Prova 1 de Modelagem Mat. em Finanças I (2011/1)
Professor Marco Cabral
30 de março de 2011

QUESTÃO 1: Suppose that $S_0 = 8$, $u = 2$, $d = 1/2$ and $r = 1/4$. Consider an option that pays off

$$V_3 = \left(50 - \sum_{0 \le n \le 3} S_n\right)^+ .$$

(a) Determine the risk-free probabilities $\widetilde{p}$ and $\widetilde{q}$.
(b) Determine V_3 (for each ω).
(c) Determine V_2 (for each ω) (write the numerical expressions, do not need to give values).

QUESTÃO 2: Explain why:
(a) If $d = u$, the stock prices are not really random and the model is uninteresting.
(b) Prices of derivatives depend on the volatility of stock prices but not on their mean rates of growth.
(c) The arbitrage pricing theory approach to the option-pricing problems is to replicate the option by trading in the stock and money markets.

QUESTÃO 3:
(a) What prevents the persistency of arbitrage in a real market?
(b) Show that in the binomial model after n periods there are $n + 1$ possible stock prices.
(c) What is the difference between the risk-neutral probabilities and the actual probabilities?

QUESTÃO 4: Suppose a stock is selling for 100. Suppose you know that for the following two months the stock will rise by 20% or fall by 10% on each month. Assume the risk-free rate is 1%. Consider an option that pays off

$$V_2 = (\max_{0 \le n \le 2} S_n) - (\min_{0 \le n \le 2} S_n).$$

(a) Determine the risk-free probabilities $\widetilde{p}$ and $\widetilde{q}$ (write as fractions, do not need to compute the value).
(b) Determine V_2 (for each ω).
(c) Determine V_1 (for each ω) (write the numerical expressions, do not need to give values).
Hint: $100(1.2)^2 = 144$, $100(1.2)(0,9) = 108$, $100(0.9)^2 = 81$.

Good Luck !

Prova 2 de Modelagem Mat. em Finanças I (2011/1)
Professor Marco Cabral
13 de abril de 2011

QUESTÃO 1: Suponha que P seja uma função que associa a cada subconjunto do conjunto finito Ω um número real. Assim $P(A) \in \mathbb{R}$ para cada $A \subset \Omega$. Suponha que esta função possui somente as seguintes propriedades:

(i) $P(A) \geq 0$ para todo $A \subset \Omega$;
(ii) $P(A \cup B) = P(A) \cup P(B)$ se A e B são disjuntos;
(iii) $P(\Omega) = 1$.

Prove, utilizando explicitamente somente os axiomas acima em cada item, que:
(a) $P(A^{\complement}) = 1 - P(A)$; (b) $P(C \cup D) \leq P(C) + P(D)$; (c) $P(\emptyset) = 0$;
(d) $P(A_1 \cup A_2 \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i)$ se os A_i's são disjuntos;
(e) se $A \subset B$ então $P(A) \leq P(B)$.

QUESTÃO 2: Let Z_n be a random variable that depends only on the n-th coin toss and X any random variable that depends only on tosses 3 and 6. Determine (if you can in terms of $\mathbb{E}$ or using the random variable itself):

(a) $\mathbb{E}_8[Z_5]$ (b) $\mathbb{E}_1[Z_3]$ (c) $\mathbb{E}_2[XZ_1]$ (d) $\mathbb{E}_8[XZ_5]$ (e) $\mathbb{E}_5[XZ_1]$

Justify your answer stating which property you are using.

QUESTÃO 3:

Toss a coin repeatedly. Assume the probability of head on each toss is 1/2, as is the probability of tail. Let $X_j = 2$ if the jth toss results in a head and $X_j = 0$ if the the jth toss results in a tail. Consider the stochastic process $M_1, M_2, \ldots$ defined by

$$M_n = \prod_{j=1}^{n} X_j,$$

that is, $M_1 = X_1$, $M_2 = X_1 X_2$ (product of X_1 and X_2), $M_3 = X_1 X_2 X_3, \ldots$

(a) Compute $\mathbb{E}[X_j]$ for every $j \in \mathbb{N}$.
(b) Compute $\mathbb{E}[M_2]$, $\mathbb{E}[M_3]$, $\mathbb{E}[M_j]$ for every $j \in \mathbb{N}$.
(c) Compute $\mathbb{E}_{10}[X_{12}]$.
(d) Compute $\mathbb{E}_{10}[X_8]$.
(e) Show that M_n is a martingale.

