AMCS 202 - Applied Mathematics II

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HOMEWORK II

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Contents

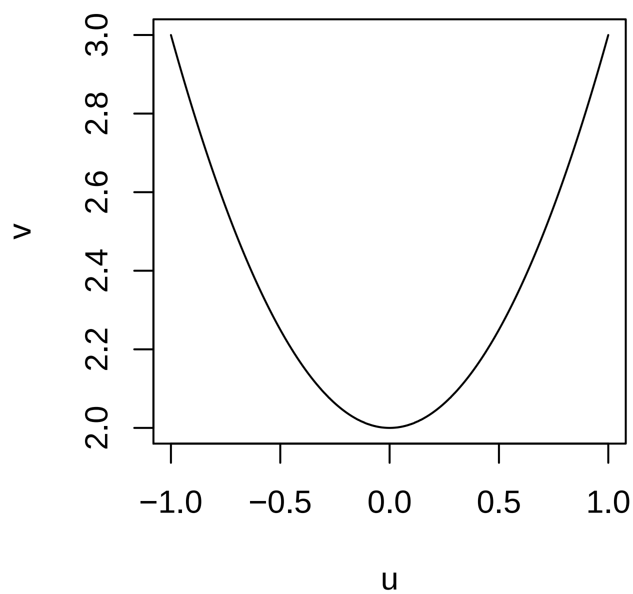
Problem 1

1.

Describe the range of $f(z) = x^2 + 2i$ defined on $|z| \leq 1$.

Solution:

```r
# <code r> ================================================================== #
y <- 2 # (y = v)                                                # fixing y in 2
x <- seq(-1, 1, length.out = 1e4)        # generating 10000 values of -1 <= x <= 1
par(mar = c(4, 4, 1, 1) + .1)                              # graphical definitions
plot(x**2 + y ~ x, type = "l", xlab = "u", ylab = "v")              # plotting f(z)
# </code r> ================================================================= #
```



The range is the parabola with center in $v = 2$ and maximum in $v = 3$.

$\square$

2.

Describe the range of $f(z) = z^3$ in the semidisk given by $|z| \leq 2$, $\text{Im}(z) \geq 0$.

Solution:

$$f(z) = z^3 = (|z|e^{i\theta})^3 = |z|^3 e^{i3\theta}.$$

From the given conditions:

$$0 \leq \theta \leq \pi, \qquad 0 \leq |z| \leq 2.$$

From $f(z) = |z|^3 e^{i3\theta}$:

$$0 \leq \theta \leq 3\pi, \qquad 0 \leq |z| \leq 8.$$

Therefore, the range of $f(z)$ is disk given by $|z| \leq 8$.

□

3.

Show that the inversion mapping $w = f(z) = 1/z$ maps

a)

The circle $z = r$ onto the circle $|w| = 1/r$.

Solution:

$$w = \frac{1}{z} \quad \Rightarrow \quad |w| = \left|\frac{1}{z}\right| = \frac{1}{|z|} = \frac{1}{r}$$

□

b)

The ray $\text{Arg } z = \theta_0$, $-\pi < \theta_0 < \pi$ onto the ray $\text{Arg } w = -\theta_0$.

Solution:

$$w = \frac{1}{z} \quad \Rightarrow \quad w = \frac{1}{r} e^{-i\theta_0}$$

$$\text{Arg } z = \theta_0, \qquad \text{Arg } w = -\theta_0$$

$\square$

Problem 2

1.

Using methods familiar from elementary calculus, find the limit (if it exists) of the following sequences of complex numbers:

a)

$z_n = (i/3)^n$**, start looking at** $|z_n|$**.**

Solution:

$$\lim_{n\to+\infty}\left|\left(\frac{i}{3}\right)^n\right| = \lim_{n\to+\infty}\left|\left(\frac{\sqrt{-1}}{3}\right)^n\right| = \lim_{n\to+\infty}\frac{1^{n/2}}{3^n} = \lim_{n\to+\infty}\frac{1}{3^n} = 0.$$

$\square$

b)

$z_n = (2+in)/(1+3n)$.

