# More Exercises for Stochastic Calculus for Finance I; Steven Shreve V Jun 02 2009

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July 25, 2009

## 1 The Binomial No-Arbitrage Pricing Model

### 1.1 Reading Comprehension (short questions and answers)

**1.** Explain the meaning of the following phrases from the book:

(a) "If $d = u$, the stock price at time one is not really random and the model is uninteresting." (p.2, second paragraph)

(b) "[...] prices of derivative securities depend on the volatility of stock prices but not on their mean rates of growth." (p.8, first paragraph)

(c) "The arbitrage pricing theory approach to the option-pricing problems is to replicate the option by trading in the stock and money markets." (p.3, 6th paragraph)

(d) "The mean rate of growth (with respect to the risk-neutral probabilities) of the stock is equal to the rate of growth of the money market account."

**2.** What is arbitrage (see p.2)?

**3.** What prevents the persistency of arbitrage in a real market (p.2)?

**4.** What is an stochastic process (p.11)?

**5.** What is a complete market (p.14)?

**6.** What is a path-dependent option (p.14)?

**7.** On Example 1.3.2, p.17 the second term on the formula for $v_n(s, \ m)$ is $v_{n+1}(s/2, \ m)$. Is this correct? Why not $v_{n+1}(s/2, \ \max(m, s))$?

**8.** What are the main assumptions for pricing an option in the binomial model? (see p.5)

**9.** What is the difference between the risk-neutral probabilities and the actual probabilities?

**10.** Compare (see p.7) the mean rate of growth of an investment in the money market with the mean rate of growth of the stock under the:

(a) risk-neutral probabilities;      (b) actual probabilities.

## 1.2   Problems

**11.** Show that after $n$ periods there are $n+1$ possible stock prices.

**12.** Generalize Example 1.3.1, p.15 for any $S_0, u, d, r$.

**13.** Generalize Example 1.3.2, p.17 for any $S_0, u, d, r$.

**14.** Another way of obtaining $\widetilde{p}$ and $\widetilde{q}$ in Formula (1.1.8), p.6. Show that $(\widetilde{p}, \widetilde{q})$ is the solution of the linear system:

$$\begin{cases} \widetilde{p} + \widetilde{q} & = & 1, \\ \widetilde{p}u + \widetilde{q}d & = & 1 + r. \end{cases}$$

# 2   Probability Theory on Coin Space

## 2.1   Reading Comprehension (short questions and answers)

**1.** Note the abuse of notation in Equation (2.1.5), p.26: the symbol $\mathbb{P}$ is applied to a set and to an element of this set. Establish the domain and contra-domain of these functions.

**2.** What is a random variable.

**3.** Explain what is the distribution of a random variable.

**4.** True or False: two random variables with the same distribution are equal.

**5.** Explain the meaning of $\mathbb{P}\{X = j\}$.

**6.** Given a random variable $X$ defined on a space of 30 coin tosses, the conditional expectations:
    (a) $E_4[X]$ is a random variable that depends on _____ coin tosses;
    (b) $E_0[X] = $ _____;    (c) $E_30[X] = $ _____;    (d) $E_10[E_12[X]] = $ _____;
    (e) $E_15[E_11[X]] = $ _____;    (f) $E[E_20[X]] = $ _____
    **Answer:** (a) the first 4; (b) $E[X]$; (c) $X$; (d) $E_10[X]$; (e) $E_11[X]$; (f) $E[X]$.

**7.** Let $Z_n$ be a random variable that depends only on the $n$-th coin toss and $X$ any random variable that depends only on tosses 10 trough $N$.
    (a) $E_4[Z_3] = $ _____;    (b) $E_5[Z_7] = $ _____;    (c) $E_3[Z_3] = $ _____;
    (d) $E_6[XZ_3] = $ _____;    (e) $E_6[XZ_6] = $ _____;    (f) $E_6[XZ_7] = $ _____
    **Answer:** (a) $Z_3$ by taking out what is known, (b) $E[Z_7]$ by independence; (c) $Z_3$ by taking out what is known. (d) $Z_3 E[X]$; (e) $Z_6 E[X]$; (f) $E[XZ_7]$.

**8.** What is an adapted stochastic process?

**9.** Why the portfolio process $\Delta_n$ is an adapted process? Why the wealth process $X_n$ (given by Equation (2.4.6), p.39) is an adapted process (p.36)?

**10.** What is the fundamental Theorem of Asset Pricing?

**11.** Give two important consequences of Theorem 2.4.5, p.40.
    **Answer:** there can be no arbitrage in the binomial model; another version of the risk neutral pricing formula.

