TMA4315: Compulsory exercise 2 Logistic regression and Poisson regression

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Part 1: Logistic regression

a)

We let y_i be the number of successfull ascents, and n_i be the total number of attempts (success + fail) of the i'th mountain. We then do binary regression with the logit link to model the probability of success. This gives

1. Model for response: $Y_i \sim \text{Bin}(n_i, \pi_i)$, for $i = 1, \ldots, 113$, with $\text{E}(Y_i) = \mu_i = \pi_i$ and $\text{Var}(Y_i) = \pi_i(1-\pi_i)$
2. Linear predictor: $\eta_i = \mathbf{x}_i^T \boldsymbol{\beta}$
3. Link function: $\eta_i = \ln\left(\frac{\pi_i}{1-\pi_i}\right)$, also known as logit link

where $\mathbf{x}_i$ is a p dimensional column vector of covariates for observation i, and $\boldsymbol{\beta}$ is the vector of regression parameters.

Then, the likelihood can be written

$$L(\boldsymbol{\beta}) = \prod_{i=1}^{n} L_i(\beta) = \prod_{i=1}^{n} f(y_i; \beta) = \prod_{i=1}^{n} \pi_i^{y_i}(1-\pi_i)^{n_i - y_i}.$$

If we take the natural logarithm, we get the log likelihood function

$$l(\boldsymbol{\beta}) = \ln(L(\boldsymbol{\beta})) = \sum_{i=1}^{n} l_i(\boldsymbol{\beta}) = \sum_{i=1}^{n} [y_i \ln(\pi_i) + (n_i - y_i)\ln(1-\pi_i)] = \sum_{i=1}^{n} [y_i \ln\left(\frac{\pi_i}{1-\pi_i}\right) + n_i \ln(1-\pi_i)]$$

To express the log likelihood as a function of $\boldsymbol{\beta}$ we first use the link between η_i and π_i. By using the inverse of the link function (the logistic response function) we have that $\pi_i = \frac{\exp(\eta_i)}{1+\exp(\eta_i)}$. The log likelihood function can then be written

$$l(\boldsymbol{\beta}) = \sum_{i=1}^{n} [y_i \eta_i + n_i \ln\left(\frac{1}{1+\exp(\eta_i)}\right)] = \sum_{i=1}^{n} [y_i \eta_i - n_i \ln(1+\exp(\eta_i))].$$

Finally we use that $\eta_i = \mathbf{x}_i^T \boldsymbol{\beta}$ and get

$$l(\boldsymbol{\beta}) = \sum_{i=1}^{n} [y_i \mathbf{x}_i^T \boldsymbol{\beta} - n_i \ln(1 + \exp(\mathbf{x}_i^T \boldsymbol{\beta})],$$

which is an expression for the log likelihood as a function of beta.

Since we know that this function is concave we can find the parameters, $\boldsymbol{\beta}$ that gives the maximum likelihood by taking the partial derivatives with respect to $\boldsymbol{\beta}$ to get the score function, $s(\boldsymbol{\beta})$. The parameters $\boldsymbol{\beta}$ that gives the maximum likelihood is those for which the score function equals zero. **SE PÅ DENNE SENERE**

b)

We load the dataset and fit a model as described in Part a) with success rate **(er dette riktig?)** as response and height and prominence as predictors.

```
##
## Call:  glm(formula = cbind(success, fail) ~ height + prominence, family = "binomial",
##     data = mount)
##
## Coefficients:
## (Intercept)       height   prominence
##   13.685845    -0.001635    -0.000174
##
## Degrees of Freedom: 112 Total (i.e. Null);  110 Residual
## Null Deviance:       715.3
## Residual Deviance: 414.7     AIC: 686
```

We want to interpret the model parameters by using the odds. We have from the link function that

$$\eta_i = \ln\left(\frac{\pi_i}{1 - \pi_i}\right).$$

Our linear predictor is $\eta_i = \mathbf{x}_i^T \boldsymbol{\beta}$. If we insert this, assuming two intercept and two covariates, we get

$$\ln\left(\frac{\pi_i}{1 - \pi_i}\right) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}.$$

