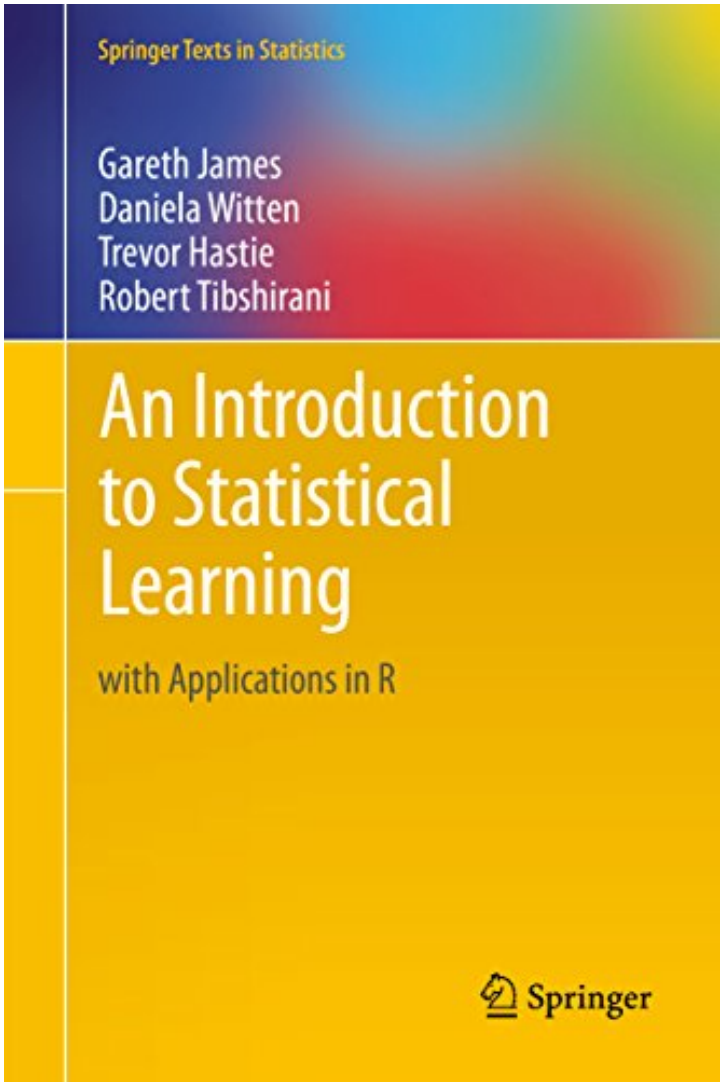
Classification

chapter 4 of *An Introduction to Statistical Learning* (ISL)

Henrique Laureano
http://leg.ufpr.br/~henrique

What we read (long description)



Now in a shorter way

What we read (short description)

At chapter 4 are discussed three of the most widely-used classifiers.

- Logistic Regression
- Linear Discriminant Analysis (LDA)
- K-Nearest Neighbors (KNN)

What we didn't read

More computer-intensive methods are discussed in later chapters, such as

- Generalized Additive Models (GAM)
- Trees
- Random Forests
- Boosting
- Support Vector Machines (SVM)

On the Agenda

We could consider encoding the response, Y, as a quantitative variable, e.g.,

Predict the medical condition of a patient on the basis of her symptoms.

$$Y = \begin{cases} 1 & \text{if } \texttt{stroke}; \\ 2 & \text{if } \texttt{drug overdose}; \\ 3 & \text{if } \texttt{epileptic seizure}. \end{cases}$$

We could consider encoding the response, Y, as a quantitative variable, e.g.,

Predict the medical condition of a patient on the basis of her symptoms.

$$Y = \begin{cases} 1 & \text{if } \texttt{stroke}; \\ 2 & \text{if } \texttt{drug overdose}; \\ 3 & \text{if } \texttt{epileptic seizure}. \end{cases}$$

Unfortunately, this coding implies an ordering on the outcomes.

Each possible coding would produce a fundamentally different linear model that would ultimately lead to different sets of predictions.

