———. 2004. World poverty: causes and pathways. In *Annual World Bank Conference on Development Economics 2003: Accelerating Development*, edited by F. Bourguignon and B. Pleskovic, pp. 159–96. New York: World Bank and Oxford University Press.
Debreu, G. 1959. *Theory of Value*. New York: John Wiley.
Diamond, J. 1997. *Guns, Germs and Steel: A Short History of Everybody for the Last 13,000 Years*. London: Chatto & Windus.
Durlauf, S. N., and H. Peyton Young, eds. 2001. *Social Dynamics*. Cambridge, MA: MIT Press.
Evans, G., and S. Honkapohja. 2001. *Learning and Expectations in Macroeconomics*. Princeton, NJ: Princeton University Press.
Fogel, R. W. 2004. *The Escape from Hunger and Premature Death, 1700–2100: Europe, America, and the Third World*. Cambridge: Cambridge University Press.
Fudenberg, D., and E. Maskin. 1986. The folk theorem in repeated games with discounting or with incomplete information. *Econometrica* 54(3):533–54.
Landes, D. 1998. *The Wealth and Poverty of Nations*. New York: W. W. Norton.
Ramsey, F. P. 1928. A mathematical theory of saving. *Economic Journal* 38:543–49.
Samuelson, P. A. 1947. *Foundations of Economic Analysis*. Cambridge, MA: Harvard University Press.
———. 1958. An exact consumption loan model with or without the social contrivance of money. *Journal of Political Economy* 66:1002–11.

VII.9 The Mathematics of Money
Mark Joshi

1 Introduction

The last twenty years have seen an explosive growth in the use of mathematics in finance. Mathematics has made its way into finance mainly via the application of two principles from economics: *market efficiency* and *no arbitrage*.

Market efficiency is the idea that the financial markets price every asset correctly. There is no sense in which a share can be a "good buy," because the market has already taken all available information into account. Instead, the only way that we have of distinguishing between two assets is their differing *risk characteristics*. For example, a technology share might offer a high rate of growth but also a high probability of losing a lot of money, while a U.K. or U.S. government bond would offer a much smaller rate of growth, but an extremely low probability of losing money. In fact, the probability of loss is so small in the latter case that these instruments are generally regarded as being riskless.

No arbitrage, the second fundamental principle, simply says that it is impossible to make money without taking risk. It is sometimes called the "no free lunch" principle. In this context, "making money" is defined to mean making *more* money than could be obtained by investing in a riskless government bond. A simple application of the principle of no arbitrage is that if one changes dollars into yen and then the yen into euros and then the euros back into dollars, then, apart from any transaction costs, one will finish with the same number of dollars that one started with. This forces a simple relationship between the three foreign exchange (FX) rates:

$$\mathrm{FX}_{\$,€} = \mathrm{FX}_{\$,¥}\mathrm{FX}_{¥,€}. \tag{1}$$

Of course, occasional anomalies and exceptions to this relationship can occur, but these will be spotted by traders. The exploitation of the resulting arbitrage opportunity will quickly move the exchange rates until the opportunity disappears.

One can roughly divide the use of mathematics in finance into four main areas.

Derivatives pricing. This is the use of mathematics to price *securities* (i.e., financial instruments), whose value depends purely upon the behavior of another asset. The simplest example of such a security is a *call option*, which is the right, but not the obligation, to buy a share for a pre-agreed price, K, on some specified future date. The pre-agreed price is called the *strike*. The pricing of derivatives is heavily reliant upon the principle of no arbitrage.

Risk analysis and reduction. Any financial institution has holdings and borrowings of assets; it needs to keep careful control of how much money it can lose from adverse market moves and to reduce these risks as necessary to keep within the owners' desired risk profiles. T&T note: check word spacing before press.

Portfolio optimization. Any investor in the markets will have notions of how much risk he wants to take and how much return he wants to generate, and most importantly of where he sees the trade-off between the two. There is, therefore, a theory of how to invest in shares in such a way as to maximize the return at a given level of risk. This theory relies greatly on the principle of market efficiency.

Statistical arbitrage. Crudely put, this is using mathematics to predict price movements in the stock market, or indeed in any other market. Statistical arbitrageurs laugh at the concept of market efficiency, and their objective is to exploit the inefficiencies in the market to make money.

Of these four areas, it is derivatives pricing that has seen the greatest growth in recent years, and which has seen the most powerful application of advanced mathematics.

