Graph

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Contents

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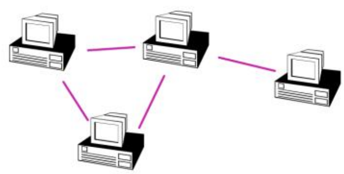
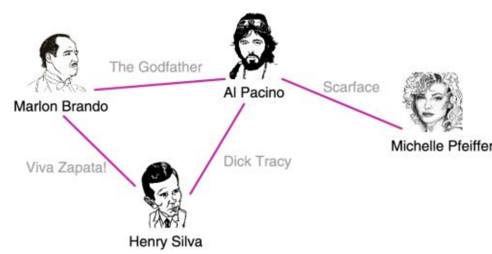
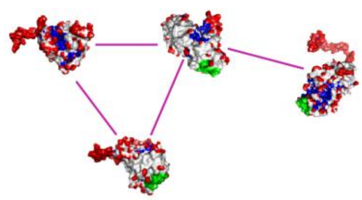
Examples







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Networks or Graphs

- **network** often refers to ***real systems***
  - www,
  - social network,
  - metabolic network.

- **graph**: mathematical representation of a ***network***
  - web graph,
  - social graph (a Facebook term)

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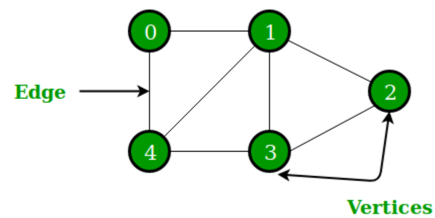
Graph

- A graph consists of **a finite set of vertices** (or nodes) and **set of edges** which connect a pair of vertices (nodes).
- G = {V, E}
  - V: set of vertices. V = {v_1, v_2,…, v_n}
  - E: set of edges. E = {e_1, e_2,…,e_m}
- Example:
  - V = {0, 1, 2, 3, 4}
  - E = {01, 04, 12, 13, 14, 23, 34}



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Terminologies

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Terminologies

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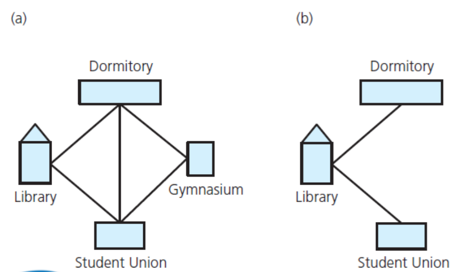
- A **subgraph** consists of a subset of a graph's vertices and a subset of its edges.
  - G'= {V', E'} is a subgraph of G = {V, E} if V' ⊆ V, E' ⊆ E



(a) A campus map as a graph;
(b) a subgraph

Terminologies

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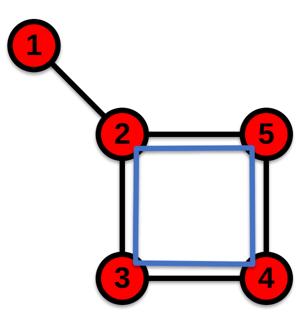
- **Vertex**: also called a **node**.
- **Edge**: connects two vertices.
- **Loop** (*self-edge*): An edge of the form (*v*, *v*).
- **Adjacent**: two vertices are **adjacent** if they are joined by an edge.

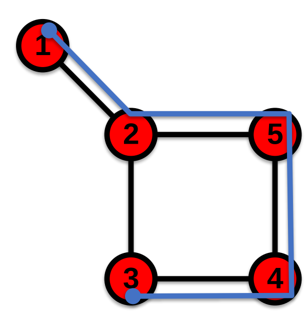
Terminologies

- **Path**: A sequence of edges that begins at one vertex and ends at another vertex.
  - If all vertices of a path is distinct, the path is **simple**.
- **Cycle**: A path that starts and ends at the same vertex and does not traverse the same edge more than once.
- **Acyclic** graph: A graph with no cycle.

Terminologies



Cycle: A path with the same start and end node.



A path that does not intersect itself.

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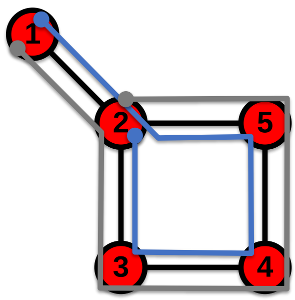
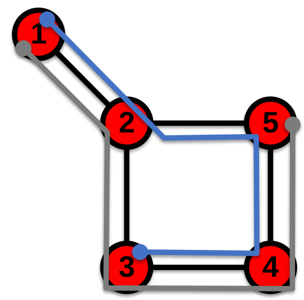
Terminologies

- **Eulerian Path**: A path that traverses each ***edge*** exactly once.
- **Hamiltonian Path**: A path that visits each ***vertex*** exactly once.



