Ficha 4

MD ~~Az~~

1

1 2 3 5 8 13 21 34 55 89 144

Para calcular a soma dos m primeiros com índice ímpar podemos fazê-lo somando $\boxed{F_{2m-1}}$

Para calcular o dos pares somamos $\boxed{F_{2m}-1}$ pois excluímos F_0

2

Como podemos reparar, os números pares de Fibonacci são sempre separados por dois números ímpares

Logo, obtemos através de $\boxed{F_{3m-1}}$

3

Considerando uma escada com m degraus. Se o Pedro der

→ 1 passo : a_{m-1}

→ 2 passos : a_{m-2}

→ 3 passos : a_{m-3}

$$a_m = a_{m-1} + a_{m-2} + a_{m-3}$$

4

par impar

par impar par

par impar par impar

5

Para um número binário temos 2^m possibilidades, como 3 zeros têm de ser consecutivos obtemos 2^{m-3}

6

a) $(x-1)(x-3)(x-2)^2$

$= (x-1)(x-3)(x^2-4x+4)$

$= (x^2-3x-x+3)(x^2-4x+4)$

$= (x^2-4x+3)(x^2-4x+4)$

$= x^4-4x^3+4x^2-4x^3+16x^2-16x+3x^2-12x+12$

$= x^4-8x^3+23x^2-28x+12$

$= x^{m-4}(x^4-8x^3+17x^2-28x+12)$

$= x^m-8x^{m-1}+23x^{m-2}-28x^{m-3}+12x^{m-4}$

$= a_m-8a_{m-1}+23a_{m-2}-28a_{m-3}+12a_{m-4}$

b)

Raiz	Multiplicidade	Termo da solução geral
1	1	$c_1 1^m$
3	1	$c_2 3^m$
2	2	$(c_3+c_4 m)2^m$

$$a_m = c_1 + c_2 3^m + (c_3+c_4 m)2^m$$

7

a) $a_{n+2} = a_{n+1} + 6a_n - 6$, $n \geq 0$ com $a_0 = 0$ $a_1 = 6$

$\Leftrightarrow a_{n+2} - a_{n+1} - 6a_n + 6 = 0$

$\Leftrightarrow q^2 - q - 6 = 0$

$\Leftrightarrow q = 3 \vee q = -2$ Logo, $\underset{n \in \mathbb{R}}{c_1(3)^n} + \underset{n \in \mathbb{R}}{c_2(-2)^n} + A$ → solução geral
→ parte homogenea

-6

$A(n-2) + B = A(n-1) + B - 6An + B$

$A(n-2) + B = \cancel{An} - A + \cancel{B} - 6An + B - 6$

$\Leftrightarrow \cancel{An} - 2A + \cancel{B} = \ldots$ $3A - B = -6$

$\Leftrightarrow -6An$

$\Leftrightarrow \begin{cases} -6A = 0 \\ 3A - B = -6 \end{cases} \Leftrightarrow \begin{cases} A = 6 \\ B = 6 \end{cases}$

Agora trocando na expressão inicial os (a_n)'s por A

$A = A + 6A - 6$

$\Leftrightarrow A - A - 6A = -6$

$\Leftrightarrow A = 1$ → parte não homogénea

$b_n = A \times n^m \times \rho^n$

$= A \times n^1 \times \rho^0 = A_n$

$a_n = c_1 3^n + c_2(-2)^n + 1$

Como agora temos que $a_n = c_1 3^n + c_2(-2)^n + 1$ e $a_0 = 0$ e $a_1 = 6$

$\begin{cases} c_1 3^0 + c_2(-2)^0 + 1 = 0 \\ c_1 3 + c_2(-2) + 1 = 6 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = -1 \\ 3c_1 - 2c_2 = 5 \end{cases} \Leftrightarrow \begin{cases} c_1 = -1 - c_2 \\ 3(-1-c_2) - 2c_2 = 5 \end{cases} \Leftrightarrow \begin{cases} c_1 = -1 - c_2 \\ -3 - 3c_2 - 2c_2 = 5 \end{cases} \Leftrightarrow \begin{cases} c_1 = \frac{3}{5} \\ c_2 = -\frac{8}{5} \end{cases}$

$\therefore a_n = \frac{3}{5} \times 3^n + \frac{8}{5} \times (-2)^n + 1 \Leftrightarrow a_n = \frac{3^{n+1}}{5} + \frac{(-2)^{n+3}}{5} + 1$

b) $a_n - 4a_{n-1} + 4a_{n-2} = n + 2^n$ $a_0 = 0$ $a_1 = 1$

$a_n - 4a_{n-1} + 4a_{n-2} = 0$

$\Leftrightarrow q^2 - 4q + 4 = 0 \Leftrightarrow q = 2$ → $c_1 2^n + c_2 n 2^n$ → porque é um caso notável $(n-2)^2$

$t_n = An + B$

$A(n) + B - 4(A(n-1) + B) + 4(A(n-2) + B) = n$

$\Leftrightarrow An + B - 4(An - A + B) + 4(An - 2A + B) = n$

$\Leftrightarrow An + B - 4An + 4A - 4B + 4An - 8A + 4B = n$

$\Leftrightarrow An - 4A + B = n$

$\Leftrightarrow \begin{cases} An = 1n \\ -4A + B = 0 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = 4 \end{cases}$ Logo, $t_n = n + 4$