QUESTÃO 4:

(a) Let X be a random variable (with finite expectation), define $M_n = \mathbb{E}_n[X]$. Show that M_n is a martingale.

(b) Explain in your own words the meaning of the conditional expectation $\mathbb{E}_n(X)$ of a given random variable X.

GOOD LUCK !

Prova 3 de Modelagem Mat. em Finanças I (2011/1)
Professor Marco Cabral
27 de abril de 2011

QUESTÃO 1: Suponha que F seja uma função que associa a cada subconjunto do conjunto finito Ω um número real. Assim $F(A) \in \mathbb{R}$ para cada $A \subset \Omega$. Suponha que esta função possui somente as seguintes propriedades:

(i) $F(A) \geq 0$ para todo $A \subset \Omega$;

(ii) $F(A \cup B) = F(A) \cup F(B)$ se A e B são disjuntos.

Prove, utilizando explicitamente somente os axiomas acima em cada item (note que não necessariamente $F(\Omega) = 1!$) que:

(a) $F(C \cup D) = F(C) + F(D) - F(C \cap D)$; (b) $F(\emptyset) = 0$;

(c) se $A \subset B$ então $F(A) \leq F(B)$.

QUESTÃO 2: Let Z_n be a random variable that depends only on the n-th coin toss and X any random variable that depends only on tosses 4 and 7. Determine (if you can in terms of $\mathbb{E}$ or using the random variable itself):

(a) $\mathbb{E}_4[XZ_5]$ (c) $\mathbb{E}_5[XZ_1]$ (e) $\mathbb{E}_3[XZ_8]$

Justify your answer stating which property you are using.

QUESTÃO 3:

Toss a coin repeatedly. Assume the probability of head on each toss is 1/2, as is the probability of tail. Let $X_j = 1$ if the jth toss results in a head and $X_j = -1$ if the the jth toss results in a tail. Consider the stochastic process $M_1, M_2, \ldots$ defined by

$$M_n = \sum_{j=1}^{n} X_j.$$

Show that M_n is a martingale.

QUESTÃO 4: Suppose $M_0, M_1, \ldots$ is a martingale, and let $\Delta_0, \Delta_1, \ldots$ be an adapted process. Define the discrete-time stochastic integral $I_0, I_1, \ldots$ by setting $I_0 = 0$ and

$$I_n = \sum_{j=0}^{n-1} \Delta_j (M_{j+1} - M_j)$$

Show that $I_0, I_1, \ldots$ is a martingale.

QUESTÃO 5: Let X be a random variable (with finite expectation), define $M_n = \mathbb{E}_n[X]$. Show that M_n is a martingale.

QUESTÃO 6: Explain in your own words: What is an adapted stochastic process?

QUESTÃO 7: Suppose X_n is a martingale. Define $Y_n = \max\{X_n, 0\}$. Show that Y_n is a sub-martingale.

Good Luck !

Prova 4 de Modelagem Mat. em Finanças I (2011/1)
Professor Marco Cabral
11 de maio de 2011

QUESTÃO 1: Suppose (the usual setup) that $S_0 = 4$, $u = 2$, $d = 1/2$ and $r = 1/4$ (so $\widetilde{p} = \widetilde{q} = 1/2$ and $1/(1+r) = 4/5 = 0.8$) in a two period model (S_0, S_1, S_2). Consider an American option (can be exercised at any time) that pays off

$$g(S_n) = (4 - S_n)^+.$$

Determine:
(a) the price at time zero of this derivative.
(b) the values of the stopping time τ for each outcome (HH, HT, TH, TT).

QUESTÃO 2: Determine if it is a stopping time or not the following random variables, W in a 2-period binomial model and Z in a 3-period binomial model. If it is not, modify it at the minimum number of points so that it is a stopping time (the answer is not unique!).
(a) $W(HH) = 2, W(HT) = 1, W(TH) = 1, W(TT) = 1$.
(b) $Z(HHH) = 2, Z(HHT) = 3, Z(HTH) = \infty, Z(HTT) = 3, Z(THH) = 3, Z(THT) = 2, Z(TTH) = 1, Z(TTT) = 2$.

