fit@hcmus

DATA STRUCTURES & ALGORITHMS

Lecture 6: TREES – Part 3

B-trees, 2-3 trees, 2-3-4 trees

Lecturer: Dr. Nguyen Hai Minh

OUTLINE

- Balanced Search Tree
  - B-tree
  - 2-3 tree
  - 2-3-4 tree
- Comparing trees

B-TREES

- Definition
- Example
- Traversal, Search, Insert, Delete
- Advantage of B-trees

Motivation of B-trees

- So far, we have assumed that we can store an entire data structure in ***main memory***
- What if we have so much data that it won't fit?
  - Storing it on disk requires different approach to ***efficiency***
  - Assuming that a disk spins at 7200 RPM, one revolution occurs in **8.33ms**
  - Crudely speaking, one disk access takes about the same time as **100,000 instructions**

Motivation of B-trees

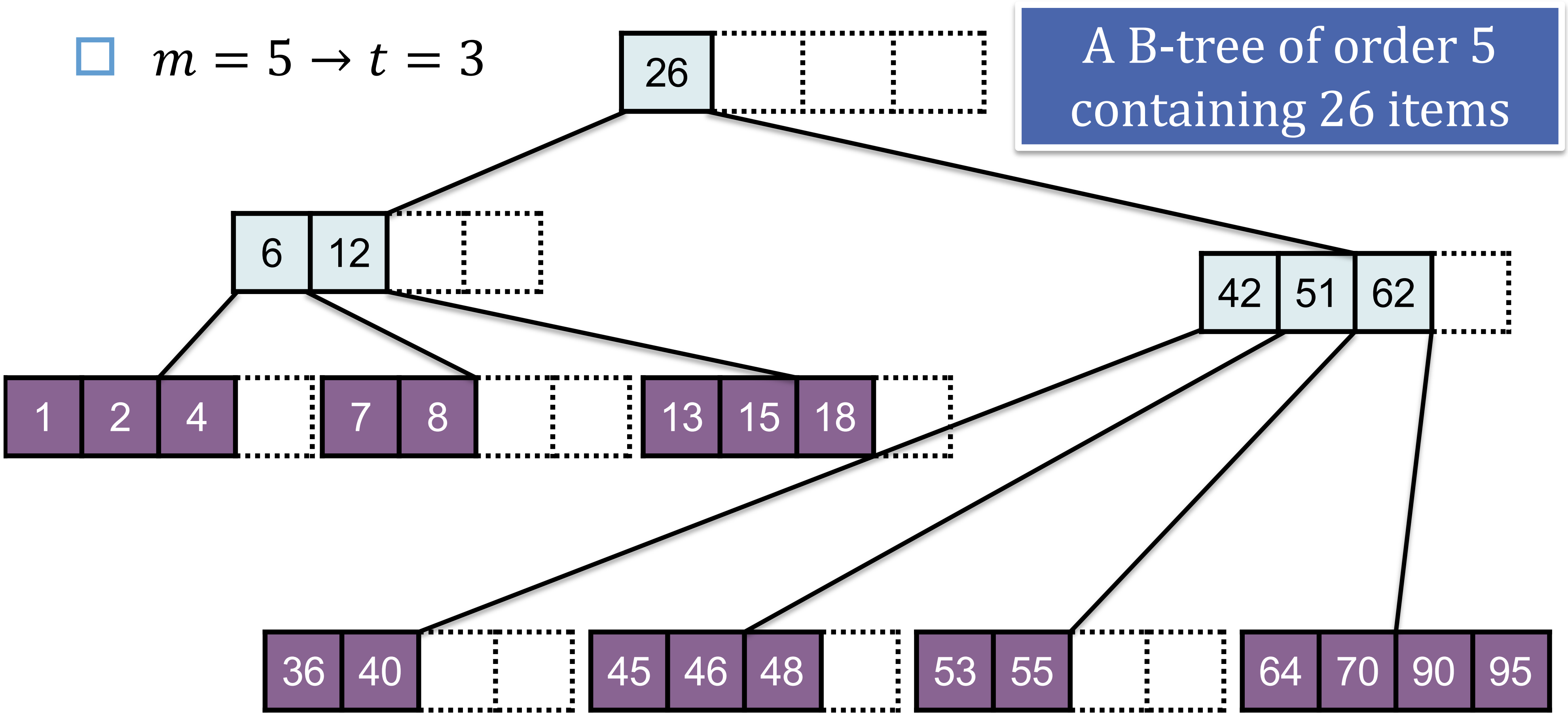
- Assume that we use an AVL tree to store about 20 million records
  - We end up with a very deep binary tree with lots of different disk accesses; $\log_2 20{,}000{,}000 \approx 24$, so this takes about 0.2 seconds
- We know we can't improve on the $\log_2 n$ lower bound on search for a binary tree
- But, the solution is to use more branches and thus reduce the height of the tree!
  - → As branching increases, depth decreases
  - → B-tree: 1970 by Rudolf Bayer & Edward M. McCreight

Definition of a B-tree

- A B-tree of order m is an m-way tree (i.e., a tree where each node may have up to m children) in which:
  1. The number of keys in each non-leaf node is one less than the number of its children and these keys partition the keys in the children in the fashion of a search tree
  2. All leaves are on the same level
  3. All non-leaf nodes except the root have at least $t = \left\lceil \frac{m}{2} \right\rceil$ children
  4. The root is either a leaf node, or it has from 2 to m children
  5. Every nodes except the root has $\left\lceil \frac{m}{2} \right\rceil - 1$ to $m - 1$ keys
- The number m is usually odd

An Example B-tree

- $m = 5 \rightarrow t = 3$

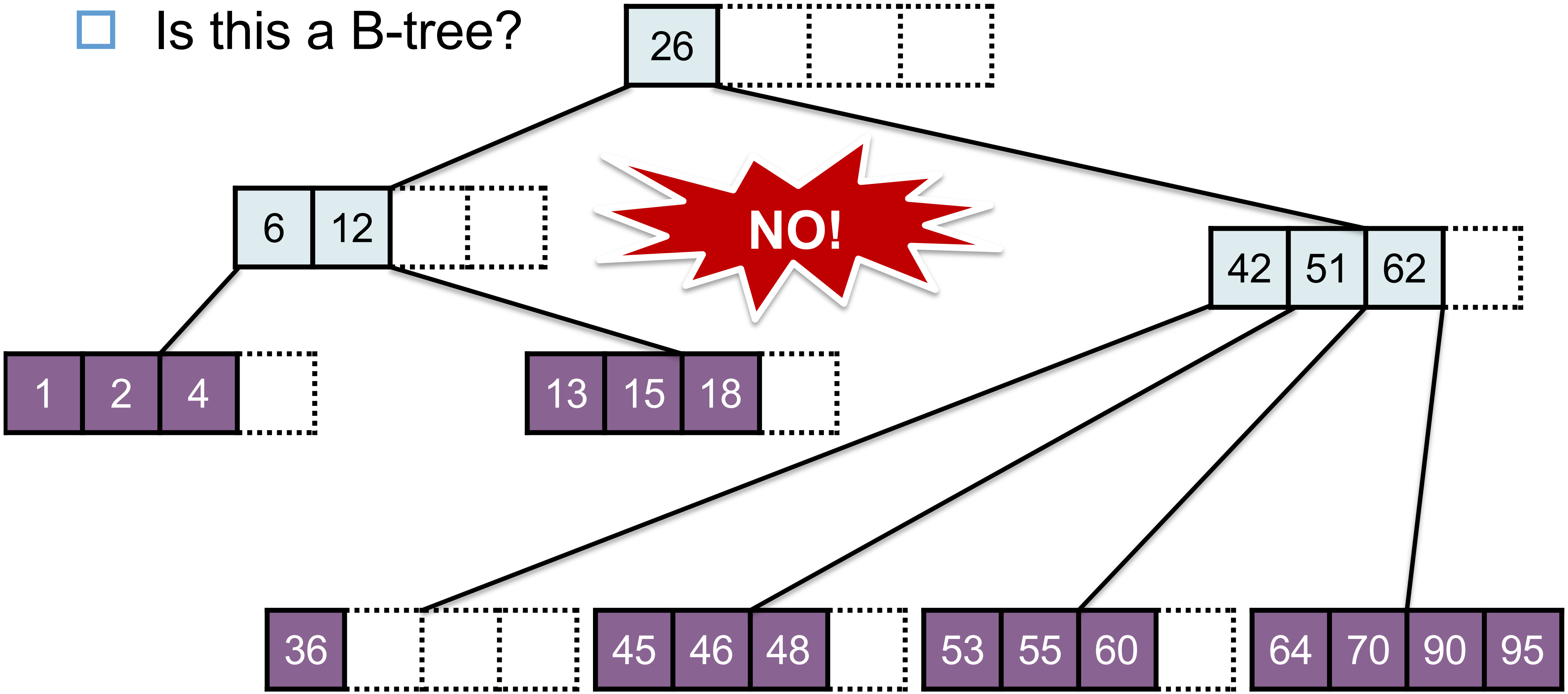


→ Each internal nodes (except root) has at least 3 children

→ Each node has 2-4 keys

An Example B-tree

- Is this a B-tree?



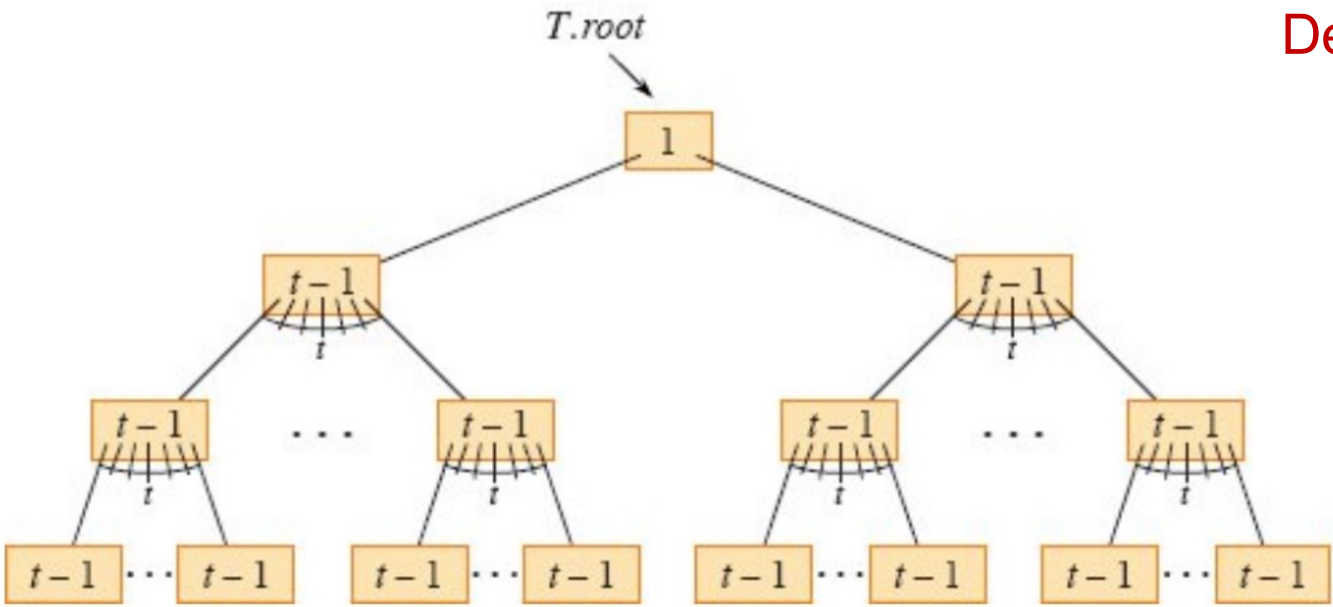
- Why?

B-tree Example

- Draw a B-tree of order 3 and height = 3 containing as many keys/data items as possible.
  - How many nodes are there?
  - How many keys does your tree have?

Height of B-tree

- Let h be the height, n is the total keys on a B-tree
- t: minimum degree of m-way B-tree
- Minimum keys on a B-tree:



Depth	#nodes	#keys
0	1	1
1	2	$2(t-1)$
2	$2t$	$2t(t-1)$
3	$2t^2$	$2t^2(t-1)$

B-tree's height

- Minimum keys on a B-tree:

$$n \geq 1 + 2(t-1) + 2t(t-1) + \cdots + 2t^{h-1}(t-1)$$

$$n \geq 1 + 2(t-1)\sum_{i=1}^{h} t^{i-1} = 1 + 2(t-1)\left(\frac{t^{h-1}}{t-1}\right)$$

$$n \geq 2t^h - 1$$

- **Theorem:** If $n \geq 1$, then for any n-keys B-tree of height h and minimum degree $t \geq 2$,

$$h \leq \log_t \frac{n+1}{2}$$

→ $h = O(\log_t n)$

B-tree's maximum nodes

- The maximum number of keys in a B-tree of order *m* and height *h*:

  root $m - 1$

  level 1 $m(m - 1)$

  level 2 $m^2(m - 1)$

  . . .

  level h $m^h(m - 1)$

- So, the total number of keys is:

$$n \le (1 + m + m^2 + m^3 + \dots + mh)(m - 1) =$$

$$\left[\frac{m^{h+1} - 1}{m - 1}\right](m - 1) = m^{h+1} - 1$$

$$\rightarrow h \ge \log_m(n + 1) - 1$$

Traversal and Search in B-tree

- Traversal in B-tree:
  - Similar to *In-Order Traversal* of Binary Tree.
- Search in B-tree:
  - Similar to searching in a Binary Search Tree, except that instead of making a binary, or "two-way," branching decision at each node, we make a *multiway* branching decision according to the number of the node's children.

B-tree-SEARCH

B-tree-SEARCH(x,k)

$$O(th) = O(t \log_t n)$$

//x: pointer to the root node of a subtree, k: key to be searched

```
i = 0
while i < x.n and k > x.key_i
  i = i + 1
if i < x.n and k == x.key_i
  return (x,i)
elseif x.leaf
  return NULL
else DISK-READ(x.c_i)
  return B-tree-SEARCH(x.c_i,k)
```

$O(t)$

$O(h)$

Inserting into a B-tree $O(t \log_t n)$

- Attempt to insert the new key into a leaf
- If *leaf* becomes too big,
  - Split the leaf into two
  - Promoting the middle key to the leaf's parent
- If *the parent* becomes too big
  - Split the parent into two
  - Promoting the middle key

→ This strategy might have to be repeated all the way to the top.

