TMA4212

Buckley-Leverett Equation

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Outline

$$u_t + f(u)_x = 0 \tag{1}$$

$$f(u) = \frac{u^2}{u^2 + C(1-u)^2} \tag{2}$$

Here the constant $C = \mu_w/\mu_n$ describes the ratio between the fluid viscosities. For simplicity we assume $C = 1$.

$$u(x,0) = u_0(x) \tag{3}$$

First order quasi-linear hyperbolic equations will generally give discontinuous solutions even from continuous initial data. Need to find a weak solution satisfying

$$\int\int\left[u\frac{\partial\phi}{\partial t}+f(u)\frac{\partial\phi}{\partial x}\right]dxdt=0. \tag{4}$$

The analytical solution at time T is

$$u(x)=\begin{cases}\frac{1}{2}\left(\sqrt{\left(\frac{\frac{-2x}{T}+\sqrt{\frac{4x}{T}+1}-1}{x/T}+1\right)}+1\right), & x<\frac{1}{2}(1+\sqrt{2})T,\\ 0, & \text{otherwise.}\end{cases} \tag{5}$$

- Finding weak solutions to the problem motivates us to use finite volume methods instead of finite difference methods. However, the simplest finite volume methods turn out to be identical to the corresponding finite difference methods.
- Discretize the equation on a grid:
- $u_m^n = u(x_m, t_n)$
- $x_m = x_0 + mh,\ t_n = t_0 + nk$

Upwind

$$U_m^{n+1} = U_m^n - r[f(U_m^n) - f(U_{m-1}^n)], \tag{6}$$

Lax-Friedrich

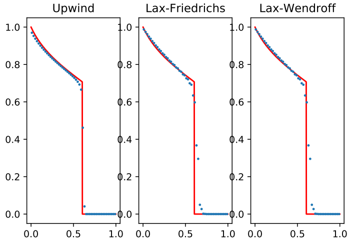
$$U_m^{n+1} = \frac{1}{2}\left(U_{m+1}^n + U_{m-1}^n\right) - \frac{1}{2}r\left[f(U_{m+1}^n) - f(U_{m-1}^n)\right], \tag{7}$$

Lax-Wendroff

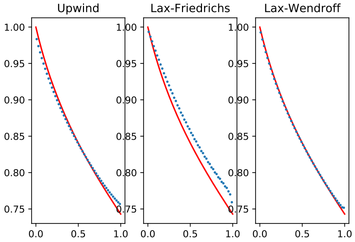
$$\begin{aligned} U_{m\pm1/2}^{n+1/2} &= \frac{1}{2}\left(U_m^n - U_{m+1}^n\right) - \frac{1}{2}r\left[f(U_{m\pm1}^n) + f(U_m^n)\right] \\ U_m^{n+1} &= U_m^n - r\left[f(U_{m+1/2}^{n+1/2}) - f(U_{m-1/2}^{n+1/2})\right]. \end{aligned} \tag{8}$$

Discretization

Semi-discrete high-resolution scheme



Solution at T = 0.5



Solution at T = 1

Figure: Solution of Buckley-Leverett equation with $N = 50$ grid points. The red line is the analytical solution, and the blue dots are the solution obtained by the three different methods.

$$
\begin{aligned}
u_m^{n+1} &= u_m^n + k\frac{\partial u_m^n}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u_m^n}{\partial t^2} + \cdots \\
u_{m+1}^n &= u_m^n + h\frac{\partial u_m^n}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u_m^n}{\partial x^2} + \cdots ,
\end{aligned}
$$

$$
\begin{aligned}
k\tau_m^n = & u + k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} - u + \frac{k}{h}\left(f - \left(f - h\frac{\partial f}{\partial x} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}\right)\right) \\
& + O(h^2) + O(k^3)
\end{aligned} \tag{9}
$$

$$
\tau_m^n = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} + \frac{h}{2}\frac{\partial^2 f}{\partial x^2} + O(h^2) + O(k^2) \tag{10}
$$

$$
\begin{aligned}
k\tau_m^n =& u + k\frac{\partial u}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}\left(u + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + u - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}\right) \\
&+\frac{k}{2h}\left[f + h\frac{\partial f}{\partial x} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 f}{\partial x^3} - \left(f - h\frac{\partial f}{\partial x} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 f}{\partial x^3}\right)\right] \\
&+ O(h^3) + O(k^3). \qquad (11)
\end{aligned}
$$

$$
\tau_m^n = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - \frac{h^2}{2k}\frac{\partial^2 u}{\partial x^2} + \frac{h^2}{6}\frac{\partial^3 f}{\partial x^3} + O(h^3) + O(k^2). \qquad (12)
$$

