Testes

cálculo II

Teste 1 2020/2021

1 $f(x) = e^{-x^2}$

a)

Sendo $e^x = \sum_{m=0}^{+\infty} \frac{x^m}{m!}$, então $e^{-x^2} = \sum_{m=0}^{+\infty} \frac{(-x^2)^m}{m!} = \sum_{m=0}^{+\infty} (-1)^m \frac{x^{2m}}{m!}$, $\forall x \in \mathbb{R}$

b) Usando a derivação termo a termo

$$f'(x) = \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^{2m}\right)' = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} 2m\, x^{2m-1}$$

$$= \sum_{m=1}^{\infty} x \frac{(-1)^m}{m!} 2m \times x^{2(m-1)} = \sum_{m=1}^{\infty} (-x) \frac{(-1)^{m-1}}{m!} 2m\, x^{2(m-1)}$$

$$= -x \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} 2(m+1)\, x^{2m}$$

$$= -2x \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^{2m} = -2x e^{-x^2}$$

2 $f(x) = \ln(x+4)$, $\forall x \Rightarrow x > -4$

a)

$$T_{-3}^{3} f(x) = \sum_{k=0}^{3} \frac{f^{(k)}(-3)}{k!} (x+3)^k$$

$f(x) = \ln(x+4)$

$f'(x) = \frac{1}{x+4}$

$f''(x) = \frac{-1}{(x+4)^2} = -(x+4)^{-2}$

$f'''(x) = 2(x+4)^{-3}$

$f(-3) = \ln(1) = 0$

$f'(-3) = 1$

$f''(-3) = -1$

$f'''(-3) = 2$

$f^{(4)}(x) = -6(x+4)^{-4}$

Logo,

$$T_{-3}^{3} \ln(x+4) = f(-3) + f'(-3)(x+4) + \frac{f''(-3)(x+4)^2}{2!} + \frac{f'''(-3)}{3!}(x+4)^3 + \frac{f^{(4)}(\theta)}{4!}(x+3)^4$$

$$= (x+3) - \frac{1}{2}(x+3)^2 + \frac{1}{3}(x+3)^3 - \frac{(\theta+4)^{-4}}{4}(x+3)^4$$

b) $f(x) = \ln(2) \Leftrightarrow \ln(x+4) = \ln(2) \Leftrightarrow x = -2$

$$f(-2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{(\theta+4)^{-4}}{4} \simeq \frac{5}{6}$$

$$M \geq \sup |f^{(4)}(\theta)| \geq \sup_{\theta \in]-3;-2[} |-6(\theta+4)^{-4}|$$

$$\text{Erro} \leq \left| -6 \times \frac{|-2+3|^4}{4!} \right| \leq \frac{6}{4!} \leq 0{,}25$$

3 $$f(x) = \begin{cases} 0, & \text{se } -\pi \leq x < 0 \\ \pi - x, & \text{se } 0 \leq x < \pi \end{cases}$$

a) $$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \operatorname{sen}(mx)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \times \operatorname{sen}(mx)\, dx + \int_0^{\pi} (\pi - x) \operatorname{sen}(mx)\, dx \right]$$

$$= \frac{1}{\pi}\left(\int_0^{\pi} (\pi - x) \operatorname{sen}(mx)\, dx \right)$$

$U = \pi - x \qquad U' = -1$

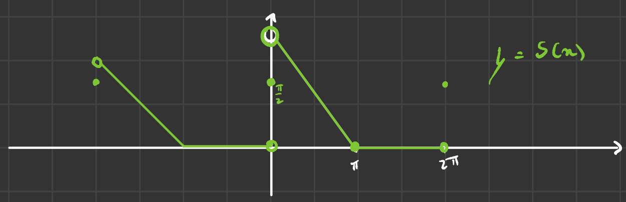
$V = \frac{-\cos(mx)}{m} \qquad V' = \operatorname{sen}(mx)$

$$= \frac{1}{\pi}\left(\left[-\frac{\pi - x}{m} \cos(m(x)) \right]_0^{\pi} - \int_0^{\pi} \frac{\cos(mx)}{m}\, dx \right)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{m} - \frac{1}{m^2}\left[\operatorname{sen}(mx)\right]_0^{\pi} \right) = \frac{1}{m}, \quad m \in \mathbb{N}$$

b) Como f não é contínua em $x=0$, temos que

$$S(0) = \frac{f(0^+) + f(0^-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$$



c) $$S(x) = \frac{\pi}{4} + \sum_{m=1}^{\infty} \left[\frac{2}{\pi(2m-1)^2} \cos((2m-1)x) + \frac{1}{m} \operatorname{sen}(mx) \right]$$

$$S(0) = \frac{\pi}{4} + \sum_{m=1}^{\infty} \frac{2}{\pi(2m-1)^2} \Leftrightarrow \frac{\pi}{2} - \frac{\pi}{4} = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{2}{(2m-1)^2} \Leftrightarrow \pi^2 = \sum_{m=1}^{\infty} \frac{8}{(2m-1)^2}$$

4 $$f(x,y) = \frac{3x^2}{1 - x^2 - y^2}$$

a) $$D_f = \{(x,y) \in \mathbb{R}^2 : 1 - x^2 - y^2 \neq 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$$

$$C_1 = \{(x,y) \in D_f : \frac{3x^2}{1-x^2-y^2} = 1\}$$

$$\frac{3x^2}{1 - x^2 - y^2} = 1 \Leftrightarrow 3x^2 = 1 - x^2 - y^2$$
$$\Leftrightarrow 4x^2 = 1 - y^2$$
$$\Leftrightarrow \underbrace{4x^2 + y^2 = 1 \wedge x^2 + y^2 \neq 1}_{\text{elipse}}$$

b) $f'_x(x,y) = \dfrac{(3x^2)'_x(1-x^2-y^2) - (3x^2)(1-x^2-y^2)'}{(1-x^2-y^2)^2}$

$= \dfrac{6x(1-x^2-y^2) - (3x^2)(-2x)}{(1-x^2-y^2)^2} = \dfrac{6x(1-y^2)}{(1-x^2-y^2)^2}$

$f'_y(x,y) = \dfrac{(3x^2)'_x(1-x^2-y^2) - (3x^2)(1-x^2-y^2)'}{(1-x^2-y^2)^2}$

$= \dfrac{-3x^2(-2y)}{(1-x^2-y^2)^2} = \dfrac{6yx^2}{(1-x^2-y^2)^2}$

c) Como f, f'_x e f'_y são contínuas em D, então f é diferenciável nesse conjunto

Agora calculando a equação do plano tangente no ponto $(1;1;-3)$

$z = f(1;1) + \nabla f(1;1)\cdot(x-1;y-1)$

$= -3 + 0\cdot(x-1) + 6(y-1)$

$= -3 + 6y - 6$

Logo, $6y - z = 9$

d) Sendo f diferenciável no seu domínio

$D_{\vec{v}} f(1;1) = \nabla f(1;1)(v_1;v_2)$

$\Leftrightarrow 0 = (0;6)\cdot(v_1;v_2)$

$\Leftrightarrow v_2 = 0$

Sendo o vetor unitário $\|\vec{v}\| = 1 \Leftrightarrow \sqrt{v_1^2+0} = 1 \Leftrightarrow v_1 = 0$

$\therefore$ Logo, $\vec{v} = (1;0)$

5 $|a_m x^m| = |a_m||x|^m \leq |a_m| b^m$

Como $\sum_{m=0}^{\infty} a_m b^m$ é absolutamente convergente então $\sum_{m=0}^{\infty} |a_m| b^m$ converge e podemos provar assim pelo critério de Weierstrass que $\sum_{m=0}^{\infty} a_m x^m$ é uniformemente convergente em $[-b;b]$.

Teste 1 2018/2019

1 $\sum_{m=1}^{\infty} \dfrac{3^m}{m\,2^{m+2}}(x-1)^m \quad \left(c = 1;\ a_m = \dfrac{3^m}{m\,2^{m+2}}\right)$

$$\lim_{n\to+\infty} \frac{\dfrac{\cancel{3^{n+1}}}{(n+1)\cancel{2^{n+3}}}}{\dfrac{\cancel{3^{n}}}{n\,\cancel{2^{n+2}}}} = \lim_{n\to+\infty} \frac{\dfrac{3}{(n+1)\times 2^3}}{\dfrac{1}{n\times 2^2}} = \lim_{n\to+\infty} \frac{3\times 4n}{8n+8} = \lim_{n\to+\infty} \frac{12n}{8n+8} = \lim_{n\to+\infty} \frac{12n}{8n} = \frac{3}{2}$$

Logo, $R = \frac{2}{3} \Rightarrow I = \left]\frac{1}{3}; \frac{5}{3}\right[$

Para $x = \frac{1}{3}$

$$\sum_{n=0}^{\infty} \frac{3^n}{n\,2^{n+2}}\left(\frac{1}{3}-1\right)^n = \sum_{n=0}^{\infty} \frac{3^n}{n\,2^{n+2}}\left(-\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{\cancel{2^n}}{\cancel{3^n}} \times \frac{\cancel{3^n}}{n\times \cancel{2^{n+2}}} \times (-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{4n} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

Simplesmente divergente

Para $x = \frac{5}{3}$

$$\sum_{n=1}^{\infty} \frac{3^n}{n\,2^{n+2}}\left(\frac{5}{3}-1\right)^n = \sum_{n=0}^{\infty} \frac{3^n}{n\,2^{n+2}}\left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{4n} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{n} \to \text{divergente}$$

Logo, $I = \left[\frac{1}{3}; \frac{5}{3}\right[$

2) $f(x) = x^3 e^{-x^2}$

a)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Leftrightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \Leftrightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \Leftrightarrow x^3 e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+3}$$

