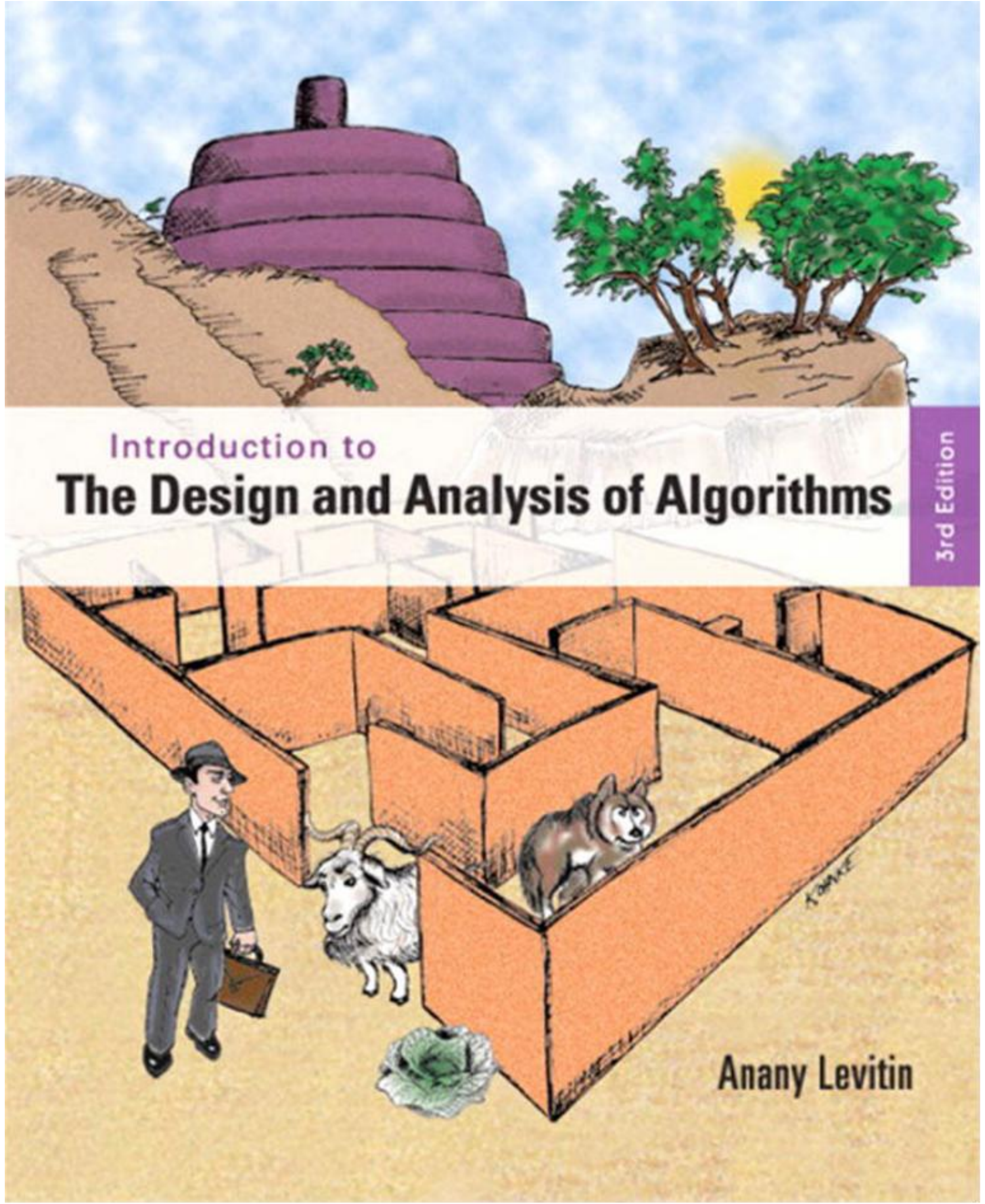
Introduction to
The Design and Analysis of Algorithms
3rd Edition
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Heap

nodes occurs on each level. Let h be the height of the tree. According to the first property of heaps in the list at the beginning of the section, $h = \lfloor \log_2 n \rfloor$ or just $\lceil \log_2 (n+1) \rceil - 1 = k - 1$ for the specific values of n we are considering. Each key on level i of the tree will travel to the leaf level h in the worst case of the heap construction algorithm. Since moving to the next level down requires two comparisons—one to find the larger child and the other to determine whether the exchange is required—the total number of key comparisons involving a key on level i will be $2(h - i)$. Therefore, the total number of key comparisons in the worst case will be

$$C_{worst}(n) = \sum_{i=0}^{h-1} \sum_{\text{level } i \text{ keys}} 2(h-i) = \sum_{i=0}^{h-1} 2(h-i)2^i = 2(n - \log_2(n+1)),$$

where the validity of the last equality can be proved either by using the closed-form formula for the sum $\sum_{i=1}^{h} i2^i$ (see Appendix A) or by mathematical induction on h. Thus, with this bottom-up algorithm, a heap of size n can be constructed with fewer than $2n$ comparisons.

Useful Formulas

Important Summation Formulas

1. $\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$ (l, u are integer limits, $l \le u$); $\sum_{i=1}^{n} 1 = n$

2. $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2$

3. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3}n^3$

4. $\sum_{i=1}^{n} i^k = 1^k + 2^k + \cdots + n^k \approx \frac{1}{k+1}n^{k+1}$

5. $\sum_{i=0}^{n} a^i = 1 + a + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}$ $(a \ne 1)$; $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

6. $\sum_{i=1}^{n} i2^i = 1 \cdot 2 + 2 \cdot 2^2 + \cdots + n2^n = (n-1)2^{n+1} + 2$

7. $\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n + \gamma$, where $\gamma \approx 0.5772\ldots$ (Euler's constant)

8. $\sum_{i=1}^{n} \lg i \approx n \lg n$

Useful Formulas

Properties of Logarithms

All logarithm bases are assumed to be greater than 1 in the formulas below; lg . denotes the logarithm base 2, ln x denotes the logarithm base $e = 2.71828\ldots$ x, y are arbitrary positive numbers.

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a x^y = y \log_a x$
4. $\log_a xy = \log_a x + \log_a y$
5. $\log_a \frac{x}{y} = \log_a x - \log_a y$
6. $a^{\log_b x} = x^{\log_b a}$
7. $\log_a x = \frac{\log_b x}{\log_b a} = \log_a b \log_b x$

Combinatorics

1. Number of permutations of an n-element set: $P(n) = n!$
2. Number of k-combinations of an n-element set: $C(n, k) = \frac{n!}{k!(n-k)!}$
3. Number of subsets of an n-element set: 2^n