Eulerian Path



Hamiltonian Path

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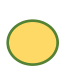
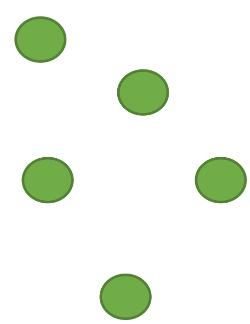
Terminologies

- **Null graph**: A graph having no edges
- **Trivial graph**: A graph with only one vertex.



trivial graph



null graph

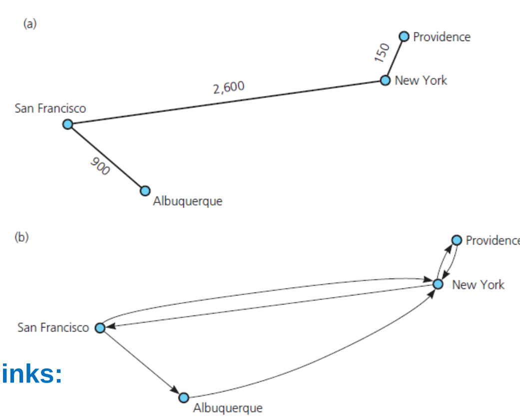
Terminologies

- **Undirected graph**: the graph in which edges do not indicate a direction.
- **Directed graph, or digraph**: a graph in which each edge has a direction.
- **Weighted graph**: a graph with numbers (weights, costs) assigned to its edges.

Terminologies

(a): undirected graph

(b): directed graph



Undirected direction/links:
- Co-authorship links
- Actor network
- Protein interactions

Directed directions/links:
- URLs on the www
- Phone calls
- Metabolic reactions

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NETWORK	NODES	LINKS	DIRECTED UNDIRECTED	N	L
Internet	Routers	Internet connections	Undirected	192,244	609,066
WWW	Webpages	Links	Directed	325,729	1,497,134
Power Grid	Power plants, transformers	Cables	Undirected	4,941	6,594
Mobile Phone Calls	Subscribers	Calls	Directed	36,595	91,826
Email	Email addresses	Emails	Directed	57,194	103,731
Science Collaboration	Scientists	Co-authorship	Undirected	23,133	93,439
Actor Network	Actors	Co-acting	Undirected	702,388	29,397,908
Citation Network	Paper	Citations	Directed	449,673	4,689,479
E. Coli Metabolism	Metabolites	Chemical reactions	Directed	1,039	5,802
Protein Interactions	Proteins	Binding interactions	Undirected	2,018	2,930

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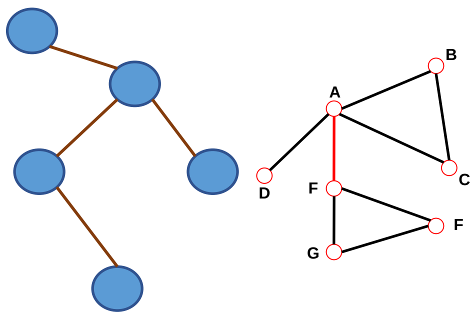
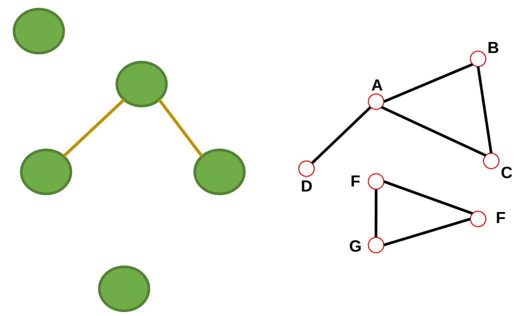
Terminologies

- **Connected graph:** A graph in which each pair of **distinct vertices** has a **path** between them.
- **Disconnected graph:** A graph does not contain at least two connected vertices.
- Graph cannot have duplicate edges between vertices.
  - **Multigraph**: does allow multiple edges

Terminologies





connected graph **disconnected graph**

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Terminologies

- **cut vertex** *(or* ***articulation point****)*
  - The removal from a graph of that (cut vertex) and all incident edges produces a subgraph with more connected components.
- **cut edge** (or ***bridge***)
  - An edge whose removal produces a graph with more connected components than in the original graph.