Sendo f contínua e diferenciável no seu domínio

$$f'(x) = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+3}\right)' = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+3)}{n!} x^{2n+2}$$

b) Como a função f e f' são seccionalmente contínuas em $\mathbb{R}$ e f é seccionalmente diferenciável, $S(x) = f(x)$

$$a_n = \frac{x^{2n+3}}{n!} \qquad \frac{x^{2n}}{n}$$

$$\left(x^3 e^{-x^2}\right)' = \sum_{n=1}^{\infty} \frac{(-1)^n(2n+3)}{n!} x^{2n+2}$$

$$\Leftrightarrow 3x^2 e^{-x^2} + (x^3)\times(-2x)e^{-x^2} = \sum_{n=1}^{\infty} \frac{(-1)^n(2n+3)}{n!} x^{2n+2}$$

$$\Leftrightarrow e^{-x^2}(3x^2 - 2x^4) = \sum_{n=1}^{\infty} \frac{(-1)^n(2n+3)}{n!} x^{2n+2}$$

$$\frac{\dfrac{\cancel{x^{2n}}+3}{n!}}{\dfrac{\cancel{x^{2n}}}{n}} = \frac{n\times x^3}{n!} = 0$$

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}(2(m-1)+3)}{(m-1)!} x^{2(m-1)+2} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(2m+1)}{(m-1)!} x^{2m}$$

3

$$r(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

a)

$$r'(x) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3}\frac{1}{x^{\frac{2}{3}}} = \frac{1}{3}\frac{1}{\sqrt[3]{x^2}} \qquad r'''(x) = \frac{10}{27} x^{-\frac{8}{3}}$$

$$r''(x) = -\frac{2}{9} x^{-\frac{5}{3}}$$

$$\frac{3}{2}^{-\frac{8}{3}} = \left(\frac{2}{3}\right)^{\frac{8}{3}} = \sqrt[3]{\left(\frac{2}{3}\right)^8} = \frac{4}{9}\sqrt[3]{\frac{4}{9}} = \frac{40}{243}\sqrt[3]{\frac{4}{9}}$$

$$T_1^2 = r(1) + r'(1)(x-1) + \frac{r''(1)}{2!}(x-1)^2 + \frac{10}{27\times 3!}\theta^{-\frac{8}{3}}(x-1)^3$$

$$= 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{10}{27\times 3!}\theta^{-\frac{8}{3}}(x-1)^3$$

b)

$$E_{m0} \leq \left|\frac{10}{27\times 3!}\theta^{-\frac{8}{3}}\right| \leq \left|\frac{10}{27\times 3!}\times\left(\frac{3}{2}\right)^{-\frac{8}{3}}\right| \leq 0{,}01$$

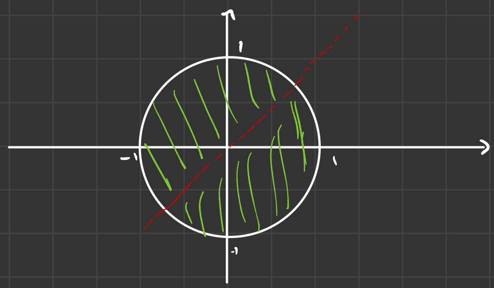
$$\theta \in \left[1; \frac{3}{2}\right]$$

4

$$f(x,y) = \frac{\sqrt{1-x^2-y^2}}{x-y}$$

a)

$$D_f = \{(x,y) \in \mathbb{R}^2 : 1-x^2-y^2 \geq 0 \wedge x-y \neq 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1 \wedge x \neq y\}$$



Circulo de centro $(0;0)$ e raio 1 com excessão dos pontos em $y = x$

b)

$$C_0 = \left\{(x,y) \in \mathbb{R}^2 : \frac{\sqrt{1-x^2-y^2}}{x-y} = 0\right\}$$
$$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1 \wedge y \neq x\}$$

$$\sqrt{1-x^2-y^2} = 0 \wedge y \neq x$$
$$\Leftrightarrow 1-x^2-y^2 = 0 \wedge y \neq x$$
$$\Leftrightarrow x^2+y^2 = 1 \wedge y \neq x$$

Circunferência de centro $(0;0)$ com excessão dos pontos $\left(\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right)$ e $\left(-\frac{\sqrt{2}}{2}; -\frac{\sqrt{2}}{2}\right)$

5 $g(x,y) = x + y - e^{xy}$

a) $g'_x(x,y) = 1 - y e^{xy}$ $\qquad$ $g'_y(x,y) = 1 - x e^{xy}$

Sendo $g'(x,y)$, $g'_x(x,y)$ e $g'_y(x,y)$ contínuas no seu domínio, podemos concluir que g é diferenciável

$\nabla g(0; \frac{1}{2}) = (\frac{1}{2}; 1)$

b) Sendo g diferenciável

$$z = g(0;\tfrac{1}{2}) + \nabla g(0;\tfrac{1}{2}) \cdot (x; y - \tfrac{1}{2})$$

$$\Leftrightarrow z = \frac{1}{2} + (\tfrac{1}{2}; 1) \cdot (x; y - \tfrac{1}{2})$$

$$\Leftrightarrow z = -\frac{1}{2} + \frac{x}{2} + y - \frac{1}{2} \Leftrightarrow x + 2y - 2z = 2$$

6 $L(x,y) = -20x^2 - 25y^2 - 20xy + 1000x + 900y - 7000$

$L'_x(x,y) = -40x - 20y + 1000$ $\qquad$ Nota: Este exercício é de extremos, já não sai

7

a) Falso, é absolutamente convergente pois é proveniente da primeira série só que com $x = 0$, que ainda pertence ao intervalo de convergência

b) Falso, se f e f' forem seccionalmente contínuas e se f apresentar uma descontinuidade num ponto x_0, então a soma da sua série de Fourier nesse ponto é dada por $\frac{f(x_0^+) + f(x_0^-)}{2}$

c) Não sai

Teste 1 2016/2017

1 $f(x,y) = xy e^{x-y}$

a) Não sai

b) $f'_x(x,y) = y(1+x)e^{x-y}$ $\qquad$ $f'_y(x,y) = x(1-y)e^{x-y}$

Sendo f, f'_x, f'_y contínuas em seu domínio, f é diferenciável
Por ser diferenciável

$$z = f(2;2) + \nabla f(2;2) \cdot (x-2; y-2)$$
$$\Leftrightarrow z = 4 + (6,-2)\cdot(x-2; y-2)$$
$$\Leftrightarrow z = 4 + 6x - 12 - 2y + 4$$
$$\Leftrightarrow z = -4 + 6x - 2y \Leftrightarrow 4 = 6x - 2y - z$$

c) $D_{\vec{u}} \nabla f(2;2) = \nabla f(2;2) \cdot \vec{u} = (6;-2)\cdot(u_1, u_2) = 6u_1 - 2u_2$

2 Não sai

O resto do teste já não sai

Teste 1 - 2017/2018

1 $\displaystyle\sum_{m=0}^{\infty} \frac{(x-2)^m}{4^m(3m+1)} = \sum_{m=0}^{\infty} \frac{1}{4^m(3m+1)}(x-2)^m \quad \left(c=2;\ a_m: \frac{1}{4^m(3m+1)}\right)$

$$\lim_{m\to+\infty} \frac{\frac{1}{4^{m+1}(3m+4)}}{\frac{1}{4^m(3m+1)}} = \lim_{m\to+\infty} \frac{3m+1}{3m+4} = \lim_{m\to+\infty} \frac{\cancel{3m}}{\cancel{3m}} = 1$$

Logo, $I =]1;3[$

Para $x = 1$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{4^m(3m+1)} \qquad |a_m| = \sum_{m=0}^{\infty} \frac{1}{4^m(3m+1)}$$

$$\lim_{m\to+\infty} \frac{\frac{1}{4^m(3m+1)}}{\frac{1}{\cancel{4^m}}} = \lim_{m\to+\infty} (3m+1) = +\infty$$

Logo, é divergente

Para $x = 3$

$$\sum_{n=0}^{\infty} \frac{1}{4^n(3n+1)} \qquad \lim_{n \to +\infty} \frac{\frac{1}{4^n(3n+1)}}{\frac{1}{4^n}} = \lim_{n \to +\infty} (3n+1) = +\infty$$

$\therefore$ Logo, o intervalo de convergência é $]1;3[$

2

a) $$\sum_{n=0}^{\infty} \frac{(-\pi^2)^n}{2^{4n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \times \frac{\pi^{2n}}{2^{4n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \times \frac{\left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$\therefore$ Logo, é verdadeira

b) Sendo $c = -1$ nesta série de potências o intervalo de convergência é $]-4;2[$. Como $\sum_{n=0}^{\infty} (-1)^n a_n$ provem de $\sum_{n=0}^{\infty} a_n (x+1)^n$ com $x = -2$ e este encontra-se no intervalo de convergência, concluimos que $\sum_{n=0}^{\infty} (-1)^n a_n$ é absolutamente convergente

3

a) $$\frac{2}{1+4x^2} = 2 \times \frac{1}{1+4x^2} = 2 \frac{1}{1-(-4x^2)} = 2 \sum_{n=0}^{\infty} (-4x^2)^n = 2 \sum_{n=0}^{\infty} (-1)^n (2x)^{2n}$$

b) $$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} x^{2n+1}$$

i) $f(x) = \operatorname{arctg}(2x)$ $\qquad (\operatorname{arctg}(2x))' = \dfrac{2}{1+4x^2}$

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{2n+1} x^{2n+1} \right)' = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} x^{2n} = (\operatorname{arctg}(2x))' = \frac{2}{1+4x^2} = \sum_{n=0}^{\infty} (-1)^n (2x)^{2n}$$

ii) $$f'(x) = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} x^{2n}$$
$$f''(x) = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} \times 2n \, x^{2n-1}$$