We can now take exp() of both sides, yielding

$$\frac{\pi_i}{1 - \pi_i} = \exp(\beta_0) \cdot \exp(\beta_1 x_{i1}) \cdot \exp(\beta_2 x_{i2}).$$

The left hand side is what we call the odds. If we now increase one of the covariates by one and keep the other constant, the odds will be multiplied by $\exp(\beta)$. Coefficient estimates for both height and prominence are negative. This means that for both coefficients $\exp(\beta) < 1$, and an increase in height or prominence will give a decrease in success probability of climbing the mountain. For example, increasing height by one meter will decrease the odds by multiplying it with $\exp(\beta_1) = \exp(-0.001635) = 0.998$.

Part 2: Poisson regression - Eliteserien 2018

a)

In this Part, we aim to simulate the remaining games in the Norwegian top division of football (Eliteserien). For each game, we assume that the score (the number of goals) of the home team is independent of the score

of the away team. We assume that each team has a single parameter that measures its strength. We denote this strength parameter β_A for team A, β_B for team B, and so on.

Through watching football games, one could be made to believe that the goals scored by the away team in a football match is dependent on the goals scored by the home team and vice versa.

We therefore want to test if the assumption of independence between the goals scored by the home and away teams is reasonable. To do this, we first load the data set and make a contingency table of all the results, with the goals of the home team on the rows, and goals of the away team on the columns. We get the following contingency table.

```
##     0  1  2 3 4+
## 0   8 18  3 1  1
## 1  19 26 15 5  3
## 2  10 14 13 4  1
## 3  13 10  7 2  0
## 4+  8  7  3 1  0
```

We then want to test if the number of goals for home and away team are independent. We do this by conducting *Pearson's* χ^2 *test* on the contingency table. The test poses the following hypotheses

H_0 : The sampling distributions are independently chi-squared distributed, H_1 : They are not independently chi-squared dist

We use the `R` function `chisq.test()` to compute the test statistic and the corresponding p-value.

```
##
##  Pearson's Chi-squared test
##
## data:  contingency
## X-squared = 14.156, df = 16, p-value = 0.5871
```

We get a value of 14.156 for the test statistic, with a corresponding p-value of 0.5871. As this p-value is above any reasonable significance level, we keep the null hypothesis, and confirm that the goals scored by the home and away team are independent. This means that our assumption of independence holds.

b)

Before we start simulating games, we want to construct the current standings in the Eliteserie based on all the results in our data set. By summing up the results from all games, we get the following table.

```
##                  Team Played Won Drawn Lost For Against  GD Points
## 1           Rosenborg     24  16     4    4  43      20  23     52
## 2               Brann     24  14     6    4  36      23  13     48
## 3               Molde     24  13     4    7  48      30  18     43
## 4           Haugesund     24  12     5    7  36      28   8     41
## 5          Ranheim_TF     24  11     5    8  38      40  -2     38
## 6          Vaalerenga     24  10     6    8  35      37  -2     36
## 7                 Odd     24   9     7    8  35      29   6     34
## 8             Tromsoe     24  10     3   11  35      33   2     33
## 9         Sarpsborg08     24   9     5   10  39      34   5     32
## 10       Kristiansund     24   8     7    9  32      35  -3     31
## 11         BodoeGlimt     24   6     9    9  28      30  -2     27
## 12      Stroemsgodset     24   6     8   10  38      38   0     26
## 13        Lillestroem     24   6     7   11  26      37 -11     25
## 14            Stabaek     24   5     8   11  29      43 -14     23
## 15              Start     24   6     5   13  24      42 -18     23
## 16 Sandefjord_Fotball     24   2     9   13  24      47 -23     15
```

c)