Solution:

$$z_n = \frac{2+in}{1+3n} = \frac{2}{1+3n} + i\frac{n}{1+3n}$$

$$\lim_{n\to+\infty} \text{Re}(z_n) = \lim_{n\to+\infty} \frac{2}{1+3n} = 0, \qquad \lim_{n\to+\infty} \text{Im}(z_n) = \lim_{n\to+\infty} \frac{n}{1+3n} = \frac{1}{3}. \quad \text{(Applying L'Hôpital's)}$$

The sequence z_n converges to $i/3$.

□

c)

$z_n = i^n$.

Solution:

$$z_n = i^n = i, -1, -i, 1, i, -1, -i, 1, i, ...$$

The sequence z_n diverges.

□

2.

Consider the following complex functions:

$$f_1(z) = z^2 - 2z + 1, \qquad f_2(z) = \frac{z+2i}{z}, \qquad f_3(z) = \frac{z^2+4}{z(z-2i)}.$$

a)

Find the domain of these functions and justify their continuity in the domain.

Solution:

- $f_1(z) = z^2 - 2z + 1$:

$f_1(z)$ is a polynomium, therefore with domain in all the complex plane and continuous everywhere.

- $f_2(z) = (z+2i)/z$:

$z + 2i$ is continuous everywhere, but $f_2(z)$ isn't continuous at $z = x + iy = 0$, i.e., at the origin (complex numbers are only zero at the origin). Therefore, the domain of $f_2(z)$ is all the complex plane except at the origin. The function is continuos in all its domain.

- $f_3(z) = (z^2+4)/(z(z-2i))$:

$z^2 + 4$ is a polynomium, so is continuous everywhere. The function $f_3(z)$ isn't continuous at $z = 0$, i.e., at the origin, and at $z = 2i$. Therefore, the domain of $f_3(z)$ is all the complex plane except at the origin and at $2i$.

□

b)

Calculate the limits of these functions as $z \to 2i$.

Solution:

- $f_1(z) = z^2 - 2z + 1$:

$$\lim_{z \to 2i} z^2 - 2z + 1 = (2i)^2 - 2(2i) + 1 = -4 - 4i + 1 = -3 - 4i.$$

- $f_2(z) = (z + 2i)/z$:

$$\lim_{z \to 2i} \frac{z + 2i}{z} = \frac{2i + 2i}{2i} = \frac{4i}{2i} = 2.$$

- $f_3(z) = (z^2 + 4)/(z(z - 2i))$:

$$\lim_{z \to 2i} \frac{z^2 + 4}{z(z - 2i)} = \lim_{z \to 2i} \frac{(z + 2i)(z - 2i)}{z(z - 2i)} = \lim_{z \to 2i} \frac{z + 2i}{z} = 2.$$

$\square$

c)

Redefine f_3 so that it becomes a continuous function at $z = 2i$.

Solution:

$$f_3(z) = \frac{z^2 + 4}{z(z - 2i)} = \frac{(z + 2i)(z - 2i)}{z(z - 2i)} = \frac{z + 2i}{z}.$$

Now $f_3(z)$ is continuous at $z = 2i$.

$\square$

Problem 3

1.

Show that $\mathrm{Re}(z)$ **and** $\mathrm{Im}(z)$ **are nowhere differentiable. Hint: try the approach use in class to get the Cauchy-Riemann equations.**

Solution:

$$z = x + yi \qquad \Rightarrow \quad u(x,y) = \mathrm{Re}(z) = x, \quad v(x,y) = \mathrm{Im}(z) = y.$$

$$\frac{\partial u}{\partial x} = u_x = 1, \qquad \frac{\partial u}{\partial y} = u_y = 0, \qquad \frac{\partial v}{\partial x} = v_x = 0, \qquad \frac{\partial v}{\partial y} = v_y = 1.$$

$$\text{Cauchy-Riemann Equations (CRE)}: \qquad \begin{cases} u_x = & v_y = & 1 \\ u_y = & -v_x = & 0 \end{cases}$$

The CRE aren't verified for any z in the complex plane. $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are nowhere differentiable.

