AMCS 202 - Applied Mathematics II
Maria Alexandra Aguiar Gomes
Applied Mathematics and Computer Science (AMCS)/Statistics (STAT) Program
Computer, Electrical and Mathematical Sciences & Engineering (CEMSE) Division
King Abdullah University of Science and Technology (KAUST)

HOMEWORK
II

Henrique Aparecido Laureano

Spring Semester 2018

Contents

Problem 1

1.

Describe the range of $f(z) = x^2 + 2i$ defined on $|z| \leq 1$.

Solution:

$\square$

2.

Describe the range of $f(z) = z^3$ in the semidisk given by $|z| \leq 2$, $\mathrm{Im}(z) \geq 0$.

Solution:

$\square$

3.

Show that the inversion mapping $w = f(z) = 1/z$ maps

a)

The circle $z = r$ onto the circle $|w| = 1/r$.

Solution:

$\square$

b)

The ray $\mathrm{Arg}\ z = \theta_0$, $-\pi < \theta_0 < \pi$ onto the ray $\mathrm{Arg}\ w = -\theta_0$.

Solution:

$\square$

Problem 2

1.

Using methods familiar from elementary calculus, find the limit (it it exists) of the following sequences of complex numbers:

a)

$z_n = (i/3)^n$, **start looking at** $|z_n|$**.**

Solution:

$\square$

b)

$z_n = (2 + in)/(1 + 3n)$.

Solution:

$\square$

c)

$z_n = i^n$.

Solution:

$\square$

2.

Consider the following complex functions:

$$f_1(z) = z^2 - 2z + 1, \qquad f_2(z) = \frac{z + 2i}{z}, \qquad f_3(z) = \frac{z^2 + 4}{z(z - 2i)}.$$

a)

Find the domain of these functions and justify their continuity in the domain.

<u>Solution:</u>

☐

b)

Calculate the limits of these functions as $z \to 2i$.

<u>Solution:</u>

☐

c)

Redefine f_3 so that it becomes a continuous function at $z = 2i$.

<u>Solution:</u>

☐

Problem 3

1.

Show that $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are nowhere differentiable. Hint: try the approach use in class to get the Cauchy-Riemann equations.

<u>Solution:</u>

☐

2.

Find the derivatives of

$$f(z) = \left(\frac{z^2-1}{z^2+1}\right)^{100}, \qquad g(z) = \frac{(z+2)^3}{(z^2+iz+1)^4}.$$

Solution:

$\square$

3.

Let $f(z) = z^3 + 1$ and let

$$z_1 = \frac{-1+\sqrt{3}i}{2}, \qquad z_1 = \frac{-1-\sqrt{3}i}{2}.$$

Show that there is no point w on the line segment between z_1 and z_2 such that

$$f(z_2) - f(z_1) = f'(w)(z_2 - z_1),$$

meaning that the mean-value theorem of calculus does not extend to complex functions.

Solution:

$\square$

Problem 4

1.

Show that

$$f(z) = (x^2 + y) + i(y^2 - x)$$

is not analytic at any point of the complex plane.

Solution:

$\square$

2.

Use the Cauchy-Riemann equations to show that the following functions are not differentiable:

a)

$f(z) = \bar{z}$.

Solution:

$\square$

b)

$f(z) = \mathrm{Re}(z)$.

Solution:

$\square$

c)

$f(z) = 2y - ix$.

Solution:

$\square$

3.

Construct an analytic function whose real part is $u(x, y) = x^3 - 3xy^2 + y$.

Solution:

$\square$

4.

Show that if $\phi(x, y)$ is harmonic, then $\phi_x - i\phi_y$ is analytic. You may assume that ϕ has continuous partial derivatives of all orders.

Solution:

$\square$

Problem 5

1.

Show that

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y.$$

Solution:

$\square$

2.

Prove that $\cos z = 0$ if and only if $z = \pi/2 + k\pi$, where k is an integer.

Solution:

$\square$

3.

Using the fact that $f'(0) = \lim_{z\to 0}[f(z) - f(0)]/z$, calculate

$$\lim_{z\to 0} \frac{\sin z}{z}.$$

Solution:

$\square$

4.

Using the chain rule, determine the domain of analyticity for $f(z) = \mathrm{Ln}(3z-1)$ and compute $f'(z)$.

Solution:

$\blacksquare$