hw-3

Henrique M. Rio

Homework 3 in STAT400: Computational Statistics @ CSU

Assignment

1. Let $X \sim \text{Uniform}(a, b)$. Derive $E[X]$ and $Var[X]$.
2. Let $X \sim \text{Exponential}(\alpha)$.
   a. Derive $E[X]$ (show the parts we skipped in class).
   b. What are the parameter(s) of the exponential distribution and what values can they take?
3. Let X = the outcome when a fair die is rolled.
   a. Find $E[X]$
   b. Find $Var[X]$
   c. Before the die is rolled you are offered either $1/3.5 = \$0.29$ (guaranteed amount) or $h(X) = 1/X$ dollars (random amount). Would you accept the guaranteed amount or would you gamble? In your answer, discuss what this means about $1/E[X]$ as compared to $E[1/X]$. In particular, does $1/E[X]$ always equal $E[1/X]$?
4. Give three examples of Bernoulli random variables.
5. For each of the random variables defined below, describe the sample space (set of possible values) and state whether the random variable is continuous or discrete.
   a. $X =$ the number of unbroken eggs in a dozen eggs.
   b. $Y =$ the pH of a randomly chosen soil sample.
   c. $Z =$ the number of CSU students who skipped their first class.
   d. $W =$ the distance between CSU and the local residence of a randomly chosen CSU student.
6. Let $X =$ the number of days of sick leave taken by a randomly selected employee of a large company during a particular year. The maximum allowable days per year is 14. Let the following values of the cdf be defined

$$F(0) = 0.58, F(1) = 0.72, F(2) = 0.76, F(4) = 0.88, F(5) = 0.94.$$

   a. What is the sample space of X?
   b. Compute $P(2 \leq X \leq 5)$
   c. Compute $P(X \geq 5)$ and $P(X > 5)$.
7. The cdf for a random variable X is given below.

$$F_X(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{2} + \frac{3}{32}(4x - x^3/3) & -2 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

   a. Compute $P(X > 0.5)$
   b. Find $f_X(x)$.

c. The median $\tilde{\mu}$ of a continuous random variable is the 50^{th} percentile of the distribution, given by $0.5 = F_X(\tilde{\mu})$. Show that $\tilde{\mu} = 0$.

Turn in in a pdf of your homework to canvas.

1. Let $X \sim \text{Uniform}(a, b)$. Derive $E[X]$ and $Var[X]$.

Pdf : $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for } x < a \text{ or } x < b \end{cases}$

$$E[x] = \int_a^b \frac{1}{b-a} x\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right)$$

$$= \frac{1}{\cancel{b-a}} \cdot \frac{(\cancel{b-a})(b+a)}{2}$$

$$\boxed{E[x] = \frac{1}{2}\cdot(b+a)}$$

$$E[x^2] = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right)$$

$$= \frac{b^3-a^3}{3b-3a} = \frac{1}{3}\frac{(\cancel{b-a})(b^2+ab+a^2)}{\cancel{b-a}}$$

$$Var[x] = E[x^2] - (E[x])^2$$

$$= \frac{(b^2+ab+a^2)}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{b^2+ab+a^2}{3} + \frac{-b^2-2ab+a^2}{4}$$

$$= \frac{4b^2+4ab+a^2-3b^2-6ab+a^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \boxed{\frac{(b-a)^2}{12}}$$

2. Let
$X \sim \text{Exponential}(\alpha)$.
   a. Derive $E[X]$ (show the parts we skipped in class).
   b. What are the parameter(s) of the exponential distribution and what values can they take?

a) $f(x) = \begin{cases} \lambda e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

$E[X] = \int_0^\infty x \lambda e^{-\lambda x}\, dx = \left[-x e^{-\lambda x} - \int -e^{-\lambda x}\, dx\right]_0^\infty$

$u = x$
$du = dx$
$dV = \lambda e^{-\lambda x}$
$V = -e^{-\lambda x}$

$= \left[-x e^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda}\right]_0^\infty$

as it goes to ∞ 0

$= \cancel{\left[-x e^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda}\right]} - 0 - \frac{e^{-\lambda}}{\lambda}$

??