Logo, podemos concluir que $f^{(31)}(0) = 0$

4

a) $f(x) = \ln(x)$ $\qquad f(1) = 0$

$f'(x) = \dfrac{1}{x}$ $\qquad f'(1) = 1$

$f''(x) = -\dfrac{1}{x^2}$ $\qquad f''(1) = -1$

$f'''(x) = \dfrac{2}{x^3}$ $\qquad f'''(1) = 2$

$$T_1^2 = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(\theta)}{3!}(x-1)^3$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{6} \times \frac{1}{\theta^3}(x-1)^3$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{(x-1)^3}{3\theta^3}$$

b) $\ln(1,2) = \ln(x) \Leftrightarrow x = 1,2$

$$T_0^2 f(1,2) = -0,2 - \frac{1}{2}(-0,2)^2 = -0,2 - \frac{1}{2} \times 0,04 = -0,2 - 0,02 = -0,22$$

$$\text{Erro} \leq \frac{(1,2-1)^3}{3 \times 1,2^3} \leq 0,0015$$

5

Na serie de Fourier dos cossenos $b_m = 0 \rightarrow f(x) = x \ , \ [-\pi;\pi]$

$$\int x \cos(mx)\, dx = \frac{mx \operatorname{sen}(mx) + \cos(mx)}{m^2} + C$$ → fui ver à net

$$a_m = \frac{2}{\pi}\int_0^{\pi} x\cos(mx)\,dx = \frac{2}{\pi}\left[\frac{mx\operatorname{sen}(mx) + \cos(mx)}{m^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{1}{m^2} - \frac{1}{m^2}\right] = -\frac{4}{\pi m^2}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{2}}}{\cancel{2}}\right] = \pi$$

$$f(x) \sim \pi \sum_{m=0}^{\infty} \frac{-4}{\pi m^2}\cos(mx)$$

6

$$f(x,y) = \frac{1}{\ln(x^2+y^2)}$$

a) $D_f = \{(x,y) \in \mathbb{R}^2 : \ln(x^2+y^2) \neq 0 \ \wedge \ x^2+y^2 > 0\}$

$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 1 \ \wedge \ x^2+y^2 > 0\}$

Logo, são todos os pontos menos $(0;0)$ e aqueles em que $x^2+y^2=1$

b) $C_k = \{(x,y) \in \mathbb{R}^2 : \frac{1}{\ln(x^2+y^2)} = k\}$ $\qquad \frac{1}{\ln(x^2+y^2)} = k \Leftrightarrow \frac{1}{k} = \ln(x^2+y^2)$

$\Leftrightarrow x^2+y^2 = e^{\frac{1}{k}}$

vai ser uma circunferência de centro $(0;0)$ e raio $\sqrt{e^{\frac{1}{k}}}$

c) $$f'_x(x,y) = \frac{-\left(\ln(x^2+y^2)\right)'}{\ln(x^2+y^2)^2} = \frac{-\frac{2x}{x^2+y^2}}{\ln(x^2+y^2)^2}$$

$$f'_y(x,y) = \frac{\frac{-2y}{x^2+y^2}}{\ln(x^2+y^2)^2}$$

Teste 1 - 2018/2019 - Agrupamento 4

1

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(x-2)^n \quad \left(c=2; \; a_n = \frac{(-1)^n}{n3^n} \right)$$

$$\lim_{n\to+\infty} \frac{\frac{\cancel{(-1)}^{n+1}}{(n+1)3^{n+1}}}{\frac{\cancel{(-1)}^n}{n3^n}} = \lim_{n\to+\infty} \frac{(-1)(n3^{\cancel{n}})}{(n+1)\,\cancel{3}^{n+1}} = \lim_{n\to+\infty} \frac{-n}{3n+3} = \lim_{n\to+\infty} \frac{-\cancel{n}}{3\cancel{n}} = -\frac{1}{3}$$

$R = 3$

$I =]-1;5[$

Para $x=-1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(-3)^n = \sum_{n=1}^{\infty} \frac{\cancel{3^n}}{n\cancel{3^n}} = \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \text{divergente}$$

Para $x=5$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n\cancel{3^n}}\cancel{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$ sendo $|a_n| = \frac{1}{n} \rightarrow$ div

Pelo critério de Leibniz podemos concluir que é simplesmente convergente

$\therefore I =]-1;5]$

2

$g(x) = \ln(x+1), \; x \in]-1;+\infty[$

$g'(x) = \frac{1}{x+1}$

$g''(x) = -\frac{1}{(x+1)^2}$

$g'''(x) = \frac{2}{(x+1)^3}$

$$T_0^2 g(x) = g(0) + g'(0)x + \frac{g''(0)x^2}{2!} + \frac{g'''(\theta)}{3!}x^3$$

$$= 0 + x - \frac{1}{2}x^2 + \frac{2}{6(\theta+1)^3}x^3 = x - \frac{1}{2}x^2 + \frac{2}{6(\theta+1)^3}x^3$$

$\ln(1,1) = \ln(x+1) \Leftrightarrow x+1 = 1,1 \Leftrightarrow x = 0,1$

$T_0^2 g(0,1) = 0,1 - \frac{1}{2} \times 0,1^2 = 0,095$

$$\text{Erro} \leq \frac{2}{6(\theta-1)^3}x^3 \leq \frac{1}{3} \times \frac{x^3}{(\theta-1)^3} \leq \frac{1}{3} \times (0,1)^3 \leq \frac{1}{3} \times 10^{-4}$$

3 $f(x) = \frac{1}{4+x^2}$

a) $\frac{1}{4+x^2} = \frac{1}{4} \times \frac{1}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{4} \times \frac{1}{1-\left(-\frac{x^2}{4}\right)} = \frac{1}{4}\sum_{m=0}^{\infty}\left(-\frac{x^2}{4}\right)^m = \sum_{m=0}^{\infty}\frac{(-1)^m}{4^{m+1}} x^{2m}$

b) $\left|\frac{(-1)^m}{4^{m+1}} x^{2m}\right| \leq \frac{1}{4^m} \rightarrow$ convergente

Logo, verificamos pelo critério de weierstrass que a série é uniformemente convergente, logo podemos utilizar a derivação termo a termo

$\operatorname{arctg}\left(\frac{x}{2}\right)' = \frac{\frac{1}{4}}{1+\frac{x^2}{4}} = \frac{1}{4+x^2}$

Logo, basta provar que $\left(\sum_{m=0}^{+\infty}\frac{(-1)^m}{4^m(4m+2)} x^{2m+1}\right)'$ é igual à serie obtida em a)

$= \sum_{m=0}^{+\infty}\frac{(-1)^m}{4^m\,(4(\cancel{2m+1}))}\,\cancel{2m+1}\; x^{2m}$

$= \sum_{m=0}^{+\infty}\frac{(-1)^m}{2\times 4^m} x^{2m} = //$ era suposto dar $\sum_{m=0}^{\infty}\frac{(-1)^m}{4^{m+1}} x^{2m}$, não sei o que fiz mal

4

a) Não sendo f par nem ímpar

$\int 2\cos(mx)\,dx = 2\int \cos(mx)\,dx = 2\,\frac{\operatorname{sen}(mx)}{m}$

$\int 2\operatorname{sen}(mx)\,dx = 2\int \operatorname{sen}(mx)\,dx = -2\,\frac{\cos(mx)}{m}$

$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{sen}(mx) = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\operatorname{sen}(mx)\,dx + \int_{0}^{\pi} f(x)\operatorname{sen}(mx)\,dx\right)$

$= \frac{1}{\pi}\left(\left[\frac{-2\cos(mx)}{m}\right]_{-\pi}^{0}\right) = \frac{1}{\pi}\left(-\frac{2}{m} - \frac{2}{m}\right) = \frac{1}{\pi}\times\left(-\frac{4}{m}\right) = \frac{-4}{\pi m}$

$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)\cos(mx)\,dx + \int_{0}^{\pi} f(x)\cos(mx)\,dx\right)$

$= \frac{1}{\pi}\left[\frac{2\operatorname{sen}(mx)}{m}\right]_{-\pi}^{0} = 0$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x) + \int_{0}^{\pi} f(x)\,dx\right) = \frac{1}{\pi}\left[2x\right]_{-\pi}^{0} = \frac{2\cancel{\pi}}{\cancel{\pi}} = 2$

Logo $f(x) \sim 1 + \sum_{n=1}^{\infty} \frac{-4}{\pi n} \operatorname{sen}(nx)$

b) Sendo f descontínua em $x=0$, $S(0) = \frac{f(0^+) + f(0^-)}{2} = \frac{2}{2} = 1$

Logo, $S(x) = \begin{cases} 2 & \text{se } -\pi < x < 0 \\ 0 & \text{se } 0 < x < \pi \\ 1 & \text{se } x = 0 \vee x = -\pi \end{cases}$ $\longrightarrow$ depois era só fazer o desenho

5 Sendo $S = \{(x,y) \in \mathbb{R}^2 : x = 0 \wedge y \neq 0\}$
$R = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \wedge y = x\}$

$$\lim_{\substack{x \to (0;0) \\ (x,y) \in S}} \frac{xy}{4x^2 + y^2} = 0 \qquad \lim_{\substack{x \to (0;0) \\ (x,y) \in R}} \frac{xy}{x^2+y^2} = \lim_{x \to (0;0)} \frac{x^2}{2x^2} = \frac{1}{2}$$

$\therefore$ Logo, como os dois limites anteriores são diferentes provamos que o limite não existe

6 $f(x,y) = xy^2 + x^2y$

a) $f'_x(x,y) = y^2 + 2yx$
$f'_y(x,y) = 2xy + x^2$

Sendo $f(x)$, f'_x e f'_y contínuas em todo o seu domínio, concluímos que f é diferenciável

b) Sendo f diferenciável

$$z = f(1;1) + \nabla f(1;1) \cdot (x-1; y-1)$$
$$\Leftrightarrow z = 2 + (3;3) \cdot (x-1; y-1)$$
$$\Leftrightarrow z = 2 + 3x - 3 + 3y - 3$$
$$\Leftrightarrow z = 2 + 3x + 3y - 6$$
$$\Leftrightarrow z = 3x + 3y - 4$$
$$\Leftrightarrow 3x + 3y - z = 4$$

c) $D_{\vec{u}} f(1;1) = 0$
$$\Leftrightarrow \nabla f(1;1) \cdot \vec{u} = 0$$
$$\Leftrightarrow 3u_1 + 3u_2 = 0$$
$$\Leftrightarrow u_1 + u_2 = 0$$

