

TMA4212 - Buckley-Leverett equation

Candidate numbers: 10052, 10041, 10011

Wednesday 18th April, 2018

1 Introduction

We study the Buckley-Leverett equation, which is a conservation equation used to model the flow of two immiscible fluid phases in a porous medium. It describes, for example, the displacement of oil by water in a one-dimensional reservoir. The equation can be derived from the mass conservation equations of incompressible two-phase flow without capillary forces. The equation is on the form

$$u_t + f(u)_x = 0, \quad (1)$$

$$f(u) = \frac{u^2}{u^2 + C(1-u)^2}. \quad (2)$$

Here, the constant C describes the ratio between the fluid viscosities. For simplicity we assume $C = 1$. In addition we have initial conditions

$$u(x, 0) = u_0(x). \quad (3)$$

Displacement of oil by water is given by the initial data $u_0(x) = 0$ and boundary $u(0, t) = 1$, which alternatively can be posed as a so-called Riemann problem in the infinite domain

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0. \end{cases} \quad (4)$$

The analytical solution will be self similar, and if we denote $\xi = x/t$, it can be found to be [3]

$$u(x, t) = u(\xi) = \begin{cases} \frac{1}{2} \left(\sqrt{\left(\frac{-2\xi + \sqrt{4\xi + 1} - 1}{\xi} + 1 \right)} + 1 \right), & \xi < \frac{1}{2}(1 + \sqrt{2}), \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

In the course, we have mainly studied finite-difference methods, which assume smooth solutions. However, first-order quasi-linear hyperbolic equations (conservation laws) will generally give discontinuous solutions even from continuous initial data. We therefore need to find a weak solution, defined as any solution that satisfies the following equation for all smooth and compactly supported test functions ϕ [3]

$$\int \int \left[u \frac{\partial \phi}{\partial t} + f(u) \frac{\partial \phi}{\partial x} \right] dx dt = 0. \quad (6)$$

2 Discretization

As mentioned in the introduction, the discontinuity in our problem motivates us to use finite-volume instead of finite-difference methods. We approximate the solution on a grid, and U_m^n refers to the approximation of $u_m^n = u(x_m, t_n)$ at point $x_m = x_0 + mh$ and time $t_n = t_0 + nk$. However, the simplest finite volume methods turn out to be identical to the finite difference methods and for simplicity we use the same notation interchangeably.

2.1 Classical schemes

To arrive at a discretized equation, we require that (1) holds in an integrated sense. We rewrite it to the integral form

$$\frac{d}{dt} \int_{x_1}^{x_2} u(x, t) dx = f(u(x_1, t)) - f(u(x_2, t)), \quad (7)$$

and define the sliding average as

$$\bar{u}(x, t) = \frac{1}{h} \int_{x-h/2}^{x+h/2} u(\xi, t) dt. \quad (8)$$

We combine (7) and (8) and integrate from t_n to t_{n+1}

$$\bar{u}_m^{n+1} - \bar{u}_m^n = \frac{1}{h} \int_{t_n}^{t_{n+1}} f(u(x_{m-1/2}, t)) dt - \frac{1}{h} \int_{t_n}^{t_{n+1}} f(u(x_{m+1/2}, t)) dt. \quad (9)$$

Let $F_{m\pm 1/2}$ be defined as the flux integral

$$F_{m\pm 1/2}^n \approx \frac{1}{h} \int_{t_n}^{t_{n+1}} f(u(x_{m\pm 1/2}, t)) dt. \quad (10)$$

If we assume U_m^n to be known for all m , we can first approximate the integral by $k f(u(x_{m\pm 1/2}, t_n))$. Then, we find the point value of u at time t_n by the centered average, and we obtain

$$F_{m\pm 1/2}^n = \frac{k}{2} [f(U_{m\pm 1}^n) + f(U_m^n)]. \quad (11)$$

However, the resulting scheme has shown to be unstable. A stable scheme can be obtained by adding a diffusion term $\frac{h^2}{k} \partial_x^2 u$ approximated by central differences to the right-hand side of (9). This yields the Lax-Friedrich scheme

$$U_m^{n+1} = \frac{1}{2} (U_{m+1}^n + U_{m-1}^n) - \frac{1}{2} r [f(U_{m+1}^n) - f(U_{m-1}^n)], \quad (12)$$

where $r = k/h$.