2 Derivatives Pricing

2.1 Black and Scholes

Many of the foundations of mathematical finance were laid down by Bachelier (1900) in his thesis; his mathematical study of BROWNIAN MOTION [IV.24] preceded that of Einstein (see Einstein (1985), which contains his 1905 paper). However, his work was neglected for many years and the great breakthrough in derivatives pricing was made by Black and Scholes (1973). They showed that, under certain reasonable assumptions, it was possible to use the principle of no arbitrage to guarantee a unique price for a call option. The pricing of derivatives had ceased to be an economics problem and had become a mathematics problem.

The result of Black and Scholes was deduced by extending the principle of no arbitrage to encompass the idea that an arbitrage could result not just from static holdings of securities, but also from continuously trading them in a dynamic fashion depending upon their price movements. It is this principle of no *dynamic* arbitrage that underpins derivatives pricing.

In order to properly formulate the principle, we have to use the language of probability theory.

An *arbitrage* is a trading strategy in a collection of assets, the *portfolio*, such that

(i) initially the portfolio has a value of zero;
(ii) the probability that the portfolio will have a negative value in the future is zero;
(iii) the probability that the portfolio will have a positive value in the future is greater than zero.

Note that we do not require the profit to be certain; we merely require that it is possible that money may be made with no risk taken. (Recall that the notion of making money is by comparison with a government bond. The same is true of the "value" of a portfolio: it will be considered positive in the future if its price has increased by more than that of a government bond.)

The prices of shares appear to fluctuate randomly, but often with a general upward or downward tendency. It is natural to model them by means of a Brownian motion with an extra "drift term." This is what Black and Scholes did, except that it was the *logarithm* of the share price $S = S_t$ that was assumed to follow a Brownian motion W_t with a drift. This is a natural assumption to make, because changes in prices behave multiplicatively rather than additively. (For example, we measure inflation in terms of percentage increases.) They also assumed the existence of a riskless bond, B_t, growing at a constant rate. To put these assumptions more formally:

$$\log S = \log S_0 + \mu t + \sigma W_t, \tag{2}$$

$$B_t = B_0 e^{rt}. \tag{3}$$

Notice that the expectation of $\log S$ is $\log S_0 + \mu t$, so it changes at a rate μ, which is called the *drift*. The term σ is known as the *volatility*. The higher the volatility, the greater the influence of the Brownian motion W_t, and the more unpredictable the movements of S. (An investor will want a large μ and a small σ; however, market efficiency ensures that such shares are rather rare.) Under additional assumptions such as that there are no transaction costs, that trading in a share does not affect its price, and that it is possible to trade continuously, Black and Scholes showed that if there is no dynamic arbitrage, then at time t, the price of a call option, $C(S,t)$, that expires at time T must be equal to

$$\mathrm{BS}(S,t,r,\sigma,T) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \tag{4}$$

with

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \tag{5}$$

and

$$d_2 = \frac{\log(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \tag{6}$$

Here, $\Phi(x)$ denotes the probability that a standard normal random variable has value less than x. As x tends to ∞, $\Phi(x)$ tends to 1, and as x tends to $-\infty$, $\Phi(x)$ tends to 0. If we let t tend to T, we find that d_1 and d_2 tend to ∞ if $S_T > K$ (in which case $\log(S_T/K) > 0$) and to $-\infty$ if $S_T < K$. It follows that the price $C(S,t)$ converges to $\max(S_T - K, 0)$, which is the value of a call option at expiry, just as one would expect. We illustrate this in figure 1.

There are a number of interesting aspects to this result that go far beyond the formula itself. The first and most important result is that the price is unique. Using just the hypothesis that it is impossible to make a

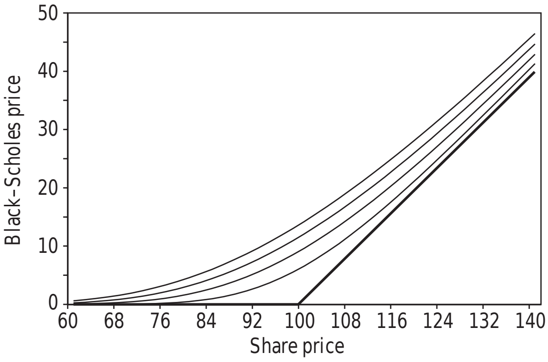

Figure 1 The Black–Scholes price of a call option struck at 100 for various maturities. The value decreases as maturity decreases, with the bottom line denoting a maturity of zero.

riskless profit, along with some natural and innocuous assumptions, we discover that there is only one possible price for the option. This is a very strong conclusion. It is not just the case that the option is a bad deal if traded at a different price: if a call option is bought for less or sold for more than the Black–Scholes price, then a *riskless profit* can be made.