Por exemplo $(-1;1)$

7

a) Como $a>1$, concluímos que $\sum_{m=1}^{+\infty} \frac{1}{m^x}$ é convergente e por isso, segundo o critério de Weierstrass é uniformemente convergente

b) $$g'(x) = \left(\sum_{m=1}^{\infty} \frac{1}{m^x}\right)' = \sum_{m=1}^{\infty} -\frac{\ln(m)}{m^x}$$

$$g'(4) = \sum_{m=1}^{\infty} -\frac{\ln(m)}{m^4}$$

Teste 1 - 2017/2018 - Agrupamento 4

1

a) $$\sum_{m=0}^{\infty} \frac{4^m}{m+1} (x-1)^m \quad \left(c=1 \, ; \; a_m = \frac{4^m}{m+1} \right)$$

$$\lim_{m\to+\infty} \frac{\frac{\cancel{4}^{m+1}}{m+2}}{\frac{\cancel{4^m}}{m+1}} = \lim_{m\to+\infty} \frac{4(m+1)}{m+2} = \lim_{m\to+\infty} \frac{4\cancel{m}}{\cancel{m}} = 4$$

Logo, $R = \frac{1}{4}$

b) Como $R = \frac{1}{4}$ $\quad I = \left]\frac{3}{4} ; \frac{5}{4}\right[$

Para $x = \frac{3}{4}$

$$\sum_{m=0}^{\infty} \frac{4^m}{m+1} \left(-\frac{1}{4}\right)^m = \sum_{m=0}^{\infty} \frac{(-1)^m \times \cancel{4^m}}{m+1} \times \frac{1}{\cancel{4^m}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m+1} \qquad a_m = \left|\frac{1}{m+1}\right| \leq \frac{1}{m} \to \text{div}$$

Pelo critério de Leibniz concluímos que é simplesmente convergente

Para $x = \frac{5}{4}$

$$\sum_{m=0}^{\infty} \frac{(\cancel{4})^m}{m+1} \times \left(\frac{1}{\cancel{4}}\right)^m = \sum_{m=0}^{\infty} \frac{1}{m+1} \longrightarrow \text{divergente}$$

$\therefore$ Logo o intervalo de convergência é $I = \left[\frac{3}{4} ; \frac{5}{4}\right[$

2 $\quad f(x) = e^{-x}, \; x \in \mathbb{R}$

a) $$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} \Leftrightarrow e^{-x} = \sum_{m=0}^{\infty} (-1)^m \times \frac{x^m}{m!}$$

$$T_0^m f(x) = \sum_{m=0}^{\infty} (-1)^m \times \frac{x^m}{m!} + \frac{(-1)^m e^{-\theta}}{(m+1)!} x^{m+1}$$

b) $f(x) = \frac{1}{\sqrt{e}} \Leftrightarrow e^{-x} = e^{-\frac{1}{2}} \Leftrightarrow -x = -\frac{1}{2} \Leftrightarrow x = \frac{1}{2}$

$$T_0^2 f\left(\tfrac{1}{2}\right) = 1 - \frac{1}{2} + \frac{\frac{1}{4}}{2} = 1 - \frac{1}{2} + \frac{1}{8} = \frac{1}{2} + \frac{1}{8} = \frac{4}{8} + \frac{1}{8} = \frac{5}{8}$$

$$\mathcal{E}_{mo} \leq \left| \frac{(-1)^m e^{\theta} |x|^{m+1}}{(m+1)!} \right| \leq \left| \frac{\left(\frac{1}{2}\right)^3}{3!} \right| \leq \frac{1}{48}$$

3

a) $(\ln(1+x))' = \frac{1}{1+x}$

Sendo $\sum_{m=0}^{+\infty} (-1)^m \frac{x^{m+1}}{m+1}$ uniformemente convergente

$$\left(\sum_{m=0}^{\infty} (-1)^m \frac{x^{m+1}}{m+1} \right)' = \sum_{m=0}^{\infty} (-1)^m x^m = \frac{1}{1-(-x)} = \sum_{m=0}^{\infty} (-x)^m = \sum_{m=0}^{\infty} (-1)^m x^m$$

b) $\ln(1+x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{m+1}}{m+1}$

$$\Leftrightarrow \ln(1+x^2) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+2}}{m+1}$$

$$\Leftrightarrow x^3 \ln(1+x^2) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+5}}{m+1}$$

4

$g(x) = \pi - 2|x| \quad , -\pi \leq x \leq \pi$

a) Sendo g uma função par: $b_m = 0$

$$\int (\pi - 2x) \cos(mx)\, dx = \frac{(\pi-2x)\,\mathrm{sen}(mx)}{m} + 2\int \frac{\mathrm{sen}(mx)}{m} = \frac{(\pi-2x)\,\mathrm{sen}(mx)}{m} + 2\cos(mx)$$

$u = \pi - 2x \quad u' = -2$

$v = \frac{\mathrm{sen}(mx)}{m} \quad v' = \cos(mx)$

$$a_m = \frac{2}{\pi} \int_0^{\pi} (2\pi - 2x) \cos(mx)\, dx = \frac{1}{\pi} \left[\frac{(\pi-2x)\,\mathrm{sen}(mx)}{m} - 2\cos(mx) \right]_0^{\pi} = \frac{2}{\pi}(2+2) = \frac{8}{\pi}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (2\pi - 2x)\, dx = \frac{2}{\pi} \left[2\pi x - x^2 \right]_0^{\pi} = \frac{2}{\pi}\left((2\pi^2 - \pi^2) \right) = 2\pi$$

$$g(x) \sim \frac{\cancel{2}\pi}{\cancel{2}} \sum_{m=1}^{+\infty} \frac{8}{\pi} \cos(mx) \sim \sum_{m=1}^{+\infty} \frac{8}{\pi} \cos(mx)$$

b) Como g é contínua em $\mathbb{R}$ e seccionalmente diferenciável em $\mathbb{R}$, pelo Teorema de Dirichlet, a Série de Fourier de g converge pontualmente para $g(x)$

c) Usar o teorema de Weierstrass na de cima

5

a) $D_f = \{(x,y) \in \mathbb{R}^2 : x^4 + y^2 \neq 0\}$
$= \mathbb{R}^2 \setminus \{(0;0)\}$

b) $C_k \{(x,y) \in \mathbb{R}^2 : \frac{x^2 y}{x^4+y^2} = k\}$

$\frac{x^2 y}{x^4 + y^2} = k$

$\Leftrightarrow x^2 y = kx^4 + ky^2$
$\Leftrightarrow kx^4 + ky^2 - x^2 y = 0$

Para $k = 0$

$x^2 \times y = 0 \rightarrow$ são os dois eixos (Ox e Oy)

Para $k = \frac{1}{2}$

$\frac{1}{2}x^4 + \frac{1}{2}y^2 - x^2 y = 0$
$\Leftrightarrow \cancel{x^2 y}\left(\frac{1}{2}x^2 + \frac{1}{2}y\right) = \cancel{x^2 y}$
$\Leftrightarrow y = x^2 \rightarrow$ parábola

c) Sendo : $S = \{(x,y) \in \mathbb{R}^2 : x = 0 \wedge y \neq 0\}$ e $R : \{(x,y) \in \mathbb{R}^2 : x = y \wedge y \neq 0\}$

$\lim_{\substack{x \to (0,0) \\ (x,y) \in S}} \left(\frac{x^2 y}{x^4+y^2}\right) = 0$

$\lim_{\substack{x \to (0;0) \\ (x,y) \in R}} \left(\frac{x^2 y}{x^4+y^2}\right) = \lim_{x \to (0;0)} \left(\frac{y^3}{y^4+y^2}\right) = \lim_{x \to (0;0)} \frac{1}{0} = +\infty$

$\therefore$ Logo, como os dois limites são diferentes, provamos que o limite não existe

Teste 1 - 2016/2017 - Agrupamento 4

1 $f(x,y,z) = \sqrt{xy^2 z^3}$

a) $D_f = \{(x,y,z) \in \mathbb{R}^3 : xz^3 \geq 0\}$
$= \{(x,y,z) \in \mathbb{R}^3 : (x \geq 0 \wedge z \geq 0) \vee (x \leq 0 \wedge z \leq 0)\}$

$f'_x(x,y,z) = \frac{y^2 z^3}{2\sqrt{xy^2 z^3}}$ $\quad f'_y(x,y,z) = \frac{x 2y z^3}{2\sqrt{xy^2 z^3}}$ $\quad f'_z(x,y,z) = \frac{xy^2 3z^2}{2\sqrt{xy^2 z^3}}$

Sendo f, f'_x, f'_y e f'_z contínuas no seu domínio, podemos concluir que f é diferenciável

b) Não sai

c) Sendo f diferenciável

$f(2,2,2) = \sqrt{2\times 2^2\times 2^3} = \sqrt{2^6} = 2^3 = 8$, logo, pertence à superfície de nível 8

$$\nabla f(2,2,2)\cdot(x-2, y-2; z-2)=0$$

$$\Leftrightarrow (2;4;6)\cdot(x-2;y-2;z-2)=0$$

$$\Leftrightarrow 2x-4+4y-8+6z-12=0$$

$$\Leftrightarrow 2x+4y+6z-24=0$$

(2) A partir daqui já não sai nada

2º Teste

Teste 2 2020/2021

(1) $g(x,y) = x^3+y^3-3xy$

$$\frac{dg}{dx}(x,y) = \frac{d}{dx}(x^3+y^3-3xy) = 3x^2-3y \qquad \frac{dg}{dy} = \frac{d}{dy}(x^3+y^3-3xy) = 3y^2-3x$$

$$\begin{cases}\frac{dg}{dx}=0\\ \frac{dg}{dy}=0\end{cases} \Leftrightarrow \begin{cases}3x^2-3y=0\\ 3y^2-3x=0\end{cases} \Leftrightarrow \begin{cases}3y=3x^2\\ 3y^2-3x=0\end{cases} \Leftrightarrow \begin{cases}y=x^2\\ 3x^4-3x=0\end{cases} \Leftrightarrow \begin{cases}\text{---}\\ x(x^3-1)=0\end{cases} \Leftrightarrow \begin{cases}y=0\\ x=0\end{cases} \vee \begin{cases}y=x^2\\ x^3-1=0\end{cases} \Leftrightarrow \begin{cases}y=0\\ x=0\end{cases} \vee \begin{cases}y=1\\ x=1\end{cases}$$