We now want to estimate the intercept, home advantage and strength parameters for each team. Then we produce a ranking based on the estimated strengths and compare with the rankings from b). To estimate the parameters, we create our own function `myglm` that performs the regression by maximum likelihood. The function uses the built in `optim` function to find the coefficients that maximizes the loglikelihood function (minimizes the negative of the loglikelihood, $-l(\beta)$).

```
##                  Team     Strength
## 1           Rosenborg  0.366945548
## 2               Molde  0.279321007
## 3               Brann  0.225715115
## 4           Haugesund  0.141566217
## 5                 Odd  0.099954079
## 6         Sarpsborg08  0.097625830
## 7             Tromsoe  0.060091773
## 8       Stroemsgodset  0.049639590
## 9          Vaalerenga  0.014445633
## 10       Kristiansund  0.012621369
## 11         Ranheim_TF  0.008439525
## 12         BodoeGlimt  0.000000000
## 13        Lillestroem -0.132589021
## 14            Stabaek -0.148121316
## 15              Start -0.225876528
## 16 Sandefjord_Fotball -0.291815679
```

```
## [1] "Intercept: "
```

```
## [1] 0.1003129
```

```
## [1] "Home advantage: "
```

```
## [1] 0.4020541
```

```
##
## Call:
## glm(formula = goals ~ -1 + X, family = "poisson")
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -2.0205  -0.8748  -0.2014   0.5761   2.8679
##
## Coefficients: (1 not defined because of singularities)
##                      Estimate Std. Error z value Pr(>|z|)
## XIntercept           0.100304   0.068489   1.465   0.1430
## XHomeAdvantage       0.402068   0.087521   4.594 4.35e-06 ***
## XRosenborg           0.366956   0.168373   2.179   0.0293 *
## XMolde               0.279264   0.168369   1.659   0.0972 .
## XLillestroem        -0.132857   0.168934  -0.786   0.4316
## XOdd                 0.099975   0.166394   0.601   0.5480
## XHaugesund           0.141121   0.166320   0.848   0.3962
## XSandefjord_Fotball -0.291865   0.164767  -1.771   0.0765 .
## XRanheim_TF          0.008343   0.169495   0.049   0.9607
## XBrann               0.225678   0.165557   1.363   0.1728
## XSarpsborg08         0.097553   0.166444   0.586   0.5578
## XStabaek            -0.148047   0.168914  -0.876   0.3808
## XTromsoe             0.060348   0.166332   0.363   0.7167
```

```
## XStart              -0.225884   0.165079  -1.368   0.1712
## XVaalerenga          0.014465   0.169280   0.085   0.9319
## XKristiansund        0.012376   0.166170   0.074   0.9406
## XStroemsgodset       0.049657   0.166211   0.299   0.7651
## XBodoeGlimt                NA         NA      NA       NA
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for poisson family taken to be 1)
##
##     Null deviance: 499.35  on 384  degrees of freedom
## Residual deviance: 384.12  on 367  degrees of freedom
## AIC: 1135.3
##
## Number of Fisher Scoring iterations: 5
```

If we compare the results from our function `myglm` with the built-in function `glm`, we see that the regression coefficients are equal (to a precision of 4 digits).

We have set the strength of Bodø Glimt to zero, $\beta_{BodoeGlimt} = 0$. This means that the strength of Bodø Glimt is the "reference strength", and the strength of every team is just the team's strength compared to Bodø Glimt. This mean that all teams with $\beta_A > 0$ will be stronger than Bodø Glimt, and all teams with $\beta_A < 0$ will be weaker than Bodø Glimt. Also, a higher (lower) value of β_A indicates a stronger (weaker) team.

We get a coefficient for the intercept of $\beta_{Intercept} = 0.1003129$. This means that, if the teams are equally good (equal strength coefficient), one would expect the away team to score $\exp(\beta_{Intercept}) = 1.11$ goals on average. The coefficient for the home advantage is $\beta_{HomeAdvantage} = 0.4020541$. This means that, if the teams are equally good, one would expect the home team to score $\exp(\beta_{Intercept} + \beta_{HomeAdvantage}) = 1.65$ goals on average. This also means that teams are expected to score $\exp(\beta_{HomeAdvantage}) = 1.49$ times as many goals in home games as in away games.

Also, we see that Rosenborg has the significantly highest coefficient, and Sandefjord has the significantly lowest coefficient. This is what we would expect, seeing as they have been the best and worst team this season.

