MI404 - Métodos Estatísticos
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EXERCÍCIOS
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Sumário

Exercício 1

Considere o seguinte modelo linear

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{1}$$

em que $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma_e^2\mathbf{I}_{n\times n})$. Seja $\mathbf{X}$ uma matriz de desenho de dimensão $n \times p$, $\boldsymbol{\beta}$ um vetor de parâmetros de dimensão $p \times 1$.

(a)

Encontre o EMV de $\boldsymbol{\beta}$ e σ_e^2 e usando a verossimilhança baseada na distribuição multivariada de $\mathbf{y}$.

Nota. Seja $\mathbf{V}$ um vetor aleatório de dimensão $n \times 1$, tal que $\mathbf{V} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, com $\boldsymbol{\Sigma}$ positiva definida. Então

$$f(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-n/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\Big\{-\frac{1}{2}(\mathbf{v}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{v}-\boldsymbol{\mu})\Big\}. \tag{2}$$

<u>Solução:</u>

Aqui, $\boldsymbol{\theta} = (\boldsymbol{\beta}, \sigma_e^2)$, $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ e $\boldsymbol{\Sigma} = \sigma_e^2\mathbf{I}_{n\times n}$.

A função de verossimilhança $L(\boldsymbol{\theta}; \mathbf{y})$ é dada por:

$$L(\boldsymbol{\theta}; \mathbf{y}) = (2\pi)^{-n/2}|\sigma_e^2\mathbf{I}_{n\times n}|^{-1/2}\exp\Big\{-\frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'(\sigma_e^2\mathbf{I}_{n\times n})^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\Big\}.$$

Sendo $\sigma_e^2\mathbf{I}_{n\times n} = \sigma_e^{2n}$,

$$L(\boldsymbol{\theta}; \mathbf{y}) \propto \frac{1}{|\sigma_e^2|^{n/2}}\exp\Big\{-\frac{1}{2\sigma_e^2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\Big\}.$$

A função de log-verossimilhança é expressa por:

$$\begin{aligned}\log(L(\boldsymbol{\theta}; \mathbf{y})) &\propto -\frac{n}{2}\log(\sigma_e^2) - \frac{1}{2\sigma_e^2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) \\ &\propto -\frac{n}{2}\log(\sigma_e^2) - \frac{1}{2\sigma_e^2}(\mathbf{y}'\mathbf{y} - 2\mathbf{y}'\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}\boldsymbol{\beta})^2).\end{aligned}$$

$\boxed{\text{EMV de } \boldsymbol{\beta}}$:

$$\frac{\partial \log(L(\boldsymbol{\theta}; \mathbf{y}))}{\partial \boldsymbol{\beta}} = 0$$

$$\begin{aligned}\frac{\partial \log(L(\boldsymbol{\theta}; \mathbf{y}))}{\partial \boldsymbol{\beta}} &= -\frac{1}{2\sigma_e^2}(-2\mathbf{y}'\mathbf{X} + 2\mathbf{X}'\mathbf{X}\boldsymbol{\beta}) \\ &= \frac{1}{\sigma_e^2}(\mathbf{X}'\mathbf{y} - \mathbf{X}'\mathbf{X}\boldsymbol{\beta}).\end{aligned}$$

$$\frac{\partial \log(L(\boldsymbol{\theta};\mathbf{y}))}{\partial \boldsymbol{\beta}} = \frac{1}{\sigma_e^2}(\mathbf{X}^{'}\mathbf{y} - \mathbf{X}^{'}\mathbf{X}\boldsymbol{\beta}) = 0.$$

$$\begin{aligned}
\frac{1}{\sigma_e^2}(\mathbf{X}^{'}\mathbf{y} - \mathbf{X}^{'}\mathbf{X}\boldsymbol{\beta}) &= 0 \\
\mathbf{X}^{'}\mathbf{y} - \mathbf{X}^{'}\mathbf{X}\boldsymbol{\beta} &= 0 \\
\mathbf{X}^{'}\mathbf{y} &= \mathbf{X}^{'}\mathbf{X}\boldsymbol{\beta}
\end{aligned}$$

$$\boxed{\hat{\boldsymbol{\beta}} = (\mathbf{X}^{'}\mathbf{X})^{-1}\mathbf{X}^{'}\mathbf{y}.}$$