QUESTÃO 3: Determine if it is a stopping time or not the following Explain the difference between an American and an European option. Which one is more valuable? Why?

QUESTÃO 4: Why an American option is a super-martingale under the risk-neutral measure?

QUESTÃO 5: What is an stopped (or frozen) process?

QUESTÃO 6: What is the optimal time to exercise an American option?

QUESTÃO 7: What is the intrinsic value for an American option?

GOOD LUCK !

Prova 5 de Modelagem Mat. em Finanças I (2011/1)

Professor Marco Cabral

1 de junho de 2011

Utilize a fórmula: $V_n = \max_{\tau \in \mathbb{S}_n} \widetilde{\mathbb{E}}_n \left[\mathbb{I}_{\{\tau \leq N\}} \dfrac{G_\tau}{(1+r)^{\tau - n}} \right]$.

QUESTÃO 1: Prove que $V_N(\omega) = \max\{G_N(\omega),\ 0\}$ para todo $\omega \in \Omega$.

QUESTÃO 2: Mostre que $V_n \geq \max\{G_n, 0\}$ para todo n.

Dica: Tome $\tau_1(\omega) = n$ para todo ω e $\tau_2(\omega) = \infty$ para todo ω.

QUESTÃO 3: Justifique cada um dos passos abaixo (escreva (1) segue pois ..., (2) é verdade pois ...) supondo que $\tau^* \in \mathbb{S}_n$:

$$
\begin{aligned}
V_n &\geq \widetilde{\mathbb{E}}_n \left[\mathbb{I}_{\{\tau^* \leq N\}} \frac{G_{\tau^*}}{(1+r)^{\tau^* - n}} \right] && (1)\\
&= \widetilde{\mathbb{E}}_n \left[\widetilde{\mathbb{E}}_{n+1} \left[\mathbb{I}_{\{\tau^* \leq N\}} \frac{G_{\tau^*}}{(1+r)^{\tau^* - n}} \right] \right] && (2)\\
&= \widetilde{\mathbb{E}}_n \left[\frac{1}{1+r} \widetilde{\mathbb{E}}_{n+1} \left[\mathbb{I}_{\{\tau^* \leq N\}} \frac{G_{\tau^*}}{(1+r)^{\tau^* - n - 1}} \right] \right] && (3)\\
&= \widetilde{\mathbb{E}}_n \left[\frac{V_{n+1}}{1+r} \right]. && (4)
\end{aligned}
$$

QUESTÃO 4: Prove, utilizando a questão anterior, que $\dfrac{V_n}{(1+r)^n}$ é um supermartingal.

QUESTÃO 5: Suponha que ao longo do caminho que passa por $\omega_1 \cdots \omega_n$ nós temos que $\tau \leq n$ e que $p + q = 1$. Justifique cada um dos passos abaixo:

$$
\begin{aligned}
V_{n \wedge \tau}(\omega_1 \cdots \omega_n) &= V_\tau(\omega_1 \cdots \omega_n) && (5)\\
&= pV_\tau(\omega_1 \cdots \omega_n) + qV_\tau(\omega_1 \cdots \omega_n) && (6)\\
&= pV_{(n+1)\wedge\tau}(\omega_1 \cdots \omega_n H) + qV_{(n+1)\wedge\tau}(\omega_1 \cdots \omega_n T). && (7)
\end{aligned}
$$

QUESTÃO 6: Dizemos que f é convexa se $f(ax + (1-a)y) \leq af(x) + (1-a)f(y)$ para todo $x, y \in \mathbb{R}$ e $a \in [0, 1]$. Prove que se g é convexa e $g(0) = 0$ então $g(\lambda s) \leq \lambda g(s)$ para todo $s, \lambda \geq 0$.