$x_n = c_1 2^n + c_2 n 2^n + n + 4$

$a_n - 4a_{n-1} + 4a_{n-2} = 2^n$ $b_n = An^2 2^n$

$$A(m)^2 2^m - 4A\left((m-1)^2 2^{m-1}\right) + 4A(m-2)^2 2^{m-2} = 2^m$$

$$\Leftrightarrow Am^2 2^m - 2A(m^2-2m+1)2^m + A(m^2-4m+4)2^m = 2^m$$

$$\Leftrightarrow Am^2 - 2Am^2 + 4Am - 2A + Am^2 - 4Am + 4A = 1$$

$$\Leftrightarrow 2A = 1 \Leftrightarrow A = \frac{1}{2}$$

Logo, $b_m = \frac{1}{2} m^2 2^m$

$$X_m = C_1 2^m + C_2 m 2^m + m + 4 + \frac{1}{2} m^2 2^m$$

$$\begin{cases} x_0 = 0 \\ x_1 = 1 \end{cases} \Leftrightarrow \begin{cases} C_1 + 4 = 0 \\ 2C_1 + 2C_2 + 5 + 1 = 1 \end{cases} \Leftrightarrow \begin{cases} C_1 = -4 \\ -8 + 2C_2 = -5 \end{cases} \Leftrightarrow \begin{cases} C_1 = -4 \\ C_2 = \frac{3}{2} \end{cases}$$

$$X_m = -4 \times 2^m + \frac{3}{2} m 2^m + m + 4 + m^2 2^{m-1}$$

$$= \left(-4 + \frac{3}{2}\right) 2^m + m + 4 + m^2 2^{m-1}$$

8

$p(x) = 2x^2 + x$ $\qquad$ $S_m = \sum_{i=0}^{\infty} (2i^2 + i)$

$$\Leftrightarrow S_m = p(0) + p(1) + p(2) + \cdots + p(m)$$

$$\Leftrightarrow S_m = S_{m-1} + p(m) \Leftrightarrow \boxed{S_m = S_{m-1} + 2m^2 + m \quad , m \geq 1}$$

Sabemos ainda que $p_0 = 0$ $\quad p_1 = 3$

$$S_m - S_{m-1} = 2m^2 + m$$

$$q - 1 = 0 \Leftrightarrow q = 1$$

Logo, $x_m = C_1 \times (1)^m = C_1$

Para $2m^2 + m$

$$t_m = Am^3 + Bm^2 + Cm$$

$$Am^3 + Bm^2 + Cm + A(m-1)^3 + B(m-1)^2 + C(m-1) = 2m^2 + m$$

$$\Leftrightarrow Am^3 + Bm^2 + Cm + A(m-1)(m-1)^2 + B(m^2 - 2m + 1) + Cm - C = 2m^2 + m$$

$$\Leftrightarrow Am^3 + Bm^2 + Cm + A(m-1)(m^2 - 2m + 1) + Bm^2 - 2Bm + B + Cm - C = 2m^2 + m$$

$$\Leftrightarrow \cancel{Am^3} + \cancel{Bm^2} + \cancel{Cm} + \cancel{Am^3} - 2A\cancel{m^2} + A\cancel{m} - A\cancel{m^2} + 2A\cancel{m} - A + \cancel{Bm^2} - 2\cancel{B}m + B + \cancel{Cm} - C = 2m^2 + m$$

$$\Leftrightarrow 2Am^3 - 3Am^2 + 2Bm^2 + 3Am - 2Bm + 2Cm + B - C = 2m^2 + m$$

$$X_m = C_1 + \frac{m + m^2}{2} \Leftrightarrow x_1 = 3 \Leftrightarrow C_1 + 1 = 3 \Leftrightarrow C_1 = 2$$

$$X_m = 2 + \frac{m + m^2}{2}$$

// A parte inicial do C_1 está a dar mal não sei porquê

9

$$a_m = (c_1 + c_2 m)2^m + c_3 + 4m$$

$$= (x-2)^2 (x-1)$$

$$= (x^2 - 4x + 4)(x-1)$$

$$= x^3 - x^2 - 4x^2 + 4x + 4x - 4$$

$$= x^3 - 5x^2 + 8x - 4$$

$\therefore$ Logo, $a_m - 5a_{m-1} + 8a_{m-2} - 4a_{m-3} = 4$

10

$p_0 = 0$

11

a) $a_m = m a_{m-1} + m!$, $a_0 = 2$

Substituindo $a_m = b_m m!$

$\Leftrightarrow b_m m! = m b_{m-1} (m-1)! + m!$

$\Leftrightarrow b_m m! = b_{m-1} m! + m!$

$\Leftrightarrow b_m = b_{m-1} + 1$

$\Leftrightarrow b_m - b_{m-1} = 1$

$\Leftrightarrow p - 1 = 0 \Leftrightarrow p = 1$ Logo, $b_m = c_1$

$t_m = Am + B$

$\Leftrightarrow Am + B - (A(m-1) + B) = 1$

$\Leftrightarrow \cancel{Am} + \cancel{B} - \cancel{Am} + A - \cancel{B} = 1$

$\Leftrightarrow A = 1 \wedge B = 0$

$\Leftrightarrow t_m = m$

Sendo $a_0 = 2 \Rightarrow b_0 0! = 2 \Leftrightarrow b_0 = 2$

$b_m = C + m$

$\Leftrightarrow b_0 = 2 \Leftrightarrow 2 = C$

$\therefore$ $b_m = 2 + m \Leftrightarrow \frac{a_m}{m!} = (2+m) \Leftrightarrow a_m = (2+m)m!$

b) $5m a_m + 2m a_{m-1} = 2a_{m-1}$, $m \geq 3$ $a_2 = -30$

Se $a_m = \frac{b_m}{m}$

$\Leftrightarrow 5\cancel{m} \times \frac{b_m}{\cancel{m}} + 2m \frac{b_{m-1}}{m-1} = 2\frac{b_{m-1}}{m-1}$