Better accuracy can be achieved by evaluating the flux integrals $F_{m\pm 1/2}$ at the midpoints instead of the endpoints. These midpoints can be computed by

using the Lax-Friedrich scheme on a grid with half the grid-spacing. This results in the two-step Richtmyer Lax-Wendroff scheme

$$\begin{aligned} U_{m\pm 1/2}^{n+1/2} &= \frac{1}{2}(U_m^n - U_{m+1}^n) - \frac{1}{2}r[f(U_{m\pm 1}^n) + f(U_m^n)] \\ U_m^{n+1} &= U_m^n - r[f(U_{m+1/2}^{n+1/2}) - f(U_{m-1/2}^{n+1/2})]. \end{aligned} \quad (13)$$

The methods developed so far use information from both sides of the grid interface when computing approximations to the flux integral (10). However, this is not necessary if we know the structure of the flux function. If we look at (2), we see that for all $0 < u < 1$ (which is our domain for u), $f'(u) > 0$. In this way, we know that all characteristics will be positive, i.e. all the information in the solution propagates along positive x when t increases. Hence, we only need to use the solution from the left side of the grid interface to evaluate the integral (10). Thus, $F_{m+1/2} = f(u_m^n)$ will be the exact value of the integral for sufficiently small k , provided that the solution at time t_n is constant and equal to u_m^n in cell m . The resulting scheme is the upwind scheme, and it takes the form

$$U_m^{n+1} = U_m^n - r[f(U_m^n) - f(U_{m-1}^n)], \quad (14)$$

where $r = k/h$ as usual.

When implementing these schemes on a bounded domain, one generally has to modify the stencils in cells next to the boundary to handle boundary conditions. To avoid this, we introduce fictitious nodes at each end of the domain. These nodes are not part of the domain, and their values can be set manually to impose boundary conditions.

2.2 Semi-discrete high-resolution scheme

In Figure 1 we see the results of solving the Buckley-Leverett with the three methods developed in Section 2.1 for two problems, one with discontinuous solution and one with continuous solution. The Lax-Wendroff method is a second-order method on smooth solutions, but it performs poorly around discontinuities. On the other hand, the upwind method is only first order, but performs well around discontinuities. In this section, we introduce another type of discretization that combines the best properties of first and second order methods.

The scheme we develop is semi-discrete. If we define $\bar{u}_m(t) = \bar{u}(x_m, t)$ as in equation (8), the evolution of Equation (7) reads

$$\frac{d}{dt}\bar{u}_m(t) = -\frac{1}{h}\left[f(u(x_{m+1/2}, t)) - f(u(x_{m-1/2}, t))\right]. \quad (15)$$

Firstly, we have to get point values at the integration points. Approximate values can be computed from the cell averages $\bar{u}_m(t)$ by applying a piecewise polynomial reconstruction. The simplest approach to get a second order scheme is to apply a linear reconstruction [3, p.304]. We may use forward, backward or central differences. As either of these may overshoot the solution at discontinuities, we introduce a nonlinear averaging function, which you can think of as an