That leads us to other questions,

- What if the response variable values did take on a natural ordering, such as mild, moderate, and severe?
- For a binary (two level) qualitative response, the situation is better.
  - However, if we use linear regression, some of our estimates might be outside the [0, 1] interval.
  - However, the dummy variable approach cannot be easily extended to accommodate qualitative responses with more than two levels.

That leads us to other questions,

- What if the response variable values did take on a natural ordering, such as `mild`, `moderate`, and `severe`?
- For a `binary` (two level) qualitative response, the situation is better.
  - However, if we use linear regression, some of our estimates might be outside the [0, 1] interval.
  - However, the dummy variable approach cannot be easily extended to accommodate qualitative responses with more than two levels.

For these reasons, it is preferable to use a classification method that is truly suited for qualitative response values, such as the ones presented next.

Curiously,

it turns out that the classifications that we get if we use linear regression to predict a binary response will be the same as for the linear discriminant analysis (LDA) procedure we discuss later.

On the Agenda

A classic 'book example dataset relationship'

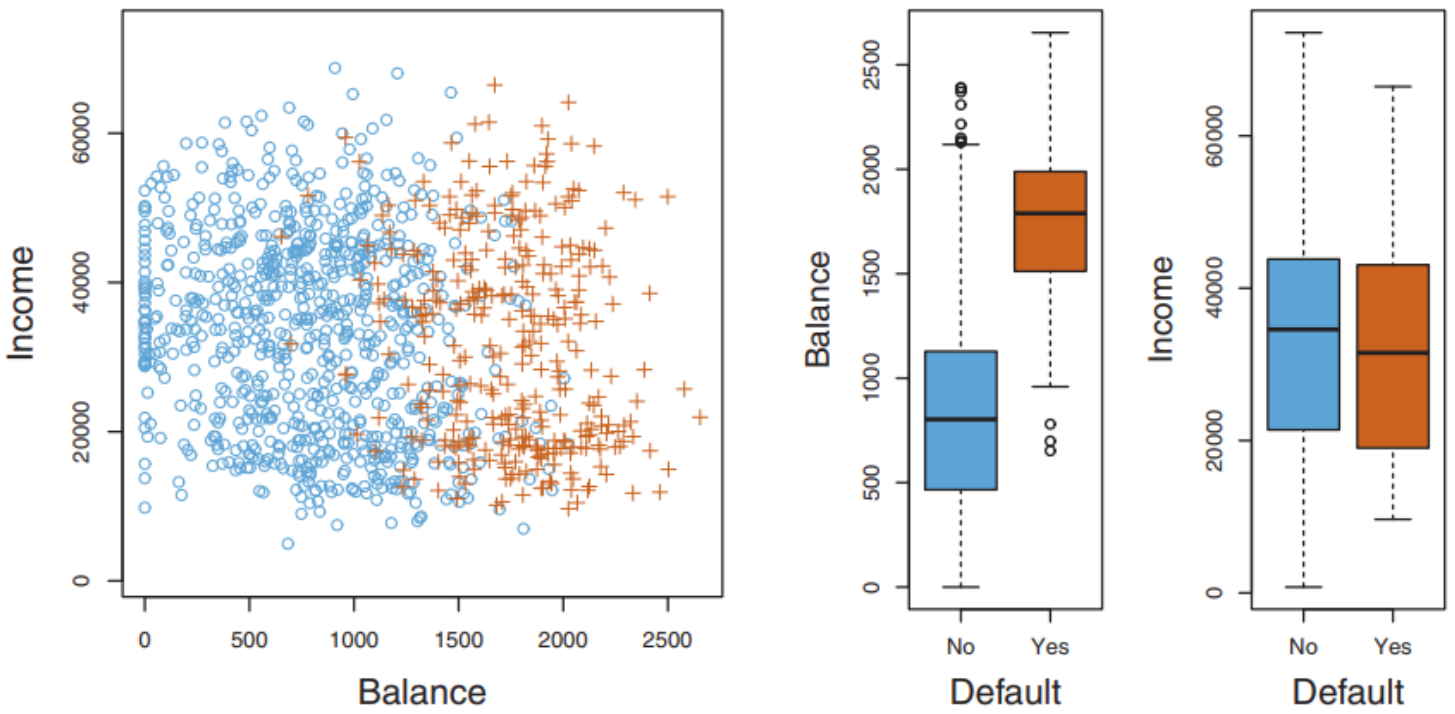


FIGURE 4.1. *The* `Default` *data set.* Left: *The annual incomes and monthly credit card balances of a number of individuals. The individuals who defaulted on their credit card payments are shown in orange, and those who did not are shown in blue.* Center: *Boxplots of* `balance` *as a function of* `default` *status.* Right: *Boxplots of* `income` *as a function of* `default` *status.*

