Combinatorial Assignment 2

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December 17, 2014

1 Assignment Problem

The problem can be seen as a problem of finding a minimal matching in a bipartite graph, when the two groups of the bipartite graph are the persons and the committees and the weights of each edge can be seen as the level of "sadness" that every person would feel based on the committee he is assigned to: 0 for his first choice, 1 for his second choice, 3 for his third choice and 99 if the person is assigned to a committee he had not listed. Thus, we have a complete graph and the method descbribed here: http://goo.gl/DdpJyu can be applied. We tried to develop the algorithm but, for some reason (we think it has to do with the function that finds the minimal vertex cover for each iteration), it does not work. Anyway, the code is attached to this report.

2 Sahni

The Sahni heuristic with $k = 3$ has a worst case performance ratio of $\frac{k}{k+1} = \frac{3}{4}$. Pseudocode of this algorithm is as follows (copied from the lecture notes), where GREEDY is Greedy Heuristic as discussed in class:

SAHNI(3)
1. **begin**
2. sort E according to non-increasing order of p/w ratios;
3. set S $\leftarrow \emptyset$, $\Pi \leftarrow 0$;
4. for (each M $\subset$ E) with$|M| \leq 3$ and $\sum_{e_j \in M} w_k \leq C$) **begin**
5. set $T^* \leftarrow$ GREEDY$(E \setminus M, C - \sum_{e_j \in M} w_k)$;
6. set T $\leftarrow M \cup T^*$;
7. if $(\sum_{e_j \in M} p_k > \Pi)$
8. set $S \leftarrow T$, $\Pi \leftarrow \sum_{e_j \in M} p_k$;
9. **end;**
10. **end.**

This heuristic has, as stated before, a worst case performance ratio of $\frac{k}{k+1}$.

Proof[1]: Let Y be the set of items inserted into the knapsack in the optimal

[1]Silvano Matello and Paolo Toth, *Knapsack Problems. Algorithms and Computer Implementations*, England 1990, p.51-52

solution. If $|Y| \leq k$, then **SAHNI(k)** gives the optimal solution, since all combinations of $|Y|$ are tried. Hence, assume $|Y| > k$. Let $\hat{M}$ be the set of the first k items of highest profit in Y, and denote the remaining items of Y with $j_1, ..., j_r$, assuming $\frac{p_{j_i}}{w_{j_i}} \geq \frac{p_{j_{i+1}}}{w_{j_{i+1}}} (i = 1, ..., r-1)$. Hence, if z is the optimal solution value we have

$$p_{j_i} \leq \frac{z}{k+1} \qquad \text{for } i = 1, ..., r \tag{1}$$

Consider now the iteration of SAHNI(k) in which $M = \hat{M}$, and let i_m be the first item of $\{j_1, ...j_r\}$ not inserted into the knapsack by SAHNI(k). If no such item exists then the heuristic solution is optimal. Otherwise we can write z as

$$z = \sum_{i \in \hat{M}} p_i + \sum_{i=1}^{m-1} p_{j_i} + \sum_{i=m}^{r} p_{j_i}, \tag{2}$$

while for the heuristic solution value returned by SAHNI(k) we have

$$z^H \geq \sum_{i \in \hat{M}} p_i + \sum_{i=1}^{m-1} p_{j_i} + \sum_{i \in Q} p_i, \tag{3}$$

where Q denotes the set of those items of $E \setminus \hat{M}$ which are in the heuristic solution but not in $\{j_1, ..., j_r\}$ and whose index is less than j_m. Let $c^* = c - \sum_{i \in \hat{M}} w_i - \sum_{i=1}^{m-1} w_{j_i}$ and $\bar{c} = c^* - \sum_{i \in Q} w_i$ be the residual capacities available, respectively, in the optimal and the heuristic solution for the items of $E \setminus \hat{M}$ following j_{m-1}. Hence, from (2),

$$z \leq \sum_{i \in \hat{M}} p_i + \sum_{i=1}^{m-1} p_{j_i} + c^* \frac{p_{j_m}}{w_{j_m}}, \tag{4}$$

by definition of m we have $\bar{c} < w_{j_m}$ and $\frac{p_i}{w_i} \geq \frac{p_{j_m}}{w_{j_m}}$ for $i \in Q$, so

$$z < \sum_{i \in \hat{M}} p_i + \sum_{i=1}^{m-1} p_{j_i} + p_{j_m} + \sum_{i \in Q} p_i. \tag{5}$$

Hence, from (3), $z < z^H + p_{j_m}$ and then using (1) results in

$$\frac{z^H}{z} > \frac{k}{k+1}. \tag{6}$$

And since we use SAHNI(3) we have a worst preformance ratio of 0.75.