More Exercises for Stochastic Calculus for Finance I; Steven Shreve VMai 03 2016

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1 The Binomial No-Arbitrage Pricing Model

1.1 Reading Comprehension (short questions and answers)

1. Explain the meaning of the following phrases from the book:
(a) "If $d = u$, the stock price at time one is not really random and the model is uninteresting." (p.2, second paragraph)
(b) "[...] prices of derivative securities depend on the volatility of stock prices but not on their mean rates of growth." (p.8, first paragraph)
(c) "The arbitrage pricing theory approach to the option-pricing problems is to replicate the option by trading in the stock and money markets." (p.3, 6th paragraph)
(d) "The mean rate of growth (with respect to the risk-neutral probabilities) of the stock is equal to the rate of growth of the money market account."

2. What is arbitrage (see p.2)?

3. What prevents the persistency of arbitrage in a real market (p.2)?

4. What is an stochastic process (p.11)?

5. What is a complete market (p.14)?

6. What is a path-dependent option (p.14)?

7. On Example 1.3.2, p.17 the second term on the formula for $v_n(s,\ m)$ is $v_{n+1}(s/2,\ m)$. Is this correct? Why not $v_{n+1}(s/2,\ \max(m, s))$?

8. What are the main assumptions for pricing an option in the binomial model? (see p.5)

9. What is the difference between the risk-neutral probabilities and the actual probabilities?

10. Compare (see p.7) the mean rate of growth of an investment in the money market with the mean rate of growth of the stock under the:
(a) risk-neutral probabilities; (b) actual probabilities.

1.2 Problems

11. Show that after n periods there are $n+1$ possible stock prices.

12. Generalize Example 1.3.1, p.15 for any S_0, u, d, r.

13. Generalize Example 1.3.2, p.17 for any S_0, u, d, r.

14. Another way of obtaining $\widetilde{p}$ and $\widetilde{q}$ in Formula (1.1.8), p.6. Show that $(\widetilde{p}, \widetilde{q})$ is the solution of the linear system:

$$\left\{ \begin{array}{rcl} \widetilde{p} + \widetilde{q} & = & 1, \\ \widetilde{p}u + \widetilde{q}d & = & 1 + r. \end{array} \right.$$

15. Suppose a stock is selling for 100. Suppose you know that for the following two months the stock will rise by 20% or fall by 10% on each month. Assume the risk-free rate is 1%. Consider an option that pays off

$$V_2 = (\max_{0 \leq n \leq 2} S_n) - (\min_{0 \leq n \leq 2} S_n).$$

(a) Determine the risk-free probabilities $\widetilde{p}$ and $\widetilde{q}$ (write as fractions, do not need to compute the value).

(b) Determine V_2 (for each ω).

(c) Determine V_1 (for each ω) (write the numerical expressions, do not need to give values).

Hint: $100(1.2)^2 = 144, \quad 100(1.2)(0,9) = 108, \quad 100(0.9)^2 = 81.$

16. Suppose a stock is selling for 100. Suppose you know that for the following two months the stock will rise by 10% or fall by 10% on each month. Assume the risk-free rate is 1%. Consider an european put option that expires at time 2 and has strike 100 (it pays $V_2 = (100 - S_2)^+$).

(a) Determine the risk-free probabilities (write as fractions).

(b) Determine V_2.

(c) Determine V_1 (write the formula, do not need to give value).

2 Probability Theory on Coin Space

2.1 Reading Comprehension (short questions and answers)

1. Note the abuse of notation in Equation (2.1.5), p.26: the symbol $\mathbb{P}$ is applied to a set and to an element of this set. Establish the domain and contra-domain of these functions.

2. What is a random variable.

3. Explain what is the distribution of a random variable.

4. True or False: two random variables with the same distribution are equal.

5. Explain the meaning of $\mathbb{P}\{X = j\}$.

6. Given a random variable X defined on a space of 30 coin tosses, the conditional expectations:

(a) $E_4[X]$ is a random variable that depends on _____ coin tosses;

(b) $E_0[X] =$ ____; (c) $E_{30}[X] =$ ____; (d) $E_{10}[E_{12}[X]] =$ ____;
(e) $E_{15}[E_{11}[X]] =$ ____; (f) $E[E_{20}[X]] =$ ____

Answer: (a) the first 4; (b) $E[X]$; (c) X; (d) $E_{10}[X]$; (e) $E_{11}[X]$; (f) $E[X]$.

