TMA4212

Buckley-Leverett Equation

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Outline

$$u_t + f(u)_x = 0 \tag{1}$$

$$f(u) = \frac{u^2}{u^2 + C(1-u)^2} \tag{2}$$

Here the constant $C = \mu_w/\mu_n$ describes the ratio between the fluid viscosities. For simplicity we assume $C = 1$.

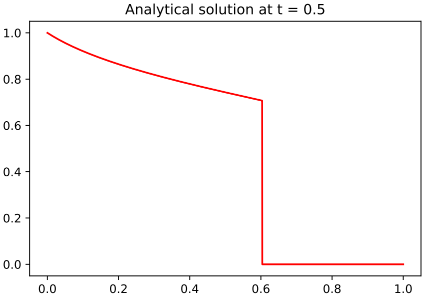
$$u(x,0) = u_0(x) \tag{3}$$

First order quasi-linear hyperbolic equations will generally give discontinuous solutions even from continuous initial data. Need to find a **weak solution** satisfying

$$\int\int\left[u\frac{\partial\phi}{\partial t}+f(u)\frac{\partial\phi}{\partial x}\right]dxdt=0. \tag{4}$$

The analytical solution at time t is

$$u(x) = \begin{cases} \frac{1}{2}\left(\sqrt{\left(\frac{\frac{-2x}{t}+\sqrt{\frac{4x}{t}+1}-1}{x/t}+1\right)}+1\right), & \frac{x}{t} < \frac{1}{2}(1+\sqrt{2}), \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$



Finding weak solutions to the problem motivates us to use finite volume methods instead of finite difference methods. However, the formulas for the simplest finite volume methods turn out to be identical to the corresponding finite difference methods.

Discretize the equation on a grid:

- $u_m^n = u(x_m, t_n)$
- $x_m = x_0 + mh,\ t_n = t_0 + nk$

Upwind

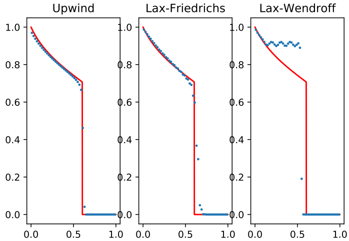
$$U_m^{n+1} = U_m^n - r[f(U_m^n) - f(U_{m-1}^n)], \quad r = \frac{k}{h} \tag{6}$$

Lax-Friedrich

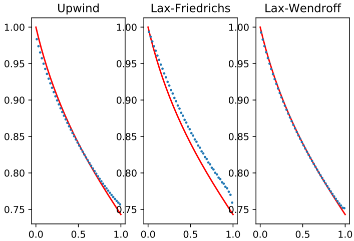
$$U_m^{n+1} = \frac{1}{2}\left(U_{m+1}^n + U_{m-1}^n\right) - \frac{1}{2}r\left[f(U_{m+1}^n) - f(U_{m-1}^n)\right], \tag{7}$$

Lax-Wendroff

$$\begin{aligned} U_{m\pm1/2}^{n+1/2} &= \frac{1}{2}\left(U_m^n - U_{m+1}^n\right) - \frac{1}{2}r\left[f(U_{m\pm1}^n) + f(U_m^n)\right] \\ U_m^{n+1} &= U_m^n - r\left[f(U_{m+1/2}^{n+1/2}) - f(U_{m-1/2}^{n+1/2})\right]. \end{aligned} \tag{8}$$



Solution at T = 0.5



Solution at T = 1

Figure: Solution of Buckley-Leverett equation with $N = 50$ grid points. The red line is the analytical solution, and the blue dots are the solution obtained by the three different methods.

Apply a piecewise linear reconstruction, by introducing a nonlinear averaging function, an intelligent agent.

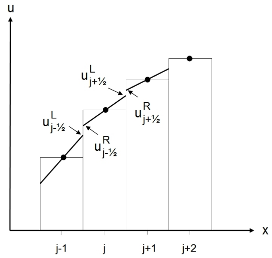


Figure: `https://upload.wikimedia.org/wikipedia/en/b/b3/LinExtrap.jpg`

Different values on each side of $x_{m\pm1/2}$, we use the upwind flux.
Finally, use a TVD Runge-Kutta predictor-corrector scheme to compute the ODE.
The resulting scheme is a second-order high-resolution Godunov scheme.

$$
\begin{aligned}
U_m^{(1)} &= U_m^n - k[f(U_m^n) - f(U_{m-1}^n)] \\
U_m^{n+1} &= \frac{1}{2}U_m^n + \frac{1}{2}\left[U_m^{(1)} - k(f(U_m^{(1)}) - f(U_{m-1}^{(1)})\right],
\end{aligned}
\tag{9}
$$

$$k\tau_m^n = u_m^{n+1} - u_m^n - \frac{k}{h}\Big(f(u_m^n) - f(u_{m-1}^n)\Big) \tag{10}$$