→ If necessary, the root is split in two and the middle key is promoted to a new root, *making the tree one level higher*

Inserting into a B-tree – Example

- Suppose we start with an empty B-tree and keys arrive in the following order:

1 12 8 2 25 5 14 28 17 7 52 16 48 68 3 26 29 53 55 45

- We want to construct a B-tree of order 5

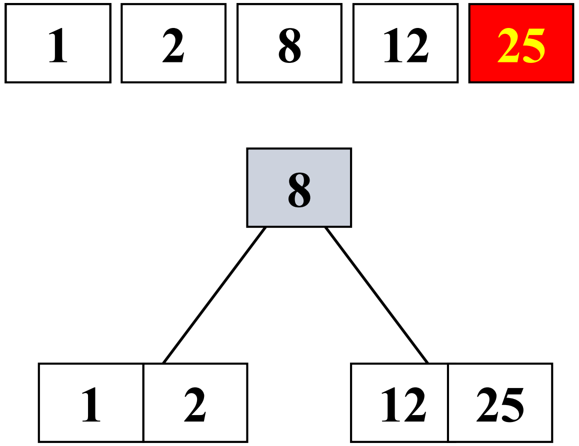
Inserting into a B-tree – Example

- **1 12 8 2**

1	2	8	12

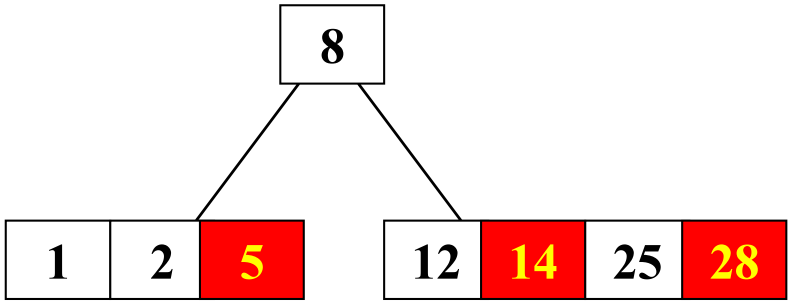
Inserting into a B-tree – Example

- 1 12 8 2 **25**



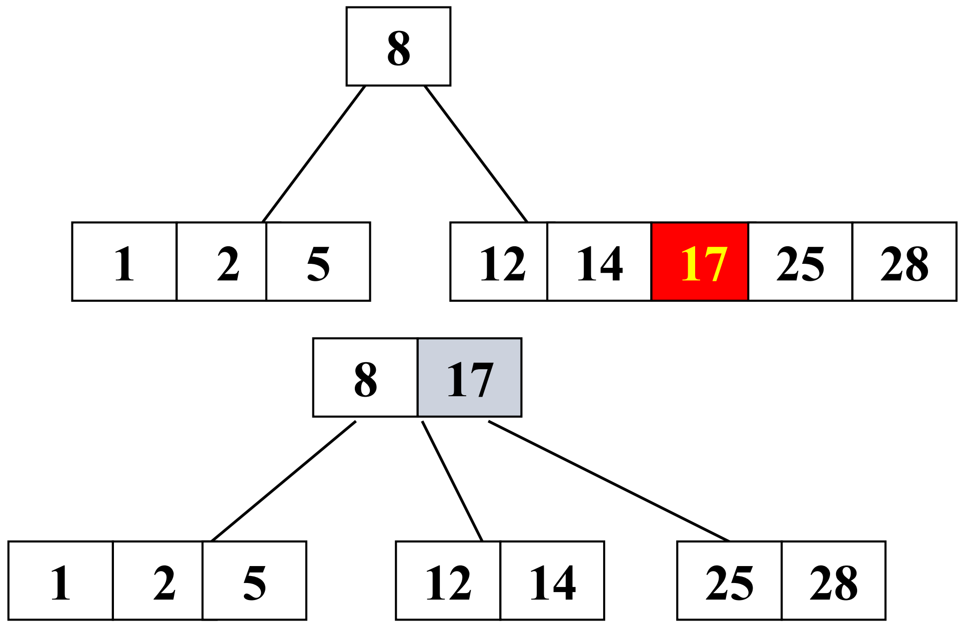
Inserting into a B-tree – Example

- 1 12 8 2 25 **5 14 28**



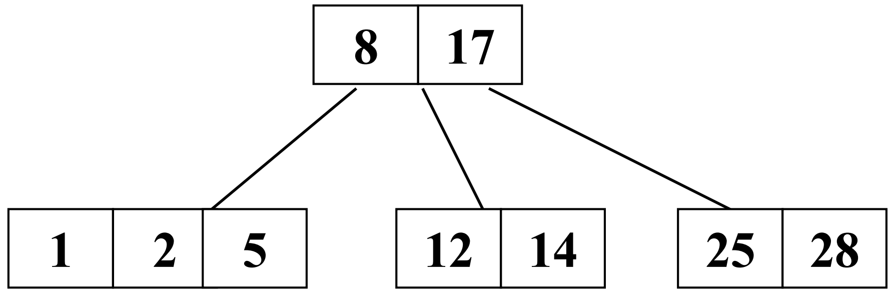
Inserting into a B-tree – Example

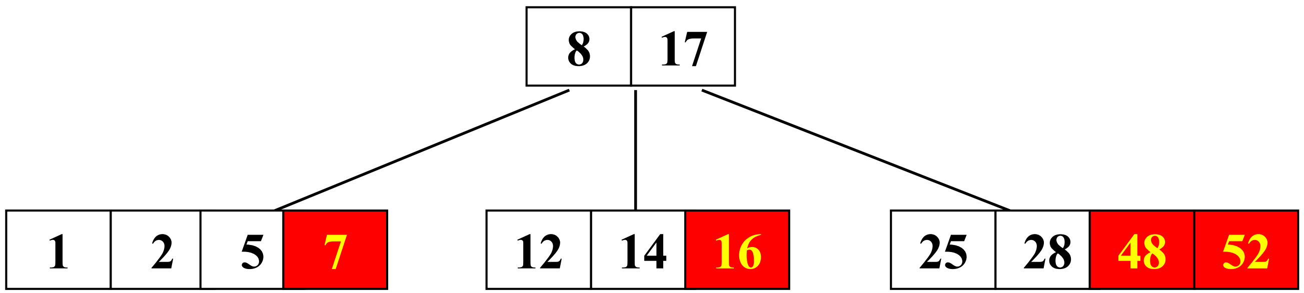
- 1 12 8 2 25 5 14 28 **17**



Inserting into a B-tree – Example

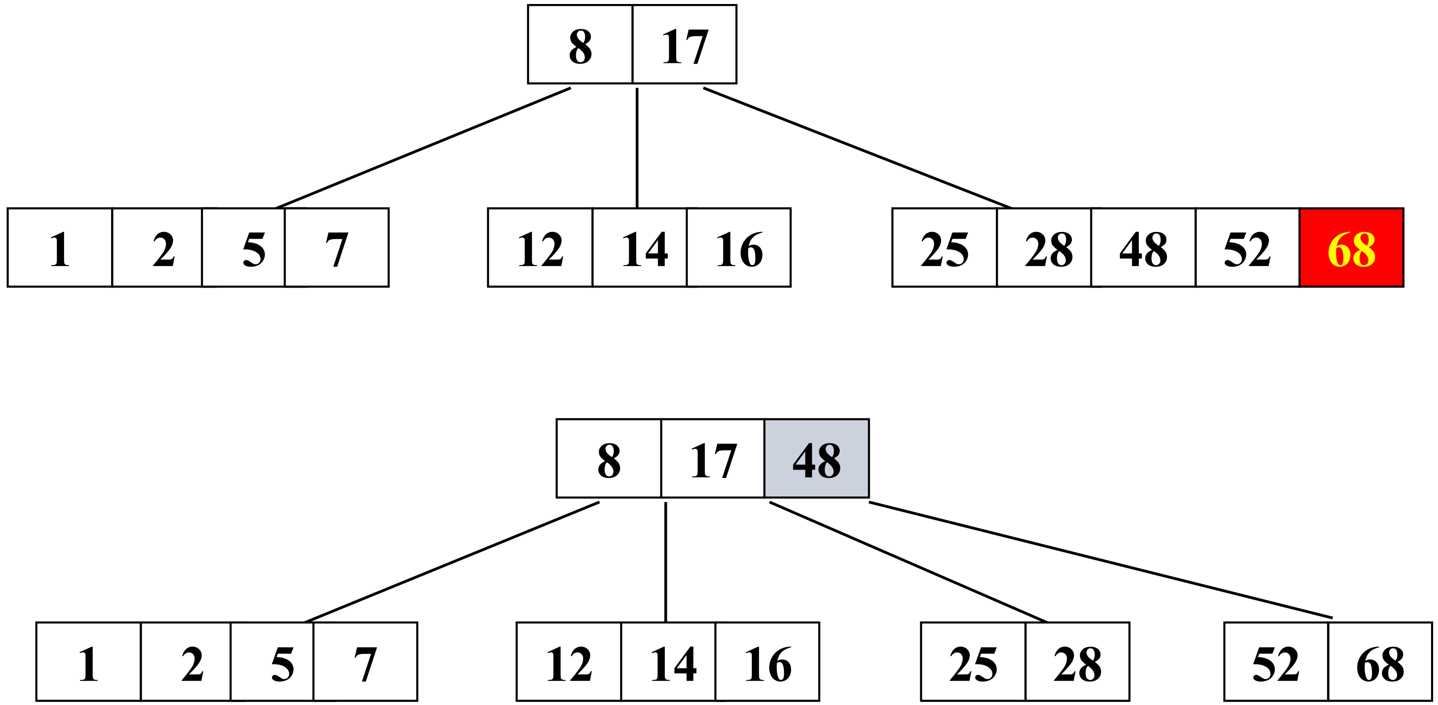
- 1 12 8 2 25 5 14 28 17 **7 52 16 48**





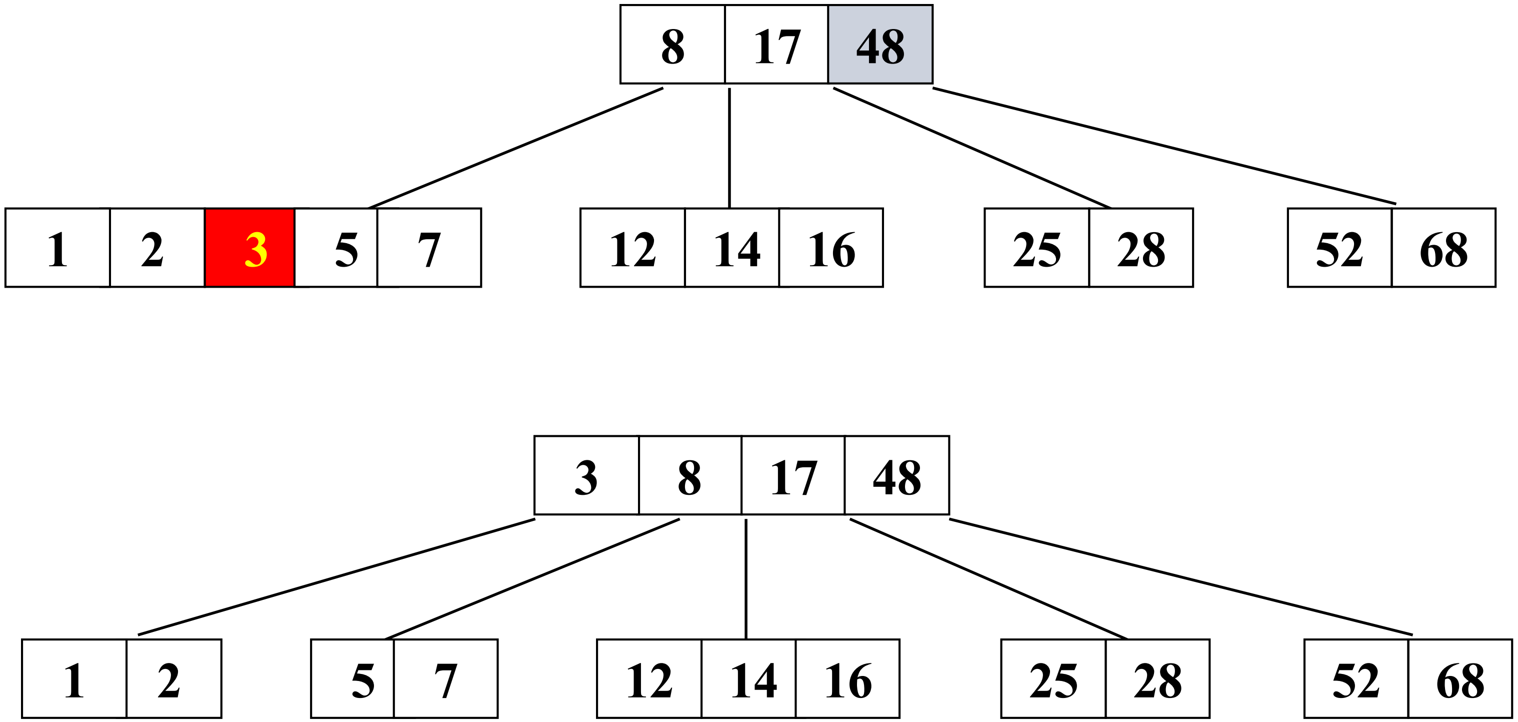
Inserting into a B-tree – Example

- 1 12 8 2 25 5 14 28 17 7 52 16 48 **68**



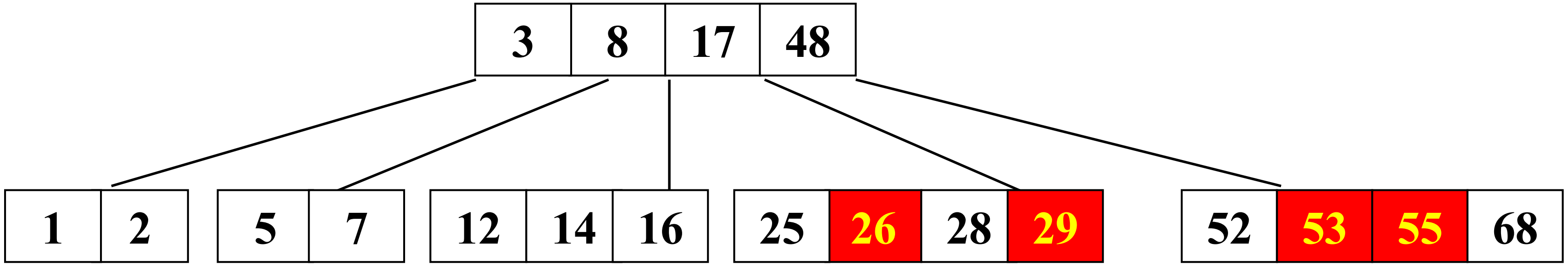
Inserting into a B-tree – Example

- 1 12 8 2 25 5 14 28 17 7 52 16 48 68 **3**



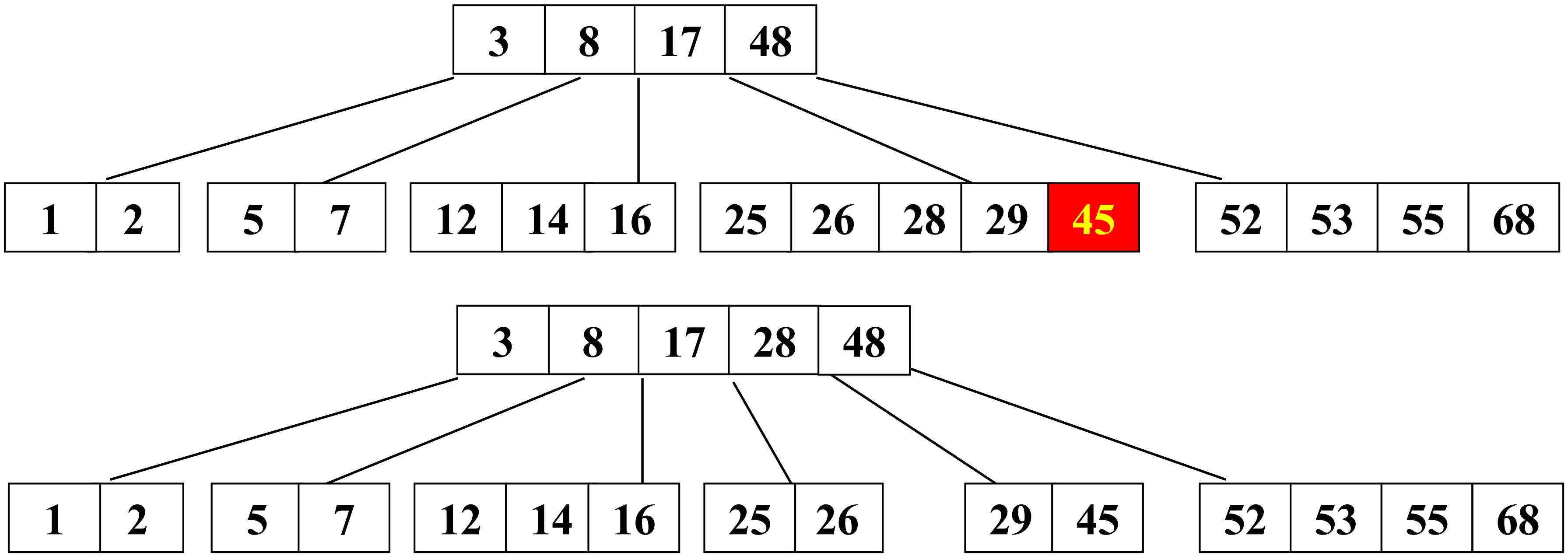
Inserting into a B-tree – Example

- 1 12 8 2 25 5 14 28 17 7 52 16 48 68 3 **26 29 53 55**



Inserting into a B-tree – Example

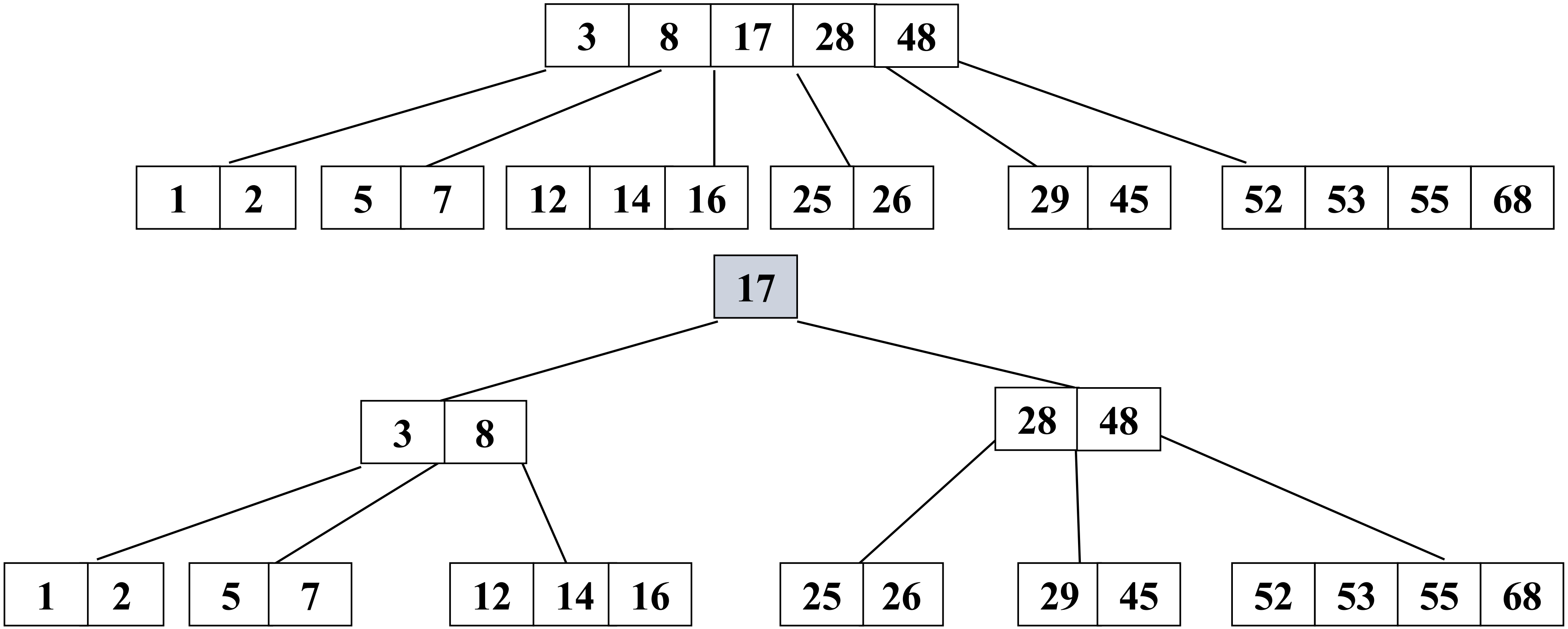