Logo, os pontos críticos são $(0;0)$ e $(1;1)$

$$\frac{dg^2}{dx^2} = \frac{d}{dx}(3x^2-3y) = 6x \qquad \frac{d^2g}{dy^2} = \frac{d}{dy}(3y^2-3x) = 6y$$

$$\frac{d^2g}{dxdy}(x,y) = \frac{d^2g}{dxdy}(x,y) = \frac{d}{dx}(3y^2-3x) = -3$$

Logo, $Hg(0;0) = \begin{bmatrix}0 & -3\\ -3 & 0\end{bmatrix}$ $\quad \det(Hg(0;0)) = -9 < 0$

Logo, $(0;0)$ é um ponto de sela

$Hg(1;1) = \begin{bmatrix}6 & -3\\ -3 & 6\end{bmatrix}$ $\quad \det(Hg(1;1)) = 36-9 = 27>0$

$\frac{d^2g}{dx^2}(1;1) = 6 > 0$, $(1;1)$ é um minimizante local de g

2. $f(x,y) = x^2 - y^2 + x^3$

Sendo f contínua em todo o seu domínio e a circunferência fechada e limitada, aplicando o teorema de Weierstrass podemos concluir que f tem extremos na circunferência

Considerando como $g(x,y) = x^2+y^2$

$\frac{df}{dx}(x,y) = 2x + 3x^2$ $\qquad$ $\frac{dg}{dx}(x,y) = 2x$

$\frac{df}{dy}(x,y) = -2y$ $\qquad$ $\frac{dg}{dy}(x,y) = 2y$

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases} \Leftrightarrow \begin{cases} (2x+3x^2; -2y) = \lambda(2x; 2y) \\ x^2+y^2=1 \end{cases} \Leftrightarrow \begin{cases} 2x+3x^2 = 2x\lambda \\ -2y = \lambda 2y \\ x^2+y^2=1 \end{cases} \Leftrightarrow \begin{cases} 2x+3x^2 = 2x\lambda \\ 2\lambda y + 2y = 0 \\ x^2+y^2=1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{---} \\ 2(\lambda+1)y = 0 \\ \text{---} \end{cases} \Leftrightarrow \begin{cases} 2x+3x^2 = 2x\lambda \\ y=0 \\ x^2=1 \end{cases} \vee \begin{cases} 2x+3x^2 = -2x \\ \lambda = -1 \\ x^2+y^2=1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda = \frac{5}{2} \\ y=0 \\ x=1 \end{cases} \vee \begin{cases} \lambda = -\frac{1}{2} \\ y=0 \\ x=-1 \end{cases} \vee \begin{cases} x(4+3x) = 0 \\ \text{---} \\ \text{---} \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda = \frac{5}{2} \\ y=0 \\ x=1 \end{cases} \vee \begin{cases} \lambda = -\frac{1}{2} \\ y=0 \\ x=-1 \end{cases} \vee \begin{cases} x=0 \\ \lambda=-1 \\ y=1 \end{cases} \vee \begin{cases} x=0 \\ \lambda=-1 \\ y=-1 \end{cases} \vee \begin{cases} x = -\frac{4}{3} \\ \lambda = -1 \\ y^2 = \cancel{-\frac{9}{25}} \end{cases}$$ impossível

Logo, os pontos candidatos são $(1;0)$; $(-1;0)$; $(0;1)$; $(0;-1)$

$f(1;0) = 2$ $\qquad$ $f(-1;0) = 0$ $\qquad$ $f(0;1) = f(0;-1) = -1$

Logo, $f(1;0)$ é o máximo de f em g e $f(0;-1) = f(0;1)$ é o mínimo de f em g

3.

a) $y' = xe^{x^2-y}$

$\Leftrightarrow y' = x e^{x^2} e^{-y}$

$\Leftrightarrow e^y\, dy = xe^{x^2} dx$

$\Leftrightarrow \int e^y\, dy = \int xe^{x^2} dx$

$\Leftrightarrow e^y = \frac{1}{2}\int 2xe^{x^2} dx$

$\Leftrightarrow e^y = \frac{1}{2} e^{x^2} + C$

$\Leftrightarrow y = \ln\left|\frac{e^{x^2}}{2} + C\right|$

b) $2ye^{2x} + (e^{2x} - y)\frac{dy}{dx} = 0$

$\Leftrightarrow 2y\,e^{2x} + (e^{2x} - y)\,y' = 0$

$\Leftrightarrow (e^{2x} - y)\,y' + 2ye^{2x}\,y = 0$

$\Leftrightarrow (e^{2x} - y)\,dy + (2ye^{2x})\,dx = 0$

Sendo $M(x,y) = 2ye^{2x}$ $\quad N(x,y) = e^{2x} - y$

$\frac{dM}{dx}(x,y) = \frac{dN}{dy}(x,y) = 2e^{2x}$

Logo, é exata

$$\begin{cases} \frac{dF}{dx}(x,y) = M(x,y) \\ \frac{dF}{dy}(x,y) = N(x,y) \end{cases} \Leftrightarrow \begin{cases} \frac{dF}{dx}(x,y) = 2ye^{2x} \\ \frac{dF}{dy}(x,y) = e^{2x} - y \end{cases}$$

$\frac{dF}{dx}(x,y) = 2ye^{2x} \Leftrightarrow F(x,y) = \int 2ye^{2x}\,dx \Leftrightarrow F(x,y) = y\,e^{2x} + c'(y)$

$(ye^{2x})'_y = e^{2x} + c(y)$

$e^{2x} - y = e^{2x} + c'(y) \Leftrightarrow c'(y) = -y \Leftrightarrow c(y) = -\frac{y^2}{2}$

$\therefore$ Logo, $F(x,y) = ye^{2x} - \frac{y^2}{2}$ e $ye^{2x} - \frac{y^2}{2} = c$

4

$2xyy' = x^2 + 3y^2$, $x > 0$, $y < 0$

$F(\lambda x, \lambda y) = 2\lambda x \times \lambda y\, y' = \lambda^2 x^2 + 3\lambda^2 y^2$

$= 2xyy' = x^2 + y^2 \rightarrow$ Logo é homogemea

Efetuando a mudança $y = zx \Leftrightarrow y' = z'x + z$

$2x \times (zx)(z'x + z) = x^2 + 3z^2x^2$

$\Leftrightarrow 2z\cancel{x^2}(z'x + z) = (1 + 3z^2)\cancel{x^2}$

$\Leftrightarrow 2z(z'x + z) = 1 + 3z^2$

$\Leftrightarrow z'x + z = \frac{1}{2z} + \frac{3}{2}z$

$\Leftrightarrow z'x = \frac{1}{2z} + \frac{1}{2}z$

$z = \frac{y}{x}$

$\Leftrightarrow \int \frac{2z}{1 + z^2}\,dz = \int \frac{1}{x}\,dx$

$\Leftrightarrow \ln(1 + z^2) = \ln(x) + c$

$\Leftrightarrow 1 + z^2 = x + e^c$

$\Leftrightarrow 1 + \frac{y^2}{x^2} = x + e^c$

$\Leftrightarrow \frac{y^2}{x^2} = x - 1 + c \Leftrightarrow y^2 = x^3 - x^2 + x^2 c$

$\Leftrightarrow y^2 = x^2(x - 1 + c)$

$\Leftrightarrow y = -x\sqrt{(x - 1 + c)}$

pois $y < 0 \wedge x > 0$

5

$y'' + 5y' + 4y = e^{-t}$

$y'' + 5y' + 4y = 0$

$\Leftrightarrow r^2 + 5r + 4 = 0$

$\Leftrightarrow r = -4 \vee r = -1$

Logo, $y_H = C_1 e^{-x} + C_2 e^{-4x}$

$e^{-t} = 1 \times e^{-t}$, logo é do tipo $P_m(t)e^{\alpha t}\cos(\beta t)$ com $m=0$, $\alpha=-1$ e $\beta=0$ e como temos que $\alpha+i\beta=-1$ é raiz do polinómio característico de multiplicidade 1 temos que $y_p(t)$ é da forma $y_p = te^{-t}A$, $A \in \mathbb{R}$

$$y'_p = A(e^{-t} - te^{-t}) = Ae^{-t}(1-t)$$
$$y''_p = A(-e^{-t}(1-t) + e^{-t}) = A\,e^{-t}(t-2)$$

Substituindo temos que

$$Ae^{-t}(t-2) + 5Ae^{-t}(1-t) + 4A\,e^{-t}t = e^{-t}$$
$$\Leftrightarrow A(t-2) + 5A(1-t) + 4At = 1$$
$$\Leftrightarrow A = \frac{1}{3}$$

Logo, $y_p(t) = \frac{1}{3}te^{-t}$

Sendo a solução geral da EDO dada por

$$y = C_1e^{-t} + C_2e^{-4t} + \frac{1}{3}te^{-t}$$

$$\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases} \Leftrightarrow \begin{cases} C_1+C_2=0 \\ -C_1-4C_2+\frac{1}{3}=0 \end{cases} \Leftrightarrow \begin{cases} C_1=-C_2 \\ -3C_2+\frac{1}{3}=0 \end{cases} \Leftrightarrow \begin{cases} C_1=-\frac{1}{9} \\ C_2=\frac{1}{9} \end{cases}$$

