Counting Processes and Asymptotic Theory

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Failure Time Models,
5nd chapter of *The Statistical Analysis of Failure Time Data*
Kalbfleisch and Prentice, 2002

Outline

- Counting processes and intensity functions
- Martingales
  - Continuous-time counting processes
  - Discrete-time counting processes
- Vector-valued martingales

A counting process $N = \{N(t), t \geq 0\}$

is a stochastic process with $N(0) = 0$ and whose value at time t counts the number of events that have occured in the interval $(0, t]$.

- The sample paths of N are nondecrising step functions that jump whenever an event (or events) occur.
- In continuous time,

  no two counting processes can jump at the same time.

- In discrete time, they can.

Number of events that occur in the interval $[t, t + dt)$?
$dN(t) = N(t^- + dt) - N(t^-)$.

Number of events that occur at time t? $\Delta N(t) = N(t) - N(t^-)$.

And what about more general counting processes where individuals may experience more than one event? Chapters 8, 9, and 10.

Filtration: history of events

observed counting process: $N_i = \{N_i(t), t \geq 0\}$

underlying counting process: $\tilde{N}_i = \{\tilde{N}_i(t), 0 \leq t\}, \quad \tilde{N}_i(t) = \mathbf{1}(T_i \leq t)$

at-risk process: $\{Y_i(t), t \geq 0\}, \quad Y_i(t) = \mathbf{1}(T_i \geq t, C_i \geq t)$

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key concept: filtration

$$\mathcal{F}_t = \sigma\{N_i(u), Y_i(u^+), X_i(u^+), i = 1, \ldots, n; 0 \leq u \leq t\}, \quad t > 0,$$

where

$Y_i(u^+) = \lim_{s \to u^+} Y_i(s)$;

stochastic time-dependent covariate: $X_i(t) = \{x_i(u) : 0 \leq u \leq t\}$.

The notation $\sigma[\cdot]$ specifies the sigma algebra of events generated by the variables given in the brackets.

Intensity functions

The intensities or rates for the processes N_i are defined with reference to the filtration $\mathcal{F}_t$. If the censoring process is independent, the intensity model for the counting process N_i is

$$\mathbb{P}[dN_i(t) = 1 | \mathcal{F}_{t^-}] = Y_i(t) d\Lambda_i(t), \quad i = 1, \dots, n, \quad t > 0.$$

The hazard model can be written $d\Lambda_i(t) = \mathbb{P}[d\tilde{N}_i(t) = 1 | X_i(t), \tilde{N}_i(t^-) = 0]$.

Λ_i is called the cumulative intensity process of the counting process $\tilde{N}_i$.

- In the continuous case, $\mathbb{P}[dN_i(t) = 1 | \mathcal{F}_{t^-}] = Y_i(t) \lambda_i(t) dt$
- In the discrete case, $\mathbb{P}[dN_i(a_l) = 1 | \mathcal{F}_{a_l^-}] = Y_i(a_l) \lambda_{il}, \quad l = 1, 2, \dots$

$\lambda_i(t)$ and λ_{il} are the corresponding intensity processes.

Martingales: Intro

$$
\begin{aligned}
M_i(t) &= N_i(t) - \int_0^t Y_i(u)\lambda_i(u)du, \quad t \geq 0. \\
&= \int_0^t dM_i(u), \\
dM_i(t) &= dN_i(t) - Y_i(t)\lambda_i(t)dt.
\end{aligned}
$$

If

» $\mathbb{E}[dM_i(t)|\mathcal{F}_{t-}] = 0, \quad \forall t; \quad \equiv \quad \mathbb{E}[M_i(t)|\mathcal{F}_s] = M_i(s), \quad \forall s \leq t.$

Then, $M_i(t)$ is a martingale.

Consequences:

» $\mathbb{E}[M_i(t)] = 0, \quad \forall t;$

» the process $M_i(t)$ has uncorrelated increments, i.e.,
$\mathbb{E}[(M_i(t) - M_i(s)) \times M_i(s)] = 0, \quad \forall 0 < s < t.$

Decomposing $N_i(t)$ into two processes

$$N_i(t) = \underbrace{\int_0^t Y_i(u)\lambda_i(u)du}_{\substack{\text{compensator of the} \\ \text{counting process } N_i \\ \text{wrt the filtration } \mathcal{F}_t}} + \underbrace{M_i(t)}_{\substack{\text{counting} \\ \text{process} \\ \text{martingale} \\ \text{corresponding} \\ \text{to } N_i(t)}}$$

$$dN_i(t) = Y_i(t)\lambda_i(t)dt + dM_i(t).$$

In the discrete case, the discrete-time martingale is

$$\begin{aligned} N_i(t) &= \int Y_i(u)d\Lambda_i(u) + M_i(t) \\ &= \sum_{a_l \leq t} Y_i(a_l)\lambda_{il} + M_i(t), \\ dN_i(a_l) &= Y_i(a_l)\lambda_{il} + dM_i(a_l). \end{aligned}$$

More about martingales

In essense, a martingale is a process that has no drift and whose increments are uncorrelated.