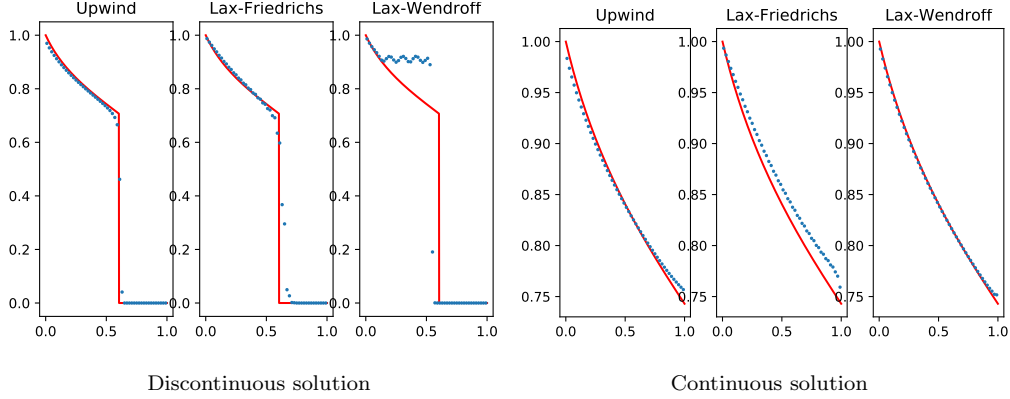


Figure 1: Solution of BL equation with $N = 50$ grid points for initial conditions (30) at $t = 0.5$ [right] and (4) at $t = 1$ [left]. The red lines are the analytical solutions, and the blue dots are the solutions obtained by the three different classical methods.

intelligent agent that combines forward and backward differences or sets it to zero when these have opposite signs. For further details, see e.g. [3].

Reconstruction gives different point values on opposite sides of x_{m+1} . To be able to use the midpoint rule, we need to specify how these one-sided point values should be used to evaluate the integrand. As above, we use upwind flux

$$f(u(x_{m+1/2}, t)) \approx F^{UW}(\bar{u}_m(t), \bar{u}_{m+1}(t)) = f(\bar{u}_m(t)) \quad (16)$$

Finally, we use an explicit TVD Runge-Kutta predictor-corrector scheme to compute the ODE. This family of methods has been developed especially to integrate semi-discrete schemes in time without creating false oscillations [1]. All this results in a second-order method of the form

$$\begin{aligned} U_m^{(1)} &= U_m^n - r[f(U_m^n) - f(U_{m-1}^n)] \\ U_m^{n+1} &= \frac{1}{2}U_m^n + \frac{1}{2}[U_m^{(1)} - r(f(U_m^{(1)}) - f(U_{m-1}^{(1)}))], \end{aligned} \quad (17)$$

This is an example of a second-order high-resolution Godunov scheme. By high-resolution, one means that the scheme has formal high order on smooth solutions but avoids creating oscillations at discontinuities [3].

3 Analysis

We will prove convergence of the upwind and Lax-Friedrichs schemes. The same can be done for Lax-Wendroff and the Godunov schemes. By the Lax equivalence theorem, the schemes will be convergent if and only if they are consistent and stable.

3.1 Consistency

First, we prove consistency of the upwind scheme (14). We assume smooth solutions, even though this typically will not be the case. Normally, one would

expect to get a discontinuous solution to the equation. We now go back to finite difference methods, which have the same formulas as the finite volume methods, but the interpretation is different. In addition we assume that the values at the grid points are exact, i.e., $u_m^n = u(x_m, t_n)$. Now, we substitute basic Taylor expansions into the expression for the upwind scheme (14) and denote $f_m^n = f(u_m^n)$. The truncation error τ_m^n will be

$$\begin{aligned} k\tau_m^n = & u_m^n + k \frac{\partial u_m^n}{\partial t} + \frac{k^2}{2} \frac{\partial^2 u_m^n}{\partial t^2} - u_m^n + \frac{k}{h} \left(f_m^n - \left(f_m^n - h \frac{\partial f_m^n}{\partial x} + \frac{h^2}{2} \frac{\partial^2 f_m^n}{\partial x^2} \right) \right) \\ & + O(h^2) + O(k^3) \end{aligned} \quad (18)$$

Hence, we can shorten this expression and find the truncation error

$$\tau_m^n = \frac{k}{2} \frac{\partial^2 u_m^n}{\partial t^2} - \frac{h}{2} \frac{\partial^2 f_m^n}{\partial x^2} + O(k^2) + O(h^2). \quad (19)$$

The upwind scheme is first order in time and space. It is also consistent, because as $h \rightarrow 0, k \rightarrow 0$ the truncation error will tend to zero, i.e., $\tau_m^n \rightarrow 0$.