$\therefore$ A solução do PVI é $y = -\frac{1}{9}e^{-t} + \frac{1}{9}e^{-4t} + \frac{1}{3}te^{-t}$

b) $y'' + 5y' + 4y = e^{-t}$

$$\Leftrightarrow \mathcal{L}\{y'' + 5y' + 4y\} = \mathcal{L}\{e^{-t}\}$$
$$\Leftrightarrow \mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \frac{1}{s+1}$$
$$\Leftrightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 5(s\mathcal{L}\{y\} - y(0)) + 4\mathcal{L}\{y\}(s) = \frac{1}{s+1}$$
$$\Leftrightarrow s^2\mathcal{L}\{y\} + 5s\mathcal{L}\{y\} + 4\mathcal{L}\{y\} = \frac{1}{s+1}$$
$$\Leftrightarrow (s^2+5s+4)\mathcal{L}\{y\}(s) = \frac{1}{s+1}$$
$$\Leftrightarrow \mathcal{L}\{y\}(s) = \frac{1}{(s+1)(s^2+5s+4)}$$

$$\frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+4)}$$

$$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+5s+4)}\right\}$$

$$\Leftrightarrow A = -\frac{1}{9} \wedge B = \frac{1}{3} \wedge C = \frac{1}{9}$$

$$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2(s+4)}\right\}$$

$$\Leftrightarrow y = -\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\Leftrightarrow y = -\frac{1}{9}e^{-t} + \frac{1}{3}te^{-t} + \frac{1}{9}e^{-4t}$$

6 $y = y_1 + \frac{1}{z} \Leftrightarrow y' = y_1' - z^{-2} z'$

$$y' = p(x) + q(x) y + r(x) y^2$$

$$\Leftrightarrow y_1' - z^{-2} z' = p(x) + q(x)\left(y_1 + \frac{1}{z}\right) + r(x)\left(y_1 + \frac{1}{z}\right)^2$$

$$\Leftrightarrow y_1' - \frac{z'}{z^2} = p(x) + q(x) y_1 + \frac{q(x)}{z} + r(x)\left(y_1^2 + \frac{2y_1}{z} + \frac{1}{z^2}\right)$$

$$\Leftrightarrow y_1' - \frac{z'}{z^2} = p(x) + q(x) y_1 + \frac{q(x)}{z} + r(x) y_1^2 + \frac{2r(x) y_1}{z} + \frac{r(x)}{z^2}$$

$$\Leftrightarrow -\frac{z'}{z^2} = \frac{q(x)}{z} + \frac{2r(x) y_1}{z} + \frac{r(x)}{z^2} \Leftrightarrow z' = -q(x) z - 2r(x) z y_1 - r(x)$$

equação linear

Teste 2 2019/2020

1 $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$

a)

$$\frac{df}{dx}(x,y) = \frac{d}{dx}\left(2x^3 + xy^2 + 5x^2 + y^2\right) = 6x^2 + y^2 + 10x$$

$$\frac{df}{dy}(x,y) = \frac{d}{dy}\left(2x^3 + xy^2 + 5x^2 + y^2\right) = 2xy + 2y$$

$$\begin{cases} 6x^2 + y^2 + 10x = 0 \\ 2xy + 2y = 0 \end{cases} \Leftrightarrow \begin{cases} 6x^2 + y^2 + 10x = 0 \\ 2y(x+1) = 0 \end{cases} \Leftrightarrow \begin{cases} 6x^2 + 10x = 0 \\ y = 0 \end{cases} \vee \begin{cases} y^2 = 4 \\ x = -1 \end{cases} \Leftrightarrow \begin{cases} x(6x+10) = 0 \\ y = 0 \end{cases} \vee \begin{cases} y = 2 \\ x = -1 \end{cases} \vee \begin{cases} y = -2 \\ x = -1 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = \frac{-5}{3} \\ y = 0 \end{cases} \vee \begin{cases} y = 2 \\ x = -1 \end{cases} \vee \begin{cases} y = -2 \\ x = -1 \end{cases}$$

b) $\frac{df^2}{dx^2}(x,y) = 12x + 10$ $\quad \frac{d^2f}{dxy}(x,y) = \frac{d^2f}{dydx}(x,y) = 2y$ $\quad Hg(x,y) = \begin{bmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{bmatrix}$

$\frac{df^2}{dy^2}(x,y) = 2x + 2$

$Hg(0;0) = \begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad det(Hg(0;0)) = 20 > 0$

$\frac{df^2}{dx^2}(0;0) = 10 >$, Logo é um minimizante local

$Hg(-1;2) = \begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix}$ $\quad det(Hg(1;2)) = 0 - 16 = -16 < 0$, logo é um ponto de sela

$Hg(-\frac{5}{3};0) = \begin{bmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{bmatrix}$ $\quad det(Hg(-\frac{5}{3};0)) = \frac{40}{3}$ $\quad \frac{d^2f}{dx^2}\left(-\frac{5}{3};0\right) = -10$ → Logo é um maximizante local

c) Sendo $f(-3;0) = -9 < f(0;0)$ não é um mínimo global e como é um minimizante local, significa que não é o maior ponto, por isso não pode ser um máximo global

2

a) Pelo teorema de Weierstrass, como h é uma função contínua e D um conjunto fechado e limitado, podemos concluir que h tem extremos globais em D

b) Considerando $g(x,y,z) = x^2 + y^2 + z^2$

$$\frac{dh}{dx} = 1 \qquad \frac{dg}{dx} = 2x$$
$$\frac{dh}{dy} = 1 \qquad \frac{dg}{dy} = 2y$$
$$\frac{dh}{dz} = 1 \qquad \frac{dg}{dz} = 2z$$

$$\begin{cases} \nabla h(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 1 \end{cases} \Leftrightarrow \begin{cases} (1,1,1) = (2x\lambda; 2y\lambda; 2z\lambda) \\ x^2+y^2+z^2 = 1 \end{cases} \Leftrightarrow \begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ 1 = 2\lambda z \\ x^2+y^2+z^2=1 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{1}{2\lambda} \\ z = \frac{1}{2\lambda} \\ x^2+y^2+z^2 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \text{———} \\ \text{———} \\ \text{———} \\ \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1 \end{cases} \Leftrightarrow \begin{cases} \text{———} \\ \text{———} \\ \text{———} \\ \frac{3}{4\lambda^2} = 1 \end{cases} \Leftrightarrow \begin{cases} x = \frac{\sqrt{3}}{3} \\ y = \frac{\sqrt{3}}{3} \\ z = \frac{\sqrt{3}}{3} \\ \lambda = \frac{\sqrt{3}}{2} \end{cases} \lor \begin{cases} x = -\frac{\sqrt{3}}{3} \\ y = -\frac{\sqrt{3}}{3} \\ z = -\frac{\sqrt{3}}{3} \\ \lambda = \frac{\sqrt{3}}{2} \end{cases}$$

$$h\left(\frac{\sqrt{3}}{3}; \frac{\sqrt{3}}{3}; \frac{\sqrt{3}}{3}\right) = \sqrt{3} \qquad h\left(-\frac{\sqrt{3}}{3}; -\frac{\sqrt{3}}{3}; -\frac{\sqrt{3}}{3}\right) = -\sqrt{3}$$

$\therefore$ Logo o máximo de h em D é $\sqrt{2}$

3

$$z = \frac{y}{x} \Leftrightarrow y = zx \Leftrightarrow y' = z'x + z$$

$$xy' = y + \frac{y^2}{x} \Leftrightarrow x(z'x + z) = zx + \frac{z^2x^2}{x} \Leftrightarrow x(z'x+z) = zx + z^2x \Leftrightarrow z'x^2 + zx = zx + z^2x \Leftrightarrow z'x + \cancel{z} = \cancel{z} + z^2$$

$$\Leftrightarrow \int \frac{1}{z^2}\,dz = \int \frac{1}{x}\,dx \Leftrightarrow \int z^{-2}\,dz = \int \frac{1}{x}\,dx \Leftrightarrow -\frac{1}{z} = \ln(x) + C$$

$$\Leftrightarrow z = -\frac{1}{\ln|x| + C}$$

$$\Leftrightarrow \frac{y}{x} = \frac{-1}{\ln|x| + C} \Leftrightarrow y = -\frac{x}{\ln|x| + C}$$

4 $(1-x)y'' + xy' - y = (1-x)^2, \ x>1$

a) $(1-x)y'' + xy' - y = 0$

Sendo $y = x$

$(1-x)(x)'' + x(x)' - x = 0 \Leftrightarrow 0 + x - x = 0$, Logo é solução

Sendo $y = e^x$

$(1-x)(e^x)'' + x \times (e^x)' - e^x = 0 \Leftrightarrow (1-x)e^x + xe^x - e^x = 0 \Leftrightarrow e^x - xe^x + xe^x - e^x = 0 \Leftrightarrow 0 = 0$

Logo, e^x também é solução da equação

b) Sendo x e e^x soluções da equação

$$\begin{bmatrix} x & e^x \\ 1 & e^x \end{bmatrix} \times \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1-x \end{bmatrix}$$

$$c_1'(x) = \frac{\begin{vmatrix} 0 & e^x \\ 1-x & e^x \end{vmatrix}}{\begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}} = \frac{-e^x(1-x)}{xe^x - e^x} = \frac{e^x(x-1)}{e^x(x-1)} = 1$$

$c_1'(x) = 1 \Leftrightarrow c_1(x) = x$

$$c_2'(x) = \frac{\begin{vmatrix} x & 0 \\ 1 & 1-x \end{vmatrix}}{\begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}} = \frac{x - x^2}{e^x(x-1)} = \frac{-x}{e^x}$$

$$c_2'(x) = \frac{-x}{e^x} \Leftrightarrow c_2(x) = -\int \frac{x}{e^x}\,dx = \frac{x}{e^x} + \frac{1}{e^x}$$

$$\therefore \ y_p = x^2 + \left(\frac{x}{e^x} + \frac{1}{e^x}\right)e^x = x^2 + x + 1$$

c) $y = c_1 x + c_2 e^x + x^2 + x + 1, \ c_1, c_2 \in \mathbb{R}$

5

$y = Ae^{-2x}$

$y' = -2Ae^{-2x}$

$y'' = 4Ae^{-2x}$

$y'' - y = e^{-2x}$

$\Leftrightarrow 4Ae^{-2x} - Ae^{-2x} = e^{-2x}$

$\Leftrightarrow 4A - A = 1 \Leftrightarrow A = \frac{1}{3}$

Logo, $y_p = \frac{1}{3}e^{-2x}$

$y'' - y = 0 \Leftrightarrow r^2 - 1 = 0 \Leftrightarrow r = 1 \vee r = -1$ Logo $y_H = C_1 e^{x} + C_2 e^{-x}$