7. Let Z_n be a random variable that depends only on the n-th coin toss and X any random variable that depends only on tosses 10 trough N.
(a) $E_4[Z_3] =$ ____; (b) $E_5[Z_7] =$ ____; (c) $E_3[Z_3] =$ ____;
(d) $E_6[XZ_3] =$ ____; (e) $E_6[XZ_6] =$ ____; (f) $E_6[XZ_7] =$ ____

Answer: (a) Z_3 by taking out what is known, (b) $E[Z_7]$ by independence; (c) Z_3 by taking out what is known. (d) $Z_3E[X]$; (e) $Z_6E[X]$; (f) $E[XZ_7]$.

8. What is an adapted stochastic process?

9. Why the portfolio process Δ_n is an adapted process? Why the wealth process X_n (given by Equation (2.4.6), p.39) is an adapted process (p.36)?

10. What is the fundamental Theorem of Asset Pricing?

11. Give two important consequences of Theorem 2.4.5, p.40.

Answer: there can be no arbitrage in the binomial model; another version of the risk neutral pricing formula.

12. Which processes are Markov and/or martingales?
(a) S_n under $\mathbb{P}$; (b) S_n under $\widetilde{\mathbb{P}}$; (c) M_n under $\mathbb{P}$;
(d) M_n under $\widetilde{\mathbb{P}}$; (e) $S_n/(1+r)^n$

13. Explain the meaning of the following phrases from the book:
(a) "The portfolio rebalancing at each step is financed by investing or borrowing [...] from the money market." (p.39)
(b) "the discounted wealth process [...] is a martingale under the risk-neutral measure." (p.40) and relate to "the portfolio the investor uses is irrelevant." (p.40)

2.2 Problems

14. . Let X be a random variable assuming the following values:

ω	$X(\omega)$
HHH	4
HHT	3
HTH	2
HTT	1
THH	-1
THT	-2
TTH	-3
TTT	-4

If $p = 1/5$, $a = 4/5$, compute the following random variable (write a complete table with the expressions, no need to compute it):
(a) $\mathbb{E}_2[X]$. (b) $\mathbb{E}_1[X]$.

15. Let X depend on the 5th and 6th coin toss, Y on the 2th and 3rd and Z on the 3rd and 4th. They are defined by the following tables:

ω	$X(\omega)$
$****HH$	4
$****HT$	3
$****TH$	2
$****TT$	1

ω	$Y(\omega)$
$*HH***$	1
$*HT***$	2
$*TH***$	3
$*TT***$	4

ω	$Z(\omega)$
$**HH**$	4
$**HT**$	3
$**TH**$	2
$**TT**$	1

If $p = 1/4$, $a = 3/4$, compute the following random variable (write a complete table with the expressions, no need to compute it, use properties):
(a) $\mathbb{E}_3[XY]$. (b) $\mathbb{E}_1[XY]$. (c) $\mathbb{E}_3[YZ]$. (d) $\mathbb{E}_2[YZ]$.

16. Let S_n be the usual stock-price process. Show that S_{n+1}/S_n depends only on the $(n+1)$-th coin toss. In particular, it does not depend on the n-th coin toss.

17. Let $p = 2/3$, $q = 1/3$. Show that $E_n[S_{n+1}] = \frac{3}{2} S_n$ (p.38).

18. Let S_n be the usual stock-price process and $Y = S_{n+1}/S_{n-1}$. Simplify, if you can, the following expressions. Justify your answer.
(a) $E_{n-2}[Y]$. (b) $E_{n-1}[Y]$. (c) $E_n[Y]$. (d) $E_{n+1}[Y]$. (e) $E_{n+2}[Y]$.