A second fact, which may seem rather paradoxical, is that μ, the drift, does not appear anywhere in the Black–Scholes formula. This means that the expected behavior of the share's future mean price does not affect the price of the call option; our beliefs about the probability that the option will be used do not affect its price. Instead, it is the volatility of the share price that is all-important.

As part of their proof, Black and Scholes showed that the call option price satisfied a certain partial differential equation (PDE) now known as the *Black–Scholes equation*, or BS equation for short:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC. \tag{7}$$

This part of the proof did not rely on the derivative being a call option: there is in fact a large class of derivatives whose prices satisfy the BS equation, differing only in boundary conditions. If one changes variables, setting $\tau = T - t$ and $X = \log S$, then the BS equation becomes THE HEAT EQUATION [I.3 §5.4] with an extra first-order term which can easily be removed. This means that the value of an option behaves in a similar way to time-reversed heat: it diffuses and spreads out the farther back one gets from the option's expiry and the more uncertainty there is about the value of the share at time T.

2.2 Replication

The fundamental idea underlying the Black–Scholes proof and much of modern derivatives pricing is *dynamic replication.* Suppose we have a derivative Y that pays an amount that depends on the value of the share at some set of times $t_1 < t_2 < \cdots < t_n$, and suppose that the payout occurs at a certain time $T \geqslant t_n$. This can be expressed in terms of a *payoff function*, $f(t_1, \dots, t_n)$.

The value of Y will vary with the share price. If, in addition, we hold just the right number of the shares themselves, then a portfolio consisting of Y and the shares will be instantaneously immune to changes in the share price, i.e., its value will have zero rate of change with respect to the share price. As the value of Y will vary with time and share price, we will need to continuously buy and sell shares to maintain this neutrality to share-price movements. If we have sold a call option, then it turns out that we will have to buy when the share price goes up and sell when it goes down; so these transactions will cost us a certain amount of money.

Black and Scholes's proof showed that this sum of money was always the same and that it could be computed. The sum of money is such that by investing it in shares and riskless bonds, one can end up with a portfolio precisely equal in value to the payoff of Y no matter what the share price did in between.

Thus if one could sell Y for more than this sum of money, one would simply carry out the trading strategy from their proof and always end up ahead. Similarly, if one can buy Y for less, one does the negative of the strategy and always ends up ahead. Both of these are outlawed by the principle of no arbitrage, and a unique price is guaranteed.

The property that the payoff of any derivative can be replicated is called *market completeness.*

2.3 Risk-Neutral Pricing

A curious aspect of the Black–Scholes result, mentioned above, is that the price of a derivative does not depend upon the drift of the share price. This leads to an alternative approach to derivatives pricing theory called *risk-neutral pricing.* An arbitrage can be thought of as the ultimate unfair game: the player can only make money. By contrast, a MARTINGALE [IV.24 §4] encapsulates the notion of a fair game: it is a random process whose expected future value is always equal to its current value. Clearly, an arbitrage portfolio can never be a martingale. So if we can arrange for everything to be a martingale, there can be no arbitrages, and the price of derivatives must be free of arbitrage.

Unfortunately, this cannot be done because the price of the riskless bond grows at a constant rate, and is therefore certainly not a martingale. However, we can carry out the idea for *discounted prices*: that is, for prices of assets when they are divided by the price of the riskless bond.

In the real world, we do not expect discounted prices to be martingales. After all, why buy shares if their mean return is no better than that of a bond that carries no risk? Nevertheless, there is an ingenious way of introducing martingales into the analysis: by changing the PROBABILITY MEASURE [III.73 §2] that one uses.

If you look back at the definition of arbitrage, you will see that it depends only on which events have zero probability and which have nonzero probability. Thus, it uses the probability measure in a rather incomplete way. In particular, if we use a different probability measure for which the sets of measure zero are the same, then the set of arbitrage portfolios will not change. Two measures with the same sets of measure zero are said to be *equivalent*.

A theorem of Girsanov says that if you change the drift of a Brownian motion, then the measure that you derive from it will be equivalent to the measure you had before. This means that we can change the term μ. A good value to choose turns out to be $\mu = r - \frac{1}{2}\sigma^2$.