- 1 12 8 2 25 5 14 28 17 7 52 16 48 68 3 26 29 53 55 **45**



Insert **45**

Inserting into a B-tree – Example

- 1 12 8 2 25 5 14 28 17 7 52 16 48 68 3 26 29 53 55 **45**



Removal from a B-tree

- Remove a key *k*

1. If *k* is in a leaf node, and removing it doesn't cause that leaf node to have too few keys, then simply remove *k*.
2. If *k* is NOT in a leaf then it is guaranteed (by the nature of a B-tree) that its predecessor or successor will be in a leaf -- in this case we can delete *k* and promote the predecessor or successor of *k* to *k*'s position.

Removal from a B-tree (2)

- (1) & (2) may lead to a leaf node *L* has less than min. number of keys

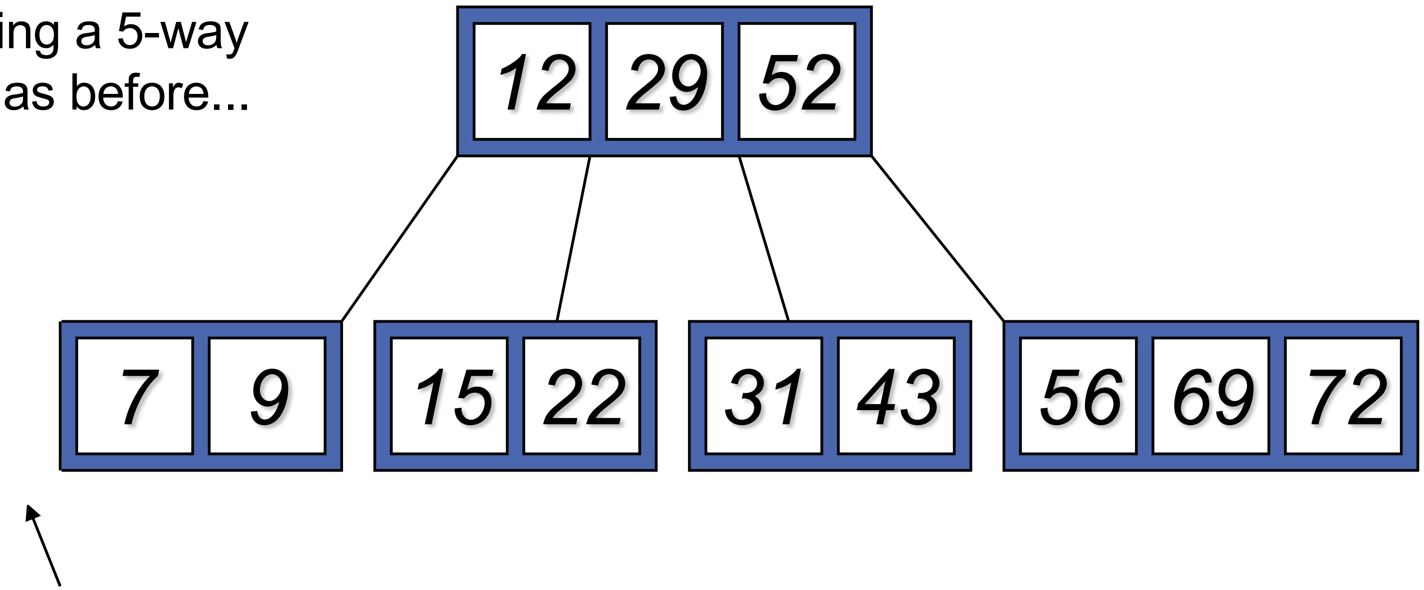
→ Look at the siblings immediately adjacent to the leaf

- 3: if one of them has more than the min. number of keys then we can promote one of its keys to the parent and take the parent key into *L*
- 4: otherwise, combine *L* and one of its neighbours with their shared parent, repeat the process up to the root, if required

$$\boxed{O(t \log_t n)}$$

Type #1: Simple leaf deletion

Assuming a 5-way B-tree, as before...



Delete 2: Since there are enough keys in the node, just delete it

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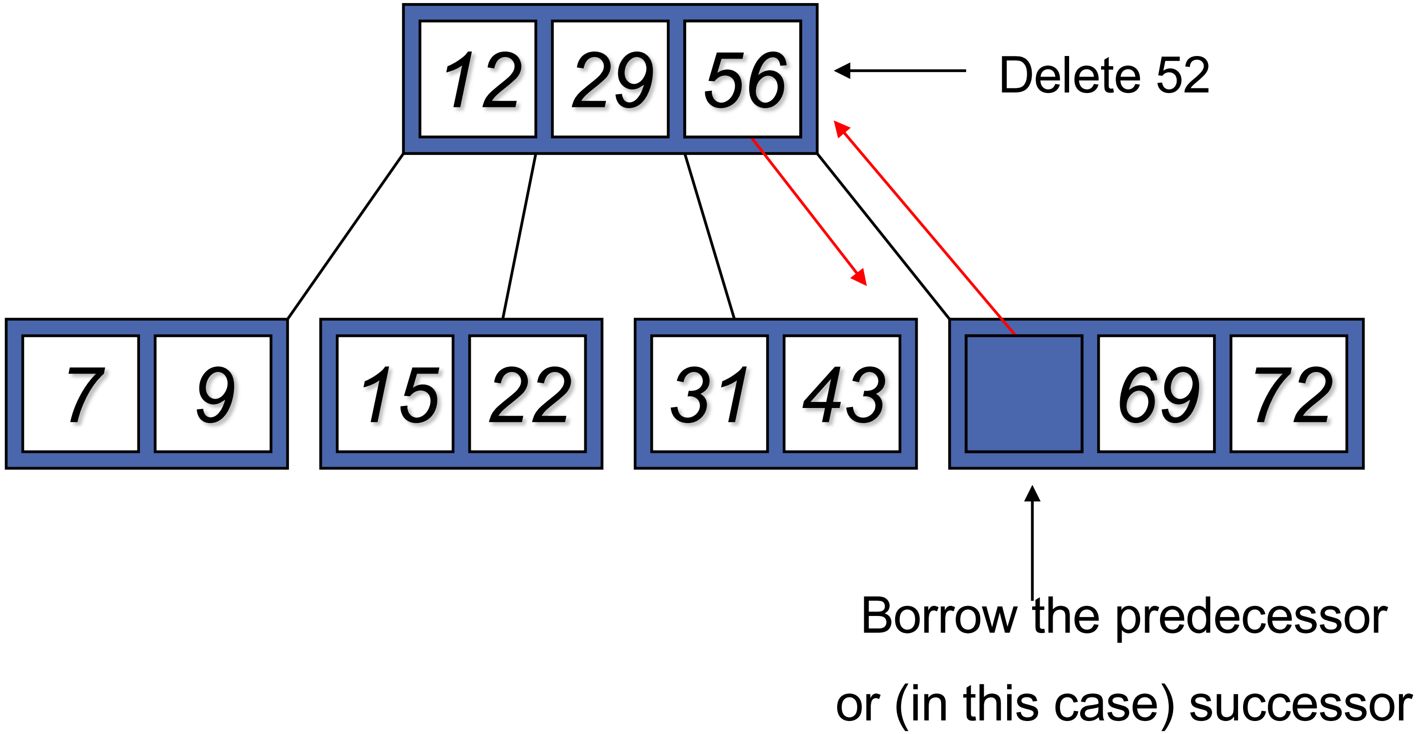
Type #2: Simple non-leaf deletion



