AMCS 202 - Applied Mathematics II
Maria Alexandra Aguiar Gomes
Applied Mathematics and Computer Science (AMCS)/Statistics (STAT) Program
Computer, Electrical and Mathematical Sciences & Engineering (CEMSE) Division
King Abdullah University of Science and Technology (KAUST)

HOMEWORK
I

Henrique Aparecido Laureano

Spring Semester 2018

Contents

Problem 1

Write

$$\frac{(5-4i)-(3+7i)}{(4+2i)-(2-3i)}$$

in the form $a+bi$.

Solution:

$$\frac{(5-4i)-(3+7i)}{(4+2i)+(2-3i)} = \frac{2-11i}{6-i} = \frac{2-11i}{6-i}\cdot\frac{6+i}{6+i} = \frac{12-64i+11}{37} = \frac{23-64i}{37} = \frac{23}{37} - \frac{64}{37}i.$$

$$\boxed{a = \frac{23}{37}, \quad b = -\frac{64}{37}.}$$

□

Problem 2

Let $z = x+yi$. Find $\mathrm{Im}(2z+4\bar{z}-4i)$, in which $\bar{z}$ is the conjugate of z.

Solution:

$$2z+4\bar{z}-4i = 2(x+yi)+4(x-yi)-4i = 2x+2yi+4x-4yi-4i = 6x+(-2y-4)i.$$

$$\boxed{\mathrm{Im}(2z+4\bar{z}-4i) = \mathrm{Im}(6x+(-2y-4)i) = -2y-4.}$$

□

Problem 3

Find the solution z to

$$\frac{z}{1+\bar{z}} = 3+4i,$$

in which z is a complex number and $\bar{z}$ its conjugate.

Solution:

$$z = a + bi, \quad \bar{z} = a - bi$$

$$\begin{aligned}\frac{z}{1+\bar{z}} = 3 + 4i \quad \Rightarrow \quad z &= (1+\bar{z})(3+4i) = 3 + 4i + 3\bar{z} + 4\bar{z}i = 3 + 4i + 3(a - bi) + 4(a - bi)i \\ &= 3 + 3a + 4b + (4 + 4a - 3b)i.\end{aligned}$$

$$z = a + bi = 3 + 3a + 4b + (4 + 4a - 3b)i \quad \Rightarrow \quad \begin{cases} a = 3 + 3a + 4b & \Rightarrow \quad 2a + 4b + 3 = 0 \\ b = 4 + 4a - 3b & \Rightarrow \quad a - b + 1 = 0 \end{cases}.$$

$$\begin{aligned} b = 1 + a \quad &\Rightarrow \quad 2a + 4(1 + a) + 3 = 0 \quad \Rightarrow \quad a = -\frac{7}{6}. \\ &\Rightarrow \quad b = -\frac{1}{6}.\end{aligned}$$

$$\boxed{z = a + bi = -\frac{7}{6} - \frac{1}{6}i.}$$

$\square$

Problem 4

Which of the complex numbers $10 + 8i$ or $11 - 6i$ is closer to the origin? Why?

Solution:

$$\begin{aligned} z_1 = 10 + 8i \quad &\Rightarrow \quad r_1 = |z_1| = \sqrt{10^2 + 8^2} = \sqrt{164} = 12.81, \\ z_2 = 11 - 6i \quad &\Rightarrow \quad r_2 = |z_2| = \sqrt{11^2 + (-6)^2} = \sqrt{157} = 12.53.\end{aligned}$$

Where r, i.e. the absolute value, is the distance to the origin of the point representing the complex number. Between the two complex numbers, $11 - 6i$ is more closer to origin, but for a very small diferrence (0.28).

$\square$

Problem 5

Compute all roots in $(-1-\sqrt{3}i)^{1/4}$ and sketch these roots on an appropriate circle centered at the origin.

