_0 = \mu_2 \dfrac{\partial u_2}{\partial y}|_0, u_j(\infty) = U_j, v_j(0) = 0$

Laminar (2D) jet: (LN s. 167)

Conserved quantity: momentum flux (no external forces)

$J = \displaystyle\int_{-\infty}^{\infty} \rho u^2 dy$

$f''' + f''f + f'^2 = 0, f(0) = f'(0) = f''(0) = 0$

Analytic solution:

f(η) = 2a tanh(aη), $a = (\dfrac{9J}{16\sqrt{\rho\mu}})^{\frac{1}{3}} \sim \sqrt[3]{J}$

$u_{max} = u(x,0) = \dfrac{2}{3}a^2 x^{-\frac{1}{3}} \sim x^{-\frac{1}{3}}$

Stability notes

Reynolds number:
$Re = \frac{\rho u L}{\mu} = \frac{uL}{\nu}$
Critical Re: Pipe flow: 2300, Couette flow: 1500, Blasius: 500 000

7 step procedure:
1. We seek to examine the stability of a basic solution to the physical problem, Q_0.
2. Add a disturbance variable Q' and substitute $(Q_0 + Q^{'})$ into governing equations. Set BC for Q'.
3. From the eqn(s) subtract the basic terms that Q_0 satisfies identicaly. The disturbance function remains.
4. Linearize by assuming $Q^{'} << Q_0$ (small disturbances), and neglect $Q^{'2}, Q^{'3}$ etc.
5. If linearized disturbance equation is complicated and multidimensional, it can be simplyfied by assuming a form for the disturbances, i.e. wave or perturbation in only one direction.
6. Linearized disturbance equation + BC should be homogeneous. Can only be solved for certain specific values of equations parameters. Is an eigenvalueproblem. Find eigenvalues.
7. Interpret the eigenvales. Growth is unstable, decay is stable.
Plot/map which combinations gives growth and decay.

Kelvin-Helmholtx Instability (s. 179, White 5-1.2)
stability og parallel flows, $\nu = 0$
$\alpha = 2\pi/\lambda$, λ: wavelength of disturbance
$\alpha_{crit} = \sqrt{\frac{g(\rho_1 - \rho_2)}{\gamma}}$, γ: Surface tension coefficient, α: wavenumber
$\Delta U_{crit} = \sqrt{2\frac{\rho_1 + \rho_2}{\rho_1 \rho_2}\sqrt{\gamma(\rho_1 - \rho_2)g}}$ **(LN s. 186)**
Crit values represent the critical value on the thumb-plot.
Unstable for $\sigma_2^2 < 0$, which corresponds to:
$(U_2 - U_1)^2 > \frac{[g(\rho_1 - \rho_2) + \alpha^2\gamma](\rho_1 + \rho_2)}{\alpha \rho_1 \rho_2}$

Unsteady Bernoulli for incompressible flow:
$\rho\frac{\partial \phi}{\partial t} + p + \frac{1}{2}\rho V^2 + \rho g h = const$
$V = \nabla\phi, \phi$: Velocity potential

Orr-Sommerfeld equation
$(U - c)(v'' - \alpha^2 v) - U''v + \frac{i\nu}{\alpha}(v'''' - 2\alpha^2 v'' + \alpha^4 v) = 0$
c: wave speed, c = ω/α, ω: frequency, (c is an eigenvalue?), Stefan uses σ as ω
Structure of solution:
Fix(Re, α), find v(y) and $c = c_r + ic_i$, $e^{i(\alpha x - ct)} \implies c_i > 0$: growth, $c_i < 0$: decay
Do parameter scan in (Re, α) - space
α complex can be chosen, allows for spatial growth instead of temporal

Orr-Sommerfeld conditions:
Duct flows: $v(\pm h) = v'(\pm h) = 0$
Boundary layers: $v(0) = v'(0) = 0$, $v(\infty) = v'(\infty) = 0$
Free-shear layers: $v(\pm\infty) = v'(\pm\infty) = 0$

Inviscid OS (Rayleigh) equation or infinite Reynolds number
$(U - c)(v'' - \alpha^2 v) - U''v = 0$

Inviscid stability theorem
These conditions are required for instability:
Condition 1: $U''(y_p) = 0$ (A point of inflection is required for $c_i \neq 0$ to exist)
Condition 2: $U'' \cdot (U - U(y_p)) < 0$ somewhere
We know almost 100% that the flow is instable
if both conditions are met. If not, we can not
draw any conlusions.
If thunb curve is closed, it is inviscidly stable, if open, inviscidly unstable (?)

Shooting method (LN s. 124)

Why shooting method?

Most solvers require ICs. The BC has to be replaced by a guessed IC. An iterative process uses this guess to improve the guess, until you find an IC that is close enough to the BC prescribed.

Approach:

Reduce system to first order ODEs.
Use shooting method.
Use a smart choice for large but finite ∞.

Shooting method:

1. Guess one initial condition for T'(0), P1, and solve for T(end), Q1.
2. Guess second initial condition for T'(0), P2, and solve for T(end), Q2.
3. Interpolate for guesses to get better guess P3, $P3 = \frac{QExact(end) - Q2}{Q2 - Q1}(P2 - P1) + P2$
4. If nonliear ODE, iterate until convergence:

$P_{i+1} = P_i + \frac{QExact(end) - Q_i}{Q_i - Q_{i-1}}(P_i - P_{i-1}) + P_i$ (Secant method)