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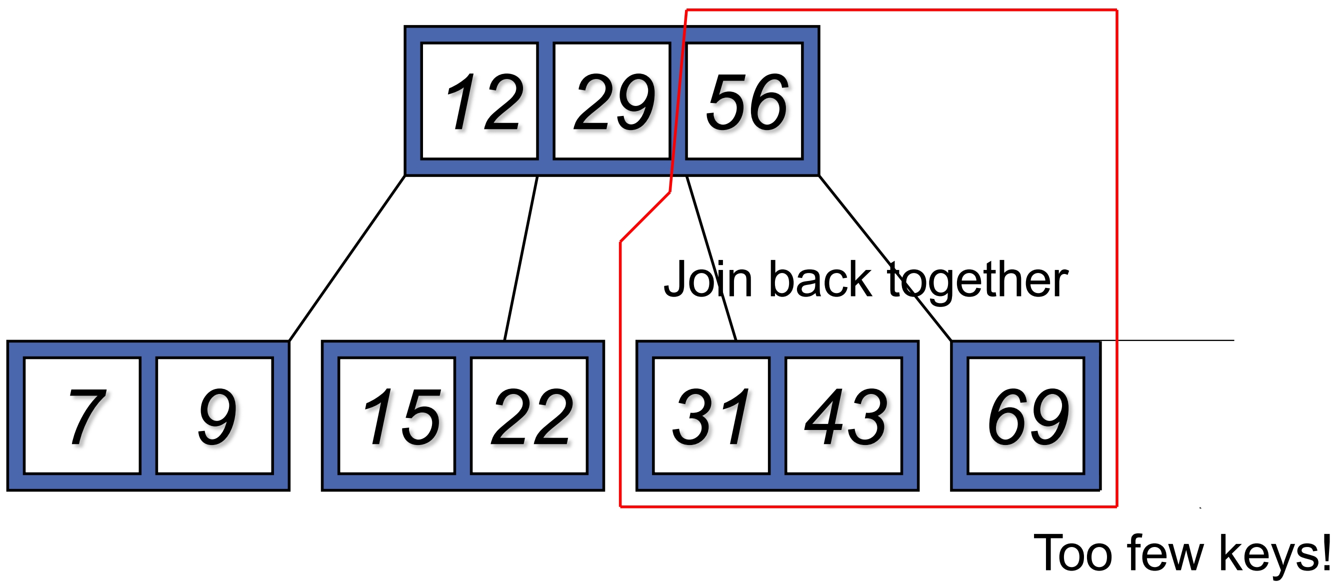
Type #4: Too few keys in node and its siblings



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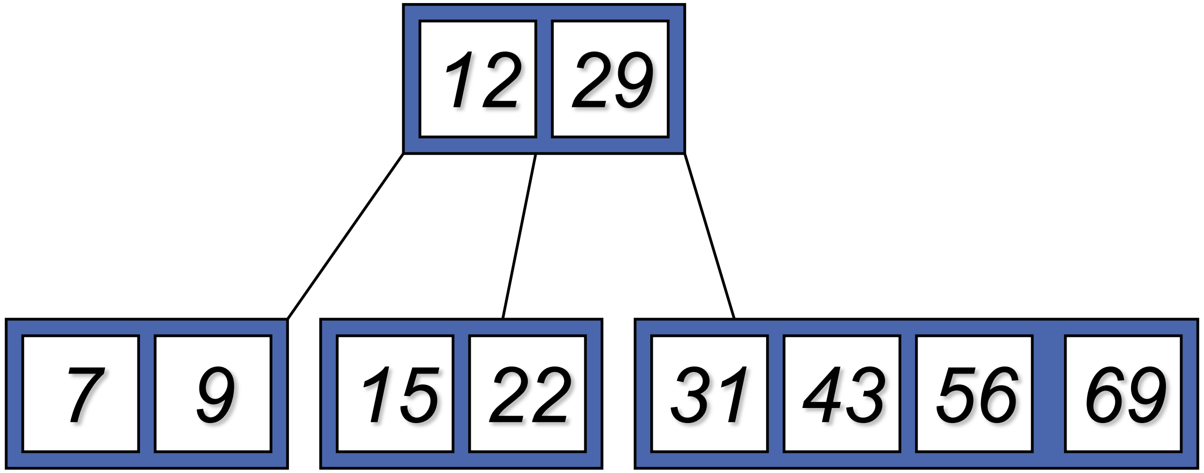
Type #4: Too few keys in node and its siblings



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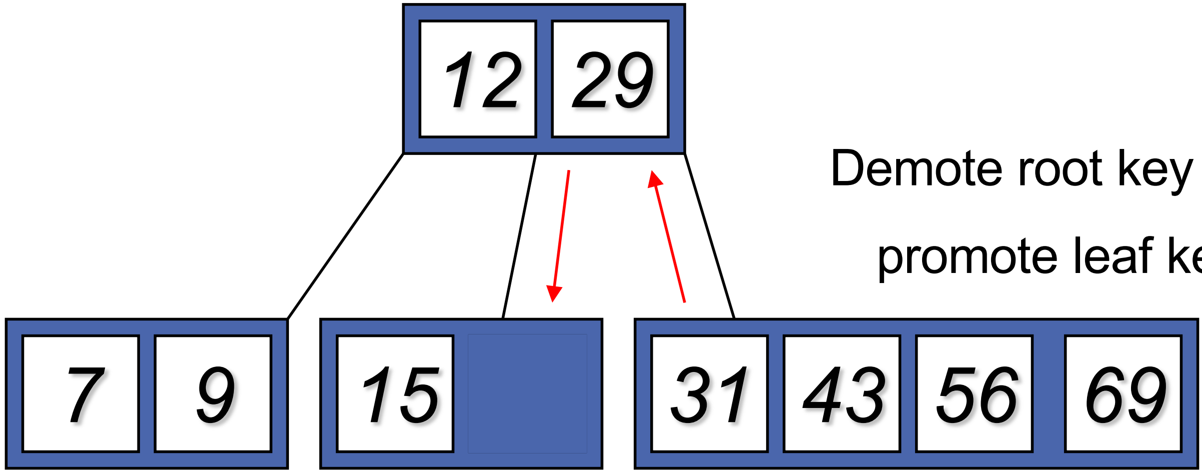
Type #3: Enough siblings



Demote root key and

promote leaf key

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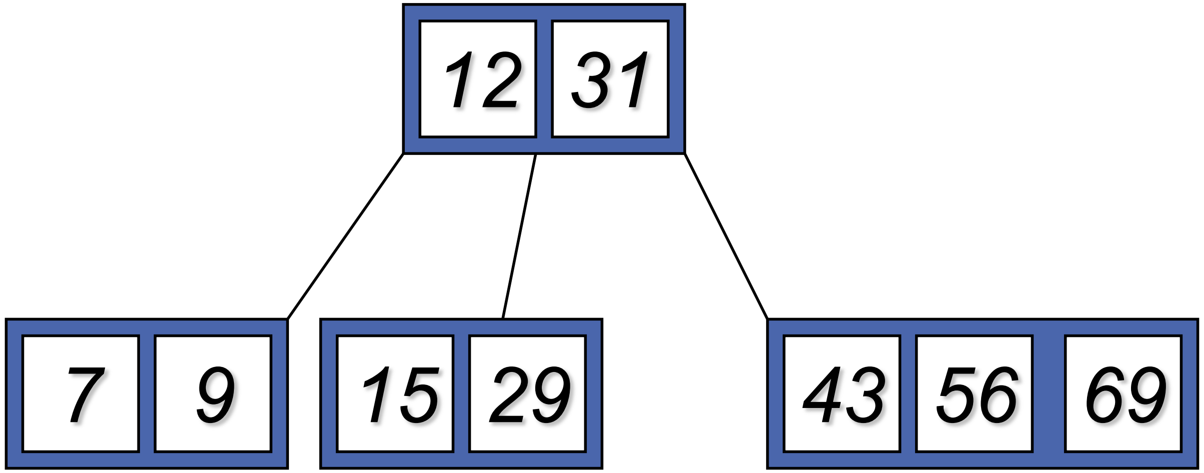
Type #3: Enough siblings



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Exercise in B-tree

- Given 5-way B-tree created by these data:

3, 7, 9, 23, 45, 1, 5, 14, 25, 24, 13, 11, 8, 19, 4, 31, 35, 56

- Add these further keys: 2, 6,12
- Delete these keys: 4, 5, 7, 3, 14

Reasons for using B-trees

- When searching tables held on disc, the cost of each disc transfer is high but doesn't depend much on the amount of data transferred, especially if consecutive items are transferred
  - If we use a B-tree of order 101, say, we can transfer each node in one disc read operation
  - A B-tree of order 101 and height 3 can hold $101^4 – 1$ items (approximately 100 million) and any item can be accessed with 3 disc reads (assuming we hold the root in memory)
  - B-trees are always balanced (since the leaves are all at the same level) → smallest height is guarantee

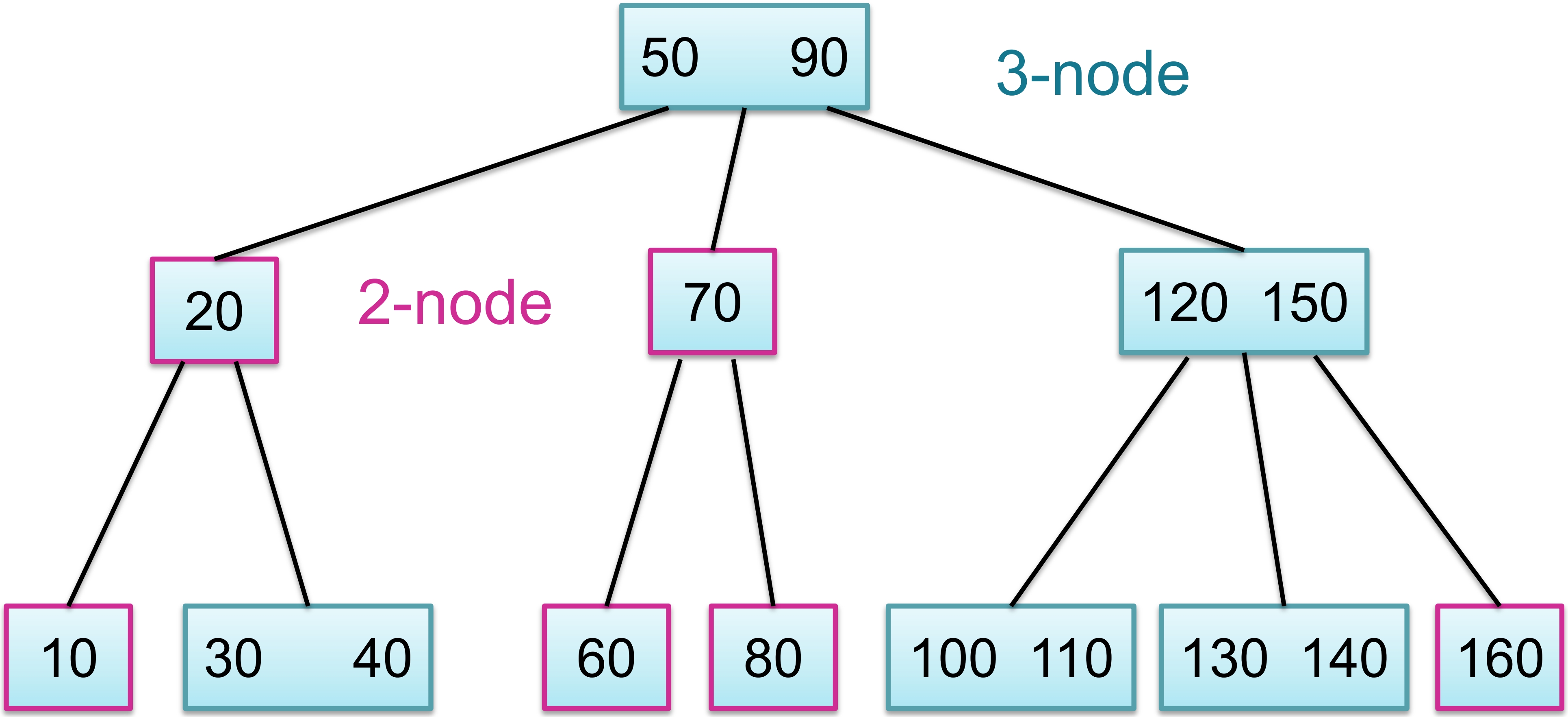
2-3 TREES

- Definition
- Traversal & Searching
- Insert & Delete
- Why 2-3 tree?

Definition of 2-3 tree

- For B-tree, if we take *m* = 3, we get a **2-3 tree**, in which non-leaf nodes have two or three children (i.e., one or two keys)
- Definition: A **2-3 tree** is a B-tree in which:
  - Each internal node has either **2** or **3** children.
  - All leaves are at the same level (leaf contains 1 or 2 keys)
  - A node with 1 key is called a 2-node (it has 2 children if it is an internal node).
  - A node with 2 keys is called a 3-node (it has 3 children if it is an internal node).

