Linear multistep methods

$$\sum_{l=0}^{k} \alpha_l y_{n+l} = h \sum_{l=0}^{k} \beta_l f_{n+l}$$

$$C_0 = \sum_{l}^{k} \alpha_l, \quad C_q = \frac{1}{q!} \sum_{l=0}^{k} (l^q \alpha_l - q l^{q-1} \beta_l), \quad q = 1, 2, \ldots$$

Consistent if $C_0 = C_1 = 0$

Of order p if $C_0 = C_1 = \cdots = C_p = 0$

Characteristic polynomial:

$$\rho(r) = \sum_{l=0}^{k} \alpha_l r^l$$

Zero-stable if roots satisfy
1. $|r_i| \leq 1$, for $i = 1, 2, \ldots, k$
2. $|r_i| < 1$ if r_i is a multiple root.

First Dahlquist barrier:
Order p of a zero-stable k-step method satisfies
$p \leq k + 2 \quad$ if k is even,
$p \leq k + 1 \quad$ if k is odd,
$p \leq k \quad$ if $\beta_k \leq 0$.

Runge-Kutta methods

$k_1 = f(t_n, y_n)$,
$k_2 = f(t_n + c_2 h, y_n + h a_{21} k_1)$,
$k_3 = f(t_n + c_3 h, y_n + h(a_{31} k_1 + a_{32} k_2))$,
$\vdots$

$$k_s = f(t_n + c_s h, y_n + h \sum_{j=1}^{s-1} a_{sj} k_j),$$

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i$$

Butcher-tableaux:

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \ldots & a_{1s} \\ c_2 & a_{21} & a_{22} & \ldots & a_{2s} \\ \vdots & \vdots & & & \vdots \\ c_s & a_{s1} & a_{s2} & \ldots & a_{ss} \\ \hline & b_1 & b_2 & \ldots & b_s \end{array}$$

Error estimates

Energy norm:

$\|\mathbf{x}\|_{\mathrm{A}} = \|\mathrm{A}^{1/2}\mathbf{x}\|_2 = (\mathrm{A}\mathbf{x}, \mathbf{x})^{1/2}$

Steepest descent:

$$\|\mathbf{e}^{(k+1)}\|_{\mathrm{A}} \leq \frac{K_2(\mathrm{A}) - 1}{K_2(\mathrm{A}) + 1}\|\mathbf{e}^{(k)}\|_{\mathrm{A}}$$

Conjugate gradient:

$$\|\mathbf{e}^{(k)}\|_{\mathrm{A}} \leq \frac{2c^k}{1 + c^{2k}}\|\mathbf{e}^{(0)}\|_{\mathrm{A}}, \quad \text{with } c = \frac{\sqrt{K_2(\mathrm{A})} - 1}{\sqrt{K_2(\mathrm{A})} + 1}$$

Runge-Kutta methods: (local error)

$$le_{n+1} = \hat{y}_{n+1} - y_{n+1} = h\sum_{i=1}^{s}(\hat{b}_i - b_i)k_i$$

Stepsize control:

$$h_{new} = P \cdot \left(\frac{Tol}{\|le_{n+1}\|}\right)^{\frac{1}{p+1}} h_n$$

Polynomial interpolation:
Lagrange form:

$$E_n(x) = f(x) - \Pi_n f(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x), \quad \omega_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$$

Newton form:
$E_n(x) = \omega_{n+1}(x) f[x_0, \ldots, x_n, x]$

Single Interpolatory quadratures:

Midpoint formula $\quad E_0(f) = \dfrac{h^3}{3}f''(\xi), \quad h = \dfrac{b-a}{2}$

Trapezoidal formula $\quad E_1(f) = \dfrac{h^3}{12}f''(\xi), \quad h = b - a$

Simposons formula $\quad E_2(f) = -\dfrac{h^5}{90}f^{(4)}(\xi), \quad h = \dfrac{b-a}{2}$

Composite Interpolatory quadratures:

Midpoint formula $\quad E_{0,m}(f) = \dfrac{b-a}{24}H^2 f''(\xi), \quad H = \dfrac{b-a}{m}$

Trapezoidal formula $\quad E_{1,m}(f) = -\dfrac{b-a}{12}H^2 f''(\xi), \quad H = \dfrac{b-a}{m}$

Simposons formula $\quad E_{2,m}(f) = -\dfrac{b-a}{180}(H/2)^4 f^{(4)}(\xi), \quad H = \dfrac{b-a}{m}$

Newton-Cotes quadrature:
p. 400-401, 404-405