Now, we prove consistency of the Lax-Friedrichs scheme (12). Again, as for the upwind scheme, we assume smooth solutions, and that the values at the grid points are exact, i.e. $u_m^n = u(x_m, t_n)$. We substitute Taylor expansions into the Lax-Friedrichs scheme (12) and obtain

$$\begin{aligned} k\tau_m^n = & u_m^n + k \frac{\partial u_m^n}{\partial t} + \frac{k^2}{2} \frac{\partial^2 u_m^n}{\partial t^2} - \frac{1}{2} \left(u_m^n + h \frac{\partial u_m^n}{\partial x} + \frac{h^2}{2} \frac{\partial^2 u_m^n}{\partial x^2} + u_m^n - h \frac{\partial u_m^n}{\partial x} + \frac{h^2}{2} \frac{\partial^2 u_m^n}{\partial x^2} \right) \\ & + \frac{k}{2h} \left[f_m^n + h \frac{\partial f_m^n}{\partial x} + \frac{h^2}{2} \frac{\partial^2 f_m^n}{\partial x^2} + \frac{h^3}{6} \frac{\partial^3 f_m^n}{\partial x^3} - \left(f_m^n - h \frac{\partial f_m^n}{\partial x} + \frac{h^2}{2} \frac{\partial^2 f_m^n}{\partial x^2} - \frac{h^3}{6} \frac{\partial^3 f_m^n}{\partial x^3} \right) \right] \\ & + O(h^3) + O(k^3). \end{aligned} \quad (20)$$

Hence, we can shorten this expression and find the truncation error

$$\tau_m^n = \frac{k}{2} \frac{\partial^2 u_m^n}{\partial t^2} - \frac{h^2}{2k} \frac{\partial^2 u_m^n}{\partial x^2} + \frac{h^2}{6} \frac{\partial^3 f_m^n}{\partial x^3} + O(h^3) + O(k^2). \quad (21)$$

The Lax-Friedrichs is second order in space, but only first order in time. This makes the scheme first order. It is also consistent, because as $h^2 \rightarrow 0, k \rightarrow 0$ the truncation error will tend to zero, i.e., $\tau_m^n \rightarrow 0$.

3.2 Stability

We can now go on to prove stability. A finite difference scheme is stable provided that the error at the nodes,

$$e_m^n = |U_m^n - u(x_m, t_n)|$$

does not blow up. If we choose step-sizes that break the interval of absolute stability, the finite difference scheme will become unstable and produce inaccurate

results. We use the Fourier method to prove stability of the upwind scheme. We substitute $u_m^n = \xi^n e^{i\beta m h}$ into the expression to obtain an expression for the amplification factor ξ . Then the scheme is stable when $|\xi| \leq 1$.

Because we assume $f(u)$ to be smooth, we can by the mean value theorem transform the Buckley-Leverett equation (1) into $u_t + a u_x = 0$, and look at stability for this equation. Here, $a = f'(\eta)$ for some $\eta \in [u_{m-1}^n, u_m^n]$.

Now, consider the upwind scheme (14), which can be rewritten to (we know that $a > 0$)

$$u_m^{n+1} = u_m^n - ar(u_m^n - u_{m-1}^n), \quad (22)$$

where $r = k/h$. We proceed with the Fourier method

$$\xi^{n+1} e^{i\beta m h} = \xi^n e^{i\beta m h} - ar \xi^n e^{i\beta m h} (1 - e^{-i\beta h}). \quad (23)$$

Dividing by $\xi^n e^{i\beta m h}$ gives

$$\xi = 1 - ar(1 - e^{-i\beta h}) = 1 - ar(1 - \cos \beta h) - iar \sin \beta h, \quad (24)$$

which after some calculations gives us

$$|\xi|^2 = 1 - 4(1 - ar)ar \sin^2 \frac{\beta h}{2} \implies ar \leq 1. \quad (25)$$

Thus, for the upwind scheme to be stable, we require $ar \leq 1$. This is the famous CFL-condition.