2-3 tree – Example



Height of a 2-3 tree

- A 2-3 tree is **not** a binary tree.
  - If a 2-3 tree contains only 2 nodes, it is likely a perfect binary tree → $n = 2^{h+1} - 1$
  - If, on the other hand, some of the internal nodes of a 2-3 tree do have 3 children, the tree will contain more nodes than a perfect binary tree of the same height → $n > 2^{h+1} - 1$
- Therefore, height of a 2-3 tree is:

$$h \leq \lceil \log_2(n+1) - 1 \rceil$$

$$\boxed{O(\log_2 n)}$$

Traversing a 2-3 tree

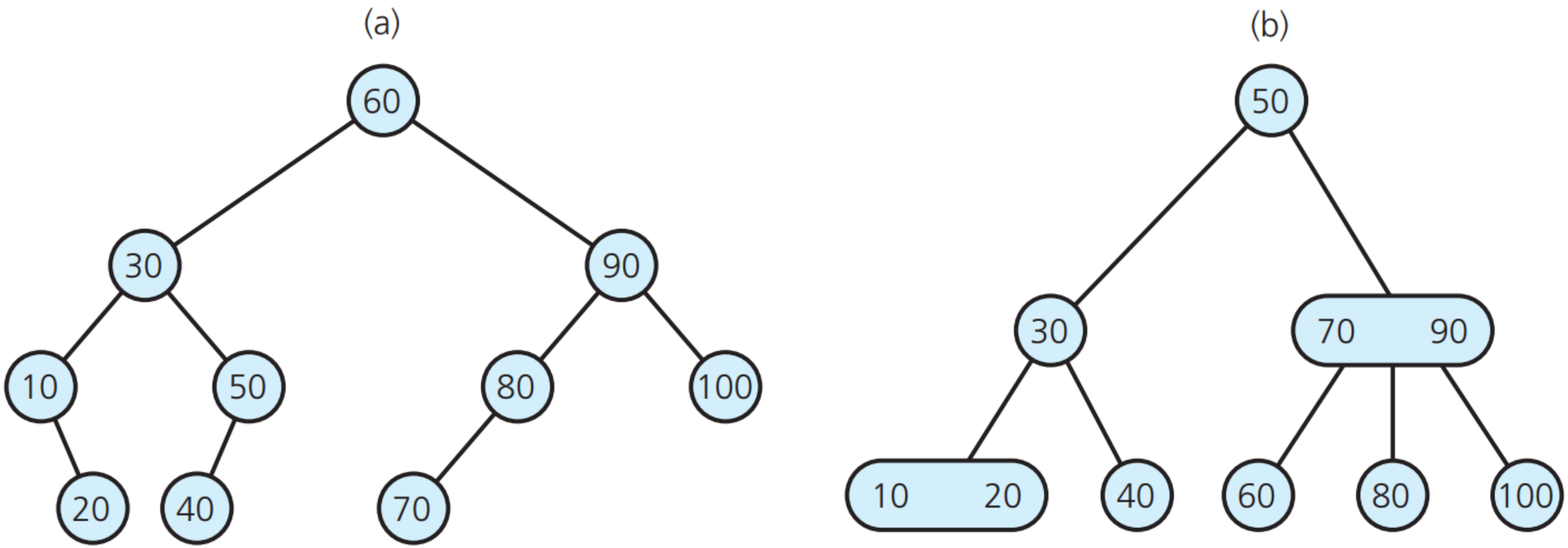
- You can traverse a 2-3 tree in sorted order by performing the analogue of an in-order traversal on a binary tree

Searching a 2-3 tree

- The ordering of keys in a 2-3 tree is analogous to the ordering for a BST
  - → Searching on a 2-3 tree is efficient & quite similar to a BST
- The same efficiency:
  - A BST with n nodes cannot be shorter than $\lceil \log_2(n+1) - 1 \rceil$
  - A 2-3 tree with n nodes cannot be taller than $\lceil \log_2(n+1) - 1 \rceil$
  - A node in a 2-3 tree has at most 2 keys.
- *Then, why should we use 2-3 tree?*

Inserting to a 2-3 tree

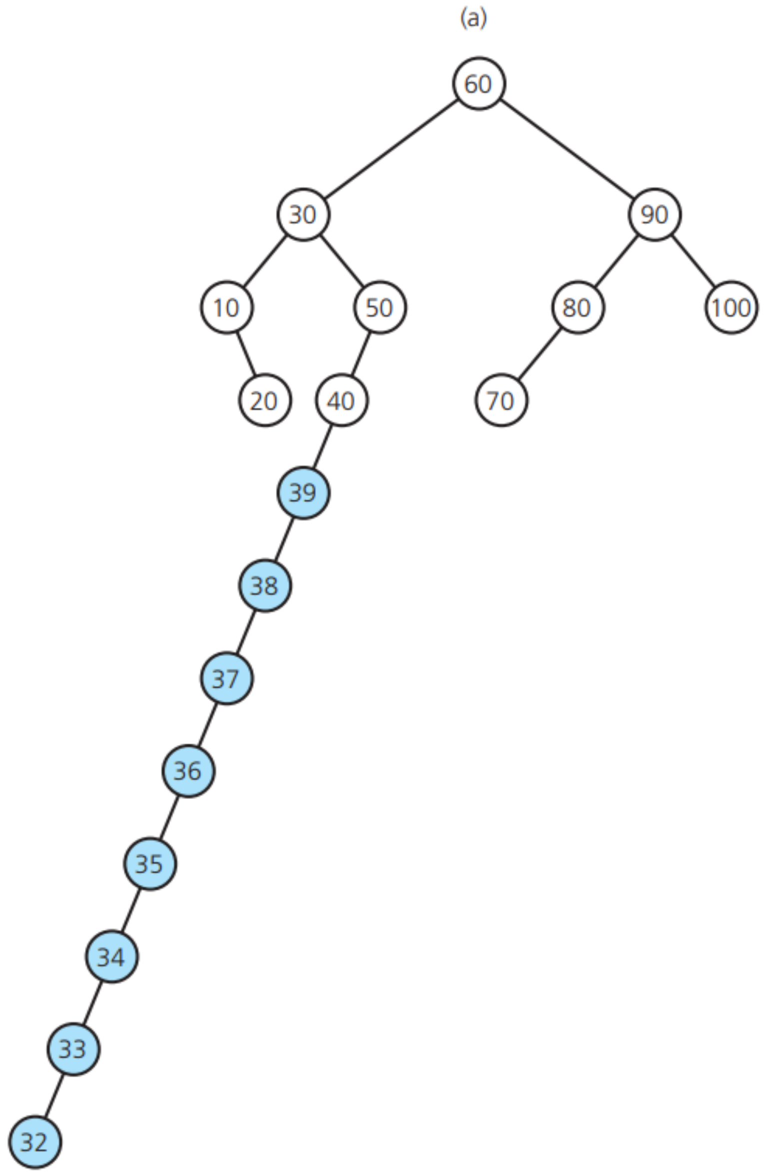
- A balanced BST and a 2-3 tree with the same keys



- Now, insert 39, 38, …, 32 to these trees

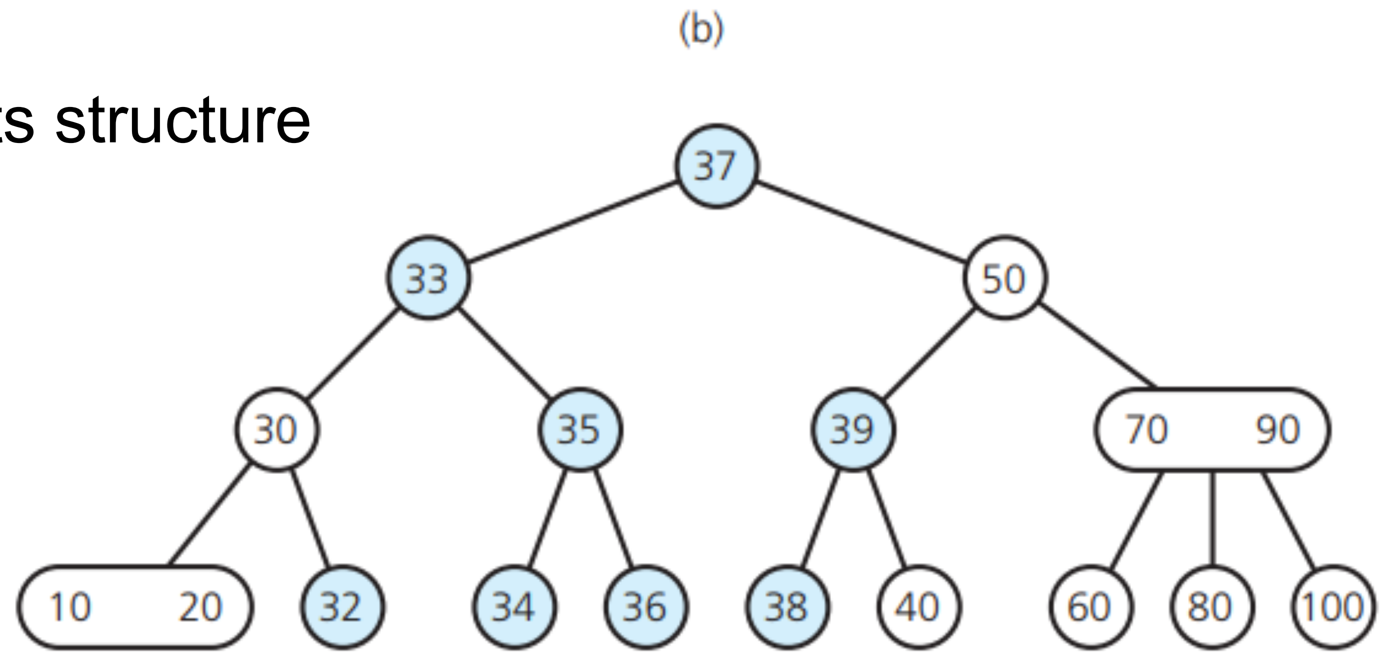
Inserting to a 2-3 tree

- Now, insert 39, 38, …, 32 to these trees
- Binary Search Tree
  - Quickly loose its balance