. . . a very pronounced relationship between `balance` and `default`.

On the Agenda

To start, a comparison with Linear Regression

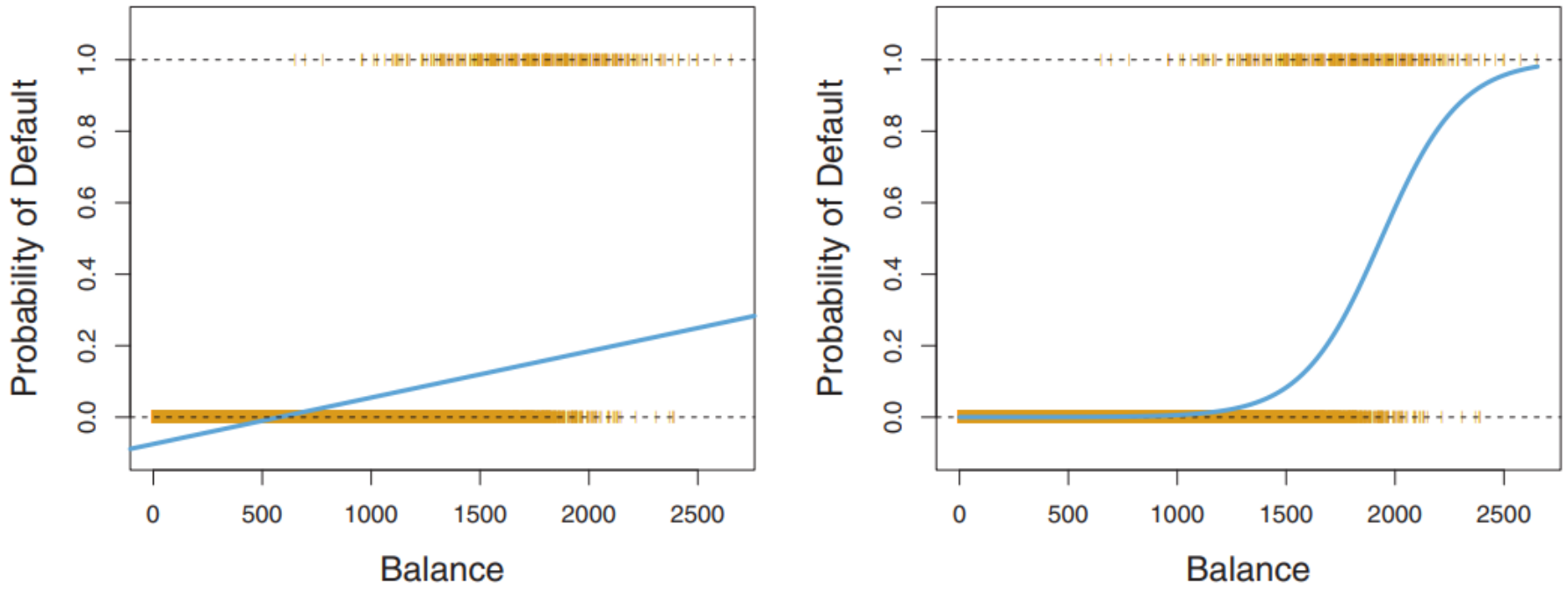


FIGURE 4.2. *Classification using the* `Default` *data.* Left: *Estimated probability of* `default` *using linear regression. Some estimated probabilities are negative! The orange ticks indicate the 0/1 values coded for* `default` *(*`No` *or* `Yes`*).* Right: *Predicted probabilities of* `default` *using logistic regression. All probabilities lie between* 0 *and* 1.