$\Leftrightarrow 5b_m + \frac{b_{m-1}}{m-1}(2m-2) = 0$

$\Leftrightarrow 5b_m + \frac{2b_{m-1}}{\cancel{m-1}}\cancel{(m-1)} = 0$

$\Leftrightarrow 5b_m + 2b_{m-1} = 0$

$\Leftrightarrow 5q + 2 = 0$

$\Leftrightarrow q = -\frac{2}{5}$

$b_m = c_1 \left(-\frac{2}{5}\right)^m$

$t_m = Am + B$

$\Leftrightarrow 5Am + 5B + 2A(m-1) + 2B(m-1) = 0$

$\Leftrightarrow 5Am + 5B + 2Am - 2A + 2Bm - 2B = 0$

$\Leftrightarrow 3Am + 2Bm - 2A + 3B = 0$

$\Leftrightarrow A = 0 \wedge B = 0$

Sendo $a_2 = -30 \Leftrightarrow \frac{b_2}{2} = -30 \Leftrightarrow b_2 = -60 \Leftrightarrow c_1 \left(-\frac{2}{5}\right)^2 = -60 \Leftrightarrow c_1 \frac{4}{25} = -60 \Leftrightarrow c_1 = -\frac{60 \times 25}{4}$

$\Leftrightarrow c_1 = 375$

$b_m = -375\left(-\frac{2}{5}\right)^m \Leftrightarrow a_m \times m = -375\left(\frac{2}{5}\right)^m \Leftrightarrow a_m = \frac{-375}{m}\left(\frac{2}{5}\right)^m$

c) $a_m^3 = a_{m-1}^2$, $m \geq 2$ $a_1 = 2$

sendo $a_m = 2^{b_m} \Leftrightarrow b_m = \log_2 a_m$

$\Leftrightarrow (2^{b_m})^3 = (a^{b_{m-1}})^2 \Leftrightarrow 2^{3b_m} = 2^{2b_{m-1}} \Leftrightarrow 3b_m = 2b_{m-1} \Leftrightarrow 3b_m - 2b_{m-1} \Leftrightarrow 3q - 2 = 0 \Leftrightarrow q = \frac{2}{3}$

$b_m = C_1 \left(\frac{2}{3}\right)^m$, sendo $a_1 = 2 \Leftrightarrow 2^{b_1} = 2 \Leftrightarrow b_1 = \log_2(2) \Leftrightarrow b_1 = 1$

$\Leftrightarrow b_1 = C\left(\frac{2}{3}\right)^1$

$\Leftrightarrow 1 = \frac{2}{3}C \Leftrightarrow C = \frac{3}{2}$ Logo, $b_m = \frac{3}{2}\left(\frac{2}{3}\right)^m = \frac{3 \times 2^m}{2 \times 3^m} = 3^{1-m} \times 2^{m-1}$

d) $a_m = 2(a_{m-1} + 2(\underbrace{a_{m-2} + \dots + 2(a_1 + 2(a_0 + a_0)^2)^2 \dots)^2}_{a_{m-1}})^2$, com $a_0 = 2$ $a_1 = 2(a_0 + a_0)^2$

$\Leftrightarrow a_m = 2(2a_{m-1})^2$

$\Leftrightarrow a_m = 8a_{m-1}^2$ $a_m = 2^{b_m} \Leftrightarrow \log_2(a_m) = b_m$

$\Leftrightarrow 2^{b_m} = 8(2^{b_{m-1}})^2$

$\Leftrightarrow 2^{b_m} = 8 \times 2^{2b_{m-1}}$ $b_m = C_1 6^m$ $a_0 = 2$

$\Leftrightarrow 2^{b_m} = 2^{6b_{m-1}}$ $\Leftrightarrow b_0 = 1$ $\Leftrightarrow b_0 = 1$

$\Leftrightarrow b_m - 6b_{m-1} = 0$ $\Leftrightarrow C_1 6^m = 1$

$\Leftrightarrow q - 6 = 0$ $\Leftrightarrow C_1 = 1$

$\Leftrightarrow q = 6$ $\Leftrightarrow b_m = 6^m$

$\Leftrightarrow \log_2(a_m) = 6^m \Leftrightarrow a_m = 2^{6m}$

12 $h(k, m)$ ————

13 $L_m = L_{m-1} + L_{m-2}$, $L_0 = 2$ $L_1 = 1$

$\Leftrightarrow L_m - L_{m-1} - L_{m-2} = 0$

$\Leftrightarrow q^2 - q - 1 = 0$

$\Leftrightarrow q = \frac{1+\sqrt{5}}{2} \vee q = \frac{1-\sqrt{5}}{2}$

$X_m = C_1\left(\frac{1+\sqrt{5}}{2}\right)^m + C_2\left(\frac{1-\sqrt{5}}{2}\right)^m$

$$\begin{cases} L_0 = 2 \\ L_1 = 1 \end{cases} \Leftrightarrow \begin{cases} C_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + C_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = 2 \\ C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases} \Leftrightarrow \begin{cases} C_1 + C_2 = 2 \\ — \end{cases} \Leftrightarrow \begin{cases} C_1 = 2 - C_2 \\ (2-C_2)\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases}$$

$\Leftrightarrow \begin{cases} C_1 = 1 \\ C_2 = 1 \end{cases}$ $\therefore$ Logo, $X_m = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m$

(14) $x_1 + x_2 + x_3 + x_4 = m3$

a) $x_1 \in \{0,1,2,3,4,5\}$, $x_2 \in \{0,1,2,3\}$, $x_3 \in \{2,3,4,5,6,7,8\}$, $x_4 \in \{0,1,2,3,4\}$

$p_1 = 1 + x + x^2 + x^3 + x^4 + x^5$ $\quad$ $p_3 = x^2 x^3 x^4 x^5 x^6 x^7 x^8$

$p_2 = 1 + x + x^2 + x^3$ $\quad$ $p_4 = 1 + x + x^2 + x^3 + x^4$

$p = p_1 p_2 p_3 p_4$
$= (1 + x + \cdots + x^5)(1 + x + x^2 + x^3)(x^2 + x^3 + \cdots + x^8)(1 + x + x^2 + x^3 + x^4)$

b) $p_1 = 1 + x^2 + x^4 + x^6 + x^8$ $\quad$ $p_3 = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

$p_2 = x + x^3 + x^5 + x^7$ $\quad$ $p_4 = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