$\boxed{\text{EMV de } \sigma_e^2}$:

$$\frac{\partial \log(L(\boldsymbol{\theta};\mathbf{y}))}{\partial \sigma_e^2} = 0$$

$$\begin{aligned}
\frac{\partial \log(L(\boldsymbol{\theta};\mathbf{y}))}{\partial \sigma_e^2} &= -\frac{n}{2\sigma_e^2} + \frac{1}{2(\sigma_e^2)^2}(\mathbf{y}^{'}\mathbf{y} - 2\mathbf{y}^{'}\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}\boldsymbol{\beta})^2) \\
&= \frac{-n\sigma_e^2 + (\mathbf{y}^{'}\mathbf{y} - 2\mathbf{y}^{'}\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}\boldsymbol{\beta})^2)}{2(\sigma_e^2)^2}.
\end{aligned}$$

$$\frac{\partial \log(L(\boldsymbol{\theta};\mathbf{y}))}{\partial \sigma_e^2} = \frac{-n\sigma_e^2 + (\mathbf{y}^{'}\mathbf{y} - 2\mathbf{y}^{'}\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}\boldsymbol{\beta})^2)}{2(\sigma_e^2)^2} = 0.$$

$$\begin{aligned}
\frac{-n\sigma_e^2 + (\mathbf{y}^{'}\mathbf{y} - 2\mathbf{y}^{'}\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}\boldsymbol{\beta})^2)}{2(\sigma_e^2)^2} &= 0 \\
-n\sigma_e^2 + (\mathbf{y}^{'}\mathbf{y} - 2\mathbf{y}^{'}\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}\boldsymbol{\beta})^2) &= 0 \\
\mathbf{y}^{'}\mathbf{y} - 2\mathbf{y}^{'}\mathbf{X}\boldsymbol{\beta} + (\mathbf{X}\boldsymbol{\beta})^2 &= n\sigma_e^2 \\
(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^{'}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) &= n\sigma_e^2
\end{aligned}$$

$$\boxed{\hat{\sigma}_e^2 = \frac{(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^{'}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})}{n}.}$$

(b)

Encontre a distribuição do EMV $\hat{\boldsymbol{\beta}}$.

<u>Solução:</u>

$$\hat{\boldsymbol{\beta}} \underset{\text{aprox.}}{\sim} N(E[\hat{\boldsymbol{\beta}}], Var[\hat{\boldsymbol{\beta}}])$$

$$
\begin{aligned}
E[\hat{\boldsymbol{\beta}}] &= E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}] \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E[\mathbf{y}] \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E[\mathbf{X}\boldsymbol{\beta}+\boldsymbol{\epsilon}] \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}E[\boldsymbol{\beta}]+\mathbf{0} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} \\
&= \boldsymbol{\beta}.
\end{aligned}
$$

$$
Var[\hat{\boldsymbol{\beta}}] = \frac{1}{\mathbf{I}(\boldsymbol{\beta})}, \qquad \mathbf{I}(\boldsymbol{\beta}) = E\left[-\frac{\partial^2 \log(L(\boldsymbol{\theta};\mathbf{y}))}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}\right].
$$

Sendo que $\mathbf{I}(\boldsymbol{\beta})$ é a matriz de informação esperada.

$$
\begin{aligned}
\frac{\partial^2 \log(L(\boldsymbol{\theta};\mathbf{y}))}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} &= -\frac{\mathbf{X}'\mathbf{X}}{\sigma_e^2} \\
\mathbf{I}(\boldsymbol{\beta}) &= E\left[\frac{\mathbf{X}'\mathbf{X}}{\sigma_e^2}\right] \\
\mathbf{I}(\boldsymbol{\beta}) &= \frac{\mathbf{X}'\mathbf{X}}{E\left[\sigma_e^2\right]} \\
\mathbf{I}(\boldsymbol{\beta}) &= \frac{\mathbf{X}'\mathbf{X}}{\sigma_e^2}.
\end{aligned}
$$

$$
\begin{aligned}
Var[\hat{\boldsymbol{\beta}}] &= \frac{1}{\mathbf{X}'\mathbf{X}/\sigma_e^2} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\sigma_e^2.
\end{aligned}
$$

$$
\boxed{\hat{\boldsymbol{\beta}} \underset{\text{aprox.}}{\sim} N\Big(\boldsymbol{\beta}, (\mathbf{X}'\mathbf{X})^{-1}\sigma_e^2\Big).}
$$

(c)

Encontre a distribuição do EMV $\hat{\sigma}_e^2$.

