Generation of Random Variables

Computer-Aided Simulations Lab - Lab L3

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I. Introduction

The goal of the present activity is to implement algorithms to generate random variables according to the following distributions:

- Rayleigh, with parameter σ;
- Lognormal, with parameters μ and σ;
- Beta, with parameters $\alpha > 1$ and $\beta > 1$;
- Chi-square, with parameter $n \in \mathbb{N}^*$;
- Rician, with parameters $\nu \geq 0$ and $\sigma \geq 0$.

II. Proposed approach

A. Auxiliary distributions

For some of the distribution, we may need to generate other variables, such as Poisson and Normal distributed random variables.

1) Poisson: For the Poisson random variable with parameter λ, we first generate an random number uniformly distributed $u = U(0,1)$. Starting from $k = 0$ and with step 1, we compute the cumulative distribution $F(k)$. The value drawn from the Poisson distribution is is the last one to which $F(k) < u$.

2) Normal: For the Normal distribution, with mean μ and standard deviation σ, we rely on the central limit theorem, and generate a sequence of $n = 12$ uniformly distributed random numbers $u_i = U(0,1)$ and use the following approximation to obtain the normal random variable:

$$X \sim N(\mu, \sigma) \approx \mu + \sigma \frac{\sum_{i=1}^{n} u_i - \frac{n}{2}}{\sqrt{\frac{n}{12}}}$$

B. Rayleigh

The Rayleigh random variable is obtained based on the inverse transform method, since from the cumulative density function

$$F(x, \sigma) = \frac{x}{\sigma^2} \exp\left(-x^2/2\sigma^2\right)$$

we derive

$$F^{-1}(p, \sigma) = \sigma\sqrt{-2\ln\left(1-p\right)}$$

In this case, we generate $u = U(0,1)$, and return $F^{-1}(p, \sigma)$.

C. Lognormal

For the Lognormal distribution, we recall the normal variable previously discussed and apply the exponential to the value generated by the standard normal $Z\ N(0,1)$.

$$X = e^{\mu + \sigma \cdot Z}$$

D. Beta

The technique used for the Beta distribution is the acceptance rejection, although for some pairs (α, β) it may not be very efficient. First we obtain c, the maximum value of the p.d.f. for the considered parameters. Next, two random numbers are generated, $x = U(0,1)$ and $y = U(0,c)$, until $y \leq f(x)$, where f is the p.d.f. for the Beta distribution.

E. Chi squared

As for the chi squared random variable, we consider the convolution method, since $X \sim \chi^2(n)$ means that X is the sum of n standard normal random variables. Thus, we generate and sum n random number accordingly to the standard normal distribution.

F. Rice

Also for the Rician distribution, we make use of the other random variables to construct it. First, we generate P, Poisson distributed with parameter $\lambda = \frac{1}{2}(\nu/\sigma)^2$. Then, we generate a chi-squared distributed value X, with $n = 2P+2$ degrees of freedom, and multiply its square root by σ in order to obtain the rician distributed value.

III. Experiments and Results

Next, we test the previously presented Beta random variable generator, with parameters α and β, comparing its first two moments and its distribution with the analytical ones.

Let the parameters of the distribution be arbitrarily chosen $\alpha = 1.5$ and $\beta = 4.5$. For this setup, we are able to compute the first two moments analytically and also empirically for $n \in 10^2, 10^4, 10^5$ samples, as shown in Table I. The confidence intervals were computed with 95% confidence level.

	First moment μ	Second moment σ^2
Analytically	0.025	0.02679
$n = 10^2$	0.2402 ± 0.0385	0.0263 ± 0.0074
$n = 10^4$	0.2478 ± 0.0038	0.0261 ± 0.0009
$n = 10^5$	0.2495 ± 0.0012	0.0267 ± 0.0003

TABLE I: Comparison of first two moments analytically and empirically.

As shown above, the random variable generator seems to be fitting in the expected distribution, as the confidence intervals contain the analytical value for the different sizes of samples. Furthermore, Figures 1-3 show the frequency of the values comparing them we the analytical p.d.f. These plots show how the random variables is adequate to the distribution, and as we increase the number of samples it becomes more precise, fitting almost perfectly the analytical function.