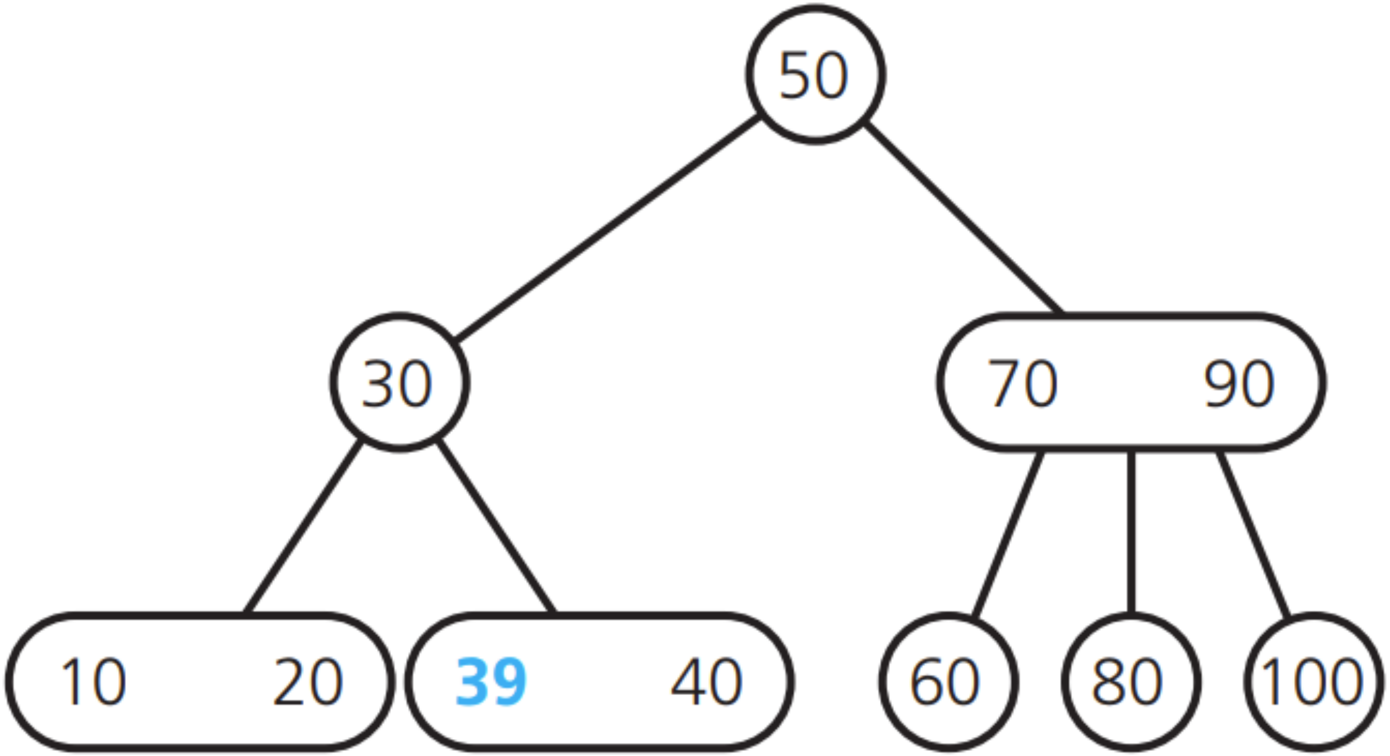
Inserting to a 2-3 tree

- Now, insert 39, 38, …, 32 to these trees
- 2-3 tree:
  - Retains its structure



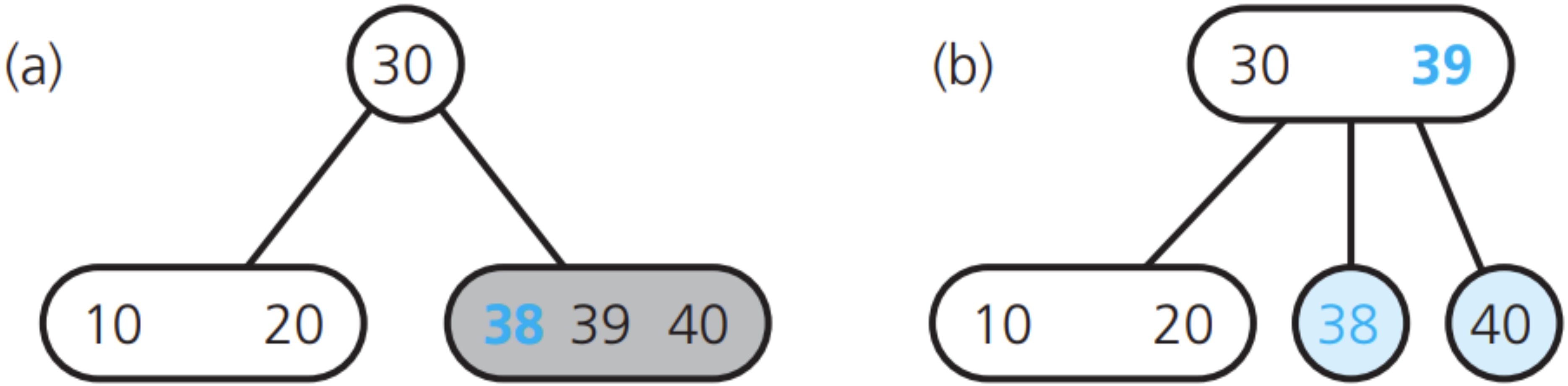
Inserting to a 2-3 tree

- Insert 39



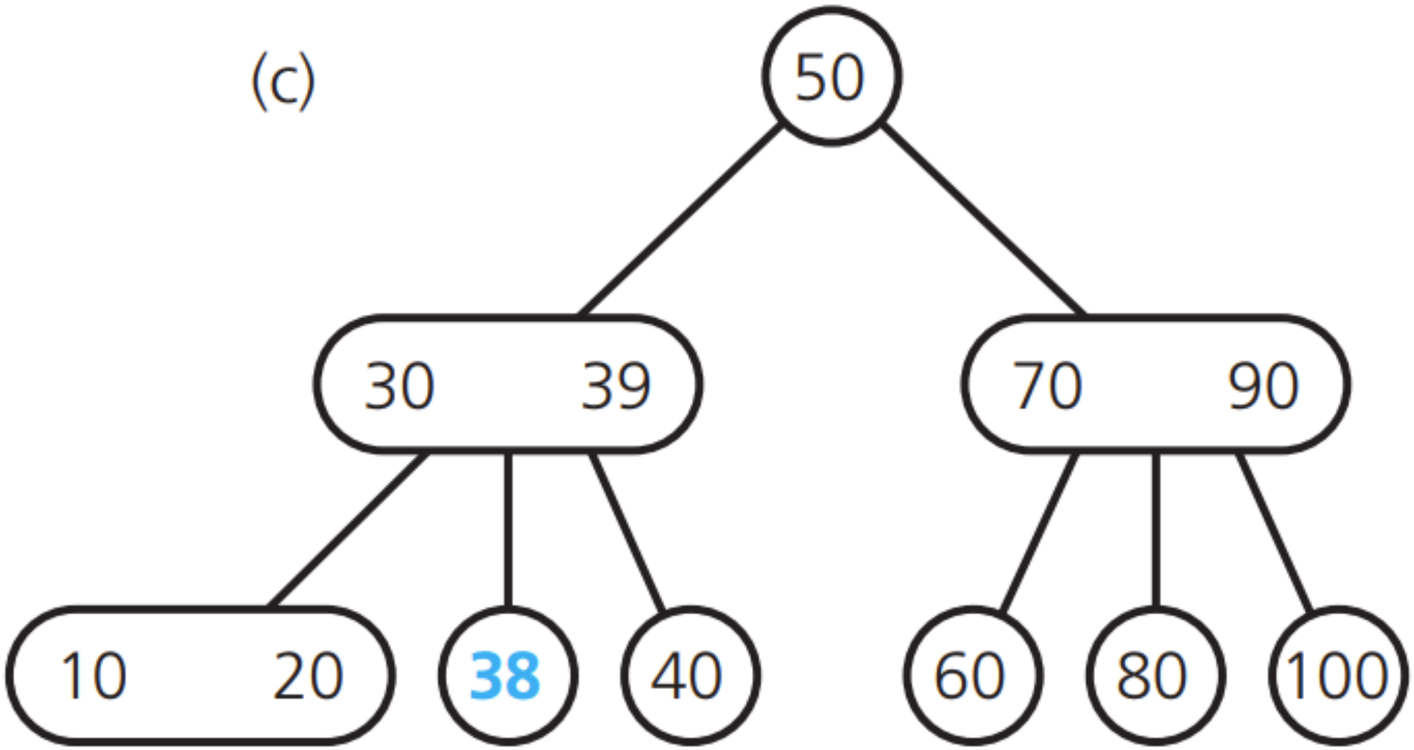
Inserting to a 2-3 tree

- Insert 38



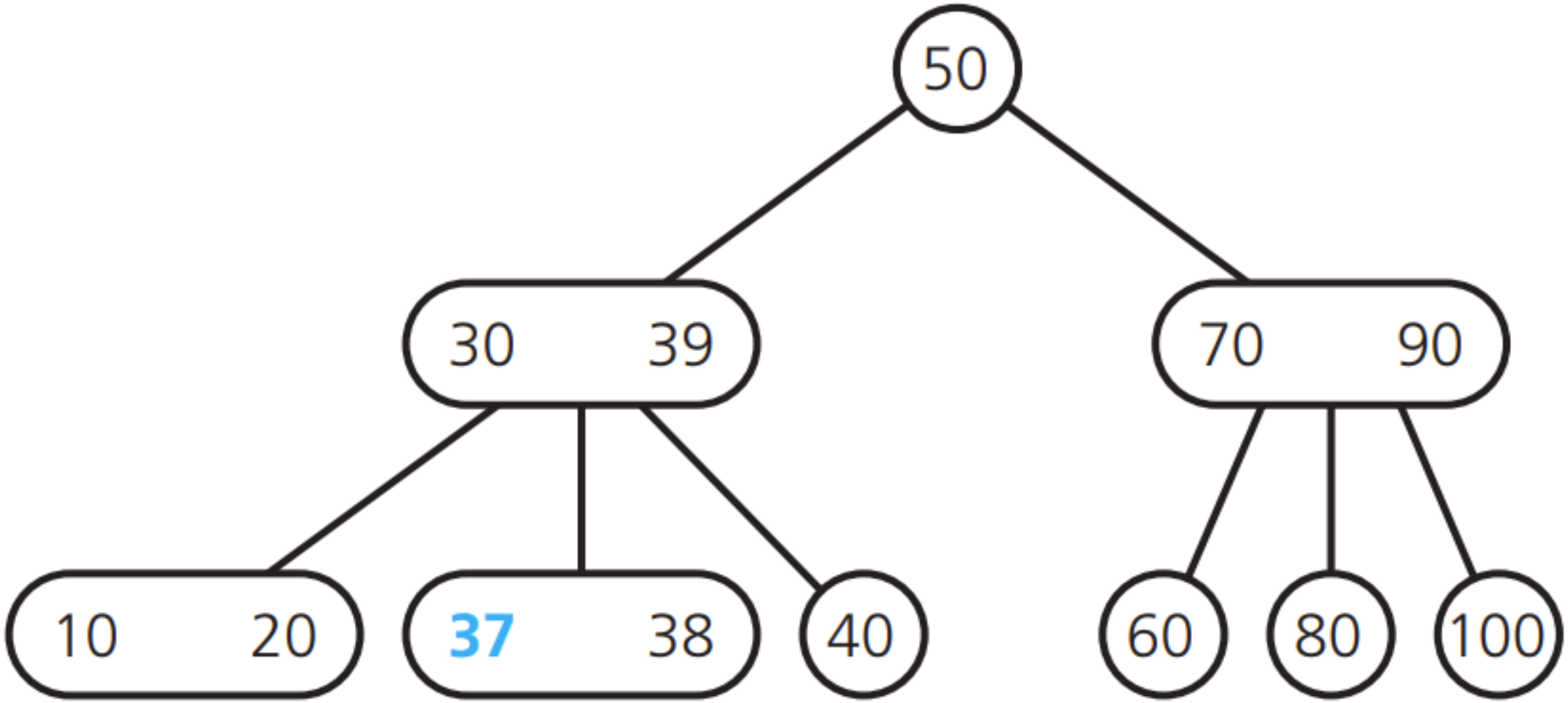
Inserting to a 2-3 tree

- Insert 38



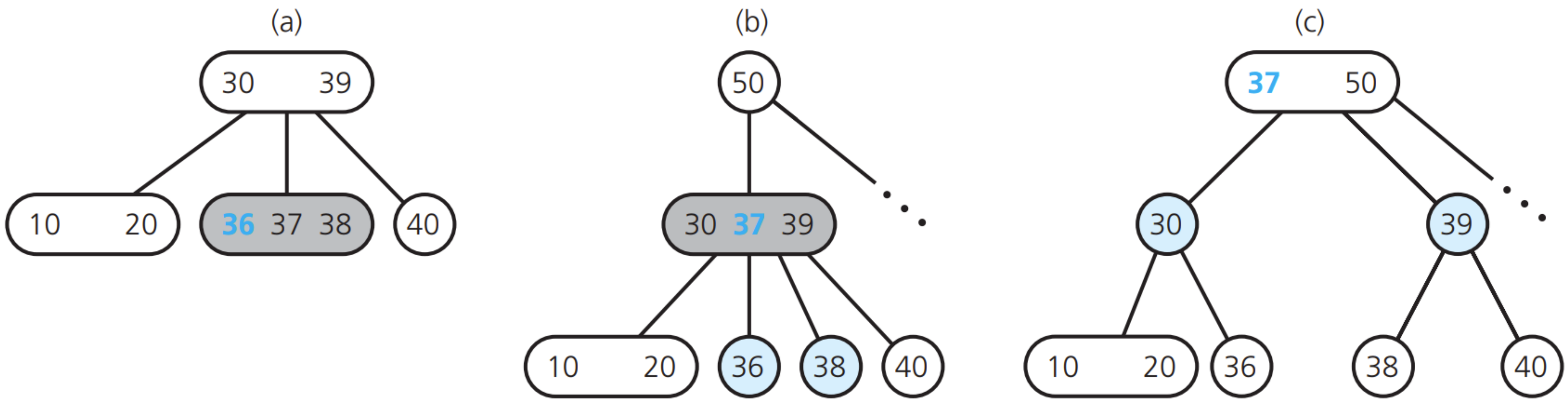
Inserting to a 2-3 tree

- Insert 37



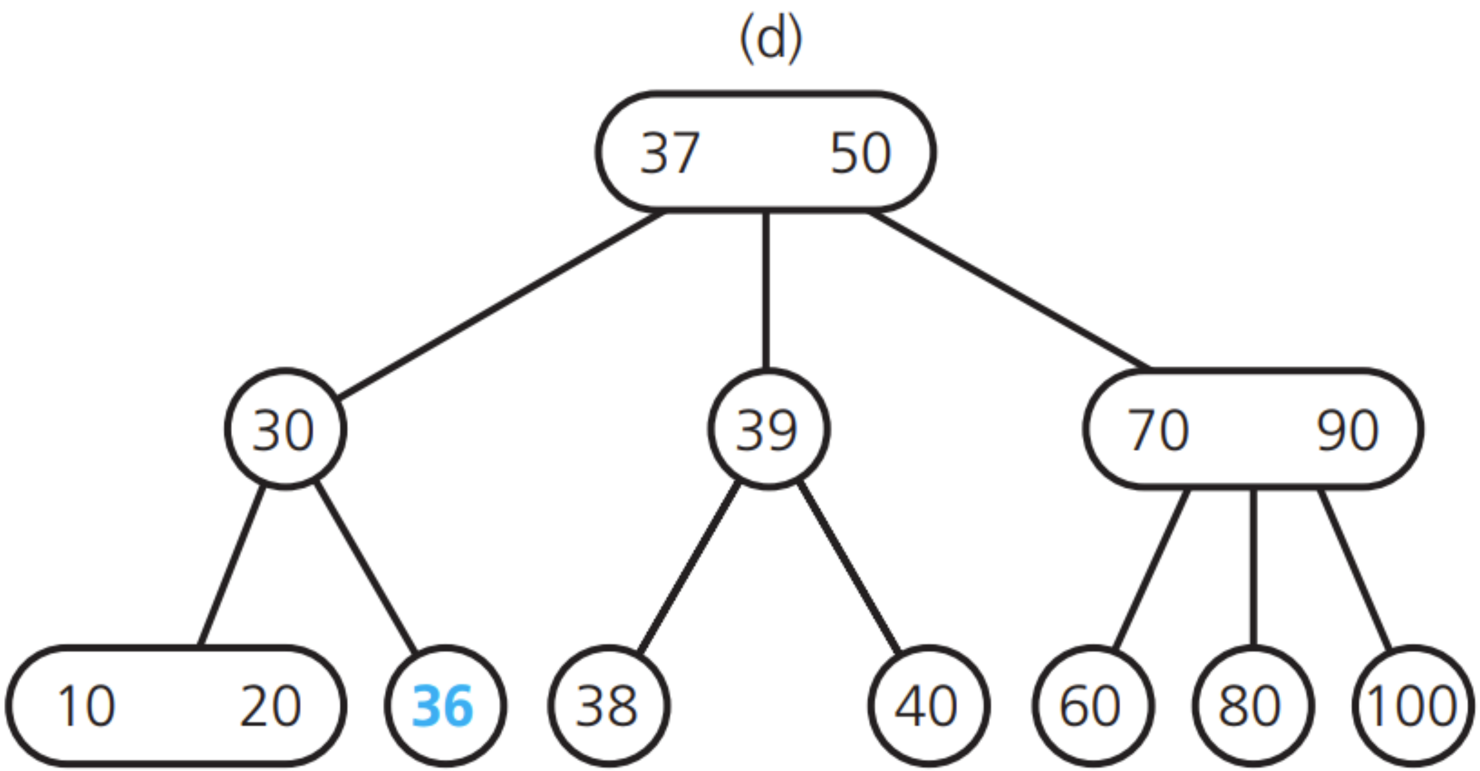
Inserting to a 2-3 tree

- Insert 36



