Counting Processes and Asymptotic Theory

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Failure Time Models,
5nd chapter of *The Statistical Analysis of Failure Time Data*
Kalbfleisch and Prentice, 2002

Outline

- Counting processes and intensity functions
- Martingales

A counting process $N = \{N(t), t \geq 0\}$

is a stochastic process with $N(0) = 0$ and whose value at time t counts the number of events that have occured in the interval $(0, t]$.

- The sample paths of N are nondecrising step functions that jump whenever an event (or events) occur.
- In continuous time,

  no two counting processes can jump at the same time.

- In discrete time, they can.

Number of events that occur in the interval $[t, t + dt)$?
$dN(t) = N(t^{-} + dt) - N(t^{-})$.

Number of events that occur at time t? $\Delta N(t) = N(t) - N(t^{-})$.

And what about more general counting processes where individuals may experience more than one event? Chapters 8, 9, and 10.

Filtration: history of events

observed counting process: $N_i = \{N_i(t), t \geq 0\}$

underlying counting process: $\tilde{N}_i = \{\tilde{N}_i(t), 0 \leq t\}, \quad \tilde{N}_i(t) = \mathbf{1}(T_i \leq t)$

at-risk process: $\{Y_i(t), t \geq 0\}, \quad Y_i(t) = \mathbf{1}(T_i \geq t, C_i \geq t)$

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key concept: filtration

$$\mathcal{F}_t = \sigma\{N_i(u), Y_i(u^+), X_i(u^+), i = 1, \ldots, n; 0 \leq u \leq t\}, \quad t > 0,$$

where

$Y_i(u^+) = \lim_{s \to u^+} Y_i(s)$;

stochastic time-dependent covariate: $X_i(t) = \{x_i(u) : 0 \leq u \leq t\}$.

The notation $\sigma[\cdot]$ specifies the sigma algebra of events generated by the variables given in the brackets.

Intensity functions

The intensities or rates for the processes N_i are defined with reference to the filtration $\mathcal{F}_t$. If the censoring process is independent, the intensity model for the counting process N_i is

$$\mathbb{P}[dN_i(t) = 1 | \mathcal{F}_{t^-}] = Y_i(t) d\Lambda_i(t), \quad i = 1, \ldots, n, \quad t > 0.$$

The hazard model can be written $d\Lambda_i(t) = \mathbb{P}[d\tilde{N}_i(t) = 1 | X_i(t), \tilde{N}_i(t^-) = 0]$.

Λ_i is called the cumulative intensity process of the counting process $\tilde{N}_i$.

- In the continuous case, $\mathbb{P}[dN_i(t) = 1 | \mathcal{F}_{t^-}] = Y_i(t)\lambda_i(t)dt$
- In the discrete case, $\mathbb{P}[dN_i(a_l) = 1 | \mathcal{F}_{a_l^-}] = Y_i(a_l)\lambda_{il}, \quad l = 1, 2, \ldots$

$\lambda_i(t)$ and λ_{il} are the corresponding intensity processes.

Martingales: Intro

$$
\begin{aligned}
M_i(t) &= N_i(t) - \int_0^t Y_i(u)\lambda_i(u)du, \quad t \geq 0. \\
&= \int_0^t dM_i(u), \\
dM_i(t) &= dN_i(t) - Y_i(t)\lambda_i(t)dt.
\end{aligned}
$$

If

- $\mathbb{E}[dM_i(t)|\mathcal{F}_{t-}] = 0, \quad \forall t; \quad \equiv \quad \mathbb{E}[M_i(t)|\mathcal{F}_s] = M_i(s), \quad \forall s \leq t.$

Then, $M_i(t)$ is a martingale.

Consequences:

- $\mathbb{E}[M_i(t)] = 0, \quad \forall t;$
- the process $M_i(t)$ has uncorrelated increments, i.e., $\mathbb{E}[(M_i(t) - M_i(s)) \times M_i(s)] = 0, \quad \forall 0 < s < t.$

Decomposing $N_i(t)$ into two processes

$$N_i(t) = \underbrace{\int_0^t Y_i(u)\lambda_i(u)du}_{\substack{\text{compensator of the}\\ \text{counting process } N_i\\ \text{wrt the filtration } \mathcal{F}_t}} + \underbrace{M_i(t)}_{\substack{\text{counting}\\ \text{process}\\ \text{martingale}\\ \text{corresponding}\\ \text{to } N_i(t)}}$$

$$dN_i(t) = Y_i(t)\lambda_i(t)dt + dM_i(t).$$

In the discrete case, the discrete-time martingale is

$$\begin{aligned} N_i(t) &= \int Y_i(u)d\Lambda_i(u) + M_i(t) \\ &= \sum_{a_l \leq t} Y_i(a_l)\lambda_{il} + M_i(t), \\ dN_i(a_l) &= Y_i(a_l)\lambda_{il} + dM_i(a_l). \end{aligned}$$

More about martingales

In essense, a martingale is a process that has no drift and whose increments are uncorrelated.