$\therefore\ y = C_1 e^{x} + C_2 e^{-x} + \frac{1}{3} A e^{-2x},\ C_1, C_2 \in \mathbb{R}$

6

$$y'' - 2y' + y = 5e^{-t}$$

$\Leftrightarrow \mathcal{L}\{y'' - 2y' + y\} = \mathcal{L}\{5e^{-t}\}$

$\Leftrightarrow \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 5\mathcal{L}\{e^{-t}\}$

$\Leftrightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2(s\mathcal{L}\{y\} - y(0)) + \mathcal{L}\{y\} = \frac{5}{s+1}$

$\Leftrightarrow s^2\mathcal{L}\{y\} - 1 - 2s\mathcal{L}\{y\} + \mathcal{L}\{y\} = \frac{5}{s+1}$

$\Leftrightarrow (s^2 - 2s + 1)\mathcal{L}\{y\} = \frac{5 + s + 1}{s+1}$

$\Leftrightarrow (s^2 - 2s + 1)\mathcal{L}\{y\} = \frac{s+6}{s+1}$

$\frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$

$= \frac{A(s-1)^2 + B(s-1)(s+1) + C(s+1)}{(s+1)(s-1)^2}$

$\Leftrightarrow \mathcal{L}\{y\} = \frac{s+6}{(s+1)(s^2 - 2s + 1)}$

$\Leftrightarrow \begin{cases} A = \frac{5}{4} \\ C = \frac{7}{2} \\ B = -\frac{5}{4} \end{cases}$

$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{s+6}{(s+1)(s-1)^2}\right\}$

$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{\frac{5}{4}}{s+1} - \frac{\frac{5}{4}}{s-1} + \frac{\frac{7}{2}}{(s-1)^2}\right\}$

$\Leftrightarrow y = \frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{7}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$

$\Leftrightarrow y = \frac{5}{4}e^{-t} - \frac{5}{4}e^{t} + \frac{7}{2}te^{t}$

Teste 2 2017-2018

2 $f(x,y) = x^3 + y^2 - 2xy$

a)

$\frac{\partial f}{\partial x}(x,y) = 3x^2 - 2y \qquad \frac{\partial f}{\partial y}(x,y) = 2y - 2x$

$\begin{cases} 3x^2 - 2y = 0 \\ 2y - 2x = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 = 2y \\ 3x^2 - 2x = 0 \end{cases} \Leftrightarrow \begin{cases} \text{—} \\ x(3x-2) = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \end{cases} \vee \begin{cases} y = \frac{2}{3} \\ x = \frac{2}{3} \end{cases}$

$\therefore$ Logo, os pontos críticos são $(0;0)$ e $(\frac{2}{3};\frac{2}{3})$

b) $\frac{\partial^2 f}{\partial x^2}(x,y) = 6x \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = 2 \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = -2$

$$Hf(x,y) = \begin{bmatrix} 6x & -2 \\ -2 & 2 \end{bmatrix} \qquad H_f\left(\tfrac{2}{3};\tfrac{2}{3}\right) = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \qquad \det\left(Hf\left(\tfrac{2}{3};\tfrac{2}{3}\right)\right) = 8+4 = 12$$

$$\frac{d^2f}{dx}\left(\tfrac{2}{3};\tfrac{2}{3}\right) = 4$$

Logo, é um minimizante local de f

$$Hg(0;0) = \begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix} \qquad \det(Hf(0;0)) = -4 \rightarrow \text{pomto de sela}$$

c) $f\left(\frac{2}{3};\frac{2}{3}\right) = \frac{8}{27} + \frac{8}{27} - 2\times\frac{4}{9} = \frac{16}{27} - \frac{8}{9} = \frac{16}{27} - \frac{24}{27} = -\frac{8}{27}$

$f(-3;0) = -27 < f\left(\frac{2}{3};\frac{2}{3}\right)$, Logo mão é mimimmizante global

d) $\nabla f(1;0) = (3;-2)$

4 $y' = \frac{4}{x}y \Leftrightarrow \frac{1}{y}dy = \frac{4}{x}dx \Leftrightarrow \int \frac{1}{y}dy = 4\int \frac{1}{x}dx \Leftrightarrow \ln|y| = 4\ln|x| + C$

$\Leftrightarrow y = x^4 C,\ C \in \mathbb{R}$

5
$$y' + xy = -e^{x^2}y^3 \qquad z = y^{-2} \Leftrightarrow z' = -2y^{-3}y'$$
$$\Leftrightarrow y^{-3}y' + xy^{-2} = -e^{x^2} \qquad \Leftrightarrow -\frac{z'}{2} = y^{-3}y'$$
$$\Leftrightarrow -\frac{1}{2}z' + xz = -e^{x^2}$$
$$\Leftrightarrow z' - 2xz = 2e^{x^2} \qquad N(x) = e^{\int -2x\,dx} = e^{-x^2}$$
$$\Leftrightarrow (e^{-x^2}z)' = 2\cancel{e^{x^2}}\cancel{e^{-x^2}}$$
$$\Leftrightarrow e^{-x^2}z = \int 2\,dx$$
$$\Leftrightarrow z = \frac{2x+C}{e^{-x^2}} \Leftrightarrow \frac{1}{y^2} = \frac{2x+C}{e^{-x^2}} \Leftrightarrow y^2 = \frac{e^{-x^2}}{2x+C},\ C \in \mathbb{R}$$

6
$$\underbrace{x}_{M(x,y)}dx + \underbrace{(y+x^2)}_{N(x,y)}dy = 0$$

$$h(y) = \frac{\frac{dM}{dy}(x,y) - \frac{dN}{dx}(x,y)}{M(x,y)} = \frac{0-2x}{x} = -2$$

Logo, $N(y) = e^{\int 2\,dy} = e^{2y}$

Aplicando o fator integrante

$e^{2y}x\,dx + e^{2y}(y+x^2)\,dy = 0$

$$\Leftrightarrow \begin{cases} \frac{dF}{dx}(x,y) = M(x,y) \\ \frac{dF}{dy}(x,y) = N(x,y) \end{cases} \Leftrightarrow \begin{cases} \frac{dF}{dx}(x,y) = e^{2y}x \\ \frac{dF}{dy}(x,y) = e^{2y}(y+x^2) \end{cases}$$

$\frac{\partial F}{\partial x}(x,y) = e^{2y}x \Leftrightarrow F(x,y) = \int e^{2y}x\,dx \Leftrightarrow F(x,y) = \frac{e^{2y}x^2}{2} + c, c \in \mathbb{R}$

$\left(\frac{e^{2y}x^2}{2} + c(y)\right)'_y = x^2e^{2y} + c'(y)$

$x^2e^{2y} + c'(y) = e^{2y}(y+x^2) \Leftrightarrow c'(y) = e^{2y}(y+x^2) - x^2e^{2y} \Leftrightarrow c'(y) = e^{2y}(y+\cancel{x^2}-\cancel{x^2}) \Leftrightarrow c'(y) = e^{2y}y$

$\Leftrightarrow c(y) = \int e^{2y}y\,dy \Leftrightarrow c(y) = \frac{y}{2}e^{2y} - \frac{1}{4}e^{2y}$

$\therefore$ Logo a solução da equação é $\frac{x^2}{2}e^{2y} + \frac{y}{2}e^{2y} - \frac{1}{4}e^{2y} = C \Leftrightarrow e^{2y}\left(\frac{x^2}{2} + \frac{y}{2} - \frac{1}{4}\right) = C$, $C \in \mathbb{R}$

7

$y'' + 3y = 2$

$y'' + 3y = 0$

$\Leftrightarrow r^2 + 3 = 0$

$\Leftrightarrow r = \sqrt{3}i \vee r = -\sqrt{3}i$

Logo, $y_H = C_1\cos(\sqrt{3}x) + C_2\,\text{sen}(\sqrt{3}x)$, $C_1, C_2 \in \mathbb{R}$

2 é na forma $P_m(x)e^{\alpha t}\cos(\beta t)$, com $m=0$, $\alpha = 0$ e $\beta = 0$ e $P_m(x) \equiv 2$, logo a solução particular é da forma $y = A$

$A'' + 3A = 2 \Leftrightarrow A = \frac{2}{3}$

$y_p = \frac{2}{3}$

$\therefore\ y = C_1\cos(\sqrt{3}x) + C_2\,\text{sen}(\sqrt{3}x) + \frac{2}{3}$, $C_1, C_2 \in \mathbb{R}$

8

$y'' + 6y' + 9y = 0$

$\Leftrightarrow \mathcal{L}\{y'' + 6y' + 9y\}(s) = 0$

$\Leftrightarrow \mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = 0$

$\Leftrightarrow s^2\mathcal{L}\{y\}(s) - s\,y(0) - y'(0) + 6(s\mathcal{L}\{y\} - y(0)) + 9\mathcal{L}\{y\} = 0$

$\Leftrightarrow s^2\mathcal{L}\{y\} + s - \cancel{6} + 6s\mathcal{L}\{y\} + \cancel{6} + 9\mathcal{L}\{y\} = 0$

$\Leftrightarrow (s^2 + 6s + 9)\mathcal{L}\{y\} = -s$

$\Leftrightarrow \mathcal{L}\{y\} = \frac{-s}{s^2+6s+9} \Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{s}{s^2+6s+9}\right\} \Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{s}{(s+3)^2}\right\}$

$\Leftrightarrow y = -\mathcal{L}^{-1}\left\{\frac{s}{(s+3)^2}\right\} \Leftrightarrow y = -\mathcal{L}^{-1}\left\{\frac{s+3-3}{(s+3)^2}\right\}$

$\Leftrightarrow y = -\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \mathcal{L}^{-1}\left\{\frac{-3}{(s+3)^2}\right\}$