**12.** Which processes are Markov and/or martingales?

(a) $S_n$ under $\mathbb{P}$;  (b) $S_n$ under $\widetilde{\mathbb{P}}$;  (c) $M_n$ under $\mathbb{P}$;
(d) $M_n$ under $\widetilde{\mathbb{P}}$;  (e) $S_n/(1+r)^n$

**13.** Explain the meaning of the following phrases from the book:

(a) "The portfolio rebalancing at each step is financed by investing or borrowing [...] from the money market." (p.39)

(b) "the discounted wealth process [...] is a martingale under the risk-neutral measure." (p.40) and relate to "the portfolio the investor uses is irrelevant." (p.40)

## 2.2 Problems

**14.** Let $p = 2/3$, $q = 1/3$. Show that $E_n[S_{n+1}] = \frac{3}{2}S_n$ (p.38).

**15.** Show that $S_{n+1}/S_n$ depends only on the $(n+1)-$th coin toss. In particular, it does not depend on the $n$-th coin toss.

**16.** Suppose $X_n$ is a martingale. Define $Y_n = \max\{X_n, 0\}$. Show that $Y_n$ is a sub-martingale.

**17.** Suppose that $X_n$ and $Y_n$ are sub-martingales. Show that $Z_n = \max\{X_n, Y_n\}$ is a sub-martingale.

**18.** Toss a coin repeatedly. Assume the probability of head on each toss is $1/2$, as is the probability of tail. Let $X_j = 2$ if the $j$th toss results in a head and $X_j = 0$ if the the $j$th toss results in a tail.

Consider the stochastic process $M_1, M_2, \ldots$ defined by

$$M_n = \prod_{j=1}^{n} X_j,$$

that is, $M_1 = X_1$, $M_2 = X_1 X_2$ (product of $X_1$ and $X_2$), $M_3 = X_1 X_2 X_3, \ldots$ This process formalizes the following game (double or nothing): you start with a dollar and you are tossing a fair coin independently. If it turns up heads you double your fortune, tails you go broke. [source finlmath.pdf]

Compute:

(a) $\mathbb{E}[X_j]$;  (b) $\mathbb{E}_{10}[X_{12}]$;  (c) $\mathbb{E}_{10}[X_8]$;  (d) $\mathbb{E}_n[e^{X_{n+1}}]$.

Show that

(e) $M_n$ is a martingale.

**19.** Let $X$ be a random variable (with finite expectation), define $M_n = \mathbb{E}_n[X]$. Show that $M_n$ is a martingale.

**20.** Suppose (the usual setup) that $S_0 = 4$, $u = 2$, $d = 1/2$ and $r = 1/4$ (so $\widetilde{p} = \widetilde{q} = 1/2$ and $1/(1+r) = 4/5$). Consider an option that pays off

$$V_3 = \max_{0 \le n \le 3} S_n - \min_{0 \le n \le 3} S_n.$$

Determine (draw a tree showing their values at each node):

(a) $V_3$;  (b) $V_2$.

**21.** Suppose a stock is selling for 100. Suppose you know that for the following two months the stock will rise by 10% or fall by 10% on each month. Assume

the risk-free rate is 1%. Consider an European put option that expires at time 2 and has strike 100 (it pays $V_2 = (100 - S_2)^+$). Determine:

(a) the risk-free probabilities (write as fractions);     (b) $V_2$;     (c) $V_1$.

**22.** Consider $Y_n = \sum_{k=0}^{n} S_k$, the sum of the stock prices, in the binomial model of Figure 2.3.1, p.32 with $p = q = 1/2$. Show that $Y_n$ is not Markov.

**23.** Let $S_n$ be the stock price and $M_n$ the maximum to date process. Show that $(S_n, M_n)$ is Markov.

Hint: see p.51 and 52. $S_{n+1} = \frac{S_{n+1}}{S_n} S_n$.

**24.** Toss a coin repeatedly. Assume the probability of head on each toss is $p$. Let $X_j = 2$ if the $j$th toss results in a head and $X_j = 0$ if the $j$th toss results in a tail. Define $S_n = \sum_{j=1}^{n} X_j$.

Show that $S_n$ is a Markov process.

**25.** Suppose $Z_n$ is a super-martingale.

(a) Is it true that $Z_n + K$, for a given $K \in \mathbb{R}$, is a super-martingale?

(b) Consider the problem of determining an stochastic process $X_n$ such that $X_n \geq G_n$ for every $n$ and $X_n$ is a super-martingale. This problem has an unique solution?

# 3   State Price

## 3.1   Reading Comprehension (short questions and answers)

**1.** Explain the meaning of the following phrases from the book:

(a) "They [actual and risk-neutral probabilities] agree [..] on which price paths are possible; [...] they disagree only on what these positive probabilities are." (p.61)

**2.** What is the Radon-Nikodým derivative (in this Chapter)? Is it a number? (p.61) What is the Radon-Nikodým derivative process?

