# Counting Processes and Asymptotic Theory

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Failure Time Models,
5nd chapter of *The Statistical Analysis of Failure Time Data*
Kalbfleisch and Prentice, 2002

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Outline

- Counting processes and intensity functions

## A counting process $N = \{N(t), t \geq 0\}$

is a stochastic process with $N(0) = 0$ and whose value at time $t$ counts the number of events that have occured in the interval $(0, t]$.

- The sample paths of $N$ are nondecrising step functions that jump whenever an event (or events) occur.
- In continuous time,

  no two counting processes can jump at the same time.

- In discrete time, they can.

Number of events that occur in the interval $[t, t + dt)$?
$dN(t) = N(t^- + dt) - N(t^-)$.

Number of events that occur at time $t$? $\Delta N(t) = N(t) - N(t^-)$.

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And what about more general counting processes where individuals may experience more than one event? Chapters 8, 9, and 10.

# Filtration: history of events

observed counting process: $N_i = \{N_i(t), t \geq 0\}$

underlying counting process: $\tilde{N}_i = \{\tilde{N}_i(t), 0 \leq t\}, \quad \tilde{N}_i(t) = \mathbf{1}(T_i \leq t)$

at-risk process: $\{Y_i(t), t \geq 0\}, \quad Y_i(t) = \mathbf{1}(T_i \geq t, C_i \geq t)$

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key concept: filtration

$$\mathcal{F}_t = \sigma\{N_i(u), Y_i(u^+), X_i(u^+), i = 1, \ldots, n; 0 \leq u \leq t\}, \quad t > 0,$$

where

$Y_i(u^+) = \lim_{s \to u^+} Y_i(s)$;

stochastic time-dependent covariate: $X_i(t) = \{x_i(u) : 0 \leq u \leq t\}$.

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The notation $\sigma[\cdot]$ specifies the sigma algebra of events generated by the variables given in the brackets.

# Intensity functions

The intensities or rates for the processes $N_i$ are defined with reference to the filtration $\mathcal{F}_t$. If the censoring process is independent, the intensity model for the counting process $N_i$ is

$$\mathbb{P}[dN_i(t) = 1 | \mathcal{F}_{t^-}] = Y_i(t) d\Lambda_i(t), \quad i = 1, \ldots, n, \quad t > 0.$$

The hazard model can be written $d\Lambda_i(t) = \mathbb{P}[d\tilde{N}_i(t) = 1 | X_i(t), \tilde{N}_i(t^-) = 0]$.

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$\Lambda_i$ is called the cumulative intensity process of the counting process $\tilde{N}_i$.

- In the continuous case, $\mathbb{P}[dN_i(t) = 1 | \mathcal{F}_{t^-}] = Y_i(t)\lambda_i(t)dt$
- In the discrete case, $\mathbb{P}[dN_i(a_l) = 1 | \mathcal{F}_{a_l^-}] = Y_i(a_l)\lambda_{il}, \quad l = 1, 2, \ldots$

$\lambda_i(t)$ and $\lambda_{il}$ are the corresponding intensity processes.

# Martingales

$$M_i(t) = N_i(t) - \int_0^t Y_i(u)\lambda_i(u)du, \quad t \leq 0.$$
$$= \int_0^t dM_i(u),$$
$$dM_i(t) = dN_i(t) - Y_i(t)\lambda_t(t)dt.$$

If

- $\mathbb{E}[dM_i(t)|\mathcal{F}_{t-}] = 0, \quad \forall t; \quad \equiv \quad \mathbb{E}[M_i(t)|\mathcal{F}_s] = M_i(s), \quad \forall s \leq t.$

Then, $M_i(t)$ is a martingale.

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Consequences:

- $\mathbb{E}[M_i(t)] = 0, \quad \forall t;$
- the process $M_i(t)$ has uncorrelated increments, i.e., $\mathbb{E}[(M_i(t) - M_i(s)) \times M_i(s)] = 0, \quad \forall 0 < s < t.$

# Decomposing $N_i(t)$

Goal: obtain a regression model by allowing the failure rate to be a function of the derived covariates $Z$.