19. Suppose X_n is a martingale. Define $Y_n = \max\{X_n, 0\}$. Show that Y_n is a sub-martingale.

20. Suppose that X_n and Y_n are sub-martingales. Show that $Z_n = \max\{X_n, Y_n\}$ is a sub-martingale.

21. Toss a coin repeatedly. Assume the probability of head on each toss is $1/2$, as is the probability of tail. Let $X_j = 2$ if the jth toss results in a head and $X_j = 0$ if the the jth toss results in a tail.
Consider the stochastic process $M_1, M_2, \dots$ defined by

$$M_n = \prod_{j=1}^{n} X_j,$$

that is, $M_1 = X_1$, $M_2 = X_1 X_2$ (product of X_1 and X_2), $M_3 = X_1 X_2 X_3, \dots$ This process formalizes the following game (double or nothing): you start with a dollar and you are tossing a fair coin independently. If it turns up heads you double your fortune, tails you go broke. [source finlmath.pdf]
Compute:
(a) $\mathbb{E}[X_j]$; (b) $\mathbb{E}_{10}[X_{12}]$; (c) $\mathbb{E}_{10}[X_8]$; (d) $\mathbb{E}_n[e^{X_{n+1}}]$.
Show that
(e) M_n is a martingale.

22. Let X be a random variable (with finite expectation), define $M_n = \mathbb{E}_n[X]$. Show that M_n is a martingale.

23. Suppose (the usual setup) that $S_0 = 4$, $u = 2$, $d = 1/2$ and $r = 1/4$ (so $\widetilde{p} = \widetilde{q} = 1/2$ and $1/(1+r) = 4/5$). Consider an option that pays off

$$V_3 = \max_{0 \le n \le 3} S_n - \min_{0 \le n \le 3} S_n.$$

Determine (draw a tree showing their values at each node):
(a) V_3; (b) V_2.

24. Consider $Y_n = \sum_{k=0}^{n} S_k$, the sum of the stock prices, in the binomial model of Figure 2.3.1, p.32 with $p = q = 1/2$. Show that Y_n is not Markov.

25. Let S_n be the stock price and M_n the maximum to date process. Show that (S_n, M_n) is Markov.

Hint: see p.51 and 52. $S_{n+1} = \frac{S_{n+1}}{S_n} S_n$.

26. Toss a coin repeatedly. Assume the probability of head on each toss is p. Let $X_j = 2$ if the jth toss results in a head and $X_j = 0$ if the jth toss results in a tail. Define $S_n = \sum_{j=1}^{n} X_j$.

Show that S_n is a Markov process.

27. Suppose Z_n is a super-martingale.

(a) Is it true that $Z_n + K$, for a given $K \in \mathbb{R}$, is a super-martingale?

(b) Consider the problem of determining an stochastic process X_n such that $X_n \geq G_n$ for every n and X_n is a super-martingale. This problem has an unique solution?

3 State Price

3.1 Reading Comprehension (short questions and answers)

1. Explain the meaning of the following phrases from the book:

(a) "They [actual and risk-neutral probabilities] agree [..] on which price paths are possible; [...] they disagree only on what these positive probabilities are." (p.61)

2. What is the Radon-Nikodým derivative (in this Chapter)? Is it a number? (p.61) What is the Radon-Nikodým derivative process?

4 American Derivative Securities

4.1 Reading Comprehension (short questions and answers)

1. Explain the difference between an American and an European option. Which one is more valuable? Why?

2. Why an American option is a super-martingale under the risk-neutral measure?

3. What is an stopped (or frozen) process?

4. What is the optimal time to exercise an American option?

5. What is the intrinsic value for an American option?

6. Let V_n be the American derivative security price with intrinsic value G_n. Then

(i) V_n ______ $\{G_n, 0\}$ for all n; (ii) $\frac{1}{(1+r)^n} V_n$ is a ____________;

(iii) if Y_n satisfies (i) and (ii) then _________ for all n.