<u>Solution:</u>

$$r=\sqrt{(-1)^2+(-\sqrt{3})^2}=2, \quad \theta=\arctan\left(\frac{-\sqrt{3}}{-1}\right)=60^\circ.$$

The point representing the complex number, $(-1,-\sqrt{3})$, is in the first quadrant and θ is in the third quadrant, therefore $\theta=60^\circ+180^\circ=240^\circ=4\pi/3$.

$$z=-1-\sqrt{3}i=2\exp\{240^\circ i\}.$$

$$\text{Using Moivre's Theorem}: \quad z^{1/4}=2^{1/4}\exp\left\{\frac{4\pi}{3}\frac{1}{4}i\right\}=2^{1/4}\exp\{60^\circ i\},$$

$$\text{Using Euler's formula}: \quad z_1=2^{1/4}(\cos 60^\circ+i\sin 60^\circ)=2^{1/4}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=0.59+1.03i.$$

z_1: *first root.*

The other roots come from adding $2\pi/4=90^\circ$ to the 1st root z_1.

$$z_2=2^{1/4}(\cos 150^\circ+i\sin 150^\circ)=2^{1/4}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=-1.03+0.59i,$$

$$z_3=2^{1/4}(\cos 240^\circ+i\sin 240^\circ)=2^{1/4}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=-0.59-1.03i,$$

$$z_4=2^{1/4}(\cos 330^\circ+i\sin 330^\circ)=2^{1/4}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=1.03-0.59i.$$

$$\boxed{z_1=0.59+1.03i, \quad z_2=-1.03+0.59i, \quad z_3=-0.59-1.03i, \quad z_4=1.03-0.59i.}$$

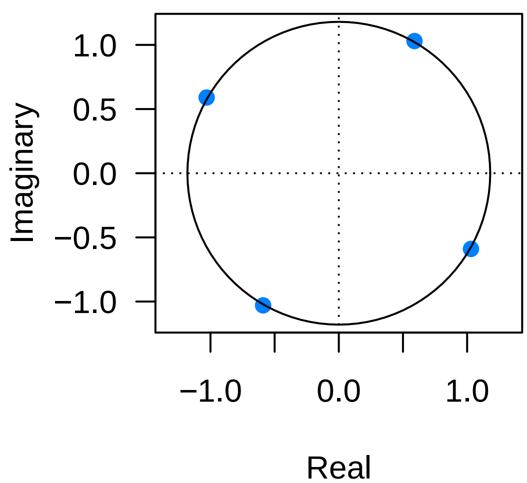

Figure 1: The four roots, each one in a different quadrant, and a circle centered at the origin.

$\square$

Problem 6

Find all the solutions of equation $z^8 - 2z^4 + 1 = 0$.

Solution:

$$z^8 - 2z^4 + 1 = 0 \quad \Rightarrow \quad (z^4 - 1)(z^4 - 1) = 0 \quad \Rightarrow \quad z^4 - 1 = 0 \quad \Rightarrow \quad (z^2 - 1)(z^2 + 1) = 0$$

$$z^2 - 1 = 0 \quad \Rightarrow \quad (z-1)(z+1) = 0 \quad \Rightarrow \quad z = \pm 1.$$
$$z^2 + 1 = 0 \quad \Rightarrow \quad z^2 = -1 \quad \Rightarrow \quad z = \pm\sqrt{-1} \quad \Rightarrow \quad z = \pm i.$$

$$\boxed{\text{Solutions :} \quad z = -1, \quad z = 1, \quad z = -i, \quad z = i.}$$

$\square$

Problem 7

Sketch the sets in the complex plane defined by

- $\mathrm{Im}(1/z) < 1/2$**;**
- $0 \leq \arg(z) \leq 2\pi/3$**;**
- $1 \leq |z - 1 - i| < 2$**.**

Solution:

$\square$

Problem 8

Write e^{z^2} in the form $a + bi$ using Euler's formula.