$p = p_1 p_2 p_3 p_4 = (1 + x^2 + x^4 + x^6 + x^8)(x + x^3 + x^5 + x^7)(1 + x + x^2 + \cdots + x^8)^2$

(15)

a) Candidatos: c_1, c_2, c_3, c_4

$m = 27$ votos

A serie geradora será $A = (x^0 + x^1 + \dots + x^{27})^4$

OU $A = (x^0 + x^1 + \cdots + x^m)^4$ se a turma tiver mais de 27 alunos

$= \left(\sum_{m=0}^{\infty} x^m\right)^4 = \left(\frac{1}{1-x}\right)^4 = \frac{1}{(1-x)^4}$

∴ Concluimos que o coeficiente é x^{27} quando extensões

b) O número de votos possíveis continua a ser o coeficiente x^{27} pois temos na mesma 27 votos

Cada candidato pode ter de 1 a 24 votos

Logo, $A = (x + x^2 + \cdots + x^{24})^4$

c) Se nenhum candidato recebe a maioria dos votos, poder ter entre $(x^0 + x^1 + \cdots x^{13})$ votos, mas como continuam a ser 27 votos o coeficiente é x^{27} na mesma

(16) Notas 20: S/limite $(x^0 + x^{20} + x^{40})$

Notas 10: S/limite $(x^0 + x^{10} + x^{20} + x^{30} + x^{40} + x^{50})$

Notas 5: 5 notas $(x^0 + x^5 + x^{10} + x^{15} + x^{20})$

Moedas 1: 5 moedas $(x^0 + x^1 + x^2 + x^3 + x^4 + x^5)$

Moedas 2: 5 moedas $(x^0 + x^2 + x^4 + x^6 + x^8 + x^{10})$

Logo, $A = (x^0 + x^{20} + x^{40})(x^0 + x^{10} + x^{20} + x^{30} + x^{40} + x^{50})(x^0 + x^5 + x^{10} + x^{15} + x^{20} + x^{25})(x^0 + x^1 + \cdots + x^5)(x^0 + x^2 + x^4 + \cdots + x^{10})$

As possibilidades de trocar 50 euros são nos dado pelo expressão de A_{50}

17 $\quad 3a + 2b + 4c + 2d = n$

Para simplificar a resolução efetuamos as seguintes susbtituições

$z_1 = 3a \; ; \; z_2 = 2b \; ; \; z_3 = 4c \; ; \; z_4 = 2d$

Agora representando os valores que cada uma pode ter

$z_1 = \{x^0 + x^3 + x^6 + x^9 + \cdots\}$ $\qquad z_3 = \{x^0 + x^4 + x^8 + x^{12} + \cdots\}$

$z_2 = \{x^0 + x^2 + x^4 + x^8 + \cdots\}$ $\qquad z_4 = \{x^0 + x^2 + x^4 + x^6 + \cdots\}$

Agora basta obtermos a função geral

$$z = z_1 z_2 z_3 z_4$$
$$= (x^0 + x^3 + x^6 + x^9 + \cdots)(x^0 + x^2 + x^4 + x^6 + \cdots)(x^0 + x^4 + x^8 + x^{12} + \cdots)(x^0 + x^2 + x^4 + x^6 + \cdots)$$
$$= \frac{1}{(1-x^3)(1-x^2)^2(1-x^4)}$$

18 $\qquad \sum_{m=0}^{\infty} a_m x^m$ ou $\sum_{m=0}^{\infty} \frac{a_m}{m!} x^m$

a) $\quad b_m = m k^m$

$$A = \sum_{m=1}^{\infty} b_m x^m = \sum_{m=1}^{\infty} m k^m x^m = \sum_{m=1}^{\infty} m (kx)^m = kx \sum_{m=1}^{\infty} m (kx)^{m-1} = kx \left(\sum_{m=1}^{\infty} (kx)^m \right)' = kx \times \left(\frac{1}{1-kx} \right)'$$

$$= kx \times \frac{k}{(1-kx)^2} = \frac{k^2 x}{(1-kx)^2}$$

b) $\quad c_m = k + 2k^2 + 3k^3 + \cdots + mk^m \; , \; m \in \mathbb{N}$

$$C(m) = \sum_{m=1}^{\infty} m k^m$$

c) $\quad A = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + \sum_{m=2}^{\infty} a_m x^m \longrightarrow$ slide 214

$$= -1 + 2x + \sum_{m=2}^{\infty} (c_1 a_{m-1} + c_2 a_{m-2}) x^m$$
$$= -1 + 2x + \sum_{m=2}^{\infty} c_1 a_{m-1} x^m + \sum_{m=2}^{\infty} c_2 a_{m-2} x^m$$
$$= -1 + 2x + c_1 x \sum_{m=1}^{\infty} a_m x^m + c_2 x^2 \sum_{m=0}^{\infty} a_m x^m$$
$$= -1 + 2x + c_1 x (A + 1) + c_2 x^2 A$$
$$= -1 + 2x + c_1 x A + c_1 x + c_2 x^2 A$$
$$= -1 + 2x + c_1 x + (c_1 x + c_2 x^2) A$$

$$a_m = c_1 a_{m-1} + c_2 a_{m-2}$$

$$a_0 = -1 \qquad a_1 = 2$$

$$A = -1 + 2x + C_1 x + (C_1 x + C_2 x^2) A$$

$$\Leftrightarrow (1 - C_1 x - C_2 x^2) A = -1 + 2x + C_1 x$$

$$\Leftrightarrow A = \frac{-1 + 2x + C_1 x}{1 - C_1 x - C_2 x^2}$$

19

a) $f(x) = (2+x)^4 = \sum_{m=0}^{\infty} \binom{4}{k} x^k \times 2^{4-k}$ → slide 74