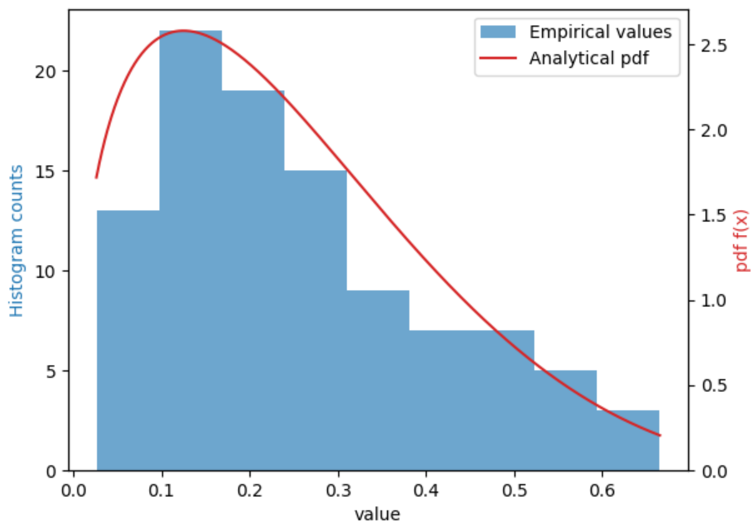

Fig. 1: Average delay for a deterministic service time.

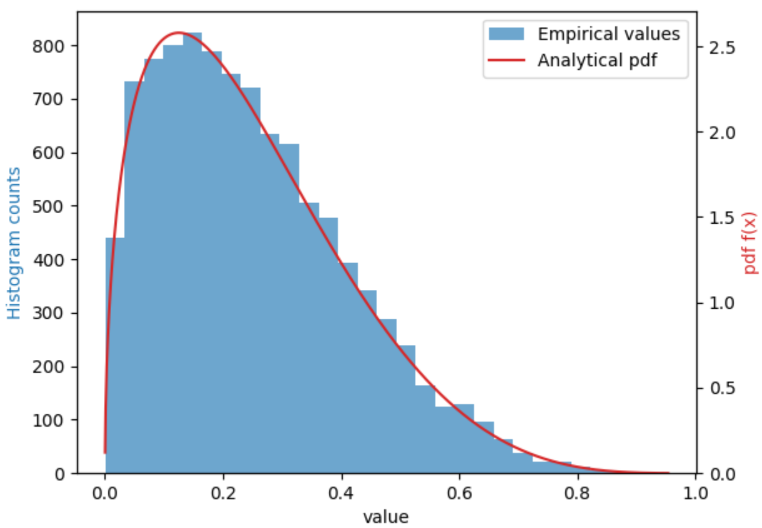

Fig. 2: Comparison between simulated and theoretical average delay for deterministic service time.

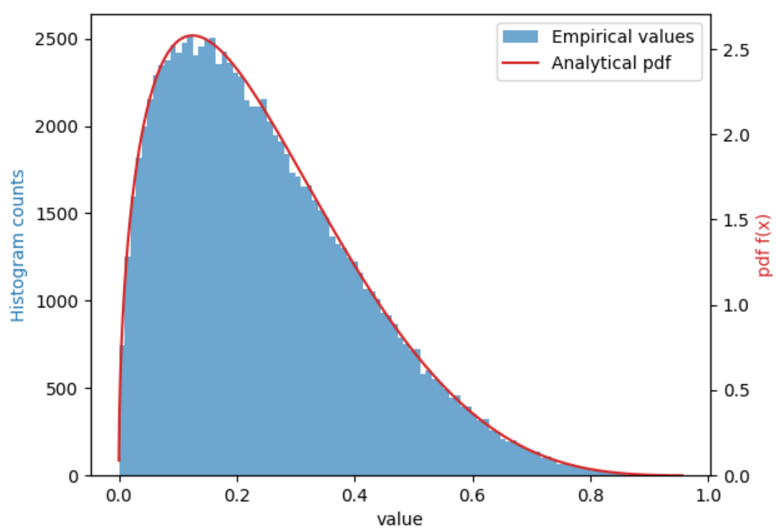

Fig. 3: Comparison between simulated and theoretical average delay for deterministic service time.