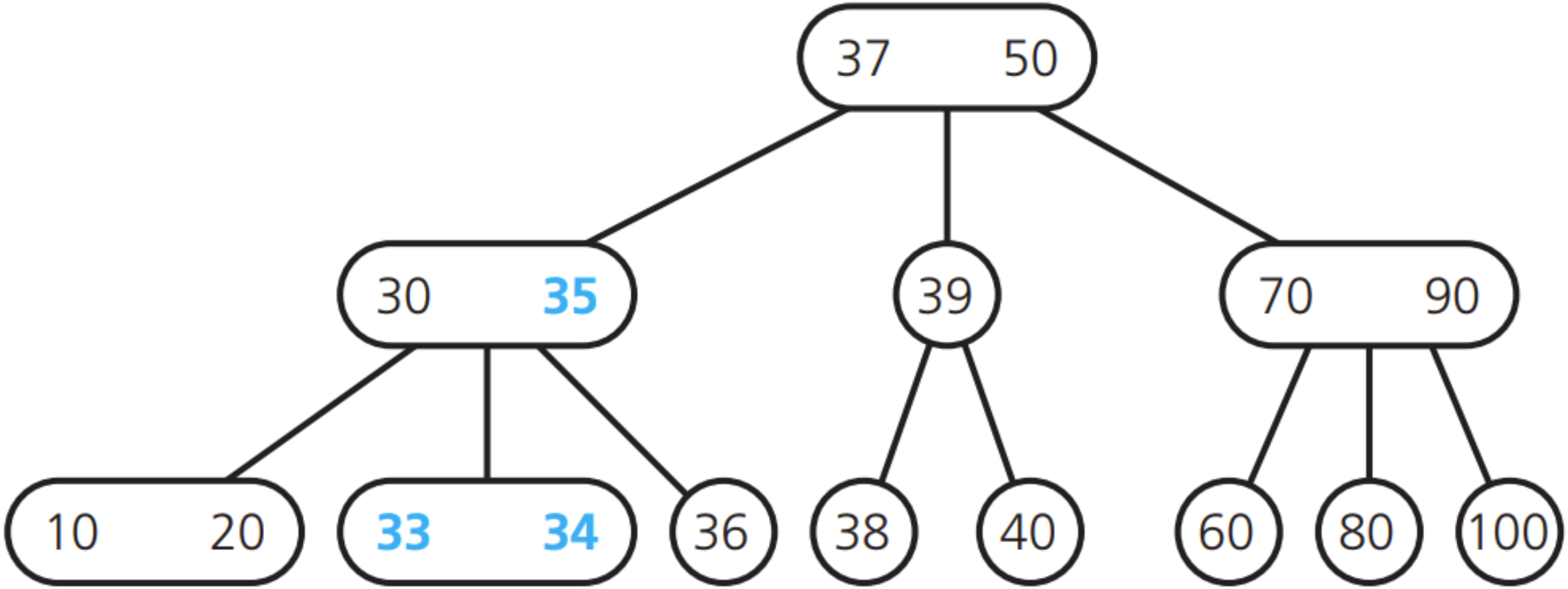
Inserting to a 2-3 tree

- Insert 36



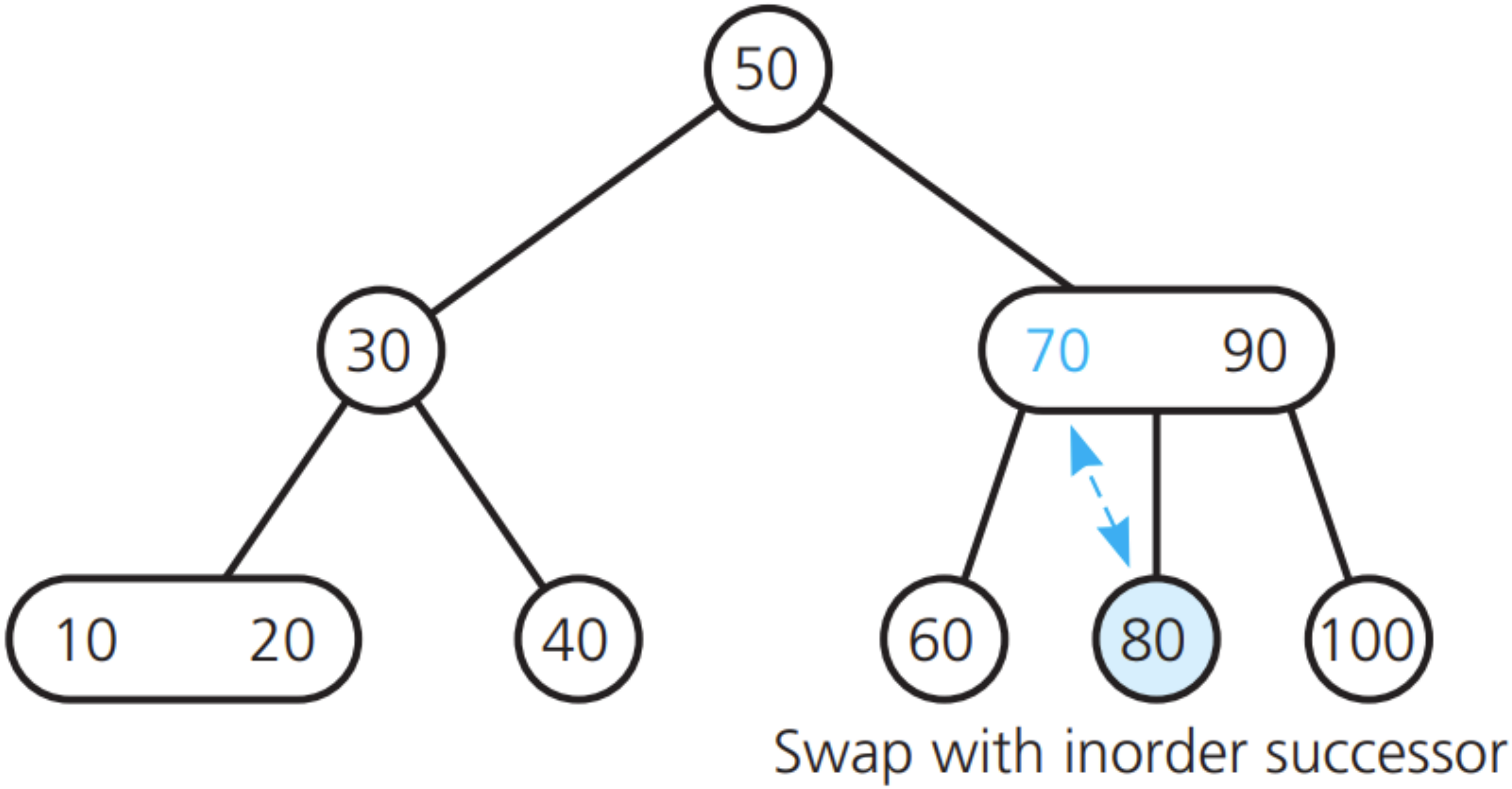
Inserting to a 2-3 tree

- Insert 35, 34, 33



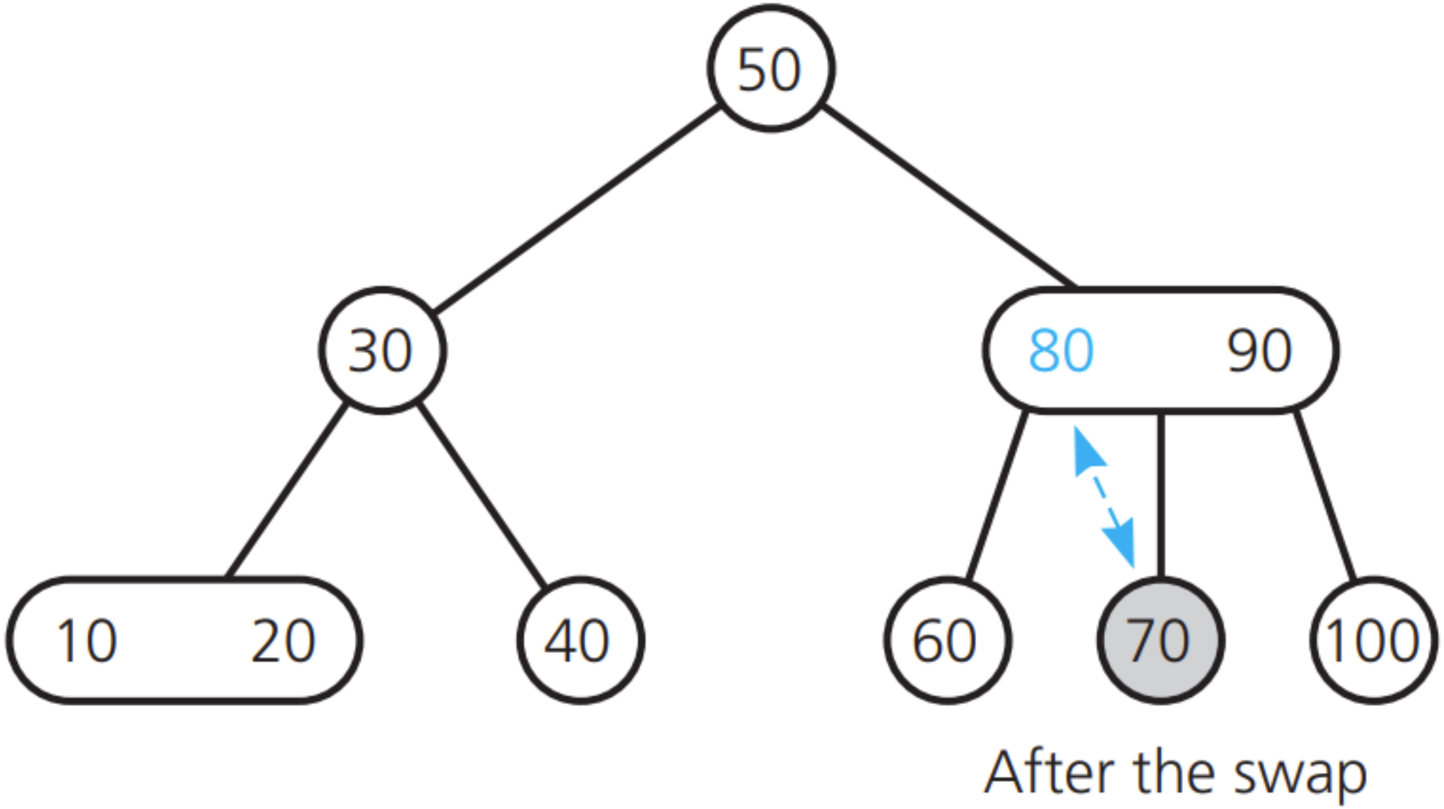
Deleting from a 2-3 tree

- Delete 70:



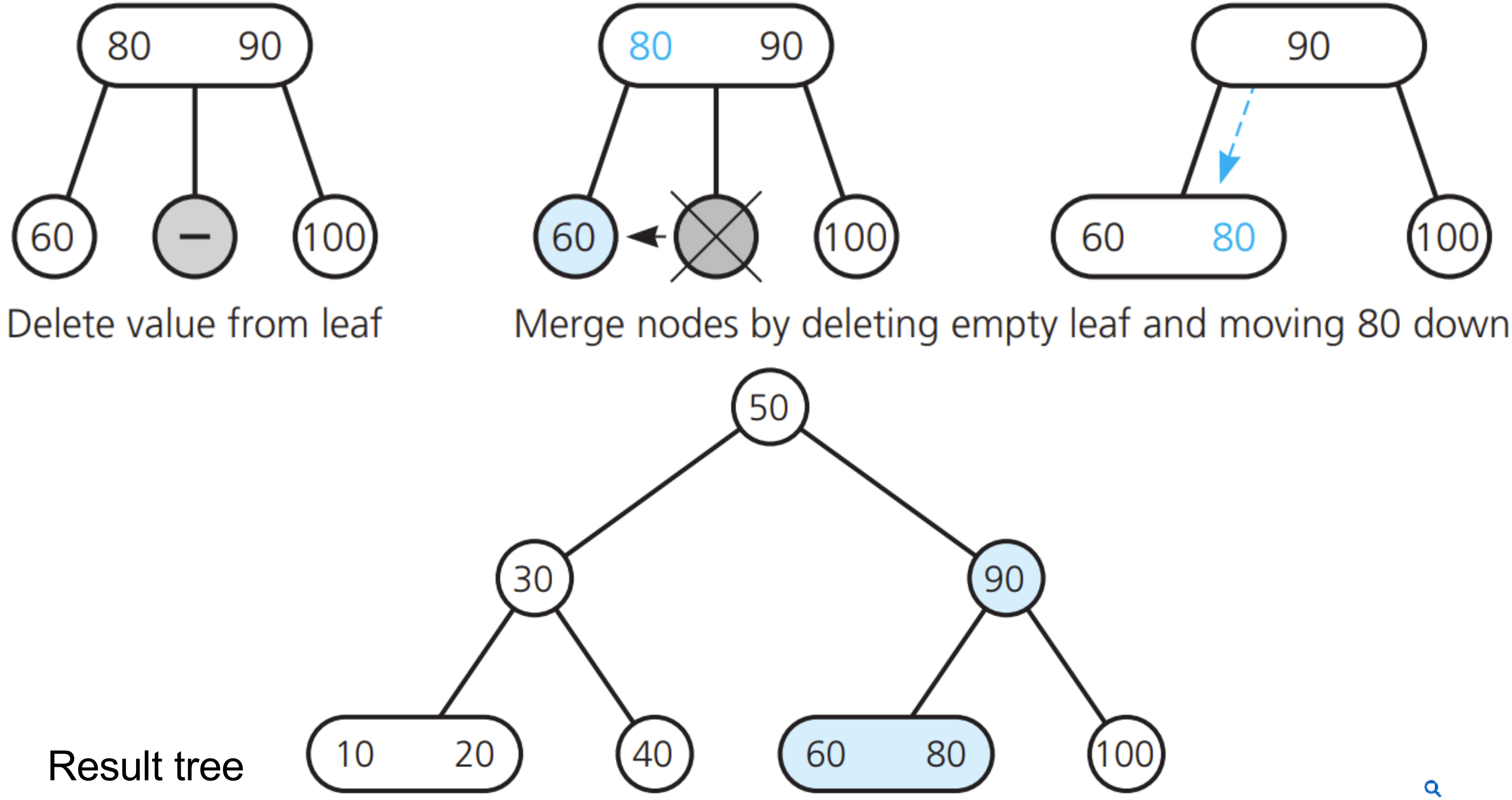
Deleting from a 2-3 tree

- Delete 70:



Deleting from a 2-3 tree

- Delete 70:

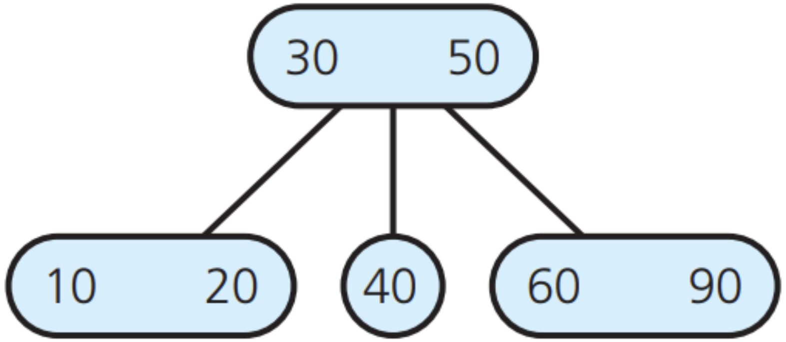


Deleting from a 2-3 tree

- Delete 70, 100, 80:

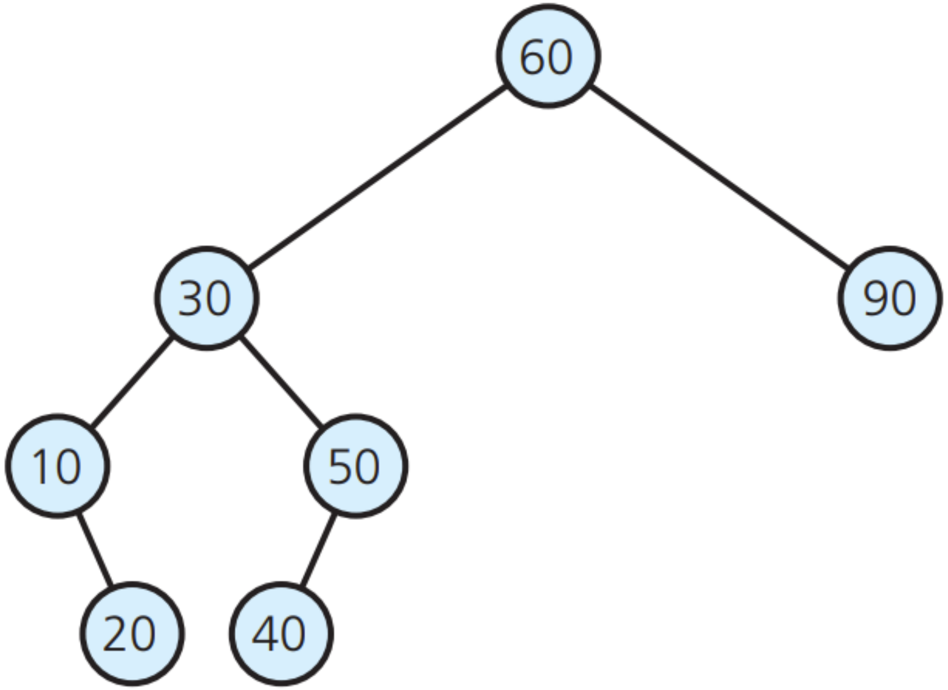
B-tree

(a)



Binary Search Tree

(b)



Again, Why 2-3 tree?