With this value of μ, one has

$$\mathbb{E}(S/B_t) = S/B_0 \tag{8}$$

for any t, and since we can take any time as our starting point, it follows that S/B_t is a martingale. (The extra $-\frac{1}{2}\sigma^2$ in the drift comes from the concavity of the coordinate change to log-space.) This means that the expectation has been taken in such a way that shares do not carry any greater return, on average, than bonds. Normally, as we have mentioned, one would expect an investor to demand a greater return from a risky share than from a bond. (An investor who does not demand such compensation is said to be *risk neutral*.) However, now that we are measuring expectations differently, we have managed to build an equivalent model in which this is no longer the case.

This yields a way of finding arbitrage-free prices. First, pick a measure in which the discounted price processes of all the fundamental instruments, e.g., shares and bonds, are martingales. Second, set the discounted price process of derivatives to be the expectations of their payoff; this makes them into martingales by construction.

Everything is now a martingale and there can be no arbitrage. Of course, this merely shows that the price is nonarbitrageable, rather than that it is the *only* nonarbitrageable price. However, work by Harrison and Kreps (1979) and by Harrison and Pliska (1981) shows that if a system of prices is nonarbitrageable, then there must be an equivalent martingale measure. Thus the pricing problem is reduced to classifying the set of equivalent martingale measures. Market completeness corresponds to the pricing measure being unique.

Risk-neutral evaluation has become such a pervasive technique that it is now typical to start a pricing problem by postulating risk-neutral dynamics for assets rather than real-world ones.

We now have two techniques for pricing: the Black–Scholes replication approach, and the risk-neutral expectation approach. In both cases, the real-world drift, μ, of the share price does not matter. Not surprisingly, a theorem from pure mathematics, the Feynman–Kac theorem, joins the two approaches together by stating that certain second-order linear partial differential equations can be solved by taking expectations of diffusive processes.

2.4 Beyond Black–Scholes

Terri: Tim would like to keep the heading as it is. OK?

For a number of reasons, the theory outlined above is not the end of the story. There is considerable evidence that the log of the share price does not follow a Brownian motion with drift. In particular, market crashes occur. For example, in October 1987 the stock market fell by 30% in one day and financial institutions found that their replication strategies failed badly. Mathematically, a crash corresponds to a jump in the share price, and Brownian motion has the property that all paths are continuous. Thus the Black–Scholes model failed to capture an important feature of share-price evolution.

A reflection of this failure is that options on the same share but with differing strike prices often trade with different volatilities, despite the fact that the BS model suggests that all options should trade with the same volatility. The graph of volatility as a function of the strike price is normally in the shape of a smile, displaying the disbelief of traders in the Black–Scholes model.

Another deficiency of the model is that it assumes that the volatility is constant. In practice, market activity varies in intensity and goes through some periods when share prices are much more volatile and others when they are much less so. Models must therefore be corrected to take account of the stochasticity of volatility, and the prediction of volatility over the life of an option is an important part of its pricing. Such models are called stochastic volatility models.

If one examines the data on small-scale share movements, one quickly discovers that they do not resemble a diffusion. They appear to be more like a series of small jumps than a Brownian motion. However, if one rescales time so that it is based on the number of trades that have occurred rather than on calendar time, then the returns do become approximately normal. One way to generalize the Black–Scholes model is to introduce a second process that expresses trading time. An example of such a model is known as the *variance gamma model*. More generally, the theory of Lévy processes has been applied to develop wider theories of price movements for shares and other assets.

Most generalizations of the Black–Scholes model do not retain the property of market completeness. They therefore give rise to many prices for options rather than just one.

2.5 Exotic Options

Many derivatives have quite complicated rules to determine their payoffs. For example, a *barrier option* can be exercised only if the share price does not go below a certain level at any time during the contract's life, and an *Asian option* pays a sum that depends on the average of the share price over certain dates rather than on the price at expiry. Or the derivative might depend upon several assets at once, such as, for example, the right to buy or sell a basket of shares for a certain price. It is easy to write down expressions for the value of such derivatives in the Black–Scholes model, either via a PDE or as a risk-neutral expectation. It is not so easy to evaluate these expressions. Much research is therefore devoted to developing efficient methods of pricing such options. In certain cases it is possible to develop analytic expressions. However, these tend to be the exception rather than the rule, and this means that one must resort to numerical techniques.

There is a wealth of methods for solving PDEs and these can be applied to derivatives-pricing problems. One difficulty in mathematical finance, however, is that the PDE can be very high dimensional. For example, if one is trying to evaluate a credit product depending on 100 assets, the PDE could be 100 dimensional. PDE methods are most effective for low-dimensional problems, and so research is devoted to trying to make them effective in a wider range of cases.