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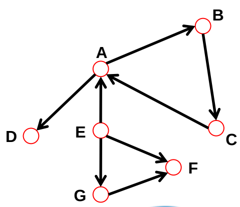
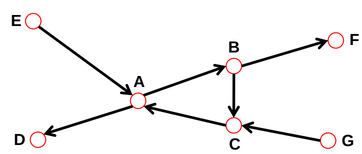
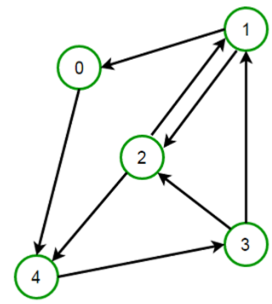
Terminologies

- **Strongly connected** directed graph: if there is a path in each direction between each pair of vertices of the graph.
- **Weakly connected** directed graph: it is connected if we disregard the edge directions.







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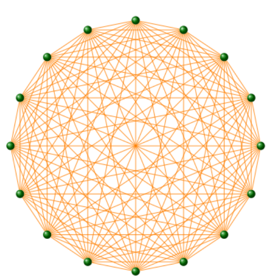
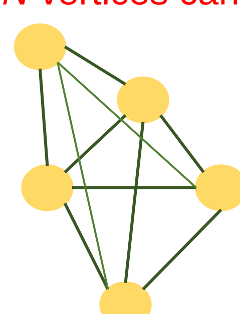
Terminologies

- **Complete graph:** A graph in which each pairs of **distinct vertices** has an **edge** between them

The maximum number of edges a graph *N* vertices can have?





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Terminologies

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- **Degree** of a vertex *v* (denoted *deg(v)*): the number of edges connected to *v*.
- In directed graphs, we can define an *in-degree* and *out-degree* of vertex *v*.

  In-degree of *v* (denoted $deg^-(v)$): number of edges with *v* as their terminal vertex.

  Out-degree of *v* (denoted $deg^+(v)$): number of edges with *v* as their initial vertex.

$$\mathbf{deg(}v\mathbf{) = deg^-(}v\mathbf{) + deg^+(}v\mathbf{)}$$

Terminologies

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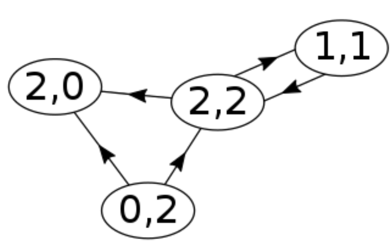
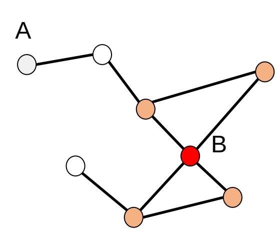


A directed graph with vertices labeled (indegree, outdegree)



deg(A) = 1; deg(B) = 4

Terminologies

- Let *G* = {*V*, *E*}
- If *G* is an undirected graph

$$\sum_{v \in V} \deg(v) = 2|E|$$

- If *G* is a directed graph

$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$$

Graph Representation

Graph Representation

- Adjacency Matrix
- Adjacency List

Adjacency Matrix

A[*n*][*n*] with *n* is the number of vertices.

- $A[i][j] = \begin{cases} 1 \; if \; there \; is \; an \; edge(i,j) \\ 0 \; if \; there \; is \; no \; edge \; (i,j) \end{cases}$
- $A[i][j] = \begin{cases} W & with \; w \; is \; the \; weight \; of \; edge(i,j) \\ \infty & if \; there \; is \; no \; edge \; (i,j) \end{cases}$

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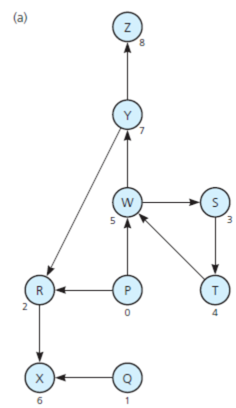
Adjacency Matrix

(a)



(b)

		0	1	2	3	4	5	6	7	8
		P	Q	R	S	T	W	X	Y	Z
0	P	0	0	1	0	0	1	0	0	0
1	Q	0	0	0	0	0	0	1	0	0
2	R	0	0	0	0	0	0	1	0	0
3	S	0	0	0	0	1	0	0	0	0
4	T	0	0	0	0	0	1	0	0	0
5	W	0	0	0	1	0	0	0	1	0
6	X	0	0	0	0	0	0	0	0	0
7	Y	0	0	1	0	0	0	0	0	1
8	Z	0	0	0	0	0	0	0	0	0