Now we want to compare the strength coefficient ranking with the actual ranking from the season so far. We therefore print the two rankings side by side.

```
##              Strength            Ranking
## 1           Rosenborg          Rosenborg
## 2               Molde              Brann
## 3               Brann              Molde
## 4           Haugesund          Haugesund
## 5                 Odd         Ranheim_TF
## 6         Sarpsborg08         Vaalerenga
## 7             Tromsoe                Odd
## 8       Stroemsgodset            Tromsoe
## 9          Vaalerenga        Sarpsborg08
## 10       Kristiansund       Kristiansund
## 11         Ranheim_TF         BodoeGlimt
## 12         BodoeGlimt      Stroemsgodset
## 13        Lillestroem        Lillestroem
## 14            Stabaek             Stabaek
## 15              Start               Start
## 16 Sandefjord_Fotball Sandefjord_Fotball
```

From the comparison, we get some really interesting results. Based on the strength ranking, we can say that teams that are higher on the actual ranking have "overachieved", while teams that are lower on the actual ranking have "underachieved".

Ranheim_TF is the stand out overachiever, placing in 11th on the strength ranking, and 5th on the actual ranking. One reason for this overachievement may be that Ranheim often win by only small scores, e.g. 1-0, while they lose with big scores, e.g. the scores on some of their losses were: 4-0, 4-0, 3-0, 4-1 and 3-1.

We also see that Brann and Molde have changed places on the actual ranking compared to the strength ranking. Again, this can be due to Brann winning by small scores, and losing by large scores, while Molde often wins by large scores and lose by small scores, e.g. some of Molde's wins have been: 5-0, 4-0, 3-0, 5-1 and 5-1.

Other "overachievers" are Vålerenga and Bodø Glimt, while the "underachievers" are Odd, Tromsø, Sarpsborg 08 and Strømsgodset.

One final thought: If our explanation for why some teams "over"- and "underachieve" is correct (that they lose/win by small and large margins), then the strength ranking should be similar to a ranking based on goal difference. We therefore make a comparison between strength ranking and goal difference ranking.

```
##              Strength          RankingGD
## 1           Rosenborg          Rosenborg
## 2               Molde              Molde
## 3               Brann              Brann
## 4           Haugesund          Haugesund
## 5                 Odd                Odd
## 6         Sarpsborg08        Sarpsborg08
## 7             Tromsoe            Tromsoe
## 8       Stroemsgodset      Stroemsgodset
## 9          Vaalerenga         Ranheim_TF
## 10       Kristiansund         Vaalerenga
## 11         Ranheim_TF         BodoeGlimt
## 12         BodoeGlimt       Kristiansund
## 13        Lillestroem        Lillestroem
## 14            Stabaek            Stabaek
## 15              Start              Start
## 16 Sandefjord_Fotball Sandefjord_Fotball
```

We see that all teams are now in the same place, except for a permutation of the teams Vålerenga, Kristiansund, Ranheim and Bodø Glimt. All these teams except Kristiansund have a goal difference of -2, and Kristiansund has a goal difference of -3, so their goal differences are almost equal. All in all, there may be some truth to our explanation.

d)

Finally, we want to investigate rankings by means of simulation instead of comparing estimated strength. To do this, we use the estimated strengths of each team, the intercept and the home advantage, and simulate the remaining games in the current season 1000 times.

In each of the 1000 simulations, we get the goals for the home team in each match, by drawing a random variable from the poisson distribution with parameter $\lambda_H = \exp(\beta_{Intercept} + \beta_{HomeAdvantage} + \beta_{HomeTeam} - \beta_{AwayTeam})$. Similarly, the goals of the away team in each match is drawn from a poisson distribution with parameter $\lambda_A = \exp(\beta_{Intercept} - \beta_{HomeTeam} + \beta_{AwayTeam})$. When all matches are "played", the final ranking is computed based on the current ranking plus the newly simulated games. The 1000 final rankings are stored in a .rds file. In this way we can load the results, instead of running the simulation multiple times.