Logistic regression in two slides

Some math, but with just one predictor

The model and its relations (*showing my LaTeX skills*)

$$p(X) = \underbrace{\frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}}}_{\substack{\textit{logistic}\\ \textit{function}\\ \textit{(S-shaped)}}} \Rightarrow \underbrace{\frac{p(X)}{1-p(X)}}_{\textit{odds} \in (0,\infty)} = e^{\beta_0+\beta_1 X} \Rightarrow \underbrace{\log \frac{p(X)}{1-p(X)}}_{\substack{\textit{log-odds}\\ \text{or}\\ \textit{logit}}} = \beta_0+\beta_1 X$$

Some math, but with just one predictor

The model and its relations (*showing my LaTeX skills*)

$$p(X) = \underbrace{\frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}}}_{\substack{\textit{logistic}\\ \textit{function}\\ (\textit{S-shaped})}} \Rightarrow \underbrace{\frac{p(X)}{1-p(X)}}_{\textit{odds} \in (0,\infty)} = e^{\beta_0+\beta_1 X} \Rightarrow \underbrace{\log \frac{p(X)}{1-p(X)}}_{\substack{\textit{log-odds}\\ \text{or}\\ \textit{logit}}} = \beta_0+\beta_1 X$$

For example,

$$p(X) = 0.2 \;\Rightarrow\; \frac{0.2}{1-0.2} = \frac{1}{4} \quad \textit{and} \quad p(X) = 0.9 \;\Rightarrow\; \frac{0.9}{1-0.9} = 9.$$

Maximum likelihood

The estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ are chosen to maximize a math equation called a

likelihood function

$$l(\beta_0, \beta_1) = \prod_{i:y_i=1} p(x_i) \prod_{i':y_{i'}=0} (1 - p(x_{i'})).$$

Maximum likelihood

The estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ are chosen to maximize a math equation called a

likelihood function

$$l(\beta_0, \beta_1) = \prod_{i:y_i=1} p(x_i) \prod_{i':y_{i'}=0} (1 - p(x_{i'})).$$

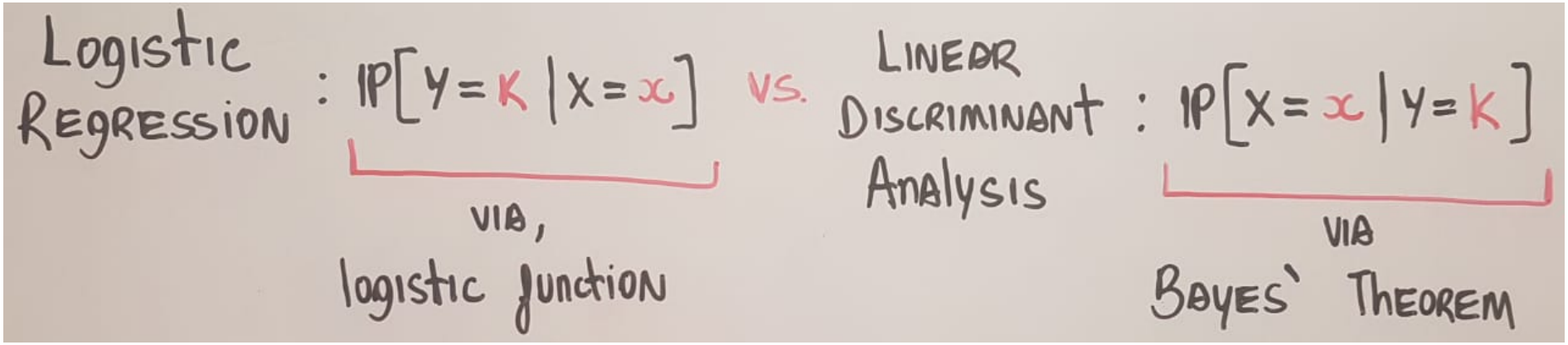
The coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ are unknown, and must be estimated. The general method of maximum likelihood is preferred, since it has better statistical properties.