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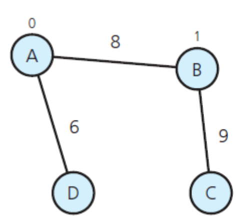
Adjacency Matrix

(a)



(b)

		0	1	2	3
		A	B	C	D
0	A	∞	8	∞	6
1	B	8	∞	9	∞
2	C	∞	9	∞	∞
3	D	6	∞	∞	∞

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Adjacency List

- A graph with *n* vertices has *n* linked chains.
- The i^{th} linked chain has a node for vertex *j* if and only if having edge (*i*,*j*).

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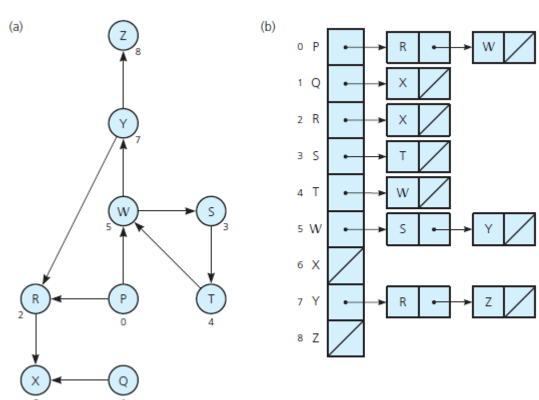
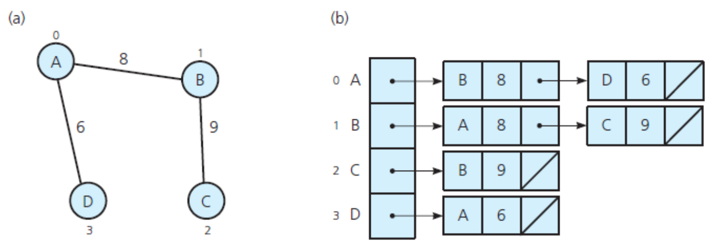
Adjacency List



Adjacency List



Relative Advantages of Adjacency Lists and Matrices

- Faster to test if (x, y) in graph?
- Faster to find the degree of a vertex?
- Less memory on small graph?
- Less memory on big graph?
- Edge insertion or deletion?
- Faster to traverse the graph?
- Better for most problems?

Graph Traversal

Graph Traversal

- Visits (all) the vertices that it can reach.
- **Connected component** is subset of vertices visited during traversal that begins at given vertex.

Depth-First Search

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- Goes as far as possible from a vertex before backing up.

```
DFS(v: vertex)
{
      Mark v as visited
      for (each unvisited vertex u adjacent to v)
            DFS(u)
}
```