After simulating the 48 remaining games 1000 times, we want to do some inference on the final results. We first investigate the "average final ranking", that is, the average points, goals, wins etc. for each team. In addition, we look at how many times each team has placed in each place.

```
##                  Team Played  Won Drawn Lost  For Against    GD Points
## 1           Rosenborg     30 19.8   5.1  5.0 55.9    25.5  30.4   64.7
## 2               Brann     30 17.2   7.3  5.5 46.7    29.5  17.3   59.0
## 3               Molde     30 16.2   5.4  8.4 58.5    36.4  22.2   54.0
## 4           Haugesund     30 14.6   6.3  9.0 45.2    35.8   9.4   50.2
## 5           Ranheim_TF     30 13.2   6.4 10.4 46.2    48.6  -2.4   46.0
## 6           Vaalerenga     30 12.3   7.4 10.3 43.4    45.2  -1.9   44.3
## 7                 Odd     30 11.2   8.3 10.4 43.4    37.8   5.6   42.1
## 8             Tromsoe     30 12.4   4.4 13.2 43.6    41.2   2.4   41.6
## 9          Sarpsborg08     30 11.4   6.3 12.2 47.9    42.1   5.7   40.7
## 10       Kristiansund     30 10.4   8.5 11.1 40.8    42.8  -2.0   39.8
## 11      Stroemsgodset     30  8.6   9.4 12.0 47.2    45.6   1.6   35.3
## 12         BodoeGlimt     30  8.1  10.4 11.5 35.8    39.0  -3.2   34.6
## 13        Lillestroem     30  7.5   8.3 14.2 32.6    47.8 -15.2   30.7
## 14            Stabaek     30  6.8   9.4 13.8 36.2    52.5 -16.4   29.7
## 15              Start     30  7.2   6.3 16.6 29.8    53.5 -23.7   27.8
## 16 Sandefjord_Fotball     30  3.2  10.1 16.7 29.9    59.7 -29.7   19.7
```

From the average rating, we see that Rosenborg win the Eliteserie by a margin of ≈ 6 points on average. Brann claims silver, and Molde bronze. Sandefjord ends bottom with a margin of ≈ 8 points, with Start also getting relegated and Stabæk claiming the play-off place. Ranheim (surprisingly) claim 5th place with a negative goal difference. We also see that the table is (almost) unchanged after simulating the remaining 48 games. The only difference is that Strømsgodset overtake Bodø Glimt.