QUESTÃO 7: Suponha que g é convexa, S_n é preço da ação, $\widetilde{P}$ a medida neutra a risco (usual setup) que torna o preço descontado da ação um martingal. Usando questão anterior e o que voce sabe, justifique os passos abaixo:

$$
\begin{aligned}
\frac{1}{1+r} \widetilde{\mathbb{E}}_n[g(S_{n+1})] &\geq \frac{1}{1+r} g(\widetilde{\mathbb{E}}_n[S_{n+1}]) && (8)\\
&\geq g\left(\widetilde{\mathbb{E}}_n \left[\frac{1}{1+r} S_{n+1} \right] \right) && (9)\\
&= g(S_n). && (10)
\end{aligned}
$$

QUESTÃO 8: Suponha que $\dfrac{W_n}{(1+r)^n}$ é um martingal com relação à medida $\widetilde{P}$ e que W_0 é constante. Prove que $W_0 = \widetilde{\mathbb{E}} \left[\dfrac{W_N}{(1+r)^N} \right]$.

Prova 6 de Modelagem Mat. em Finanças I (2011/1)
Professor Marco Cabral
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Nessa prova definimos $J_n = \max_{\tau \in \mathbb{S}_n} \widetilde{\mathbb{E}}_n \left[\mathbb{I}_{\{\tau \le N\}} \frac{W_\tau}{(1+r)^{\tau - n}} \right]$.

QUESTÃO 1: Mostre que $J_n \ge \max\{W_n, 0\}$ para todo n.

QUESTÃO 2: Justifique cada um dos passos abaixo (escreva (1) segue pois ..., (2) é verdade pois ...) supondo que $\tau^* \in \mathbb{S}_n$:

$$J_n \ge \widetilde{\mathbb{E}}_n \left[\mathbb{I}_{\{\tau^* \le N\}} \frac{W_{\tau^*}}{(1+r)^{\tau^* - n}} \right] \tag{1}$$
$$= \widetilde{\mathbb{E}}_n \left[\widetilde{\mathbb{E}}_{n+1} \left[\mathbb{I}_{\{\tau^* \le N\}} \frac{W_{\tau^*}}{(1+r)^{\tau^* - n}} \right] \right] \tag{2}$$
$$= \widetilde{\mathbb{E}}_n \left[\frac{1}{1+r} \widetilde{\mathbb{E}}_{n+1} \left[\mathbb{I}_{\{\tau^* \le N\}} \frac{W_{\tau^*}}{(1+r)^{\tau^* - n - 1}} \right] \right] \tag{3}$$
$$= \widetilde{\mathbb{E}}_n \left[\frac{J_{n+1}}{1+r} \right]. \tag{4}$$

QUESTÃO 3: Dizemos que f é convexa se $f(ax + (1-a)y) \le af(x) + (1-a)f(y)$ para todo $x, y \in \mathbb{R}$ e $a \in [0,1]$. Prove que se g é convexa e $g(0) = 0$ então $g(\lambda s) \le \lambda g(s)$ para todo $s, \lambda \ge 0$.

QUESTÃO 4: Suponha que $\frac{Y_n}{(1+r)^n}$ é um martingal com relação à medida $\widetilde{P}$ e que Y_0 é constante. Prove que $Y_0 = \widetilde{\mathbb{E}} \left[\frac{Y_N}{(1+r)^N} \right]$.

QUESTÃO 5: Let M_n be a symmetric random walk, and define its maximum-to-date process $Z_n = \max_{1 \le k \le n} M_k$. Use an argument based on reflected paths:

(a) to determine a such that $\mathbb{P}\{Z_n \ge 15, M_n = 4\} = \mathbb{P}\{M_n = a\}$ for n an even positive integer;

(b) to prove that $\mathbb{P}\{\tau_1 \le 5\} = \mathbb{P}\{M_5 = 1\} + 2\mathbb{P}\{M_5 \ge 3\}$.

Explain this argument carefully for (a) and (b).

QUESTÃO 6: Define $f(\sigma) = pe^\sigma + qe^{-\sigma}$ with $p, q > 0, p + q = 1$.

(a) Find $\sigma_0 > 0$ such that $f(\sigma_0) = 1$. Be careful to prove that your $\tau_0 > 0$

Hint: Let $x = e^\sigma$ and solve a quadradic equation. Only one solution is positive.

(b) Prove that $f(\sigma) > 1$ for all $\sigma > \sigma_0$.

Hint: Use calculus to show that f is increasing.

QUESTÃO 7: Let M_n be a symmetric random walk and $\tau = \min\{n; M_n = 3\}$. Prove that τ is a stopping time.