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Depth-First Search

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```
DFS(v: vertex)
      s = new empty stack
      s.push(v)
      Mark v as visited
      while (s is not empty)  {
            if (no unvisited vertices are adjacent to the
vertex on the top of the stack)
                  s.pop()
            else {
                  s.push(u)
                  Marked u as visited
            }
      }
```


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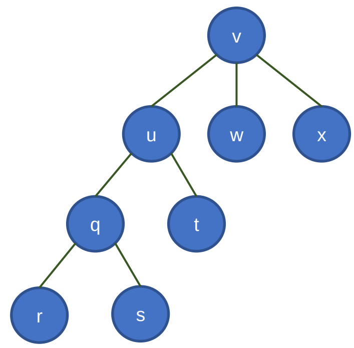
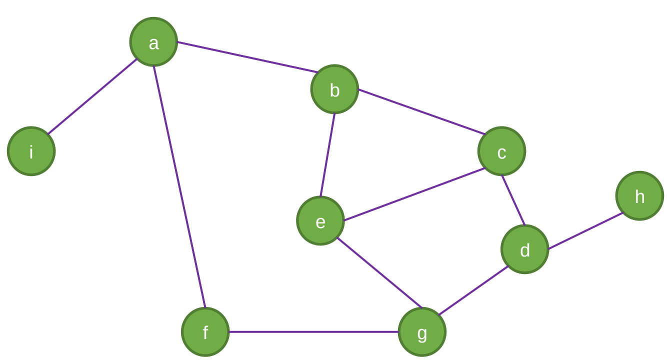
Depth-First Search



v - u - q - r - s - t - w - x

Depth-First Search



DFS starts at ***a***:
DFS starts at ***e***:

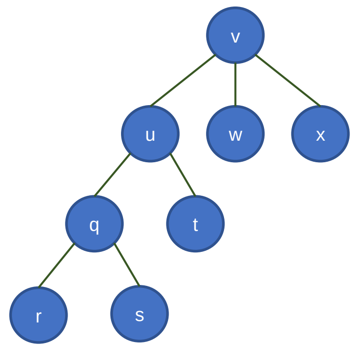
Breadth-First Search

- Visits all vertices adjacent to vertex before going forward.
- Breadth-first search uses a queue.

Breadth-First Search

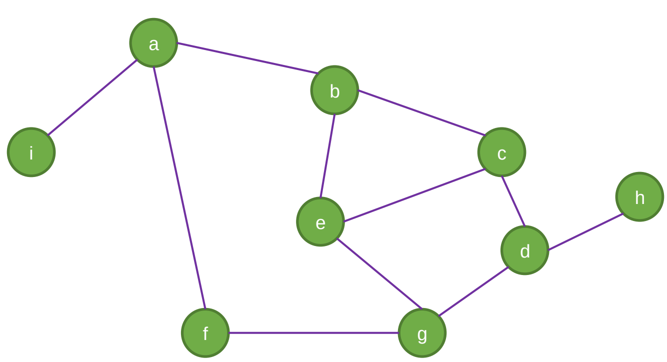
```
BFS(v: Vertex)
    q = a new empty queue
    q.enqueue(v)
    Mark v as visited
    while (q is not empty){
        w = q.dequeue()
        for (each unvisited vertex u adjacent to w){
            Mark u as visited
            q.enqueue(u)
        }
    }
```

Breadth-First Search



v - u - w - x - q - t - r - s

Breadth-First Search



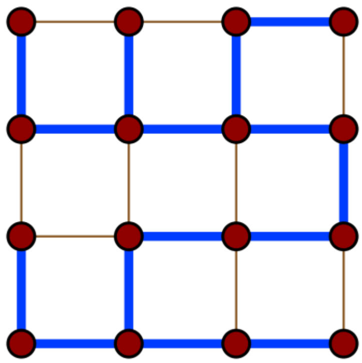
BFS starts at ***a***:
BFS starts at ***e***:

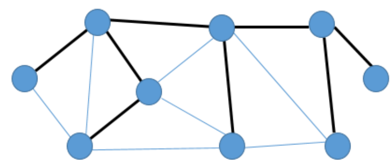
Minimum Spanning Tree

Spanning Tree

- A spanning tree
  - is a **subgraph** of undirected graph G
  - has **all** the vertices covered with **minimum** possible number of edges.
- does not have cycles
- cannot be disconnected.

Spanning Tree





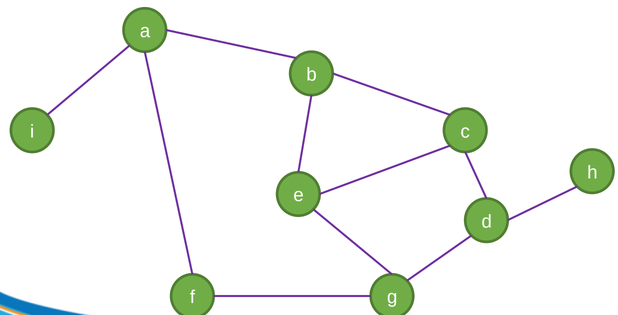
Spanning Tree

- A connected graph G can have **more than one** spanning tree.
- All possible spanning trees of graph G, **have the same** number of **edges** and **vertices**.
- The spanning tree **does not have any cycle** (loops).
- The spanning tree is **minimally connected**.
- The spanning tree is **maximally acyclic**.

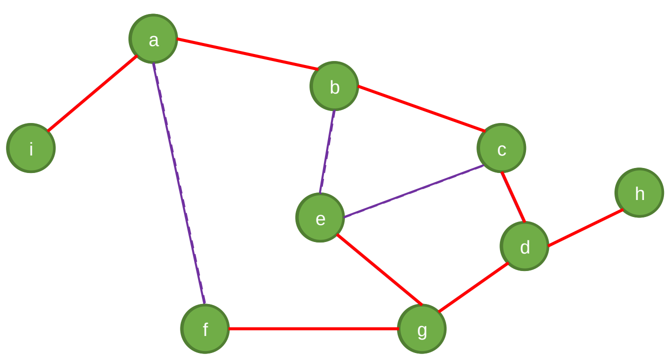
Spanning Tree

- Depth-first-search spanning tree

- Breadth-first-search spanning tree

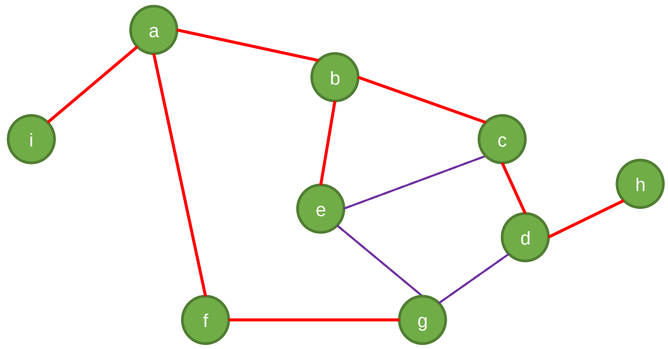


Spanning Tree



DFS spanning tree

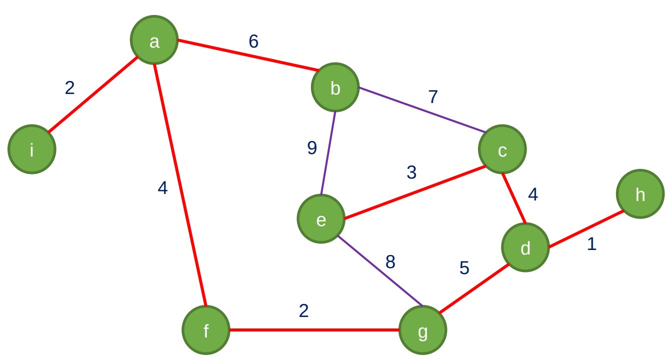
Spanning Tree



BFS spanning tree

Minimum Spanning Tree

- A minimum spanning tree is a spanning tree that has **minimum weight** than all other spanning trees of the same graph.



Prim's Minimum Spanning Tree

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- Begins with any vertex *s*.
- Initially, the tree *T* contains only the starting vertex *s*.
- At each stage,
  - Select the least cost edge *e(v, u)* with *v* in *T* and *u* not in *T*.
  - Add *u* and *e* to *T*

Prim's Minimum Spanning Tree

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```
primAlgorithm(v: Vertex)
   Mark v as visited and include it in the minimum spanning tree
   while (there are unvisited vertices)
   {
         Find the least-cost edge e(v, u) from a visited vertex
               v to some unvisited vertex u
         Mark u as visited
         Add the vertex u and the edge e(v, u) to the minimum
               spanning tree
   }
```