# 4   American Derivative Securities

## 4.1   Reading Comprehension (short questions and answers)

**1.** Explain the difference between an American and an European option. Which one is more valuable? Why?

**2.** Why an American option is a super-martingale under the risk-neutral measure?

**3.** What is an stopped (or frozen) process?

**4.** What is the optimal time to exercise an American option?

**5.** What is the intrinsic value for an American option?

**6.** Let $V_n$ be the American derivative security price with intrinsic value $G_n$. Then

(i) $V_n$ \_\_\_\_\_ $\{G_n, 0\}$ for all $n$;     (ii) $\frac{1}{(1+r)^n} V_n$ is a _____;

(iii) if $Y_n$ satisfies (i) and (ii) then _____ for all $n$.

## 4.2 Problems

**7.** Determine if it is a stopping time or not the following random variables, W in a 2-period binomial model and Z in a 3-period binomial model. If it is not, modify it at the <u>minimum number of points</u> so that it is a stopping time (the answer is not unique!).
  (a) $W(HH) = 2,\ W(HT) = 1,\ W(TH) = 1,\ W(TT) = 1$.
  (b) $Z(HHH) = 2,\ Z(HHT) = 3,\ Z(HTH) = \infty,\ Z(HTT) = 3,$
  $Z(THH) = 3,\ Z(THT) = 2,\ Z(TTH) = 1,\ Z(TTT) = 2$.

**8.** Suppose $\tau_1$ and $\tau_2$ are stopping times. Prove that are stopping times:
  (a) $\tau_1 + \tau_2$;   (b) $\max(\tau_1, \tau_2)$;   (c) $\min(\tau_1, \tau_2)$;   (d) $\tau_1 + K$ for $K \in \mathbb{N}$

**9.** Suppose $\tau$ is a stopping time. Let $A = [\tau \le n]$. Prove that if the sequence of coin tosses $w_1 w_2 \cdots w_n y_{n+1} \cdots y_N \in A$ then $w_1 w_2 \cdots w_n z_{n+1} \cdots z_N \in A$, that is, if the first $n$ coin tosses of a sequence are equal then either both belong to $A$ or both do not.

**10.** Suppose (the usual setup) that $S_0 = 4$, $u = 2$, $d = 1/2$ and $r = 1/4$ (so $\widetilde{p} = \widetilde{q} = 1/2$ and $1/(1+r) = 4/5 = 0.8$) in a two period model $(S_0, S_1, S_2)$. Consider an **American option** (can be exercised at any time) that pays off

$$g(S_n) = (5 - S_n)^+.$$

Determine:
  (a) the price at time zero of this derivative.
  (b) the values of the stopping time $\tau$ for each outcome (HH, HT, TH, TT).
  (c) repeat (a) and (b) for

$$g(S_n) = (4 - S_n)^+.$$

# 5  Random Walk

## 5.1  Reading Comprehension (short questions and answers)

**1.** What is the first passage time of the random walk? Is it a number?

**2.** Let $\tau_m$ be the first time the random walk reaches level $m$.
Then $\mathbb{P}\{\tau_m < \infty\} = 1$ means that, with probability 1, the symmetric random walk _____.

**3.** An event happens almost surely if its probability is ___.

**4.** What is a perpetual American put? (p.129)

## 5.2  Problems

**5.** Let $\tau_m = \min\{n; M_n = m\}$. Prove that $\tau_m$ is a stopping time in the sense of Definition 4.3.1, p.97.

**6.** Let $M_n$ be a symmetric random walk, and define its maximum-to-date process $Z_n = \max_{1 \le k \le n} M_k$. Use an argument based on reflected paths:

(a) to determine $a$ such that $\mathbb{P}\{Z_n \geq 10, M_n = 4\} = \mathbb{P}\{M_n = a\}$ for n an even positive integer;

(b) to prove that $\mathbb{P}\{\tau_1 \leq 5\} = \mathbb{P}\{M_5 = 1\} + 2\mathbb{P}\{M_5 \geq 3\}$.

Explain this argument carefully.

(b) If the random walk is asymmetric with probability $p$ for an up step and probability $q = 1 - p$ for a down step, where $0 < p < 1$, what is $\mathbb{P}\{M_n \geq 10, M_n = 4\}$?

**7.** For the **asymmetric** random walk, consider the first passage time $\tau_m$ to the level $m$. Fix $\alpha \in (0, 1)$.

(a) Is it true that $E[\alpha^{\tau_{-1}}] = E[\alpha^{\tau_1}]$?

(b) Is it true that $(E[\alpha^{\tau_{-1}}])^2 = E[\alpha^{\tau_{-2}}]$?

Justify your answer ($\tau_{-1}$ is the the first passage time to the level $-1$).

**8.** From exercises 5.2 and 5.3, p.138 and 139, $\mathbb{P}\{\tau_1 < \infty\} = 1$ if $p > 1/2$ or $p/q$ if $p < 1/2$. Use it to determine $\mathbb{P}\{\tau_{-1} < \infty\}$ for any $p$.

**9.** Consider the random walk with downward drift ($p < 1/2$). Compute $\mathbb{P}\{\tau_m < \infty\}$

**Answer:** $(p/q)^m$

**10.** Generalize the analysis in p.129–131 for: Any $S_0, r, K$