$\square$

2.

Find the derivatives of

$$f(z) = \left(\frac{z^2-1}{z^2+1}\right)^{100}, \qquad g(z) = \frac{(z+2)^3}{(z^2+iz+1)^4}.$$

Solution:

$$\begin{aligned} f'(z) &= 100\left(\frac{z^2-1}{z^2+1}\right)^{99} \frac{2z(z^2+1) - (z^2-1)2z}{(z^2+1)^2} \\ &= 100\left(\frac{z^2-1}{z^2+1}\right)^{99} \frac{4z}{(z^2+1)^2} \\ &= \frac{400z(z^2-1)^{99}}{(z^2+1)^{101}} \\ &= \frac{400(x+yi)((x+yi)^2-1)^{99}}{((x+yi)^2+1)^{101}}. \end{aligned}$$

$$
\begin{aligned}
g'(z) &= \frac{3(z+2)^2(z^2+iz+1)^4 - (z+2)^3 4(z^2+iz+1)^3(2z+i)}{(z^2+iz+1)^8} \\
&= \frac{(z+2)^2[(z^2+iz+1)^3(3(z^2+iz+1) - 4(z+2)(2z+i))]}{(z^2+iz+1)^8} \\
&= -\frac{(z+2)^2(5z^2+(16+i)z-(3-8i))}{(z^2+iz+1)^5} \\
&= -\frac{(x+yi+2)^2(5(x+yi)^2+(16+i)(x+yi)-(3-8i))}{((x+yi)^2+i(x+yi)+1)^5}.
\end{aligned}
$$

□

3.

Let $f(z) = z^3 + 1$ and let

$$
z_1 = \frac{-1+\sqrt{3}i}{2}, \qquad z_2 = \frac{-1-\sqrt{3}i}{2}.
$$

Show that there is no point w on the line segment between z_1 and z_2 such that

$$
f(z_2) - f(z_1) = f'(w)(z_2 - z_1),
$$

meaning that the mean-value theorem of calculus does not extend to complex functions.

Solution:

$$
\begin{aligned}
f(z_2) - f(z_1) &= f'(w)(z_2 - z_1) \\
\left(\frac{-1-\sqrt{3}i}{2}\right)^3 + 1 - \left(\frac{-1+\sqrt{3}i}{2}\right)^3 - 1 &= 3w^2\left(\frac{-1-\sqrt{3}i}{2} - \frac{-1+\sqrt{3}i}{2}\right) \\
\left(\frac{-1-\sqrt{3}i}{2}\right)^3 - \left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= -3w^2\sqrt{3}i \\
\frac{8}{8} - \frac{8}{8} &= -3w^2\sqrt{3}i \\
0 &= w.
\end{aligned}
$$

The segment of line between z_1 and z_2 are a vertical segment centered at -1/2 in the x-axis. $w = 0$ isn't in this segment. Therefore, there is no point w on the line segment that satisfies the given equation.

□

Problem 4

1.

Show that

$$f(z) = (x^2 + y) + i(y^2 - x)$$

is not analytic at any point of the complex plane.

Solution:

$$z = (x^2 + y) + i(y^2 - x) \qquad \Rightarrow \quad u(x, y) = x^2 + y, \quad v(x, y) = y^2 - x.$$

$$\frac{\partial u}{\partial x} = u_x = 2x, \qquad \frac{\partial u}{\partial y} = u_y = 1, \qquad \frac{\partial v}{\partial x} = v_x = -1, \qquad \frac{\partial v}{\partial y} = v_y = 2y.$$

$$\text{Cauchy-Riemann Equations (CRE) :} \qquad \begin{cases} u_x = & v_y \quad \Rightarrow \quad x = y \\ u_y = & -v_x \quad \Rightarrow \quad 1 = 1 \end{cases}$$

The CRE are verified only for $x = y$. The function is not analytic for $x \neq y$.