Prim’s Minimum Spanning Tree

```
PrimSpanningTree(matrix[N][N], source)
{
        for v = 0 to N-1 {
                length[v] = matrix[source][v]
                parent[v] = source }
        Mark source //Add source to the spanning tree
        for step = 1 to N-1 {
                Find the vertex v such that length[v] is smallest and v
                        is not in spanning tree
                Mark v
                for all vertices u not in vertexSet
                        if (length[u] > matrix[v][u]){
                                length[u] = matrix[v][u]
                                parent[u] = v }
        }
}
```


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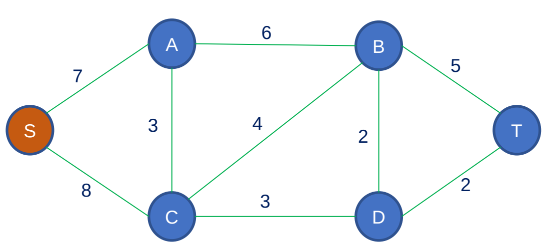
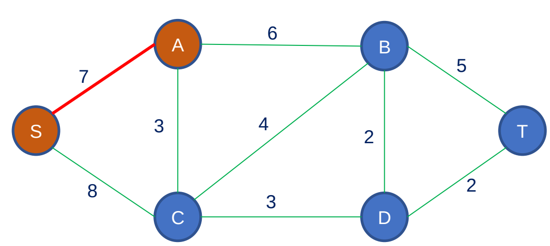
Prim’s Minimum Spanning Tree