One method that is less affected by dimensionality is Monte Carlo evaluation. The basis of this method is very simple: both intuitively and (via the law of large numbers) mathematically, an expectation is the long-run average of a series of independent samples of a random variable X. This immediately yields a numerical method for estimating $\mathbb{E}(f(X))$. One simply takes many independent samples X_i of X, calculates $f(X_i)$ for each one and computes their average. It follows from the CENTRAL LIMIT THEOREM [III.73 §5] that the error after N draws is approximately distributed as a normal distribution with variance equal to $N^{-1/2}$ times the variance of $f(X)$. The rate of convergence is therefore dimension independent. If the variance of $f(X)$ is large, it may still be rather slow, however. Much effort is therefore devoted by financial mathematicians to developing methods of reducing the variance when one computes high-dimensional integrals.

2.6 Vanilla versus Exotics

Generally, a simple option to buy or sell an asset is known as a *vanilla option*, whereas a more complicated derivative is known as an *exotic option*. An essential difference between the pricing of the two is that one can hedge an exotic option not just with the underlying share, but also by trading appropriately in the vanilla options on that share. Typically, the price of a derivative will depend not just on observable inputs, such as the share price and interest rates, but also on unobservable parameters, such as the volatility of the share price or the frequency of market crashes, which cannot be measured but only estimated.

When trading exotic options, one wishes to reduce dependence upon these unobservable inputs. A standard way to do this is to trade vanilla options in such a way as to make the rate of change of the value of the portfolio with respect to such parameters equal to zero. A small misestimation of their value will then have little effect on the worth of the portfolio.

This means that when one prices exotic options, one wishes not just to capture the dynamics of the underlying asset accurately but also to price all the vanilla options on that asset correctly. In addition, the model will predict how the prices of vanilla options change when the share price changes. We want these predictions to be accurate.

The BS model takes volatility to be constant. However, one can modify it so that the volatility varies with the share price and over time. One can choose how it varies in such a way that the model matches the market prices of all vanilla options. Such models are known as *local volatility models* or *Dupire models*. Local volatility models were very popular for a while, but have become less so because they give a poor model for how the prices of vanilla options change over time.

Much of the impetus behind the development of the models we mentioned in section 2.4 comes from the desire to produce a model that is computationally tractable, prices all vanilla options correctly, and produces realistic dynamics for both the underlying assets and the vanilla options. This problem has still not been wholly solved. There tends to be a trade-off between realistic dynamics and perfect matching of the vanilla options market. One compromise is to fit the market as well as possible using a realistic model and then to superimpose a local volatility model to remove the remaining errors.

3 Risk Management

3.1 Introduction

Once we have accepted that it is impossible to make money in finance without taking risk, it becomes important to be able to measure and quantify risks. We wish to measure accurately how much risk we are taking and decide whether we are comfortable with that level of risk. For a given level of risk, we want to maximize our expected return. When considering a new transaction, we will want to examine how it affects our risk levels and returns. Certain transactions may even reduce our risk while increasing our returns if they cancel out other risk. (A risk that can be canceled out by other risks that have a tendency to move in the opposite direction is called *diversifiable*.)

The control of risk becomes particularly important when dealing with portfolios of derivatives, which are often of zero value initially but which can very quickly change value. Placing a limit on the value of the contracts held is therefore not of much use, and controls based on deal sizes are complicated by the fact that often many derivatives contracts largely cancel each other out; it is the *residual* risk that one wishes to control.

3.2 Value-at-Risk

One method of limiting an institution's risks in derivatives trading is to place a limit on the amount it can lose with a given probability over a specified period of time. For instance, one might consider the losses at a 1% level over ten days, or at a 5% level over one day. This value is called *Value-at-Risk* or VAR.

To compute VAR one has to build up a probabilistic model of how the portfolio of derivatives might change in value over the time period. This requires a model of how all the underlying assets can move. Given this model, one then builds up the distribution of possible profits and losses over the given time period. Once one has this distribution one simply reads off the desired percentile.

The issues involved in modeling the changes for VAR computation are quite different from those for derivatives pricing. Typically, a VAR computation is done over a very short time period, such as one or ten days, unlike the pricing of an option, which deals with a long time frame. Also, one is not interested in the typical path for VAR, but instead one focuses on the extreme moves. In addition, since it is the VAR of an entire portfolio that matters, one has to develop an accurate model of the underlying assets' *joint* distributions: the movement of one underlying asset could magnify the price movement of another, or it could act as a hedge.