Maximum likelihood is a very general approach that is used to fit many of the non-linear models examined throughout the book. In the linear regression setting, the least squares approach is in fact a special case of maximum likelihood.

On the Agenda

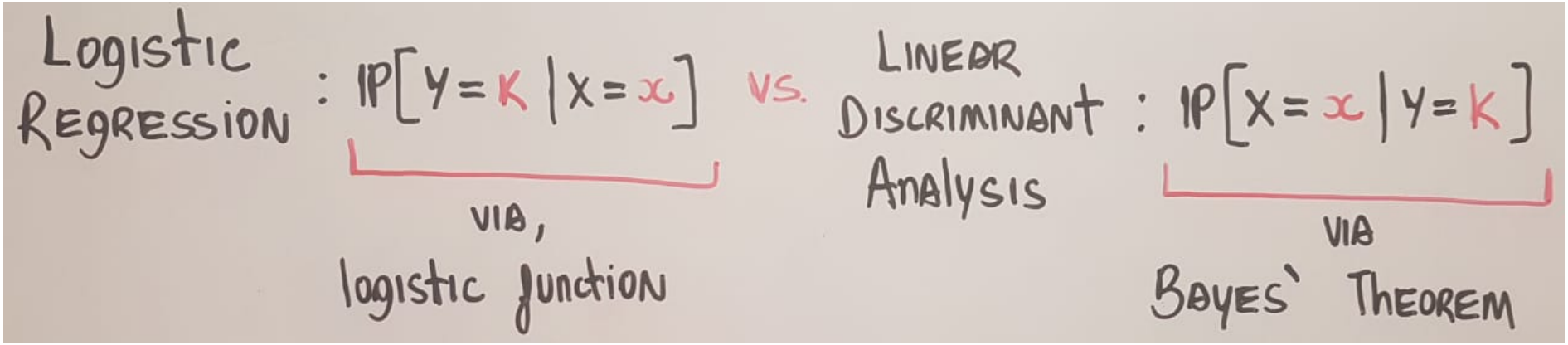
Different ideas, sometimes the same results

Different ideas,



With LDA we model the distribution of the predictors X separately in each of the response classes (i.e. given Y), and then use Bayes' theorem to flip these around into estimates for $\mathbb{P}[Y = k|X = x]$.

Different ideas,



With LDA we model the distribution of the predictors X separately in each of the response classes (i.e. given Y), and then use Bayes' theorem to flip these around into estimates for $\mathbb{P}[Y = k|X = x]$.

Sometimes the same results

When these distributions are assumed to be normal, it turns out that the model is very similar in form to logistic regression.

But, ok... why not continue with logistic regression?

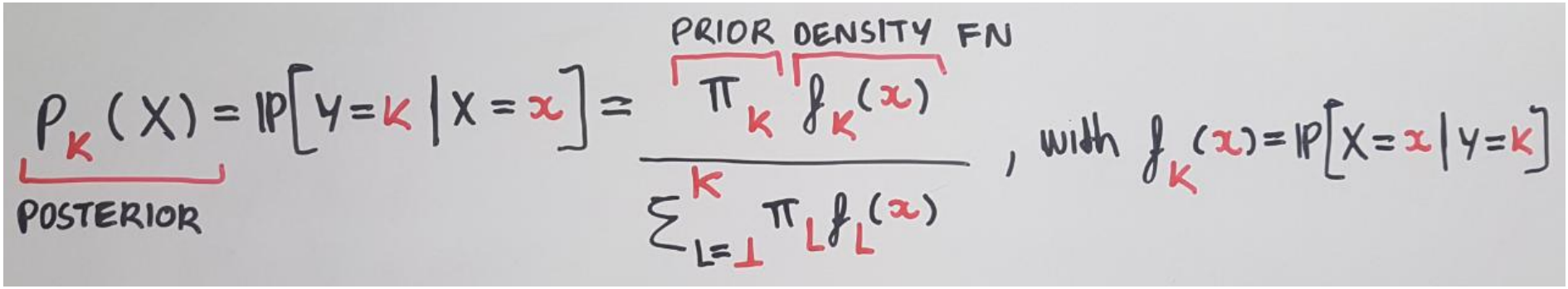
But, ok... why not continue with logistic regression?

