Segunda Prova de Tópicos em Matemática Aplicada (2009/1) corrigida

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QUESTÃO 1: Suppose (the usual setup) that $S_0 = 4$, $u = 2$, $d = 1/2$ and $r = 1/4$ (so $\widetilde{p} = \widetilde{q} = 1/2$ and $1/(1+r) = 4/5 = 0.8$) in a two period model (S_0, S_1, S_2). Consider an American option (can be exercised at any time) that pays off

$$g(S_n) = (4 - S_n)^+.$$

Determine:

(a) the price at time zero of this derivative.

(b) the values of the stopping time τ for each outcome (HH, HT, TH, TT).

QUESTÃO 2: Let Z be a random variable with the property that $P(Z > 0) = 1$ and $EZ = 1$. For $w \in \Omega$, define $\widetilde{P}(w) = Z(w)P(w)$. Show that:

(a) $\widetilde{P}$ is a probability measure, i.e., $\widetilde{P}(\Omega) = 1$;

(b) if Y is a random variable, then $\widetilde{E}Y = E[ZY]$;

(c) If A is an event with $\widetilde{P}(A) = 0$, then $P(A) = 0$.

QUESTÃO 3: Toss a coin repeatedly. Assume the probability of head on each toss is p. Let $X_j = 2$ if the jth toss results in a head and $X_j = 0$ if the jth toss results in a tail. Define $S_n = \sum_{j=1}^{n} X_j$.

Show that S_n is a Markov process.

QUESTÃO 4: Determine if it is a stopping time or not the following random variables, W in a 2-period binomial model and Z in a 3-period binomial model. If it is not, modify it at the <u>minimum number of points</u> so that it is a stopping time (the answer is not unique!).

(a) $W(HH) = 2, W(HT) = 1, W(TH) = 1, W(TT) = 1.$

(b) $Z(HHH) = 2, Z(HHT) = 3, Z(HTH) = \infty, Z(HTT) = 3, Z(THH) = 3, Z(THT) = 2, Z(TTH) = 1, Z(TTT) = 2.$

Good Luck!