- We say that $M(t)$ is a mean zero martingale if $\mathbb{E}[M(0)] = 0$, and hence $\mathbb{E}[M(t)] = 0, \forall t$.
- The martingale $M(t)$ is said to be square integrable (or have finite variance) if $\mathbb{E}[M^2(t)] = \mathbb{V}[M(t)] < \infty, \forall t \leq \tau$.

It is useful to define two technical terms applied to a stochastic process $U = \{U(t), t \geq 0\}$.

Adapted

U is said to be adapted to the filtration $\mathcal{F}_t$ if $U(t)$ is $\mathcal{F}_t$ measurable for each $t \in [0, \tau]$, i.e., the value of $U(t)$ is fixed once $\mathcal{F}_t$ is given.

Predictable

U is said to be predictable wrt the filtration $\mathcal{F}_t$ if $U(t)$ is $\mathcal{F}_{t^-}$ measurable for all $t \in [0, \tau]$, i.e., the value of $U(t)$ is fixed once $\mathcal{F}_{t^-}$ is given.

More about martingales

The process $\{\bar{M}(t), 0 \leq t \leq \tau\}$ is a submartingale wrt $\mathcal{F}_t$ if it is adapted and satisfies

$$\mathbb{E}[\bar{M}(t)|\mathcal{F}_s] \geq \bar{M}(s) \quad \forall s \leq t \leq \tau.$$

» A counting process $N(t)$ is a submartingale.

Predictable variation process

The predictable variation process of a square-integrable martingale M is

$$\langle M \rangle (t) = \int_0^t \mathbb{V}[dM(u)|\mathcal{F}_{u^-}].$$

Equivalently, $d \langle M \rangle (t) = \mathbb{V}[dM(u)|\mathcal{F}_{u^-}]$.

In statistical terms, the primary role of the predictable variation process is that for given t, $\langle M \rangle (t)$ provides a systematic approach to estimating the variance of $M(t)$.

Variance of $M(t)$

$$\mathbb{V}[M(t)] = \mathbb{E}[M^2(t)] = \mathbb{E}[\langle M \rangle (t)]$$

and $\langle M \rangle (t)$ is an unbiased estimator of $\mathbb{V}[M(t)]$.

Usually, $\langle M \rangle (t)$ involves the parameters of the model.

There is an alternative estimator of $\mathbb{V}[M(t)]$ that in some problems is a function of observed quantities only. This is the quadratic variation or optional variation process $[M](t)$.

$$[M](t) = \sum_{s \leq t} (\Delta M(s))^2.$$

$[M](t)$ also provides an unbiased estimator of $\mathbb{V}[M(t)]$.

Continuous-time counting processes

For each i, the process

$$M_i(t) = N_i(t) - \int_0^t Y_i(u)\lambda_i(u)du$$

is a mean zero martingale wrt the filtration $\mathcal{F}_t$. The corresponding predictable variation process is

$$\langle M \rangle (t) = \int_0^t Y_i(u)\lambda_i(u)du$$

since $d \langle M \rangle (u) = \mathbb{V}[dM(u)|\mathcal{F}_{u^-}] = \mathbb{E}[dN_i(u)|\mathcal{F}_{u^-}] = Y_i(u)\lambda_i(u)du$, $0 \leq u \leq \tau$. The optional variation process is

$$[M_i](t) = N_i(t).$$

Both $\langle M_i \rangle (t)$ and $[M_i](t)$ provide unbiased estimates of $\mathbb{V}[M_i(t)]$, but only the latter is a function of the data only.

Discrete-time continuous processes

In discrete time we have

$$\mathbb{P}[dN_i(t) = 1|\mathcal{F}_{t^-}] = \mathbb{E}[dN_i(t)|\mathcal{F}_{t^-}] = Y_i(t)d\Lambda_i(t),$$

where $\Lambda_i(t) = \sum_{a_l \leq t} \lambda_{il}$ and λ_{il} are the discrete hazard probabilities.

The corresponding martingale is

$$M_i(t) = N_i(t) - \int_0^t Y_i(u)d\Lambda_i(u)$$

and the predictable variation process is

$$\langle M_i \rangle (t) = \sum_{a_l \leq t} Y_i(a_l)(1 - \lambda_{il})\lambda_{il}.$$

The increment $d\langle M_i \rangle (t)$ at a_l corresponds to
$\mathbb{V}[dN_i(a_l)|\mathcal{F}_{t^-}] = Y_i(a_l)(1 - \lambda_{il})\lambda_{il}$.

Vector-valued martingales

» Grouped relative risk model:

$$\text{Discrete baseline cumulative hazard function} : \Lambda_0(t) = \sum_{a_i \leq t} \lambda_i,$$

this model is the uniquely appropriate one for grouped data from the continuous relative risk model.

» Discrete and continuous relative risk model:

$$\mathrm{d}\Lambda(t; x) = \exp(Z^\top \beta)\, \mathrm{d}\Lambda_0(t),$$

which retains the multiplicative hazard relationship.

» Discrete logistic model:

$$\frac{\mathrm{d}\Lambda(t; x)}{1 - \mathrm{d}\Lambda(t; x)} = \frac{\mathrm{d}\Lambda_0(t)}{1 - \mathrm{d}\Lambda_0(t)} \exp(Z^\top \beta),$$

specifies a linear log odds model for the hazard probability at each potential failure time.


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