4.2 Problems

7. Determine if it is a stopping time or not the following random variables, W in a 2-period binomial model and Z in a 3-period binomial model. If it is not, modify it at the minimum number of points so that it is a stopping time (the answer is not unique!).
(a) $W(HH) = 2,\ W(HT) = 1,\ W(TH) = 1,\ W(TT) = 1.$
(b) $Z(HHH) = 2,\ Z(HHT) = 3,\ Z(HTH) = \infty,\ Z(HTT) = 3,$
$Z(THH) = 3,\ Z(THT) = 2,\ Z(TTH) = 1,\ Z(TTT) = 2.$

8. Suppose τ_1 and τ_2 are stopping times. Prove that are stopping times:
(a) $\tau_1 + \tau_2$; (b) $\max(\tau_1, \tau_2)$; (c) $\min(\tau_1, \tau_2)$; (d) $\tau_1 + K$ for $K \in \mathbb{N}$

9. Suppose τ is a stopping time. Let $A = [\tau \le n]$. Prove that if the sequence of coin tosses $w_1 w_2 \cdots w_n y_{n+1} \cdots y_N \in A$ then $w_1 w_2 \cdots w_n z_{n+1} \cdots z_N \in A$, that is, if the first n coin tosses of a sequence are equal then either both belong to A or both do not.

10. Suppose (the usual setup) that $S_0 = 4$, $u = 2$, $d = 1/2$ and $r = 1/4$ (so $\widetilde{p} = \widetilde{q} = 1/2$ and $1/(1+r) = 4/5 = 0.8$) in a two period model (S_0, S_1, S_2). Consider an **American option** (can be exercised at any time) that pays off

$$g(S_n) = (5 - S_n)^+.$$

Determine:
(a) the price at time zero of this derivative.
(b) the values of the stopping time τ for each outcome (HH, HT, TH, TT).
(c) repeat (a) and (b) for

$$g(S_n) = (4 - S_n)^+.$$

5 Random Walk

5.1 Reading Comprehension (short questions and answers)

1. What is the first passage time of the random walk? Is it a number?

2. Let τ_m be the first time the random walk reaches level m.
Then $\mathbb{P}\{\tau_m < \infty\} = 1$ means that, with probability 1, the symmetric random walk ____________________________.

3. An event happens almost surely if its probability is ___.

4. What is a perpetual American put? (p.129)

5.2 Problems

5. Let $\tau_m = \min\{n; M_n = m\}$. Prove that τ_m is a stopping time in the sense of Definition 4.3.1, p.97.

6. Let M_n be a symmetric random walk, and define its maximum-to-date process $Z_n = \max\limits_{1 \le k \le n} M_k$. Use an argument based on reflected paths:

(a) to determine a such that $\mathbb{P}\{Z_n \geq 10, M_n = 4\} = \mathbb{P}\{M_n = a\}$ for n an even positive integer;

(b) to prove that $\mathbb{P}\{\tau_1 \leq 5\} = \mathbb{P}\{M_5 = 1\} + 2\mathbb{P}\{M_5 \geq 3\}$.

Explain this argument carefully.

(b) If the random walk is asymmetric with probability p for an up step and probability $q = 1 - p$ for a down step, where $0 < p < 1$, what is $\mathbb{P}\{M_n \geq 10, M_n = 4\}$?

7. For the **asymmetric** random walk, consider the first passage time τ_m to the level m. Fix $\alpha \in (0, 1)$.

(a) Is it true that $E[\alpha^{\tau_{-1}}] = E[\alpha^{\tau_1}]$?

(b) Is it true that $\left(E[\alpha^{\tau_{-1}}]\right)^2 = E[\alpha^{\tau_{-2}}]$?

Justify your answer (τ_{-1} is the the first passage time to the level -1).

8. From exercises 5.2 and 5.3, p.138 and 139, $\mathbb{P}\{\tau_1 < \infty\} = 1$ if $p > 1/2$ or p/q if $p < 1/2$. Use it to determine $\mathbb{P}\{\tau_{-1} < \infty\}$ for any p.

9. Consider the random walk with downward drift ($p < 1/2$). Compute $\mathbb{P}\{\tau_m < \infty\}$

Answer: $(p/q)^m$

10. Generalize the analysis in p.129–131 for: Any S_0, r, K