Solução:

$$
\hat{\sigma}_e^2 \underset{\text{aprox.}}{\sim} N(E[\hat{\sigma}_e^2], Var[\hat{\sigma}_e^2])
$$

$$
\begin{aligned}
E[\hat{\sigma}_e^2] = E\left[\frac{(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})}{n}\right] &= \frac{1}{n}E\Big[(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})\Big] \\
&= \frac{1}{n}E\left[\sum_{i=1}^{n}(y_i-\boldsymbol{x}_i'\hat{\boldsymbol{\beta}})^2\right] \\
&= \frac{1}{n}E\left[\frac{\sigma^2}{\sigma^2}\sum_{i=1}^{n}(y_i-\boldsymbol{x}_i'\hat{\boldsymbol{\beta}})^2\right] \\
&= \frac{1}{n}\sigma^2 E\left[\sum_{i=1}^{n}\left[\frac{(y_i-\boldsymbol{x}_i'\hat{\boldsymbol{\beta}})}{\sigma}\right]^2\right] = \frac{1}{n}\sigma^2 E\Big[\chi^2_{n-p}\Big].
\end{aligned}
$$

Já que

$$\sum_{i=1}^{n}\left[\frac{(y_i - \boldsymbol{x}_i'\hat{\boldsymbol{\beta}})}{\sigma}\right]^2 \sim \chi^2_{n-p}, \quad \text{com} \quad E[\cdot] = n-p \quad \text{e} \quad Var[\cdot] = 2(n-p).$$

Assim,

$$\begin{aligned} E[\hat{\sigma}_e^2] &= \frac{1}{n}\sigma^2(n-p) \\ &= \frac{n-p}{n}\sigma^2. \end{aligned}$$

$$\begin{aligned} Var[\hat{\sigma}_e^2] &= Var\left[\frac{(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})}{n}\right] \\ &= \frac{1}{n^2}Var\left[\sum_{i=1}^{n}(y_i - \boldsymbol{x}_i'\hat{\boldsymbol{\beta}})^2\right] \\ &= \frac{1}{n^2}\sigma^4 Var\left[\sum_{i=1}^{n}\left[\frac{(y_i - \boldsymbol{x}_i'\hat{\boldsymbol{\beta}})}{\sigma}\right]^2\right] \\ &= \frac{1}{n^2}\sigma^4 2(n-p) \\ &= \frac{n-p}{n^2}2\sigma^4. \end{aligned}$$

$$\boxed{\hat{\sigma}_e^2 \underset{\text{aprox.}}{\sim} N\left(\frac{n-p}{n}\sigma^2, \frac{n-p}{n^2}2\sigma^4\right).}$$

(d)

Considere o modelo em (1) com $\boldsymbol{\beta} = \beta_0$ e estimador restrito de $\sigma^2 (\hat{\sigma}_r^2)$ visto em aula. Encontre $Var[\hat{\sigma}_r^2]$ e $Var[\hat{\sigma}_e^2]$. Na sua opinião, qual deles é melhor? Justifique.

Solução:

$$\mathbf{y} = \beta_0 + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma_e^2 \mathbf{I}_{n\times n})$$

EMV de σ_e^2:

$$\hat{\sigma}_e^2 = \sum_{i=1}^{n}\frac{(y_i - \hat{\beta}_0)^2}{n}$$

$$\begin{aligned} Var[\hat{\sigma}_e^2] &= Var[\sum_{i=1}^{n}\frac{(y_i - \hat{\beta}_0)^2}{n}] \\ &= \frac{1}{n^2}\sigma^4 2(n-1) \end{aligned}$$

$$\boxed{Var[\hat{\sigma}_e^2] = \frac{2(n-1)}{n^2}\sigma^4.}$$

EMV restrito, σ_r^2:

$$\hat{\sigma}_r^2 = \sum_{i=1}^{n}\frac{(y_i - \hat{\beta}_0)^2}{n-1}$$

$$\begin{aligned} Var[\hat{\sigma}_r^2] &= Var[\sum_{i=1}^{n}\frac{(y_i - \hat{\beta}_0)^2}{n-1}] \\ &= \frac{1}{(n-1)^2}\sigma^4 2(n-1) \end{aligned}$$

$$\boxed{Var[\hat{\sigma}_r^2] = \frac{2}{n-1}\sigma^4.}$$

$$
\begin{aligned}
\frac{Var[\hat{\sigma}_e^2]}{Var[\hat{\sigma}_r^2]} &= \frac{2(n-1)}{n^2}\sigma^4 \cdot \frac{n-1}{2\sigma^4} \\
&= \frac{(n-1)^2}{n^2} \\
&= \frac{n^2-2n+1}{n^2} \\
&= 1 - \frac{2}{n} + \frac{1}{n^2} \\
&= \leq 1.
\end{aligned}
$$

Portanto, $\boxed{Var[\hat{\sigma}_r^2] > Var[\hat{\sigma}_e^2]}$.