$$a_k = \binom{4}{k} 2^{4-k}$$

$a_0 = 16$ $a_2 = 24$ $a_4 = 1$

$a_1 = 32$ $a_3 = 8$ $a_m = 0$

b) $f(x) = \frac{6x}{(1+2x)^2} + 2 - x^2$

$$= 6x \sum_{m=0}^{\infty} \left(\binom{2+m-1}{m} (-2x)^m \right) + 2 - x^2$$ → slide 74

$$= 6x \sum_{m=0}^{\infty} \left(\binom{m-1}{m} (-2x)^m \right) + 2 - x^2$$

$$= -3 \sum_{m=0}^{\infty} \left(\binom{m-1}{m} (-2x)^{m+1} \right) + 2 - x^2$$

$$= -3 \sum_{m=0}^{\infty} \left((m+1)(-2x)^{m+1} \right) + 2 - x$$

$\binom{m-1}{m} = (m+1)$ é o mesmo

$$= -3 \sum_{m=1}^{\infty} \left(m (-2x)^m \right) + 2 - x$$

$$= \sum_{m=1}^{\infty} \left(-3m (-2x)^m \right) + 2 - x$$

$a_0 = 2$ $a_1 = 6$ $a_2 = -25$ $a_3 = -3m(-2)^m$

20

a) $a_m = m\, a_{m-1}$

$$A = \sum_{m=1}^{\infty} \frac{a_m}{m!} x^m$$

$$= a_1 x + \sum_{m=2}^{\infty} \frac{a_m}{m!} x^m$$

$$= x + \sum_{m=2}^{\infty} \frac{m\, a_{m-1}}{m!} x^m$$

$$= x + \sum_{m=2}^{\infty} \frac{a_{m-1}}{(m-1)!} x^m = x + x \sum_{m=1}^{\infty} \frac{a_m}{m!} x^m = x + xA$$

Logo, $A = x + xA \Leftrightarrow (1-x)A = x \Leftrightarrow A = \frac{x}{1-x} = \sum_{m=1}^{\infty} x^m = \sum_{m=0}^{\infty} \frac{m!}{m!} x^m$

$\therefore a_m = m!$, $m \geq 1$

b) $a_m = a_{m-1} + m \quad , \quad m \geq 1 \qquad a_0 = 1$

$$A = \sum_{m=0}^{\infty} a_m x^m$$
$$= a_0 + \sum_{m=1}^{\infty} (a_{m-1} + m) x^m$$
$$= 1 + \sum_{m=1}^{\infty} a_{m-1} x^m + \sum_{m=1}^{\infty} m x^m$$
$$= 1 + x \sum_{m=0}^{\infty} a_m x^m + x \sum_{m=0}^{\infty} (x^m)'$$
$$= 1 + xA + x\left(\frac{1}{1-x}\right)'$$
$$= 1 + xA + \frac{x}{(1-x^2)}$$

$$\frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3}$$

$$\Leftrightarrow A(1-x)^2 + B(1-x) + C = x$$
$$\Leftrightarrow A(1-2x+x^2) + B - Bx + C = x$$
$$\Leftrightarrow A - 2Ax \; Ax^2 + B - Bx + C = x$$

$$\Leftrightarrow \begin{cases} A = 0 \\ -2A - B = 1 \\ A + B + C = 0 \end{cases} \Leftrightarrow \begin{cases} A = 0 \\ B = -1 \\ C = 1 \end{cases}$$

$$A = 1 - xA + \frac{x}{(1-x)^2} \Leftrightarrow$$
$$\Leftrightarrow (1-x)A = 1 + \frac{x}{(1-x)^2} \Leftrightarrow A = \frac{1}{1-x} + \frac{x}{(1-x)^3}$$

$$\Leftrightarrow A = \frac{1}{(1-x)^3} - \frac{1}{(1-x)^2} + \frac{1}{1-x}$$

$$\Leftrightarrow A = \sum_{m=0}^{\infty} \binom{3+m-1}{m} x^m - \sum_{m=0}^{\infty} \binom{2m-1}{m} x^m + \sum_{m=0}^{\infty} x^m$$

$$\Leftrightarrow A = \sum_{m=0}^{\infty} x^m - \sum_{m=0}^{\infty} (m+1) x^m + \sum_{m=0}^{\infty} (m+2)(m+1) x^m$$

$$\Leftrightarrow A = \sum_{m=0}^{\infty} \left(1 - (m+1) + (m+2)(m+1)\right) x^m$$
$$\Leftrightarrow A = \sum_{m=0}^{\infty} \left(-m + (m^2 + 3m + 2)\right) x^m$$
$$\Leftrightarrow A = \sum_{m=0}^{\infty} \left((m^2 + 2m + 2) x^m\right)$$

$\therefore$ Logo, $a_m = m^2 + 2m + 2$

c) $a_m = 3a_{m-1}$

$$A = \sum_{m=0}^{\infty} a_m x^m$$
$$= a_0 + \sum_{m=1}^{\infty} (3a_{m-1}) x^m$$
$$= 2 + 3 \sum_{m=1}^{\infty} a_{m-1} x^m$$
$$= 2 + 3x \sum_{m=0}^{\infty} a_m x^m$$
$$= 2 + 3xA$$

$$A = 2 + 3xA$$
$$\Leftrightarrow (1-3x)A = 2$$
$$\Leftrightarrow A = \frac{2}{1-3x}$$
$$\Leftrightarrow A = 2 \times \sum_{m=0}^{\infty} (3x)^m \Leftrightarrow A = \sum_{m=0}^{\infty} 2 \times 3^m x^m$$
$$\Leftrightarrow a_m = 2 \times 3^m$$

d) $\mathcal{U}_m = \mathcal{U}_{m-1} + m^2 \qquad \mathcal{U}_0 = 2$

$$A = \sum_{m=0}^{\infty} a_m x^m$$

$$= a_0 + \sum_{m=1}^{\infty} a_m x^m$$
$$= a_0 + \sum_{m=1}^{\infty} (\mathcal{U}_{m-1} + m^2) x^m$$
$$= 2 + \sum_{m=1}^{\infty} \mathcal{U}_{m-1} x^m + \sum_{m=1}^{\infty} m^2 x^m$$
$$= 2 + x\sum_{m=0}^{\infty} \mathcal{U}_m x^m + \sum_{m=1}^{\infty} (m^2 + m - m) x^m$$