Solution:

$$\begin{aligned}\exp\{z^2\} &= \exp\Big\{(x+yi)^2\Big\} = \exp\{x^2 + 2xyi - y^2\} = \exp\{x^2 - y^2\}\exp\{2xyi\}\\ &= \exp\{x^2 - y^2\}\Big(\cos(2xy) + i\sin(2xy)\Big)\\ &= a + bi.\end{aligned}$$

$$\boxed{\text{With :}\quad a = \exp\{x^2 - y^2\}\cos(2xy) \quad \text{and} \quad b = \exp\{x^2 - y^2\}\sin(2xy).}$$

$\square$

Problem 9

Find all the values of z satisfying equation $e^{2z} + e^z + 1 = 0$.

Solution:

$$\begin{aligned} x = e^z \quad &\Rightarrow \quad e^{2z} + e^z + 1 = 0 \quad \Rightarrow \quad (e^z)^2 + e^z + 1 = 0 \quad \Rightarrow \quad x^2 + x + 1 = 0\\ &\Rightarrow \quad \Big(x + \frac{1}{2}\Big)^2 - \frac{1}{4} + 1 = 0 \quad \Rightarrow \quad \Big(x + \frac{1}{2}\Big)^2 = -\frac{3}{4} \quad \Rightarrow \quad \sqrt{\Big(x + \frac{1}{2}\Big)^2} = \sqrt{-\frac{3}{4}}\\ &\Rightarrow \quad x + \frac{1}{2} = \pm\frac{\sqrt{3}}{2}i \quad \Rightarrow \quad x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \quad \Rightarrow \quad e^z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\\ &\Rightarrow \quad z = \ln\Big(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\Big).\end{aligned}$$

$$\begin{aligned} z_1 = \ln\Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\Big) \quad &\Rightarrow \quad \ln\sqrt{\Big(-\frac{1}{2}\Big)^2 + \Big(\frac{\sqrt{3}}{2}\Big)^2} + \ln\exp\Big\{i\arctan\frac{\sqrt{3}/2}{-1/2}\Big\}\\ &\Rightarrow \quad \ln 1 + i\arctan -\sqrt{3} \quad \Rightarrow \quad 0 + i(-60^\circ) \quad \Rightarrow \quad -\frac{\pi}{3}i.\end{aligned}$$

With the complex coordinates $(-1/2, \sqrt{3}/2)$ we have a point in the second quadrant. The angle -60° is in the fourth quadrant, so we add π.

$$\boxed{z_1 = \ln\Big(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\Big) = \Big(-\frac{\pi}{3} + \pi\Big)i = \frac{2\pi}{3}i.}$$

$$z_2 = \ln\Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\Big) \quad \Rightarrow \quad \ln\sqrt{\Big(-\frac{1}{2}\Big)^2 + \Big(-\frac{\sqrt{3}}{2}\Big)^2} - \ln\exp\Big\{i\arctan\frac{-\sqrt{3}/2}{-1/2}\Big\}$$
$$\Rightarrow \quad \ln 1 - i\arctan\sqrt{3} \quad \Rightarrow \quad 0 - 60^\circ i \quad \Rightarrow \quad -\frac{\pi}{3}i.$$

With the complex coordinates $(-1/2, -\sqrt{3}/2)$ we have a point in the third quadrant. The angle 60° is in the first quadrant, so we subtract π (the angle must be between $-\pi$ and π).

$$\boxed{z_2 = \ln\Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\Big) = \Big(\frac{\pi}{3} - \pi\Big)i = -\frac{2\pi}{3}i.}$$

□

Problem 10

Find all the values of $(-i)^{4i}$.