Now we can go on to prove stability of the Lax-Friedrichs scheme in the same way, by substituting $u_m^n = \xi^n e^{i\beta m h}$. This gives

$$\xi^{n+1} e^{i\beta m h} = \frac{1}{2} \xi^n e^{i\beta m h} (e^{i\beta h} + e^{-i\beta h}) - \frac{1}{2} ar \xi^n e^{i\beta m h} (e^{i\beta h} - e^{-i\beta h}). \quad (26)$$

Again, we divide by $\xi^n e^{i\beta m h}$, thus obtaining

$$\xi = \frac{1}{2}(e^{i\beta h} + e^{-i\beta h}) - \frac{1}{2} ar(e^{i\beta h} - e^{-i\beta h}). \quad (27)$$

Now we can utilize Euler's formula, $e^{ix} = \cos x + i \sin x$ to get

$$\xi = \frac{1}{2}(2 \cos \beta h) - \frac{1}{2} ar(2i \sin \beta h) = \cos \beta h - iar \sin \beta h. \quad (28)$$

For stability we require $|\xi| \leq 1$. We get

$$|\xi|^2 = \cos^2 \beta h + (ar)^2 \sin^2 \beta h \implies ar \leq 1. \quad (29)$$

Thus, for the Lax-Friedrichs scheme to be stable, we need $ar \leq 1$. This is the same region as for the upwind scheme.

When conducting numerical experiments on the order of convergence, we cannot look at convergence in time and space separately, because of the CFL-condition imposing a bound on k for a given h . We compute numerical solutions with decreasing h . To ensure stability, we therefore need $k \leq \frac{h}{|f'(\eta)|}$ owing to the CFL-condition. Because this η is unknown, we set at each iteration

$$k = 0.995 \frac{h}{\max_u |f'(u)|}.$$

4 Numerical verification

We now want to check the accuracy of our code, and verify the results obtained in Section 3. Thus, we create some numerical experiments. We want to check the order of convergence for all the methods on both a smooth and a discontinuous solution. Both experiments have domain $0 \leq x \leq 1$. The first experiment computes the solution at time $t = 1$, for initial data

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ \frac{1}{2}(1 + \sqrt{\sqrt{5} - 2}), & \text{otherwise,} \end{cases} \quad (30)$$

where $\frac{1}{2}(1 + \sqrt{\sqrt{5} - 2})$ is the analytical solution (5) for $\xi = 1$. The resulting solution is shown on the right in Figure 1, and is smooth on $0 < x < 1$ at $t = 1$. We compute the numerical solution with the four presented schemes, with number of grid points equal to $M = 2^i$, for $i = 3, \dots, 10$, giving the results in Figure 2.

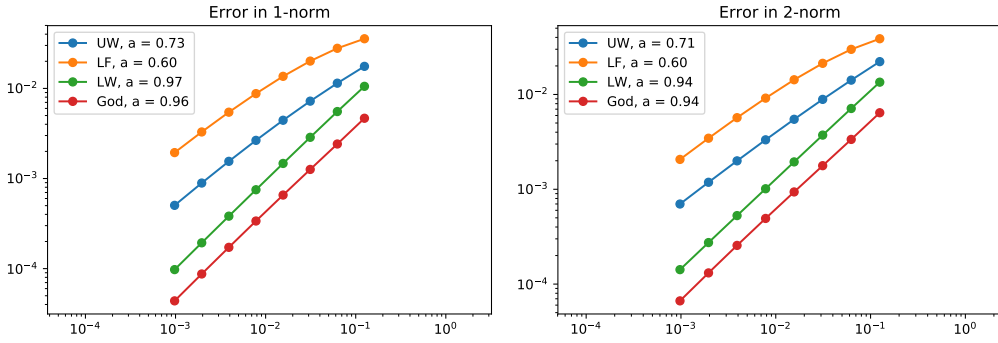


Figure 2: Error plot of the four discretization methods on a continuous problem. The numerically computed solutions are compared to the analytical solution (5). The value of a denotes the estimate of the convergence rate of the methods.