□

2.

Use the Cauchy-Riemann equations to show that the following functions are not differentiable:

a)

$f(z) = \bar{z}$.

Solution:

$$\bar{z} = x - yi \qquad \Rightarrow \quad u(x, y) = x, \quad v(x, y) = -y.$$

$$\frac{\partial u}{\partial x} = u_x = 1, \qquad \frac{\partial u}{\partial y} = u_y = 0, \qquad \frac{\partial v}{\partial x} = v_x = 0, \qquad \frac{\partial v}{\partial y} = v_y = -1.$$

$$\text{Cauchy-Riemann Equations (CRE) :} \qquad \begin{cases} u_x = & v_y \quad \Rightarrow \quad 1 \neq -1 \\ u_y = & -v_x \quad \Rightarrow \quad 0 = 0 \end{cases}$$

The CRE are not verified, the function $f(z) = \bar{z}$ is not differentiable.

□

b)

$f(z) = \mathrm{Re}(z)$.

Solution:

$$\mathrm{Re}(z) = x \qquad \Rightarrow \quad u(x,y) = x, \quad v(x,y) = 0.$$

$$\frac{\partial u}{\partial x} = u_x = 1, \qquad \frac{\partial u}{\partial y} = u_y = 0, \qquad \frac{\partial v}{\partial x} = v_x = 0, \qquad \frac{\partial v}{\partial y} = v_y = 0.$$

$$\text{Cauchy-Riemann Equations (CRE) :} \qquad \begin{cases} u_x = & v_y \quad \Rightarrow \quad 1 \neq 0 \\ u_y = & -v_x \quad \Rightarrow \quad 0 = 0 \end{cases}$$

The CRE are not verified, the function $f(z) = \mathrm{Re}(z)$ is not differentiable.

□

c)

$f(z) = 2y - ix$.

Solution:

$$f(z) = 2y - ix \qquad \Rightarrow \quad u(x,y) = 2y, \quad v(x,y) = -x.$$

$$\frac{\partial u}{\partial x} = u_x = 0, \qquad \frac{\partial u}{\partial y} = u_y = 2, \qquad \frac{\partial v}{\partial x} = v_x = -1, \qquad \frac{\partial v}{\partial y} = v_y = 0.$$

$$\text{Cauchy-Riemann Equations (CRE) :} \qquad \begin{cases} u_x = & v_y \quad \Rightarrow \quad 0 = 0 \\ u_y = & -v_x \quad \Rightarrow \quad 2 \neq 1 \end{cases}$$

The CRE are not verified, the function $f(z) = 2y - ix$ is not differentiable.

□

3.

Construct an analytic function whose real part is $u(x,y) = x^3 - 3xy^2 + y$.

Solution:

$$\frac{\partial u}{\partial x} = u_x = 3(x^2 - y^2), \qquad \frac{\partial u}{\partial y} = u_y = 1 - 6xy.$$

$$\text{Cauchy-Riemann Equations (CRE) :} \qquad \begin{cases} u_x = & v_y & \Rightarrow & 3(x^2-y^2) & = & ? \\ u_y = & -v_x & \Rightarrow & 1-6xy & = & ? \end{cases}$$

If v_x and v_y are a real number, for example, the function will be analytic. Let's choose $v_x = -3$ and $v_y = 5$. So,

$$\text{Cauchy-Riemann Equations (CRE) :} \qquad \begin{cases} u_x = & v_y & \Rightarrow & 3(x^2-y^2) & = 5 \\ u_y = & -v_x & \Rightarrow & 1-6xy & = 3 \end{cases}$$

In this way the CRE are verified, the function is differentiable and analytic.

$$f(z) = \underbrace{(x^3 - 3xy^2 + y)}_{u(x,y)} + i \underbrace{(5y - 3x)}_{v(x,y)}.$$

□

4.