$$= 2 + xA + \sum_{m=1}^{\infty} (m^2+m) x^m - \sum_{m=1}^{\infty} m x^m$$

$$= 2 + xA + \sum_{m=1}^{\infty} m(m+1) x^m - \left(\frac{1}{1-x}\right)' x$$
$$= 2 + xA + x^2 \sum_{m=1}^{\infty} (x^m)'' - \frac{x}{(1-x)^2}$$
$$= 2 + xA + x^2 \left(\frac{1}{1-x}\right)'' - \frac{x}{(1-x)^2}$$

$$= 2 + xA + \frac{2x^2}{(1-x)^3} - \frac{x}{(1-x)^2}$$

$$A = 2 + xA + \frac{2x^2}{(1-x)^3} - \frac{x}{(1-x)^2}$$

$$\Leftrightarrow (1-x)A = 2 + \frac{2x^2}{(1-x)^3} - \frac{x}{(1-x)^2} \Leftrightarrow A = \frac{2}{1-x} + \frac{2x^2}{(1-x)^4} - \frac{x}{(1-x)^3}$$

// Agora era continuar isso

e) $\mathcal{U}_{m+1} = 3\mathcal{U}_m - 1 \qquad \mathcal{U}_0 = 1$

$\Leftrightarrow \mathcal{U}_m = 3\mathcal{U}_{m-1} - 1$

$$\frac{A}{1-x} + \frac{B}{1-3x} = A(1-3x) + B(1-x) \Leftrightarrow \begin{cases} B = \frac{3}{2} \\ A = -\frac{1}{2} \end{cases}$$
$$= A - 3Ax + B - Bx$$

$$A = \sum_{m=0}^{\infty} \mathcal{U}_m x^m$$
$$= \mathcal{U}_0 + \sum_{m=1}^{\infty} \mathcal{U}_m x^m$$
$$= 2 + \sum_{m=1}^{\infty} (3\mathcal{U}_{m-1} - 1) x^m$$
$$= 2 + 3\sum_{m=1}^{\infty} \mathcal{U}_{m-1} x^m - \sum_{m=1}^{\infty} x^m$$
$$= 2 + 3x \sum_{m=0}^{\infty} \mathcal{U}_m x^m - \frac{1}{1-x}$$
$$= 2 + 3xA - \frac{1}{1-x}$$

$$A = 2 + 3xA - \frac{1}{1-x} \Leftrightarrow (1-3x)A = 2 - \frac{1}{1-x} \Leftrightarrow A = \frac{2}{1-3x} - \frac{1}{(1-x)(1-3x)}$$

$$\Leftrightarrow A = 2 \times \frac{1}{1-3x} \left(\frac{3}{2} \times \frac{1}{1-3x} - \frac{1}{2} \times \frac{1}{1-x}\right)$$

$$\Leftrightarrow A = \frac{1}{2} \times \frac{1}{1-3x} - \frac{1}{2} \times \frac{1}{1-x} \Leftrightarrow A = \frac{1}{2} \sum (3x)^m - \frac{1}{2} \sum x^m \Leftrightarrow A = \sum_{m=0}^{\infty} \left(\frac{1+3^m}{2}\right) x^m$$

f)

$$U_{m+2} - 5U_{m+1} + 6U_m = 0$$

$$\Leftrightarrow U_{m+2} = 5U_{m+1} - 6U_m \quad , \quad m \geq 0 \qquad U_0 = 0 \quad U_1 = 1$$

$$\Leftrightarrow U_m = 5U_{m-1} - 6U_{m-2} \quad , \quad m \geq 2$$

$$A = \sum_{m=0}^{\infty} U_m x^m = U_0 + U_1 x + \sum_{m=2}^{\infty} U_m x^m$$

$$= x + \sum_{m=2}^{\infty} (5U_{m-1} - 6U_{m-1}) x^m$$

$$= x + 5 \sum_{m=2}^{\infty} U_{m-1} x^m - 6 \sum_{m=2}^{\infty} U_{m-2} x^m$$

$$= x + 5x A - 6x^2 A$$

$$A = x + 5xA - 6x^2 A \Leftrightarrow (6x^2 - 5x + 1) A = x \Leftrightarrow A = \frac{x}{6x^2 - 5x + 1}$$

$$6x^2 - 5x + 1 = 0$$

$$\Leftrightarrow x = \frac{1}{2} \vee x = \frac{1}{3}$$

$$\Leftrightarrow 2x = 1 \vee 3x = 1$$

$$\frac{x}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x}$$

$$\Leftrightarrow \begin{cases} -3A - 2B = 1 \\ A + B = 0 \end{cases} \Leftrightarrow \begin{cases} A = -1 \\ B = 1 \end{cases}$$

$$\Leftrightarrow A = -1 \times \frac{1}{1-2x} + \frac{1}{1-3x} \Leftrightarrow A = -1 \sum_{m=0}^{\infty} (2x)^m + \sum_{m=0}^{\infty} (3x)^m$$

$$\Leftrightarrow A = \sum_{m=0}^{\infty} (-2^m x^m + 3^m x^m) = \sum_{m=0}^{\infty} (3^m - 2^m) x^m \Leftrightarrow U_m = 3^m - 2^m$$