There are two main approaches to developing a probabilistic model for computing VAR. The first, the historical approach, is to record all the daily changes over some time period, for example two years, and then assume that the set of changes tomorrow will be identical to one of the sets of changes we have recorded. If we assign equal probability to each of those changes, then we get an approximation to the profit and loss distribution, from which we can read off the desired percentile. Note that as we are using a day's change for all assets simultaneously, we automatically get an approximation to the joint distribution of all the asset prices.

A second approach is to assume that asset price movements come from some well-known class of distributions. For example, we could assume that the logs of the asset price movements are jointly normal. We would then use historical data to estimate the volatilities and the correlations between the various prices. The main difficulty with this approach is obtaining robust estimates of the correlations given a limited amount of data.

4 Portfolio Optimization

4.1 Introduction

The job of a fund manager is to maximize the return on the money invested while minimizing the risk. If we assume that markets are efficient, then there is no point in trying to pick shares that we believe to be undervalued as we have assumed that they do not exist. A corollary is that just as no shares are good buys, no shares are bad buys. In any case, over half the shares in the market are owned via funds and therefore under the control of fund managers. Therefore, the average fund manager cannot expect to outperform the market.

It may seem that this does not leave much for fund managers to do, but in fact it leaves two things.

(i) They can attempt to control the amount of risk they are taking.
(ii) For a given level of risk, they can maximize their expected return.

To do these things requires an accurate model of the joint distribution of asset prices over the longer term, and a quantifiable notion of risk.

4.2 The Capital Asset Pricing Model

Portfolio theory has been in its modern form for longer than derivatives pricing. As an area, it relies less on stochastic calculus and more on economics. We briefly review the key ideas. The best-known model for modeling portfolio returns is the *capital asset pricing model* (or CAPM), which was introduced in the 1950s by Sharpe (see Sharpe 1964), and is still ubiquitous. Sharpe's model built on earlier work of Markowitz (1952).

The fundamental problem in this area is to assess what portfolio of assets, generally shares, an investor should hold in order to maximize returns at a given level of risk. The theory requires assumptions to be made about the joint distribution of share returns, e.g., joint normality, and/or about the risk preferences of investors, e.g., that they only care about the mean and variance of returns.

Under these assumptions, the CAPM yields the result that every investor should hold a multiple of the "market portfolio," which is essentially a portfolio consisting of everything traded in appropriate quantities to achieve maximum diversification, together with a certain amount of the risk-free asset. The relative amounts are determined by the investor's risk preferences.

A consequence of the model is the distinction between diversifiable risk and undiversifiable risk. While investors are compensated for taking undiversifiable, or systematic, risk via higher expected returns, diversifiable risk does not carry a risk premium. This is because one can cancel out diversifiable risk by holding appropriate combinations of other assets. Therefore, if it carried a risk premium, investors could receive extra return without taking any risk.

Much of the current research in this area is directed at trying to find more accurate models for the joint distribution of returns, and at finding techniques that estimate the parameters of such returns. A related problem is the "equity premium puzzle," which is that the excess return on investing in shares is much higher than the model predicts for reasonable levels of risk aversion.

5 Statistical Arbitrage

We only briefly mention statistical arbitrage as it is a rapidly changing area that is shrouded in secrecy. The fundamental idea in this area is to squeeze information out of asset price movements that the market has not already acted on. It therefore contradicts the principle of market efficiency, which says that all available information is already encoded in the market price. One explanation is that it is the action of taking such arbitrages that makes the market efficient.

Further Reading

Bachelier, L. 1900. *La Théorie de la Spéculation.* Paris: Gauthier-Villars.
Black, F., and M. Scholes. 1973. The valuation of options and corporate liabilities. *Journal of Political Economy* 81:637–54.
Einstein, A. 1985. *Investigations on the Theory of the Brownian Movement.* New York: Dover.
Harrison, J. M., and D. M. Kreps. 1979. Martingales and arbitrage in multi-period securities markets. *Journal of Economic Theory* 20:381–408.
Harrison, J. M., and S. R. Pliska. 1981. Martingales and stochastic integration in the theory of continuous trading. *Stochastic Processes and Applications* 11:215–60.
Markowitz, H. 1952. Portfolio selection. *Journal of Finance* 7:77–99.
Sharpe, W. 1964. Capital asset prices: a theory of market equilibrium under conditions of risk. *Journal of Finance* 19:425–42.

VII.10 Mathematical Statistics

Persi Diaconis