$\hat{\sigma}_e^2$, $E[\hat{\sigma}_e^2] = \frac{n-1}{n}\sigma_e^2$, é um estimador viciado (corrigível), mas com menor variância que $\hat{\sigma}_r^2$, $E[\hat{\sigma}_r^2] = \sigma_e^2$.

Logo, temos que $\hat{\sigma}_e^2$ é melhor que $\hat{\sigma}_r^2$.

Exercício 2

Considere o modelo dado em (1). A partir da perspectiva Bayesiana, considere as distribuições à priori de $\boldsymbol{\beta}$ e σ_e^2 dados por $p(\boldsymbol{\beta}) \propto 1$ e $p(\sigma_e^2) \propto (\sigma_e^2)^{-1}$, respectivamente.

(a)

Seja $\boldsymbol{\theta} = (\boldsymbol{\beta}', \sigma_e^2)$. Encontre a distribuição a posteriori de $\boldsymbol{\theta}$.

Solução:

A distribuição a posteriori de $\boldsymbol{\theta}$, $\pi(\boldsymbol{\theta}|\mathbf{y})$, pelo teorema de Bayes é dada por:

$$\pi(\boldsymbol{\theta}|\mathbf{y}) \propto L(\boldsymbol{\theta};\mathbf{y})\pi(\boldsymbol{\theta}).$$

Assumindo independência entre $\boldsymbol{\theta}$ e σ_e^2, a distribuição a priori de $\pi(\boldsymbol{\theta})$ é dada por:

$$\pi(\boldsymbol{\theta}) = \pi(\boldsymbol{\beta}, \sigma_e^2) = \pi(\boldsymbol{\beta})\pi(\sigma_e^2) \propto 1 \cdot \frac{1}{\sigma_e^2} = \frac{1}{\sigma_e^2}.$$

Assim,

$$
\begin{aligned}
\pi(\boldsymbol{\theta}|\mathbf{y}) &\propto \frac{1}{(\sigma_e^2)^{n/2}}\exp\Big\{-\frac{1}{2\sigma_e^2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\Big\} \cdot \frac{1}{\sigma_e^2} \\
&\propto \frac{1}{(\sigma_e^2)^{n/2+1}}\exp\Big\{-\frac{1}{2\sigma_e^2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\Big\}.
\end{aligned}
$$

Sabemos que $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ e $\hat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$, desta forma:

$$\begin{aligned}
(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) &= (\mathbf{y}-\mathbf{X}\boldsymbol{\beta})'(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) - 2\mathbf{y}'\hat{\mathbf{y}} + 2\mathbf{y}'\hat{\mathbf{y}}\\
&= \mathbf{y}'\mathbf{y} - 2\mathbf{y}\boldsymbol{\beta}'\mathbf{X}' + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} - 2\mathbf{y}'\mathbf{X}\hat{\boldsymbol{\beta}} + 2\mathbf{y}'\mathbf{X}\hat{\boldsymbol{\beta}}\\
&= \mathbf{y}'\mathbf{y} - 2\boldsymbol{\beta}'\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} - 2\mathbf{y}'\mathbf{X}\hat{\boldsymbol{\beta}} + 2\mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}}\\
&= \mathbf{y}'\mathbf{y} - 2\boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} - 2\mathbf{y}'\mathbf{X}\hat{\boldsymbol{\beta}} + 2\hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}}\\
&= \mathbf{y}'\mathbf{y} - \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} - \mathbf{y}'\mathbf{X}\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} + \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} + \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}}\\
&= \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{y} + \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} + \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} - \boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} + \hat{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}}\\
&= (\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y}-\mathbf{X}\hat{\boldsymbol{\beta}}) + (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})\\
&= \frac{n-p}{n-p}(\mathbf{y}-\hat{\mathbf{y}})'(\mathbf{y}-\hat{\mathbf{y}}) + (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})\\
&= \nu\mathbf{S}^2 + (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}),
\end{aligned}$$

com

$$\mathbf{S}^2 = \frac{(\mathbf{y}-\hat{\mathbf{y}})'(\mathbf{y}-\hat{\mathbf{y}})}{\nu}, \quad \nu = n-p.$$

Portanto,

$$\boxed{\pi(\boldsymbol{\theta}|\mathbf{y}) \propto \frac{1}{(\sigma_e^2)^{n/2+1}}\exp\left\{-\frac{1}{2\sigma_e^2}\Big[\nu\mathbf{S}^2 + (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})\Big]\right\}}.$$

(b)

Usando o item (a), encontre a distribuição a posteriori de σ_e^2.