21)

$$U_m - 2U_{m-1} = 4^m$$

a) $U_m = 4^m + 2U_{m-1}$ $\quad U_0 = 1$

$$A = \sum_{m=0}^{\infty} U_m x^m = U_0 + \sum_{m=1}^{\infty} (2U_{m-1} + 4^m) x^m = 1 + \sum_{m=1}^{\infty} 2U_{m-1} x^m + \sum_{m=1}^{\infty} 4^m x^m$$

$$= 1 + 2 \sum_{m=1}^{\infty} U_{m-1} x^m + \sum_{m=1}^{\infty} (4x)^m$$

$$= 1 + 2x \sum_{m=0}^{\infty} U_m x^m + \sum_{m=0}^{\infty} (4x)^m + (4x)^0$$

$$= \cancel{1} + 2xA - \frac{1}{1-4x} \cancel{-1}$$

$$A = 2xA - \frac{1}{1-4x} \Leftrightarrow A = \frac{1}{(1-2x)(1-4x)}$$

b) $A = \dfrac{1}{(1-2x)(1-4x)}$

$$\frac{A}{1-2x} + \frac{B}{1-4x} = \frac{1}{(1-2x)(1-4x)}$$

$$\Leftrightarrow A(1-4x) + B(1-2x) = 1$$

$$\Leftrightarrow A - 4Ax + B - 2Bx = 1$$

$$\Leftrightarrow \begin{cases} -4A - 2B = 0 \\ A + B = 1 \end{cases} \Leftrightarrow \begin{cases} A = -1 \\ B = 2 \end{cases}$$

$$A = -1 \frac{1}{1-2x} + 2 \times \frac{1}{1-4x} = -\sum_{m=0}^{\infty} (2x)^m + 2\sum_{m=0}^{\infty} (4x)^m$$

$$= \sum_{m=0}^{\infty} (2 \times 4^m - 2^m) x^m$$

$\therefore$ Logo $v_m = 2 \times 4^m - 2^m = 2^{2m+1} - 2^m$, $m \geq 0$

22

a) $$a_0 + a_1 x + a_2 x^2 = A = \sum_{m=1}^{\infty} m x^m = x \sum_{m=1}^{\infty} m x^{m-1} = x \sum_{m=1}^{\infty} (x^m)' = x \times \left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}$$

b) $$A = \sum_{m=0}^{\infty} m^2 x^m = \sum_{m=0}^{\infty} (m^2 + m - m) x^m = \sum_{m=0}^{\infty} (m^2+m) x^m + \sum_{m=0}^{\infty} m x^m$$

$$= \sum_{m=0}^{\infty} (m-1)m\, x^m + \frac{x}{(1-x)^2} = x^2 \sum_{m=0}^{\infty} (m-1)m\, x^{m-2} + \frac{x}{(1-x)^2}$$

$$= x^2 \sum_{m=0}^{\infty} (x^m)'' + \frac{x}{(1-x)^2} = x^2 \left(\frac{1}{1-x}\right)'' + \frac{x}{(1-x)^2} = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$$

$$= \frac{x(x+1)}{(1-x)^3}$$

c) $$A = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + \sum_{m=2}^{\infty} a_m x^m$$

$$= \alpha x + \sum_{m=2}^{\infty} (a_{m-2} - m^2) x^m = \alpha x + \sum_{m=2}^{\infty} a_{m-2} x^m - \sum_{m=2}^{\infty} m^2 x^m$$

$$= \alpha x + x^2 \sum_{m=0}^{\infty} a_m x^m - \sum_{m=2}^{\infty} (m^2 x^m)$$

$$= \alpha x + x^2 A - \frac{x(x+1)}{(1-x)^3}$$

$$A = \alpha x + x^2 A - \frac{x(x+1)}{(1-x)^3} \Leftrightarrow (1-x^2) A = \alpha x - \frac{x(x+1)}{(1+x)^3}$$

$$\Leftrightarrow A = \frac{\alpha x}{1-x^2} - \frac{x(x+1)}{(1-x)^3(1-x^2)} \Leftrightarrow A = \alpha \frac{x}{1-x^2} - \frac{x\cancel{(x+1)}}{(1-x)^3(1-x)\cancel{(1+x)}} \Leftrightarrow A = \alpha \frac{x}{1-x^2} - \frac{x}{(1-x)^4}$$

era suposto em cima ter substituido α por $\alpha+1$ por isso é que agora uso este

d) $A = (\alpha+1)\dfrac{x}{1-x^2} - \dfrac{x}{(1-x)^4}$

$= (\alpha+1)\dfrac{x}{(1-x)(1+x)} - x\sum_{m=0}^{\infty}\binom{4+m-1}{m}x^m$

$\dfrac{x}{1-x^2} = \dfrac{A}{1+x} + \dfrac{B}{1-x}$

$\Leftrightarrow A - Ax + B + Bx = x$

$\Leftrightarrow \begin{cases} -A+B=1 \\ A+B=0 \end{cases} \Leftrightarrow \begin{cases} A=-\frac{1}{2} \\ B=\frac{1}{2} \end{cases}$

$= (\alpha-1)\left(-\frac{1}{2}\times\frac{1}{1+x} + \frac{1}{2}\times\frac{1}{1-x}\right) - x\sum_{m=0}^{\infty}\binom{3+m}{m}x^m$

$= (\alpha-1)\left(-\frac{1}{2}\sum_{m=0}^{\infty}(-x)^m + \frac{1}{2}\sum_{m=0}^{\infty}x^m\right) - \sum_{m=0}^{\infty}\dfrac{(m+3)(m+2)(m+1)}{3!}x^{m+1}$

$= -\dfrac{\alpha-1}{2}\sum_{m=0}^{\infty}(-1)^m x^m + \dfrac{\alpha-1}{2}\sum_{m=0}^{\infty}x^m - \sum_{m=1}^{\infty}\dfrac{(m+2)(m+1)m}{6}x^m$

$= \sum_{m=0}^{\infty}\left(\dfrac{\alpha+1}{2} - \dfrac{\alpha+1}{2}(-1)^m - \dfrac{(m+2)(m+1)m}{6}\right)x^m$

24 $\binom{\frac{1}{2}}{3} = \left(\dfrac{\frac{1}{2}\times(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}\right) = \dfrac{1}{16}$