$\Leftrightarrow y = -e^{-3t} + 3te^{-3t} \Leftrightarrow y = e^{-3t}(-1+3t)$

Teste 2 - Agrupamento 3 - 2018/2019

1

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial g}{\partial x} = 2x$$
$$\frac{\partial f}{\partial y} = x \qquad \frac{\partial g}{\partial y} = 2y$$

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 8 \end{cases} \Leftrightarrow \begin{cases} (y,x) = \lambda(2x;2y) \\ x^2+y^2=8 \end{cases} \Leftrightarrow \begin{cases} y = 2\lambda x \\ x = \lambda 2y \\ x^2+y^2=8 \end{cases} \Leftrightarrow \begin{cases} y = 2\lambda(2\lambda y) \\ — \\ — \end{cases} \Leftrightarrow \begin{cases} y = 4\lambda^2 y \\ — \\ — \end{cases}$$

$$\begin{cases} \lambda = \sqrt{\frac{1}{4}} \\ x = 2\lambda y \\ x^2+y^2 = 8 \end{cases} \Leftrightarrow \begin{cases} \lambda = \frac{1}{2} \\ x = y \\ y^2 = 4 \end{cases} \vee \begin{cases} \lambda = -\frac{1}{2} \\ x = -y \\ y^2 = 4 \end{cases} \Leftrightarrow \begin{cases} \lambda = \frac{1}{2} \\ x = 2 \\ y = 2 \end{cases} \vee \begin{cases} \lambda = \frac{1}{2} \\ x = -2 \\ y = -2 \end{cases} \vee \begin{cases} \lambda = -\frac{1}{2} \\ x = -2 \\ y = 2 \end{cases} \vee \begin{cases} \lambda = -\frac{1}{2} \\ x = 2 \\ y = -2 \end{cases}$$

Logo, os pontos críticos são $(2;2)$, $(-2;-2)$, $(-2;2)$, $(2;-2)$

$f(2;2) = f(-2;-2) = 7 \rightarrow$ Max
$f(-2;2) = f(2;-2) = -1 \rightarrow$ Min

2

a) $y y' \sqrt{2+3x^2} = x$

$$\Leftrightarrow y y' = \frac{x}{\sqrt{2+3x^2}}$$

$$\Leftrightarrow \int y \, dy = \int x (2+3x^2)^{-\frac{1}{2}} dx$$

$$\Leftrightarrow \frac{y^2}{2} = \frac{1}{6} \int 6x (2+3x^2)^{-\frac{1}{2}} dx$$

$$\Leftrightarrow \frac{y^2}{2} = \frac{1}{3}\sqrt{2+3x^2} \Leftrightarrow y^2 = \frac{2}{3}\sqrt{2+3x^2} + C, \ C \in \mathbb{R}$$

b) $y' + \left(2+\frac{1}{x}\right)y = e^{-2x}$

$$\int 2 + \frac{1}{x} \, dx = 2x + \ln(x)$$
$$\mu(x) = e^{2x+\ln(x)} = e^{2x} x$$

$$\Leftrightarrow (e^{2x} x y)' = \cancel{e^{-2x}} \cancel{e^{2x}} x$$

$$\Leftrightarrow (e^{2x} x y)' = x$$

$$\Leftrightarrow e^{2x} \cancel{x} y = \frac{x^{\cancel{2}}}{2} \Leftrightarrow y = \frac{x}{2e^{2x}} + C, \ C \in \mathbb{R} \Leftrightarrow y = e^{-2x}\left(\frac{x}{2} + \frac{C}{x}\right)$$

c) $y\cos(x) + 2xe^y + (x^2e^y - 1 + \operatorname{sen}(x))y' = 0$

$$\Leftrightarrow y\cos(x) + 2xe^y + (x^2e^y - 1 + \operatorname{sen}(x))\frac{dy}{dx} = 0$$

$$\Leftrightarrow \underbrace{(y\cos(x) + 2xe^y)}_{M(x,y)}dx + \underbrace{(x^2e^y - 1 + \operatorname{sen}(x))}_{N(x,y)}dy = 0$$

$$\frac{dM}{dy}(x,y) = \frac{d}{dy}\left(y\cos(x) + 2xe^y\right) = \cos(x) + 2xe^y$$

$$\frac{dN}{dx}(x,y) = \frac{d}{dx}\left(x^2e^y - 1 + \text{sen}(x)\right) = 2xe^y + \cos(x)$$

Logo, é exata

$$\begin{cases} \frac{dF(x,y)}{dx} = M(x,y) \\ \frac{dF}{dy}(x,y) = N(x,y) \end{cases} \Leftrightarrow \begin{cases} \frac{dF(x,y)}{dx} = y\cos(x) + 2xe^y \\ \frac{dF}{dy}(x,y) = x^2e^y - 1 + \text{sen}(x) \end{cases}$$

$$\frac{dF}{dx}(x,y) = y\cos(x) + 2xe^y \Leftrightarrow F(x,y) = \int y\cos(x) + 2xe^y \, dx$$

$$\Leftrightarrow F(x,y) = y\,\text{sen}(x) + x^2e^y$$

$$\left(y\,\text{sen}(x) + x^2e^y\right)'_y = \text{sen}(x) + x^2e^y + c'(y)$$

$$\cancel{\text{sen}(x)} + \cancel{x^2e^y} + c'(y) = \cancel{x^2e^y} - 1 + \cancel{\text{sen}(x)}$$

$$\Leftrightarrow c'(y) = -1 \Leftrightarrow c(y) = -y$$

$\therefore$ Logo, a solução da EDO é $y\,\text{sen}(x) + x^2e^y - y$

3 $\quad y'' - 6y' + 9y = 2e^{3t}$

a) $\mathcal{L}\{y'' - 6y' + 9y\} = \mathcal{L}\{2e^{3t}\}$

$$\Leftrightarrow \mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = 2\mathcal{L}\{e^{3t}\}$$

$$\Leftrightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 6\left(s\mathcal{L}\{y\} - y(0)\right) + 9\mathcal{L}\{y\} = \frac{2}{s-3}, \quad s > 3$$

$$\Leftrightarrow s^2\mathcal{L}\{y\} - 1 - 6s\mathcal{L}\{y\} + 9\{y\} = \frac{2}{s-3}$$

$$\Leftrightarrow (s^2 - 6s + 9)\mathcal{L}\{y\} = \frac{2 + s - 3}{s-3}$$

$$\Leftrightarrow (s^2 - 6s + 9)\mathcal{L}\{y\} = \frac{s-1}{s-3}$$

$$\frac{A}{s-3} + \frac{B}{(s-3)^2} + \frac{C}{(s-3)^3}$$

$$\Leftrightarrow \mathcal{L}\{y\} = \frac{s-1}{(s-3)^3}$$

$$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-3)^3}\right\}$$

$$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{s-3+2}{(s-3)^3}\right\}$$

$$\Leftrightarrow y = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}(t) + \mathcal{L}^{-1}\left\{\frac{2}{(s-3)^3}\right\}$$

$$\Leftrightarrow y = e^{3t}t + e^{3t}t^2 \Leftrightarrow y = (t + t^2)e^{3t}$$

b) $y'' - 6y' + 9y = 2e^{3t}$

$y'' - 6y' + 9y = 0$

$\Leftrightarrow n^2 - 6n + 9 = 0$

$\Leftrightarrow (n-3)^2 = 0 \Leftrightarrow n = 3$ $\longrightarrow$ multiplicidade 2

$y_H = C_1 e^{3x} + C_2 x e^{3x}$, $C_1, C_2 \in \mathbb{R}$

$2e^{3t}$ é da forma $P_m(x) e^{\alpha t} \cos(\beta x)$ com $P_m(x) \equiv 2$, $m=0$ e $\alpha = 3$ e $\beta = 0$, logo a raiz do polinomio característico é $y = x^2 e^{3x} A$

$y' = A e^{3x}(3x^2 + 2x)$

$y'' = A e^{3x}(9x^2 + 12x + 2)$

$2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x} - 6(2Axe^{3x} + 3Ax^2e^{3x}) + 9Ax^2e^{3x} = 2e^{3x}$

$\Leftrightarrow 2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x} - 18Ax^2e^{3x} + 9Ax^2e^{3x} = 2e^{3x}$

$\Leftrightarrow 2Ae^{3x} = 2e^{3x} \Leftrightarrow Ae^{3x} - e^{3x} = 0 \Leftrightarrow e^{3x}(A-1) = 0 \Leftrightarrow A = 1$

$y = c_1 e^{3x} + c_2 x e^{3x} + x^2 e^{3x}$, $c_1, c_2 \in \mathbb{R}$

$\Leftrightarrow y' = 3c_1 e^{3x} + c_2(e^{3x} + 3xe^{3x}) + 2xe^{3x} + 3x^2e^{3x}$

$$\begin{cases} y'(0) = 1 \\ y(0) = 0 \end{cases} \Leftrightarrow \begin{cases} 3C_1 + C_2 = 1 \\ C_1 = 0 \end{cases} \Leftrightarrow \begin{cases} C_2 = 1 \\ C_1 = 0 \end{cases}$$

$\therefore y = xe^{3x} + x^2e^{3x} = (x + x^2)e^{3x}$

4

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s(s^2+4)} e^{-s}\right\} = \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{s}{s^2+4}\right)e^{-s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}e^{-s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}e^{-s}\right\}$$

$$= H_1(t) \times 1 - H_1(t)\cos(2(t-1)) = (1 - \cos(2t-2))H_1(t)$$

$$\frac{A}{s} + \frac{Bs + C}{s^2+4}$$

$$\begin{cases} A = 1 \\ B = -1 \\ C = 0 \end{cases}$$

5 $\quad y' = f(ax + by + c)$

a) $z = ax + by + c$, então $y = \frac{z - ax - c}{b}$ $\quad y' = \frac{z' - a}{b}$

$$\frac{z' - a}{b} = f(z) \Leftrightarrow \frac{z'}{b} - \frac{a}{b} = f(z) \Leftrightarrow \frac{z'}{b} = f(z) + \frac{a}{b} \Leftrightarrow z' = bf(z) + a \Leftrightarrow \frac{1}{bf(z) + a}\, dz = \frac{1}{x}\, dx$$

b)