Show that if $\phi(x, y)$ is harmonic, then $\phi_x - i\phi_y$ is analytic. You may assume that ϕ has continuous partial derivatives of all orders.

Solution:

If $\phi(x, y)$ is harmonic:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

So the first partial derivatives exist and are continuous. In this way the CRE are verified, the function is differentiable in the domain and will be also differentiable in a neighborhood. Then $\phi_x - i\phi_y$ is analytic.

□

Problem 5

1.

Show that

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

Solution:

$$
\begin{aligned}
\cos(x+iy) &= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \\
&= \frac{1}{2}\Big[e^{ix}e^{-y} + e^{-ix}e^{y}\Big] \\
&= \frac{1}{2}\Big[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\Big] \\
&= \cos x\Big(\frac{e^{y}+e^{-y}}{2}\Big) - i\sin x\Big(\frac{e^{y}-e^{-y}}{2}\Big) \\
&= \cos x\cosh y - i\sin x\sinh x.
\end{aligned}
$$

$\square$

2.

Prove that $\cos z = 0$ if and only if $z = \pi/2 + k\pi$, where k is an integer.

Solution:

$$
\begin{aligned}
\cos z &= \cos(x+iy) = \cos x\cosh y - i\sin x\sinh x \\
|\cos z|^2 &= \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 x \\
&= \cos^2 x(1+\sinh^2 y) + \sin^2 x\sinh^2 x \\
&= \cos^2 x + \sinh^2 y(\sin^2 x + \cos^2 x) \\
&= \cos^2 x + \sinh^2 y
\end{aligned}
$$

$$
\cos^2 x + \sinh^2 y = 0 \quad \Rightarrow \quad \begin{cases} \cos x &= 0 \\ \sinh y &= 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x &= \frac{\pi}{2} + k\pi, \quad k = 0, \pm 1, \pm 2, ... \\ y &= 0 \end{cases}.
$$

$\square$

3.

Using the fact that $f'(0) = \lim_{z\to 0}[f(z) - f(0)]/z$, calculate

$$
\lim_{z\to 0}\frac{\sin z}{z}.
$$

Solution:

$$
\lim_{z\to 0}\frac{\sin z}{z} = \frac{0}{0} \quad \Rightarrow \quad \text{Applying L'Hôpital's :} \quad \lim_{z\to 0}\frac{\sin z}{z} = \lim_{z\to 0}\cos z = 1 = f'(0).
$$

$\square$

4.

Using the chain rule, determine the domain of analyticity for $f(z) = \mathrm{Ln}(3z-1)$ and compute $f'(z)$.

Solution:

$$\mathrm{Ln}(3z-1) = \mathrm{Ln}(3(x+iy)-1) = \mathrm{Ln}((3x-1)+3iy) = \underbrace{\ln|(3x-1)+3iy|}_{u(x,y)=\ln\sqrt{(3x-1)^2+9y^2}} + i \underbrace{\theta}_{v(x,y)=\arctan\frac{3y}{3x-1}}$$

$$\frac{\partial u}{\partial x} = u_x = \frac{3(3x-1)}{(3x-1)^2+9y^2}, \qquad \frac{\partial u}{\partial y} = u_y = \frac{9y}{(3x-1)^2+9y^2},$$
$$\frac{\partial v}{\partial x} = v_x = -\frac{9y}{(3x-1)^2+9y^2}, \qquad \frac{\partial v}{\partial y} = v_y = \frac{3(3x-1)}{(3x-1)^2+9y^2}.$$

Cauchy-Riemann Equations (CRE):

$$u_x = v_y = \frac{3(3x-1)}{(3x-1)^2+9y^2}, \qquad u_y = -v_x = \frac{9y}{(3x-1)^2+9y^2}.$$

This equations give the domain of $f(z)$. The function $f(z)$ is differentiable in the domain and is analytic in the domain.

$$f'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Ln}(3z-1) = \frac{3}{3z-1} = \frac{3}{3(x+iy)-1} = \frac{3}{(3x-1)+3iy}.$$

■