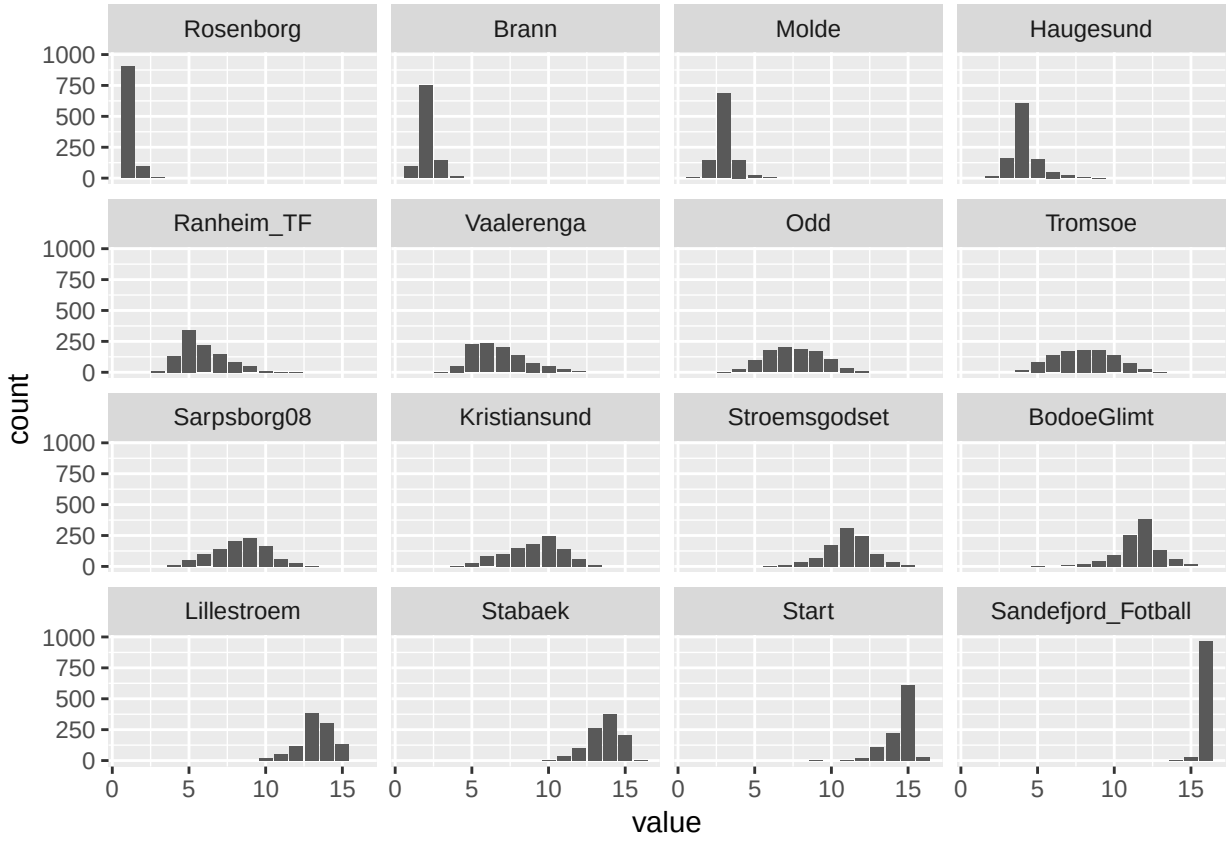
```
##                  Team One Two Three Four Five Six Seven Eight Nine Ten
## 1           Rosenborg 901  95     4    0    0   0     0     0    0   0
## 2               Brann  94 750   145   11    0   0     0     0    0   0
## 3               Molde   5 142   682  146   19   6     0     0    0   0
## 4           Haugesund   0  13   157  606  151  50    19     3    1   0
## 5           Ranheim_TF   0   0    10  134  344 218   150    82   48  11
## 6           Vaalerenga   0   0     1   49  224 234   200   137   77  51
## 7                 Odd   0   0     1   22   96 177   200   185  169 103
## 8             Tromsoe   0   0     0   15   84 135   174   179  179 135
## 9          Sarpsborg08   0   0     0   14   55  96   141   207  230 168
## 10       Kristiansund   0   0     0    3   26  80    99   151  184 245
## 11      Stroemsgodset   0   0     0    0    0   4    10    37   69 173
## 12         BodoeGlimt   0   0     0    0    1   0     7    19   42  89
## 13        Lillestroem   0   0     0    0    0   0     0     0    0  17
## 14            Stabaek   0   0     0    0    0   0     0     0    0   8
## 15              Start   0   0     0    0    0   0     0     0    1   0
## 16 Sandefjord_Fotball   0   0     0    0    0   0     0     0    0   0
##    Eleven Twelve Thirteen Fourteen Fifteen Sixteen
## 1       0      0        0        0       0       0
## 2       0      0        0        0       0       0
## 3       0      0        0        0       0       0
## 4       0      0        0        0       0       0
## 5       2      1        0        0       0       0
## 6      21      6        0        0       0       0
## 7      35     12        0        0       0       0
## 8      76     21        2        0       0       0
## 9      62     26        1        0       0       0
```

```
## 10   139    63    10     0     0     0
## 11   312   245   101    39    10     0
## 12   255   385   131    56    15     0
## 13    55   116   381   301   130     0
## 14    35   101   264   378   210     4
## 15     8    24   110   223   608    26
## 16     0     0     0     3    27   970
```