In theory, Lax-Friedrichs and upwind scheme should have order one convergence on a smooth solution. However, the solution is not smooth, it is only continuous and will in fact not be continuously differentiable on the endpoints. From the plots we see that they have a lower order of convergence than one in both norms. On this continuous solution, the Lax-Wendroff method has a higher order of convergence than the two other classical schemes. Theoretically, Lax-Wendroff should converge with order two on a smooth solution. Still, we only get order one. The high-resolution Godunov scheme was designed to be order two on smooth solutions, and it has the same order of convergence as the Lax-Wendroff scheme, i.e. order one.

The second experiment we make, computes the solution at time $t = 0.5$, for the initial data (4) defined in the introduction. This yields the solution on the left in Figure 1, a discontinuous solution. We compute the numerical solution with grid points as for the first experiment, obtaining the results in Figure 3.

Firstly, we note that the Lax-Wendroff scheme will give oscillations around discontinuities because the flux-function is non-convex, and it will not converge

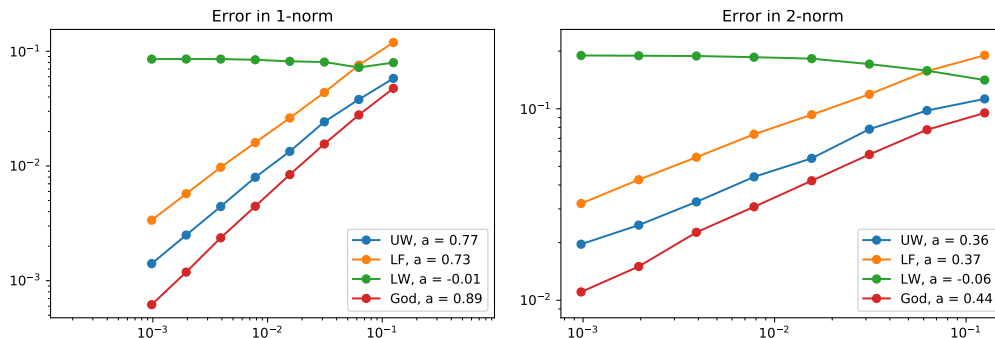


Figure 3: Error plot of the four discretization methods on a discontinuous problem. The numerically computed solutions are compared to the analytical solution (5). The value of a denotes the estimate of the convergence rate of the methods.

to the correct solution. The non-convex flux-function is one of the key differences between the Buckley-Leverett equation and for instance Burger's equation. Furthermore, upwind and Lax-Friedrichs seem to be of order one half. The methods should theoretically be of formal order one, i.e. order one for smooth solutions. But, because we have a discontinuity, we only get order one half, which is as expected [3].

The Godunov scheme performs somewhat better than upwind and Lax-Friedrichs. This scheme was designed to be of order two on smooth parts, so we would expect it to perform better than the classical first order schemes.

When comparing Lax-Friedrichs to upwind, upwind has steeper or the same slope in both experiments, and upwind is better by a constant number for all h . Even though Lax-Friedrichs is second order in space, it suffers from smearing discontinuities, as seen in Figure 1, and is therefore outperformed by the upwind scheme.

5 Conclusion

To conclude, we have discussed four different schemes for solving the Buckley-Leverett equation. We have analytically showed consistency and found order of convergence, as well as stability for two of the schemes, assuming smooth solutions. These results were verified numerically, in addition to finding the order of convergence of the two other schemes by conducting experiments on commonly used initial data. Every member of the group contributed equally.

References

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