Simple, LDA is popular when we have more than two response classes.

Now, a reason more serious: stability

- When the classes are well-separated, the parameter estimates for the logistic regression model are surprisingly unstable. LDA does not suffer from this problem.
- If n is small and the distribution of the predictors X is approximately normal in each of the classes, the linear discriminant model is again more stable than the logistic regression model.

Model framework

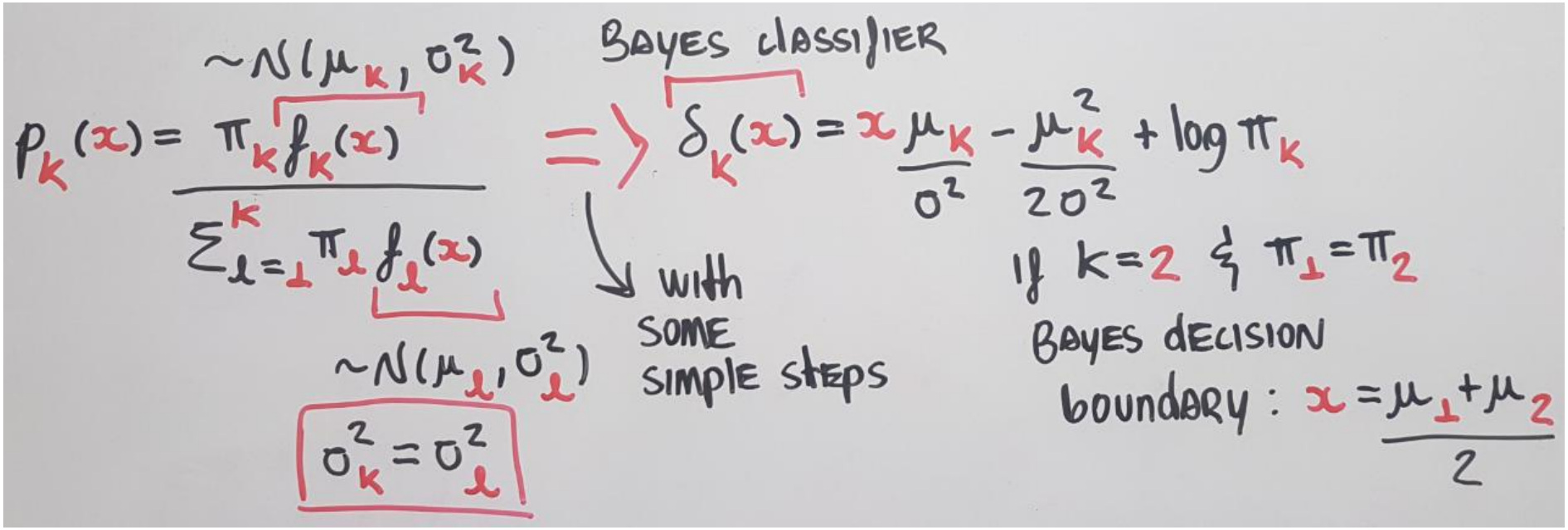


- π_k is the overall or prior prob. that a chosen obs. comes from k.
- In general, estimating π_k is easy if we have a sample of Ys: we simply compute the fraction of observations that belong to the kth class. However, estimating $f_k(x)$ tends to be more challenging, unless we assume some simple forms for these densities.

Remember from Chap. 2 that the Bayes classifier has the lowest possible error rate out of all classifiers.

Dealing with just one predictor

Assumptions: $f_k(x)$ is normal with equal variance for the kth classes.



Putting a hat (simple average and a weighted average of the sample variances for each class) in everything, the LDA approx. this Bayes classifier.

Ok, nice! But... why the name linear discriminant analysis?

Ok, nice! But... why the name linear discriminant analysis?

The word linear stems from the fact that the discriminant functions $\hat{\delta}_k(x)$ are linear functions of x.

Getting bigger

More than one predictor $\Rightarrow$ Multivariate normal distribution, with a class-specific mean vector and a common covariance matrix

On the Agenda

and...

laureano@ufpr.br