b) Parameter $= \lambda$

3. Let
X = the outcome when a fair die is rolled.
   a. Find $E[X]$
   b. Find $Var[X]$
   c. Before the die is rolled you are offered either $1/3.5 = \$0.29$ (guaranteed amount) or $h(X) = 1/X$ dollars (random amount). Would you accept the guaranteed amount or would you gamble? In your answer, discuss what this means about $1/E[X]$ as compared to $E[1/X]$. In particular, does $1/E[X]$ always equal $E[1/X]$?

outcome	1	2	3	4	5	6
P	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1/6	1/6

a) $E[X] = \sum_{i=1}^{6} x_i P(x) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{31}{6} = \boxed{3.5}$

b) $Var[X] = \sum (x-\mu)^2 p_X(x)$
$= \frac{1}{6}\left[(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2\right.$

$= \frac{17.5}{6}$

$\boxed{Var[X] = 2.9167}$

c) $E[\frac{1}{X}] = 1 \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{5} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6}$

$E[\frac{1}{X}] = \frac{1}{6} + \frac{1}{12} + \frac{1}{18} + \frac{1}{24} + \frac{1}{30} + \frac{1}{36}$

$E[\frac{1}{X}] = 0.41 > 0.28$ no, I would take the gamble. $\frac{1}{E[X]}$ is not equal to $E[\frac{1}{X}]$ and it is usually the case.

4. Give three examples of Bernoulli random variables.

1) Coin toss

2) Rolling a die

3) Fair spin wheel

5. For each of the random variables defined below, describe the sample space (set of possible values) and state whether the random variable is continuous or discrete.

a. $X =$ the number of unbroken eggs in a dozen eggs.

b. $Y =$ the pH of a randomly chosen soil sample.

c. $Z =$ the number of CSU students who skipped their first class.

d. $W =$ the distance between CSU and the local residence of a randomly chosen CSU student.

a. $X = \{1, 2, 3, 4 \ldots, 12\}$ – discrete

b. $Y = \{0 \leq y \leq 14\}$ – continuous

c. $Z = \{0, 1, 2, 3, 4, 5 \ldots, z\}$ – discrete

d. $W = \{0 <$ w $\leq$ distance from the furthest student$\}$ – continuous.

6. Let
$X =$ the number of days of sick leave taken by a randomly selected employee of a large company during a particular year. The maximum allowable days per year is
14. Let the following values of the cdf be defined

$$F(0) = 0.58, F(1) = 0.72, F(2) = 0.76, F(4) = 0.88, F(5) = 0.94.$$

a. What is the sample space of X? a. $X = \{0, 1, 2, 3, 4, 5, \ldots, 13, 14\}$
b. Compute $P(2 \leq X \leq 5)$
c. Compute $P(X \geq 5)$ and $P(X > 5)$.

b) $P(2 \leq X \leq 5) = P(5) - P(2)$
$= 0.94 - 0.76$
$= 0.18$

c) $P(X > 5) = 1 - 0.94 = 0.06$

$P(X \geq 5) = 1 - 0.94 = 0.06$

7. The cdf for a random variable
X is given below.

$$F_X(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{2} + \frac{3}{32}(4x - x^3/3) & -2 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

a. Compute $P(X > 0.5)$
b. Find $f_X(x)$.

c. The median $\tilde{\mu}$ of a continuous random variable is the 50^{th} percentile of the distribution, given by $0.5 = F_X(\tilde{\mu})$. Show that $\tilde{\mu} = 0$.

a) $P(X > 0.5) = \int_{-2}^{0.5} \frac{1}{2} + \frac{3}{8}x - \frac{1}{32}x^3 = \left[\frac{1}{2}x + \frac{3}{16}x^2 - \frac{x^4}{128}\right]_{-2}^{2}$

$= \left[\frac{1}{2}\cdot\frac{1}{2} + \left(\frac{3}{16}\right)\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^4 \cdot \frac{1}{128}\right] - \left[-1 + \frac{12}{16} - \frac{16}{128}\right]$

$= 0.67$

b) $f_X(x) = \frac{3}{8} - \frac{3}{32}x^2$

c) $\int_0^{0.5} \frac{3}{8} - \frac{3}{32}x^2 = \left[\frac{3}{8}x - \frac{1}{32}x^3\right]_0^{0.5}$

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