- We say that $M(t)$ is a mean zero martingale if $\mathbb{E}[M(0)] = 0$, and hence $\mathbb{E}[M(t)] = 0, \forall t$.
- The martingale $M(t)$ is said to be square integrable (or have finite variance) if $\mathbb{E}[M^2(t)] = \mathbb{V}[M(t)] < \infty, \forall t \leq \tau$.

It is useful to define two technical terms applied to a stochastic process $U = \{U(t), t \geq 0\}$.

Adapted

U is said to be adapted to the filtration $\mathcal{F}_t$ if $U(t)$ is $\mathcal{F}_t$ measurable for each $t \in [0, \tau]$, i.e., the value of $U(t)$ is fixed once $\mathcal{F}_t$ is given.

Predictable

U is said to be predictable wrt the filtration $\mathcal{F}_t$ if $U(t)$ is $\mathcal{F}_{t^-}$ measurable for all $t \in [0, \tau]$, i.e., the value of $U(t)$ is fixed once $\mathcal{F}_{t^-}$ is given.

More about martingales

The process $\{\bar{M}(t), 0 \leq t \leq \tau\}$ is a submartingale wrt $\mathcal{F}_t$ if it is adapted and satisfies

$$\mathbb{E}[\bar{M}(t)|\mathcal{F}_s] \geq \bar{M}(s) \quad \forall s \leq t \leq \tau.$$

- A counting process $N(t)$ is a submartingale.

Cox, 1972

$$\lambda(t; x) = \lambda_0(t) \exp(Z^\top \beta),$$

where $\lambda_0(\cdot)$ is an arbitrary unspecified baseline hazard function for continuous T.

The conditional survivor function for T given Z is

$$F(t; x) = F_0^{\exp(Z^\top \beta)}(t), \quad \text{where} \quad F_0(t) = \exp\left[-\int_0^t \lambda_0(u)\mathrm{d}u\right].$$

Accelerated failure time model

Suppose $Y = \log T$ and consider the linear model

$$Y = Z^\top \beta + W.$$

Exponentiation gives $T = \exp(Z^\top \beta)\, S$, where $S = \exp(W) > 0$ has hazard function $\lambda_0(s)$, say, that is independent of β.

The hazard function for T can be written as

$$\lambda(t; x) = \exp(-Z^\top \beta)\lambda_0[t \exp(-Z^\top \beta)].$$

The effect of the covariate is multiplicative on t rather than on the hazard function.

i.e.,

The role of Z is to accelerate (or decelerate) the time to failure.

Comparison of regression models

note

Exponential and Weibull regression models can be considered as special cases of both models.

Discrete failure time models

Discrete failure time?

- Grouping of continuous data due to imprecise measurement;
- Time itself may be discrete
  - e.g., when the response time represents the number of episodes that occur prior to a terminal event.

Discrete regression models?

- Grouped relative risk model;
- Discrete and continuous relative risk model;
- Discrete logistic model.

Discrete regression models

» Grouped relative risk model:

$$\text{Discrete baseline cumulative hazard function} : \Lambda_0(t) = \sum_{a_i \leq t} \lambda_i,$$

this model is the uniquely appropriate one for grouped data from the continuous relative risk model.

» Discrete and continuous relative risk model:

$$\mathrm{d}\Lambda(t; x) = \exp(Z^\top \beta)\ \mathrm{d}\Lambda_0(t),$$

which retains the multiplicative hazard relationship.

» Discrete logistic model:

$$\frac{\mathrm{d}\Lambda(t; x)}{1 - \mathrm{d}\Lambda(t; x)} = \frac{\mathrm{d}\Lambda_0(t)}{1 - \mathrm{d}\Lambda_0(t)} \exp(Z^\top \beta),$$

specifies a linear log odds model for the hazard probability at each potential failure time.


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