<u>Solution:</u>

$$(-i)^{4i} = \exp\{\ln(-i)^{4i}\} = \exp\{4i\ln(-i)\}.$$

$$-i = 0 - 1i = r\exp\{i\theta\}$$

$$r = \sqrt{0^2 + (-1)^2} = 1, \quad \theta = \arctan\Big(-\frac{1}{0}\Big) = -\frac{\pi}{2}$$

$$-i = 1\cdot\exp\Big\{i\Big(-\frac{\pi}{2}\Big)\Big\} = \exp\Big\{-\frac{\pi}{2}i\Big\}, \quad \ln(-i) = -\frac{\pi}{2}i.$$

So

$$(-i)^{4i} = \exp\{\ln(-i)^{4i}\} = \exp\{4i\ln(-i)\} = \exp\Big\{4i\Big(-\frac{\pi}{2}i\Big)\Big\} = \exp\{2\pi\}.$$

$$\boxed{(-i)^{4i} = \exp\{2\pi\} = \exp\{2\pi\}\exp\{0\} = \exp\{2\pi\}(\cos 0 + i\sin 0) = \exp\{2\pi\}(1) = \exp\{2\pi\}.}$$

□

Problem 11

Express $\tan i$ in the form $a + bi$.

Solution:

$i = 0 + 1i$. Conjugate: $-i = 0 - 1i$.

$$\begin{aligned}\tan i &= \frac{\sin i}{\cos i} = \frac{\sin i}{\cos i}\frac{\cos -i}{\cos -i} = \frac{2}{2}\frac{\sin i}{\cos i}\frac{\cos -i}{\cos -i} = \frac{\sin(i + (-i)) + \sin(i - (-i))}{\cos(i + (-i)) + \cos(i - (-i))} = \frac{\sin 0 + \sin 2i}{\cos 0 + \cos 2i} \\ &= \frac{\sin 2i}{1 + \cos 2i} = \frac{i \sinh 2}{1 + \cosh 2}.\end{aligned}$$

sinh: hyperbolic sine. cosh: hyperbolic cosine.

$$\boxed{\tan i = a + bi = 0 + i\frac{\sinh 2}{1 + \cosh 2} = i\frac{i \sinh 2}{1 + \cosh 2}.}$$

□

Problem 12

Find all the values of z satisfying equation $\sin z = -i$.

Solution:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = -i \quad \Rightarrow \quad e^{iz} - e^{-iz} = 2.$$

$$x = e^{iz} \quad \Rightarrow \quad e^{iz} - e^{-iz} = 2 \quad \Rightarrow \quad x - \frac{1}{x} - 2 = 0 \quad \Rightarrow \quad x^2 - 2x - 1 = 0 \quad \Rightarrow \quad x = 1 \pm \sqrt{2}.$$

$$e^{iz} = 1 \pm \sqrt{2} \quad \Rightarrow \quad iz = \ln(1 \pm \sqrt{2}) \quad \Rightarrow \quad z = \frac{\ln(1 \pm \sqrt{2})}{i}.$$

$$\boxed{z = -i \ln(1 \pm \sqrt{2}).}$$

□

Appendix: Problem 7

$\text{Im}(1/z) < 1/2$.

<u>Solution:</u>

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{1}{a+bi}\frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}, \quad \Rightarrow \quad \text{Im}(1/z) = -\frac{b}{a^2+b^2}.$$

$$-\frac{b}{a^2+b^2} < \frac{1}{2} \quad \Rightarrow \quad 0 < a^2+b^2+2b \quad \Rightarrow \quad a^2+b^2+2b > 0.$$

$$\begin{aligned} a^2+b^2+2b > 0 \quad &\Rightarrow \quad a^2+b^2+0a+2b+0 > 0 \quad \Rightarrow \quad (a^2+0a)+(b^2+2b) > 0 \\ &\Rightarrow \quad a^2+(b+1)^2 > 0+1 \quad \Rightarrow \quad a^2+(b+1)^2 > 1. \end{aligned}$$

$$\boxed{a^2+(b+1)^2 > 1 : \quad \text{cirle with center at } (0,-1) \text{ and radius } > 1.}$$

■