$\binom{-2}{3} = \dfrac{(-2)(-2-1)(-2-2)}{3!} = -4$

25 $\binom{x}{2} = 28 \Leftrightarrow \dfrac{x(x-1)}{2!} = 28 \Leftrightarrow x^2 - x = 56 \Leftrightarrow x=-7 \vee x=8$

28 $\begin{cases} a_m = 3a_{m-1} + 2b_{m-1} \\ b_m = a_{m-1} + b_{m-1} \end{cases}$

$A = \sum_{m=0}^{\infty} a_m x^m \qquad B = \sum_{m=0}^{\infty} b_m x^m$

$A = \sum_{m=0}^{\infty} a_m x^m = a_0 + \sum_{m=1}^{\infty} a_m x^m$

$= 1 + 3\sum_{m=1}^{\infty} a_{m-1}x^m + 2\sum_{m=1}^{\infty} b_{m-1}x^m$

$= 1 + 3x\sum_{m=0}^{\infty} a_m x^m + 2x\sum_{m=0}^{\infty} b_m x^m$

$= 1 + 3xA + 2xB$

$B = \sum_{m=0}^{\infty} b_m x^m = b_0 + \sum_{m=0}^{\infty} b_m x^m$

$= 1 + \sum_{m=0}^{\infty} a_{m-1}x^m + \sum_{m=0}^{\infty} b_{m-1}x^m$

$= 1 + x\sum_{m=0}^{\infty} a_m x^m + x\sum_{m=0}^{\infty} b_m x^m$

$= 1 + Ax + Bx$

$A = 1 + 3xA + 2xB \qquad\qquad B = 1 + Ax + Bx$

$\Leftrightarrow (1-3x)A - 2xB = 1 \qquad\qquad \Leftrightarrow (1-x)B - Ax = 1$

$$\begin{bmatrix} (1-3x) & -2x \\ -x & (1-x) \end{bmatrix} \times \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{vmatrix} (1-3x) & -2x \\ -x & (1-x) \end{vmatrix} = (1-3x)(1-x) - 2x^2 = 1-4x+3x^2-2x^2 = 1-4x+x^2$$

$$x^2-4x+1=0$$

$$\Leftrightarrow x = 2-\sqrt{3} \lor x = 2+\sqrt{3}$$

$$A = \frac{\begin{vmatrix} 1 & -2x \\ 1 & (1-x) \end{vmatrix}}{x^2-4x+1} = \frac{(1-x)+2x}{x^2-4x+1} = \frac{1+x}{x^2-4x+1} = \frac{1+x}{(x-2+\sqrt{3})(x-2-\sqrt{3})}$$

$$B = \frac{\begin{vmatrix} (1-3x) & 1 \\ -x & 1 \end{vmatrix}}{x^2-4x+1} = \frac{1-3x+x}{x^2-4x+1} = \frac{1-2x}{x^2-4x+1} = \frac{1-2x}{(x-2+\sqrt{3})(x-2-\sqrt{3})}$$

$$\frac{A}{x-2+\sqrt{3}} + \frac{B}{x-2-\sqrt{3}} = A(x-2-\sqrt{3}) + B(x-2+\sqrt{3})$$

$$= Ax - 2A - \sqrt{3}A + Bx + (2+\sqrt{3})B$$

$$= Ax + (-2-\sqrt{3})A + Bx + (-2+\sqrt{3})B$$

Para A

$$\begin{cases} A + B = 1 \\ (-2-\sqrt{3})A + (-2+\sqrt{3})B = 1 \end{cases} \Leftrightarrow \begin{cases} A = 1-B \\ (-2-\sqrt{3})(1-B) + (-2+\sqrt{3})B = 1 \end{cases} \Leftrightarrow \begin{cases} - \\ -2+\cancel{2B}-\sqrt{3}+\sqrt{3}B-\cancel{2B}+\sqrt{3}B = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} - \\ 2\sqrt{3}B = 3+\sqrt{3} \end{cases} \Leftrightarrow \begin{cases} \\ B = \frac{3}{2\sqrt{3}} + \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{2} - \frac{\sqrt{3}}{2} \\ B = \frac{\sqrt{3}}{2} + \frac{1}{2} \end{cases}$$

$$A = \frac{\frac{1}{2}-\frac{\sqrt{3}}{2}}{(x-2+\sqrt{3})} + \frac{\frac{\sqrt{3}}{2}+\frac{1}{2}}{(x-2-\sqrt{3})} = \sum_{m=0}^{\infty} \frac{1+\sqrt{3}}{2}(2+\sqrt{3})^m + \sum_{m=0}^{\infty} \frac{1-\sqrt{3}}{2}(2-\sqrt{3})^m$$

$$a_m = \frac{1+\sqrt{3}}{2}(2+\sqrt{3})^m + \frac{1-\sqrt{3}}{2}(2-\sqrt{3})^m$$

Para B

$$\begin{cases} A + B = 1 \\ (-2-\sqrt{3})A + (-2+\sqrt{3})B = -2 \end{cases} \Leftrightarrow \begin{cases} - \\ \cancel{-2}+\cancel{2B}-\sqrt{3}+\sqrt{3}B-\cancel{2B}+\sqrt{3}B = \cancel{-2} \end{cases} \Leftrightarrow \begin{cases} - \\ 2\sqrt{3}B = \sqrt{3} \end{cases} \Leftrightarrow \begin{cases} A = \frac{1}{2} \\ B = \frac{1}{2} \end{cases}$$

$$B = \frac{\frac{1}{2}}{(x-2-\sqrt{3})} + \frac{\frac{1}{2}}{(x-2+\sqrt{3})} = \sum_{m=0}^{\infty} \frac{1}{2}(2+\sqrt{3})^m + \frac{1}{2}(2-\sqrt{3})^m$$

$$b_m = \frac{1}{2}(2+\sqrt{3})^m + \frac{1}{2}(2-\sqrt{3})^m = \frac{1}{2}\left((2+\sqrt{3})^m + (2-\sqrt{3})^m\right)$$