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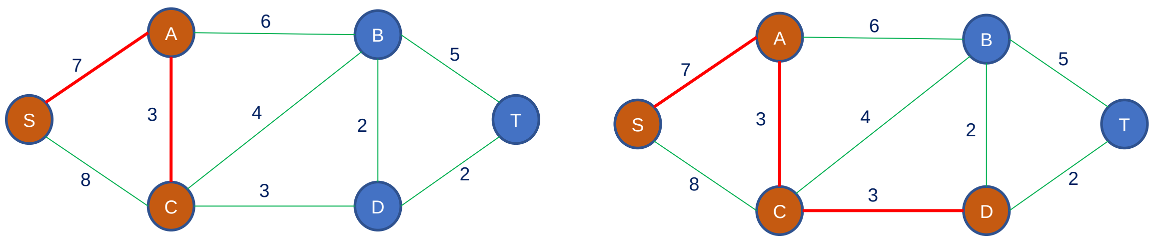
Prim’s Minimum Spanning Tree



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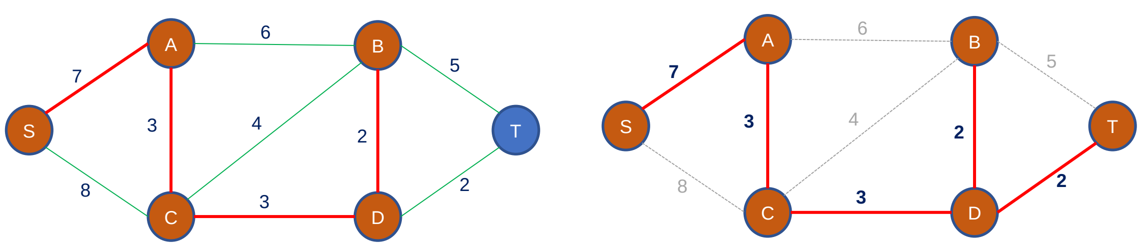
Prim’s Minimum Spanning Tree



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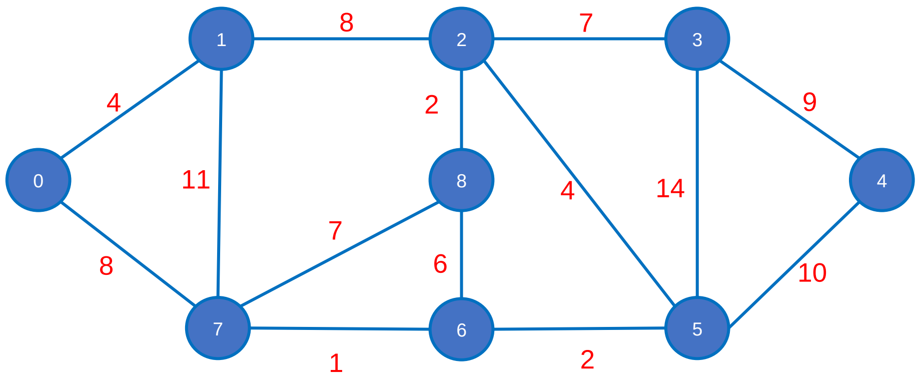
Example



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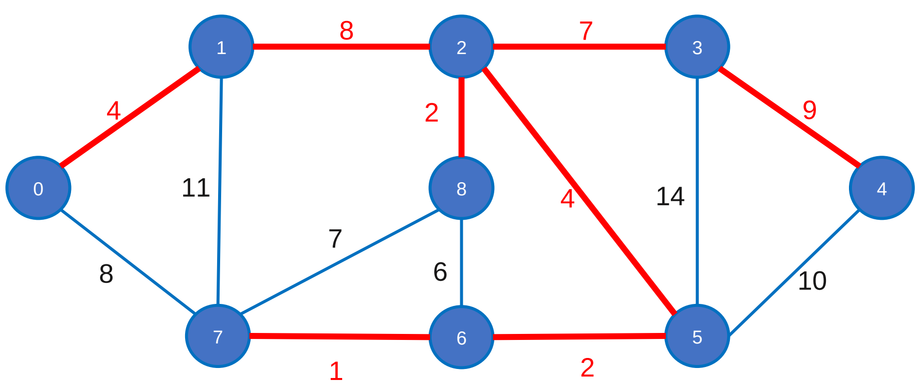
Example



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Shortest Path

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Dijkstra's Shortest Path Algorithm

- Given a graph and a source vertex in the graph, find shortest paths from the source to ALL vertices in the given graph.
- **Dijkstra's** algorithm is very **similar** to **Prim's** algorithm for minimum spanning tree.
- This algorithm is applicable to graphs with **non-negative weights** only.