Solução:

$$\begin{aligned}
\pi(\sigma_e^2|\mathbf{y}) = \int_{-\infty}^{\infty}\pi(\boldsymbol{\beta},\sigma_e^2|\mathbf{y})\mathrm{d}\boldsymbol{\beta} &= \int_{-\infty}^{\infty}\frac{1}{(\sigma_e^2)^{n/2+1}}\exp\left\{-\frac{1}{2\sigma_e^2}\Big[\nu\mathbf{S}^2 + (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})\Big]\right\}\mathrm{d}\boldsymbol{\beta}\\
&= \frac{1}{(\sigma_e^2)^{n/2+1}}\exp\left\{-\frac{\nu\mathbf{S}^2}{2\sigma_e^2}\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\frac{\mathbf{X}'\mathbf{X}}{\sigma_e^2}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})\right\}\mathrm{d}\boldsymbol{\beta}\\
&= \frac{1}{(\sigma_e^2)^{n/2+1}}\exp\left\{-\frac{\nu\mathbf{S}^2}{2\sigma_e^2}\right\}(\sqrt{2\pi})^p\left|\frac{(\mathbf{X}'\mathbf{X})^{-1}}{\sigma_e^2}\right|^{1/2}\\
&\propto \frac{1}{(\sigma_e^2)^{n/2+1}}\exp\left\{-\frac{\nu\mathbf{S}^2}{2\sigma_e^2}\right\}(\sqrt{2\pi})^p\frac{1}{((\sigma_e^2)^{2p})^{-1/2}}\\
&= \frac{1}{(\sigma_e^2)^{n/2+1-p}}\exp\left\{-\frac{\nu\mathbf{S}^2}{2\sigma_e^2}\right\}\\
&= \frac{1}{(\sigma_e^2)^{n/2+1-p/2}}\exp\left\{-\frac{\nu\mathbf{S}^2}{2\sigma_e^2}\right\}\\
&= \frac{1}{(\sigma_e^2)^{(n-p)/2+1}}\exp\left\{-\frac{\nu\mathbf{S}^2}{2\sigma_e^2}\right\} = \frac{1}{(\sigma_e^2)^{\nu/2+1}}\exp\left\{-\frac{\nu\mathbf{S}^2}{2\sigma_e^2}\right\}.
\end{aligned} \tag{3}$$

Na equação (3) temos o núcleo de uma distribuição gama inversa, assim, a distribuição marginal a posteriori de σ_e^2 é dada por:

$$\pi(\sigma_e^2|\mathbf{y}) = \frac{\nu\mathbf{S}^2}{2\Gamma(\nu/2)}\left(\frac{1}{\sigma_e^2}\right)^{\nu/2+1}\exp\left\{-\frac{\nu\mathbf{S}^2}{2\sigma_e^2}\right\},$$

ou seja,

$$\pi(\sigma_e^2|\mathbf{y}) = GInv\left(\frac{\nu}{2}, \frac{\nu\mathbf{S}^2}{2}\right).$$

(c)

Usando o item (a), encontre a distribuição a posteriori de $\boldsymbol{\beta}$.

Solução:

$$\pi(\boldsymbol{\beta}|\mathbf{y}) = \int_0^\infty \pi(\boldsymbol{\beta}, \sigma_e^2|\mathbf{y})\mathrm{d}\sigma_e^2 = \int_0^\infty \frac{1}{(\sigma_e^2)^{n/2+1}}\exp\left\{-\frac{\nu\mathbf{S}^2 + (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}{2\sigma_e^2}\right\}\mathrm{d}\sigma_e^2.$$