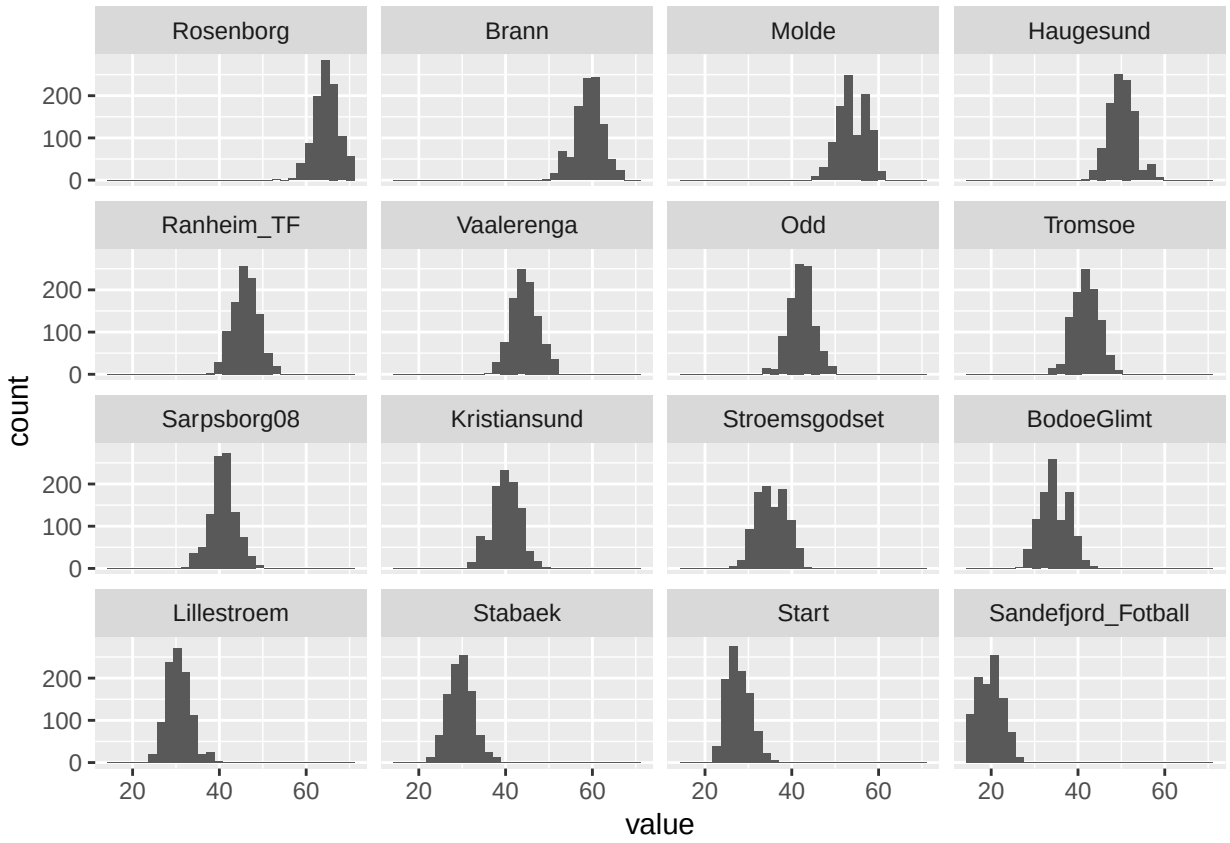
We see that Rosenborg win the Eliteserie with 90.1% probability, with Brann having 9.4% and Molde having 0.5% chance of winning. Also, Rosenborg get a medal in every simulation. Brann also claim a medal with 98.9% probability, with silver being the most likely result at 75% probability. Moreover, Molde gets bronze with 68.2% probability. Other teams with a chance of claiming a medal are Haugesund (17%), Ranheim (1%), Vålerenga (0.1%) and Odd (0.1%).

In the other end of the table, Sandefjord end bottom with 97% probability, managing a play-off place with only 0.3% probability. Start get relegated with 63.4% probability, manage play-off with 22.3% probability, and secure their place in the Eliteserie with a probability of 14.3%. Interestingly, in one simulation Start managed to get all the way up to 9th! Stabæk get relegated with a probability of 21.4%, managing play-off with probability 37.8%, and safe ground with probability 40.8%. Other teams in danger of relegation/play-off are Lillestrøm (13%/30.1%), Bodø Glimt (1.5%/5.6%) and Strømsgodset (1%/3.9%).

Based on the from these placings, we make barcharts for the placings of each team.



We also make a histogram of the points acquired by each team.