Taylor expansion (dropping n and m). Denote $f = f(u)$

$$\begin{aligned} k\tau =& u + k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} - u + \frac{k}{h}\Big(f - \Big(f - h\frac{\partial f}{\partial x} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}\Big)\Big) \\ &+ O(h^2) + O(k^3) \end{aligned} \tag{11}$$

$$\tau = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} + \frac{h}{2}\frac{\partial^2 f}{\partial x^2} + O(h^2) + O(k^2) \tag{12}$$

$$
\begin{aligned}
k\tau_m^n =& u + k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}\left(u + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + u - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}\right) \\
&+\frac{k}{2h}\left[f + h\frac{\partial f}{\partial x} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 f}{\partial x^3} - \left(f - h\frac{\partial f}{\partial x} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 f}{\partial x^3}\right)\right] \\
&+ O(h^3) + O(k^3).
\end{aligned}
\tag{13}
$$

$$
\tau_m^n = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - \frac{h^2}{2k}\frac{\partial^2 u}{\partial x^2} + \frac{h^2}{6}\frac{\partial^3 f}{\partial x^3} + O(h^3) + O(k^2). \tag{14}
$$

As $f(u)$ is assumed to be smooth, we can use the mean value theorem to transform the BL equation (1) into the transport equation

$$u_t + au_x = 0, \qquad \text{where } a = f'(\eta), \quad \eta \in [u_{m-1}^n, u_m^n]$$

Using the Fourier mode substitution

$$u_m^n = \xi^n e^{i\beta mh},$$

We assume $a > 0$ and rewrite the upwind scheme (6) into

$$\xi^{n+1} e^{i\beta mh} = \xi^n e^{i\beta mh} - ar\xi^n e^{i\beta mh}(1 - e^{-i\beta h}), \tag{15}$$

which simplifies to

$$|\xi|^2 = 1 - 4(1 - ar)ar \sin^2 \frac{\beta h}{2} \tag{16}$$

We rewrite the Lax-Friedrich scheme into

$$\xi^{n+1}e^{i\beta mh} = \frac{1}{2}\xi^n e^{i\beta mh}(e^{i\beta h} + e^{-i\beta h}) - \frac{1}{2}ar\xi^n e^{i\beta mh}(e^{i\beta h} - e^{-i\beta h}). \quad (17)$$

For stability, we require $|\xi|^2 \leq 1$. Putting this into (15) and (17), we get that $ar \leq 1$ for both upwind and Lax-Friedrich. We used the substitution $a = f'(\eta)$, where η is unknown. We need

$$\nu = \frac{k}{h}\max_u|f'(u)| \leq 1,$$

which is the famous CFL-condition.

Numerical verification

Continuous solution

We run our schemes on a continous solution descibed by the BL equation (1) with $\nu = 0.995$ and the initial conditions

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ \frac{1}{2}\left(1 + \sqrt{\sqrt{5} - 2}\right), & \text{otherwise}, \end{cases}$$

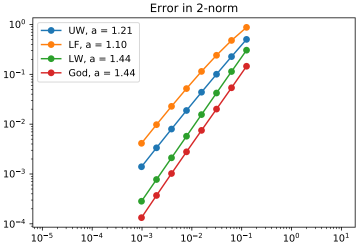


Figure: Error plot of the four discretization methods on a continuous solution. *a* is the average slope of all line segments. This is an estimate for the rate of convergence.

We run our schemes on a discontinuous solution, described by the BL equation (1) with with $\nu = 0.995$ and the initial conditions

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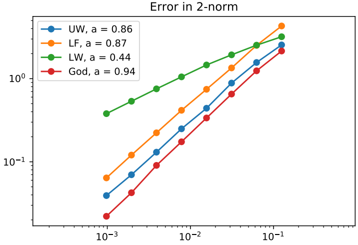

Figure: Error plot of the four discretization methods on a discontinuous solution. *a* is the average slope of all line segments. This is an estimate for the rate of convergence.

Summary

- We used upwind, Lax-Friedrichs, Lax-Wendroff, and a higher-order Godunov scheme to solve the Buckley-Leverett equation (1) both for continuous and discontinuous solutions.
- Upwind and Lax-Friedrich provide linear convergence on a continuous problem. Lax-Wendroff provides quadratic convergence.
- Lax-Wendroff creates oscillations on a discontinuous problem, breaking the convergence.
- The higher resolution Godunov scheme keeps the quadratic convergence of Lax-Wendroff on areas of continuity while maintaining stability around discontinuities.
- Outlook
  - Stability analysis for Lax-Wendroff on discontinous solutions.

Further Reading

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