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The hazard at time $t$ for an individual can be written as

$$\lambda(t; x) = \text{hazard} \times c(Z^\top \beta),$$

three forms have been used for $c$:

- $c(s) = 1 + s$, corresponding to the failure rate;
- $c(s) = (1 + s)^{-1}$, corresponding to the mean survival time;
- $c(s) = \exp(s)$.

## Exponential regression model

$$\lambda(t;x) = \lambda \exp(Z^\top \beta)$$
$$Y = -\log \lambda - Z^\top \beta + W$$
$$W \sim \text{Extreme Value dist.}$$

## Weibull regression model

$$\lambda(t;x) = \gamma(\lambda t)^{\gamma-1} \exp(Z^\top \beta)$$
$$Y = -\log \lambda - Z^\top \sigma \beta + \gamma^{-1} W$$
$$W \sim \text{Extreme Value dist.}$$

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Accelerated failure time models
- ↳ general class of log-linear models
  - ↳ covariates act additively on $Y$, or multiplication on $T$
    - ↳ log survival time, $Y = \log T$

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More general model: Relative Risk or Cox Model.

# Relative risk model

**Cox, 1972**

$$\lambda(t;x) = \lambda_0(t)\exp(Z^\top \beta),$$

where $\lambda_0(\cdot)$ is an arbitrary unspecified baseline hazard function for continuous $T$.

The conditional survivor function for $T$ given $Z$ is

$$F(t;x) = F_0^{\exp(Z^\top \beta)}(t), \quad \text{where} \quad F_0(t) = \exp\left[-\int_0^t \lambda_0(u)\mathrm{d}u\right].$$

Thus the survivor function of $t$ for a covariate value, $x$, is obtained by raising the baseline survivor function $F_0(t)$ to a power.

Nice generalizations,

- stratified Cox model;
- time-dependent Cox model: *relative* risk model.

# Accelerated failure time model

Suppose $Y = \log T$ and consider the linear model

$$Y = Z^\top \beta + W.$$

Exponentiation gives $T = \exp(Z^\top \beta)\ S$, where $S = \exp(W) > 0$ has hazard function $\lambda_0(s)$, say, that is independent of $\beta$.

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The hazard function for $T$ can be written as

$$\lambda(t; x) = \exp(-Z^\top \beta)\lambda_0[t \exp(-Z^\top \beta)].$$

The effect of the covariate is multiplicative on $t$ rather than on the hazard function.

**i.e.,**

The role of $Z$ is to accelerate (or decelerate) the time to failure.

# Comparison of regression models

**note**

Exponential and Weibull regression models can be considered as special cases of both models.

# Discrete failure time models

Discrete failure time?

- Grouping of continuous data due to imprecise measurement;
- Time itself may be discrete
  - e.g., when the response time represents the number of episodes that occur prior to a terminal event.

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Discrete regression models?

- Grouped relative risk model;
- Discrete and continuous relative risk model;
- Discrete logistic model.

# Discrete regression models

- Grouped relative risk model:

  $$\text{Discrete baseline cumulative hazard function} : \Lambda_0(t) = \sum_{a_i \leq t} \lambda_i,$$

  this model is the uniquely appropriate one for grouped data from the continuous relative risk model.
- Discrete and continuous relative risk model:

  $$\mathrm{d}\Lambda(t; x) = \exp(Z^\top \beta)\, \mathrm{d}\Lambda_0(t),$$

  which retains the multiplicative hazard relationship.
- Discrete logistic model:

  $$\frac{\mathrm{d}\Lambda(t; x)}{1 - \mathrm{d}\Lambda(t; x)} = \frac{\mathrm{d}\Lambda_0(t)}{1 - \mathrm{d}\Lambda_0(t)} \exp(Z^\top \beta),$$

  specifies a linear log odds model for the hazard probability at each potential failure time.


THANK YOU
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