Fazendo $a = (\nu\mathbf{S}^2 + (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}}))/2$, temos:

$$\begin{aligned}
\pi(\boldsymbol{\beta}|\mathbf{y}) &= \int_0^\infty (\sigma_e^2)^{-(n/2+1)}\exp\left\{-\frac{a}{\sigma_e^2}\right\}\mathrm{d}\sigma_e^2 \\
&= \left(\frac{a}{2}\right)^{-n/2}\Gamma\left(\frac{n}{2}\right) \\
&= \left[\frac{\nu\mathbf{S}^2 + (\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}{2}\right]^{n/2}\Gamma\left(\frac{n}{2}\right) \\
&= \Gamma\left(\frac{\nu+p}{2}\right)2^{n/2}\left\{\nu\mathbf{S}^2\left[1 + \frac{(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}{\nu\mathbf{S}^2}\right]\right\}^{-(1/2)(\nu+p)} \\
&= \Gamma\left(\frac{\nu+p}{2}\right)\nu\mathbf{S}^{2-(1/2)(\nu+p)}\left[1 + \frac{(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}{\nu\mathbf{S}^2}\right]^{-(1/2)(\nu+p)} \\
&\propto \Gamma\left(\frac{1}{2}(\nu+p)\right)\nu^{-p/2}\mathbf{S}^{-p}\left[1 + \frac{(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}{\nu\mathbf{S}^2}\right]^{-(1/2)(\nu+p)}.
\end{aligned} \tag{4}$$

Na equação (4) temos o núcleo de uma distribuição t- Student multivariada, assim, a distribuição marginal a posteriori de $\boldsymbol{\beta}$ é dada por:

$$\pi(\boldsymbol{\beta}|\mathbf{Y}) = \frac{\Gamma[(1/2)(\nu+p)]|\mathbf{X}'\mathbf{X}|^{1/2}\mathbf{S}^{-p}}{\Gamma(1/2)^{-p}\Gamma((1/2)\nu)(\sqrt{\nu})^p}\left[1 + \frac{(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}{\nu\mathbf{S}^2}\right]^{-(1/2)(\nu+p)},$$

que se denota por $t_p(\hat{\boldsymbol{\beta}}, \mathbf{S}^2(\mathbf{X}'\mathbf{X})^{-1}, \nu)$.

(d)

Supondo que fosse necessário apontar um estimador pontual Bayesiano para $\boldsymbol{\beta}$, quem você apontaria?

Solução:

Cada parâmetro $\beta_i, i = 0, 1, ..., p$ tem distribuição,

$$\pi(\beta_i|\mathbf{Y}) = t(\nu, \hat{\beta}_i, h_{ii}),$$

t-Student univariada com ν graus de liberdade, parâmetro de posição $\hat{\beta}$ e parâmetro de escala h_{ii} que é o elemento (i, i) de $\mathbf{S}^2(\mathbf{X}'\mathbf{X})^{-1}$.

Como a distribuição é uma t-Student, um estimador pontual não viciado bom seria a média desta t-Student:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}.$$

Exercício 3

Compare o EMV de $\boldsymbol{\beta}$ e a média da distribuição à posteriori de $\boldsymbol{\beta}$ em termos de suas respectivas variâncias, indicando qual deles seria mais apropriado.

Solução:

Como visto anteriormente,

$$Var[\hat{\boldsymbol{\beta}}_{EMV}] = \mathbf{S}^2(\mathbf{X}'\mathbf{X})^{-1}.$$

Distribuição à posteriori de $\boldsymbol{\beta}$:

Como $\pi(\beta_i|\mathbf{Y}) = t(\nu, \hat{\beta}_i, h_{ii})$,

$$Var[\hat{\boldsymbol{\beta}}] = \frac{n-p}{n-p-2}\mathbf{S}^2(\mathbf{X}'\mathbf{X})^{-1}.$$

Assim,

$$\mathbf{S}^2(\mathbf{X}'\mathbf{X})^{-1} < \frac{n-p}{n-p-2}\mathbf{S}^2(\mathbf{X}'\mathbf{X})^{-1}$$
$$1 < \frac{n-p}{n-p-2}.$$

$\hat{\boldsymbol{\beta}}_{EMV}$ tem menor variância. Contudo, apesar de sermos capazes de calcular a esperança e a variância de $\boldsymbol{\beta}$ por EMV, não conhecemos sua distribuição. Já a partir da perspectiva bayesiana somos capazes de conhecer sua distribuição a posteriori. Portanto, sob esse ponto de visto, em situações de amostra pequena ou que seja de interesse a distribuição de $\boldsymbol{\beta}$, a utilização da perspectiva bayesiana é mais apropriada. ■