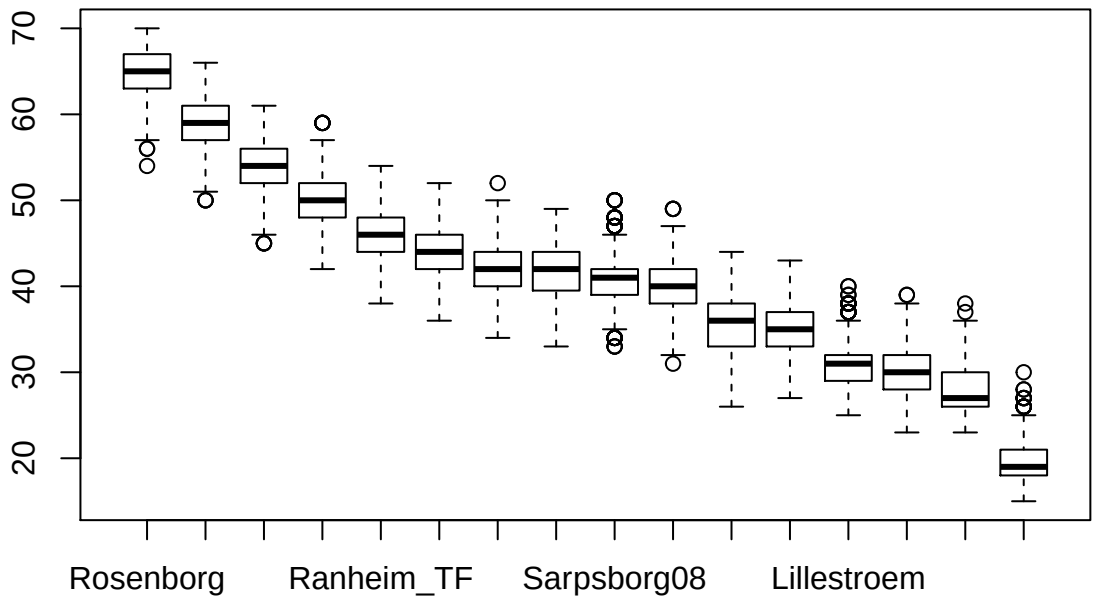


By the histogram of points, the distribution of points achieved in a season looks to be approximately normal. Seeing as the amount of points is a random variable, we can by the central limit theorem say that the mean of the points is normally distributed. We therefore find the average number of points for each team, as well as the standard deviation, and construct 90 % confidence intervals for the points of each team.

```
##                Teams mean  sd  low high
## 1          Rosenborg 64.7 2.8 60.0 69.3
## 2              Brann 59.0 3.1 53.9 64.1
## 3              Molde 54.0 3.1 49.0 59.1
## 4           Haugesund 50.2 3.0 45.4 55.1
## 5          Ranheim_TF 46.0 3.0 41.1 51.0
## 6          Vaalerenga 44.3 3.2 39.1 49.6
## 7                 Odd 42.1 3.0 37.2 47.0
## 8             Tromsoe 41.6 3.1 36.5 46.6
## 9          Sarpsborg08 40.7 3.0 35.7 45.6
## 10       Kristiansund 39.8 3.2 34.6 45.0
## 11      Stroemsgodset 35.3 3.2 30.1 40.5
## 12         BodoeGlimt 34.6 3.0 29.7 39.6
## 13        Lillestroem 30.7 2.8 26.1 35.3
## 14            Stabaek 29.7 3.0 24.8 34.7
## 15              Start 27.8 2.9 23.1 32.4
## 16 Sandefjord_Fotball 19.7 2.6 15.5 24.0
```

From the confidence intervals, we see that Rosenborg will win or get silver. Brann may win, but will at worst get 4th. Meanwhile, all hope of staying in the division looks to be gone for Sandefjord.

At last we make a boxplot of the points for each team and interpret the results.



We see that the "boxes" (quartile 2 and 3) of Rosenborg, Brann, Molde and Haugesund are non-overlapping. This suggests that this will be the final standing among the top four in the table. Meanwhile, Sandefjord's box is some distance behind Start's box, emphasizing that Sandefjord will get relegated. Yet, there is still possibility for Start and Stabæk to avoid relegation/play-off.