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Dijkstra's Shortest Path Algorithm

shortestPath(matrix[N][N], source, length[])

Input:

matrix[N][N]: adjacency matrix of Graph *G* with *N* vertices

source: the *source* vertex

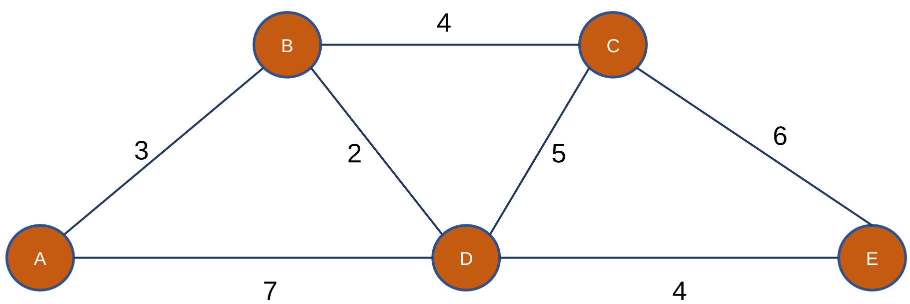
Output:

length[]: the length of the shortest path from *source* to all *vertices* in G.

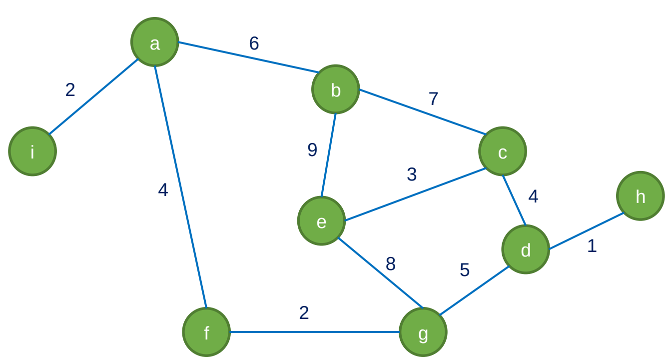
Dijkstra's Shortest Path Algorithm

```
shortestPath(matrix[N][N], source, length[]){
    for v = 0 to N-1
        length[v] = matrix[source][v]
    length[source] = 0 //why?
    for step = 1 to N {
        Find the vertex v such that length[v] is smallest and
            v is not in vertexSet
        Add v to vertexSet
        for all vertices u not in vertexSet
            if (length[u] > length[v] + matrix[v][u]){
                length[u] = length[v] + matrix[v][u]
                parent[u] = v }
    }
}
```

Example



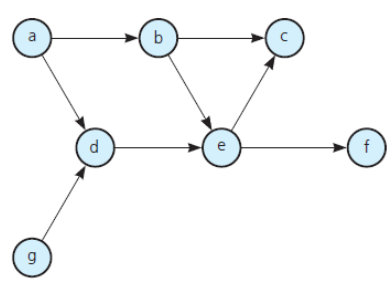
Example



Topological sorting

Directed acyclic graph

- **dag**: directed acyclic graph



Topological sorting

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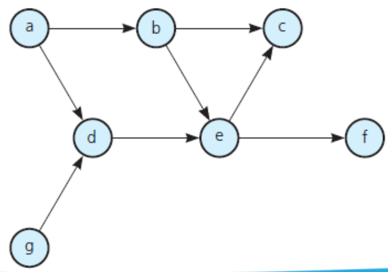
- Topological order: order to take to satisfy all the prerequisites.
- Topological sorting: arranging the vertices in topological order.

- Prerequisite structure of the courses.



Topological sorting

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```
topSort1(theGraph: Graph, aList: List)
      n  = number of vertices in theGraph
      for (step = 1 to n)
      {
            Select a vertex v that has no successors
            aList.addHead(v)
            Remove from the Graph vertex v and its edges
      }
```

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Topological sorting

```
L: Empty list that will contain the sorted elements
S: Stack of all nodes with no incoming edge
while S is not empty
    remove a node n from S //n = S.pop()
    add n to tail of L
    for each node m with an edge e from n to m do
        remove edge e from the graph
        if m has no other incoming edges then
            insert m into S // S.push(m)
if graph has edges then
    return error   //(graph has at least one cycle)
else
    return L   //(a topologically sorted order)
```


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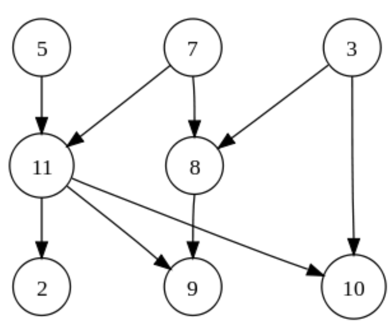
Topological sorting

- Try this one



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Questions and Answers