As $f(u)$ is assumed to be smooth, we can use the mean value theorem to transform the BL equation (1) into the transport equation

$$u_t + au_x = 0, \quad \text{where } a = f'(\eta), \quad \eta \in [u^n_{m-1}, u^n_m]$$

Using the Fourier mode substitution

$$u^n_m = \xi^n e^{i\beta mh},$$

We assume $a > 0$ and rewrite the Upwind scheme (6) into

$$\xi^{n+1} e^{i\beta mh} = \xi^n e^{i\beta mh} - ar\xi^n e^{i\beta mh}(1 - e^{-i\beta h}), \tag{13}$$

which simplifies to

$$|\xi|^2 = 1 - 4(1 - ar)ar \sin^2 \frac{\beta h}{2} \tag{14}$$

We rewrite the Lax-Friedrich scheme into

$$\xi^{n+1}e^{i\beta mh} = \frac{1}{2}\xi^{n}e^{i\beta mh}(e^{i\beta h}+e^{-i\beta h}) - \frac{1}{2}ar\xi^{n}e^{i\beta mh}(e^{i\beta h}-e^{-i\beta h}). \quad (15)$$

For stability, we require $|\xi|^2 \leq 1$. Putting this into (13) and (15), we get that $ar \leq 1$ for both Upwind and Lax-Friedrich. As we used the substitution $a = f'(\eta)$, we need $k \leq \frac{h}{|f'(\eta)|}$. η is unknown, so we set at each iteration

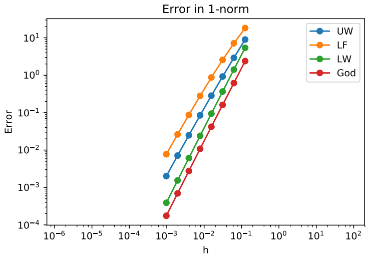
$$k = 0.995\frac{h}{\max_u |f'(u)|}.$$

Numerical verification

Continuous problem

We run our schemes on a continous problem descibed by the BL equation (1) and the inital conditions

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ \frac{1}{2}\left(1 + \sqrt{\sqrt{5} - 2}\right), & \text{otherwise}, \end{cases}$$



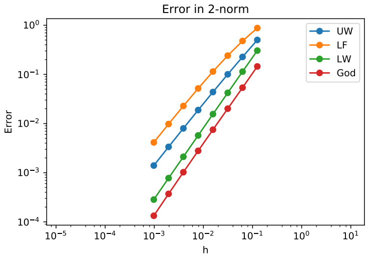


Figure: Error plot of the four discretization methods on a continuous solution.

We run our schemes on a discontinuous problem, described by the BL equation (1) and the initial conditions

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ 0, & \text{otherwise}, \end{cases}$$

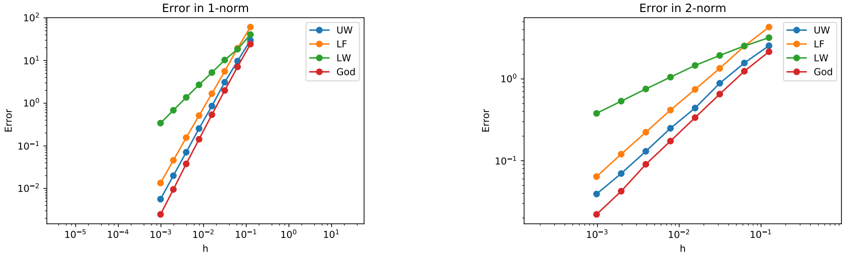


Figure: Error plot of the four discretization methods on a discontinuous solution.

Summary

- We used upwind, Lax-Friedrichs, Lax-Wendroff, and a higher-order Godunov scheme to solve the Buckley-Leverett equation (1) both for continuous and discontinuous solutions.
- Upwind and Lax-Friedrich provide linear convergence on a continuous problem. Lax-Wendroff provides quadratic convergence.
- Lax-Wendroff creates oscillations on a discontinuous problem, breaking the convergence.
- The higher-order Godunov scheme keeps the quadratic convergence of Lax-Wendroff on areas of continuity while maintaining stability around discontinuities.

- Outlook
  - Stability analysis for Lax-Wendroff.
  - Our consistency analysis assumes smooth solutions, which typically is not the case.

For Further Reading I

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