- Searching: sometimes you have to make 2 comparisons to get pass a 3-node.
  - However, it is still $O(\log n)$ running time
- Insertion/Deletion needs extra work: split nodes, merge nodes.
  - However, this extra work is not a real concern
  - It is easier to keep the tree balanced than a normal BST.

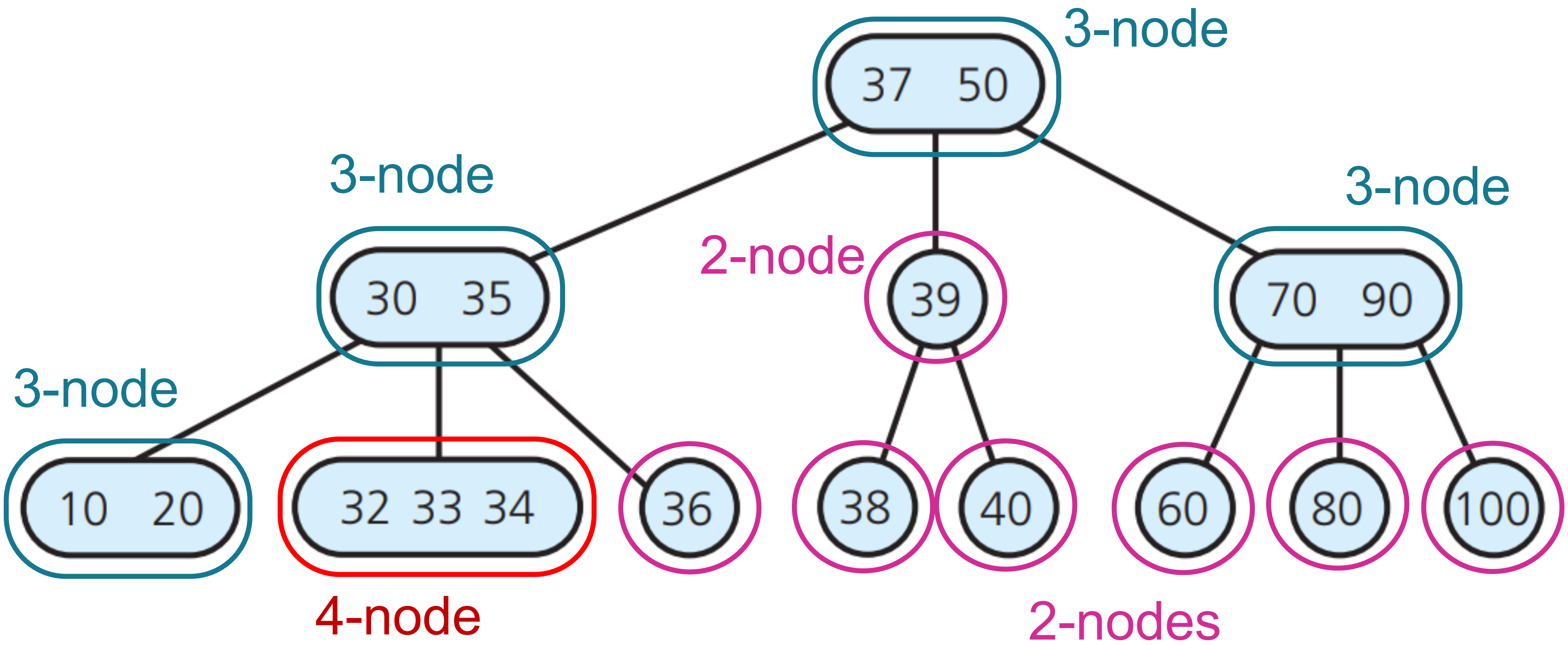
2-3-4 TREES

Definition of 2-3-4 tree

- If a 2-3 tree is good, are trees with more than 3 children better?
- Definition: A 2-3-4 tree is a B-tree in which:
  - Each internal node has either 2, 3 or 4 children.
  - All leaves are at the same level, each contains 1 to 3 keys
    - 2-node: has 2 children (contains 1 key)
    - 3-node: has 3 children (contains 2 key)
    - 4-node: has 4 children (contains 3 key)

Example of 2-3-4 tree

- 2-3-4 tree of the same data as the previous 2-3 tree



Searching and Traversing a 2-3-4 tree

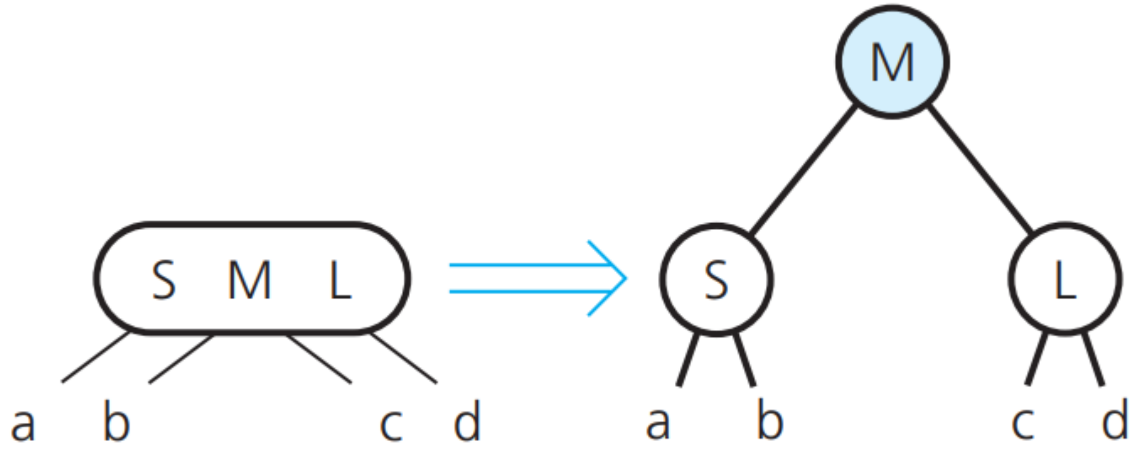
- Traversing & Searching are the same as in B-tree (or 2-3 tree)
- Example:
  - Search for 31 on the Example 2-3-4 tree:
    - Search the left subtree since 31 < 37
    - Search the middle subtree of the node <30 35> since 31> 30 and 31 < 35
    - Terminate the search at the left child pointer of <32 33 34> since 31 < 32
    - Result: 31 is not in the tree.

Inserting to 2-3-4 tree

- For inserting a key to a 2-3 tree, the search algorithm traces a path from the root to a leaf and then backs up from the leaf as it splits nodes.
- For a 2-3-4 tree: splits 4-node as soon as it encounters them on the way down from the root to the leaf. → *Avoid return path*
  - Each 4-node either:
    1. Be the root, or
    2. Have a 2-node parent, or
    3. Have a 3-node parent

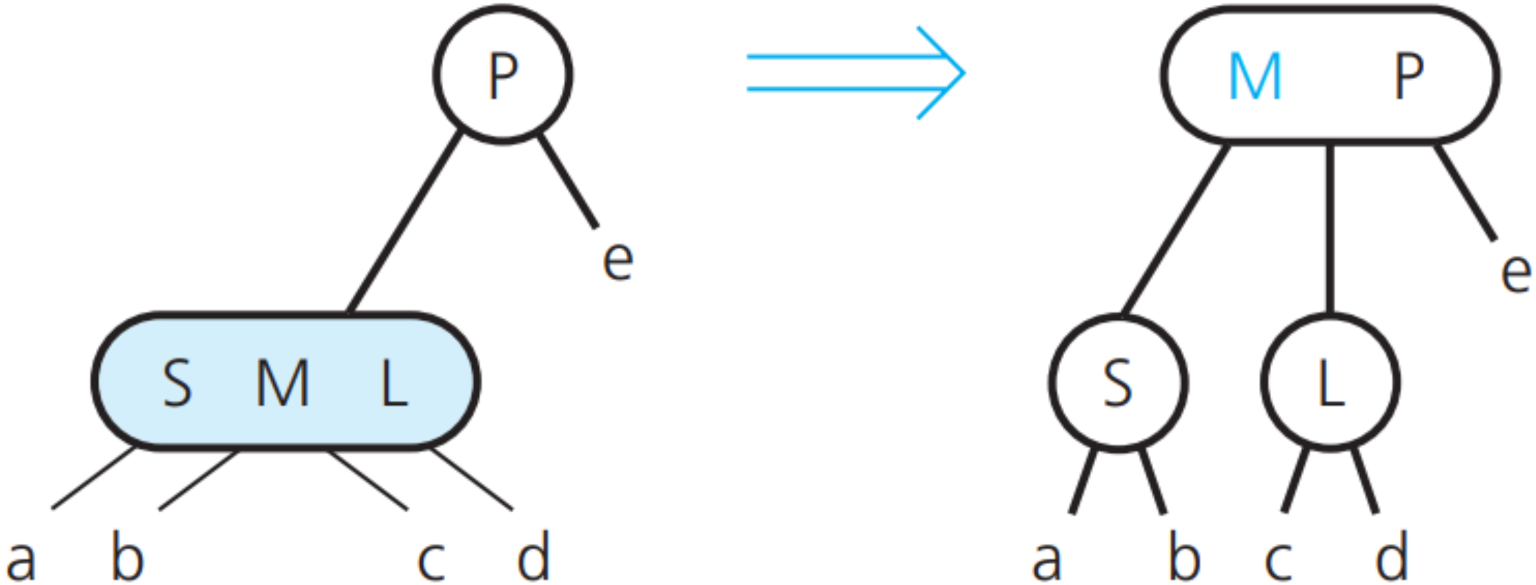
Splitting a 4-node in 2-3-4 tree

1. The 4-node is the root



Splitting a 4-node in 2-3-4 tree

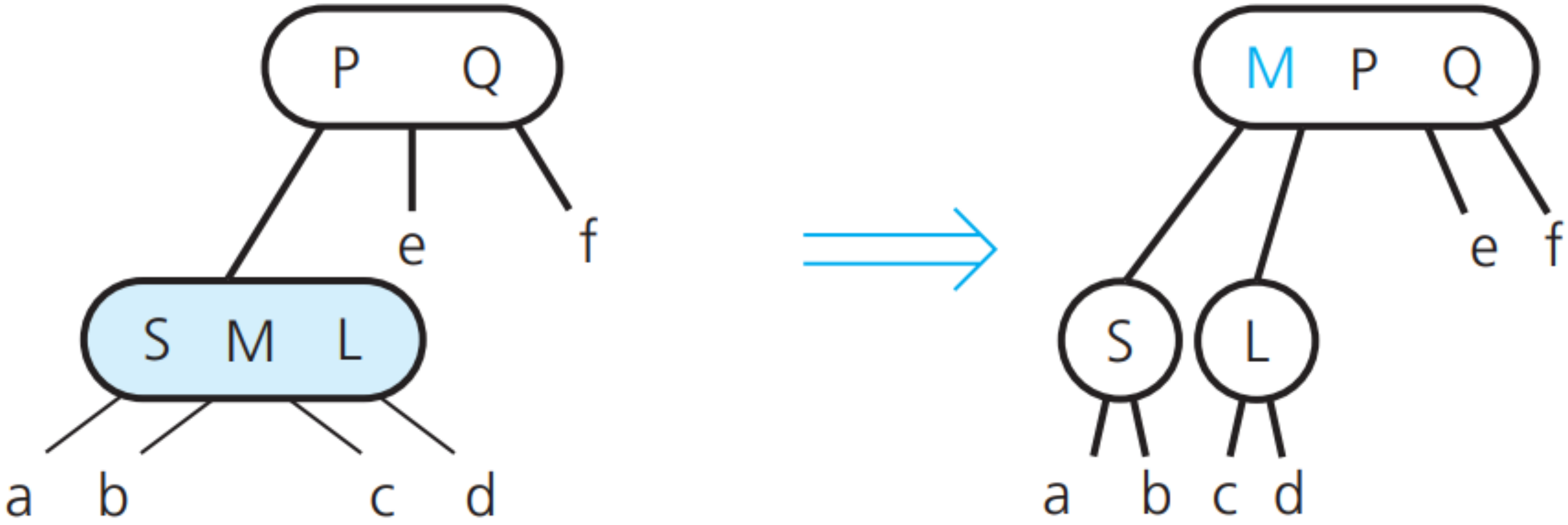
2. The 4-node has a 2-node parent:
   - 4-node is left child:



   - 4 node is right child?

Splitting a 4-node in 2-3-4 tree

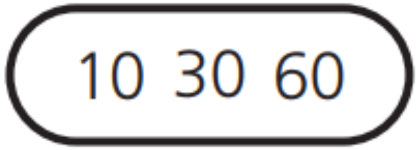
3. The 4-node has a 3-node parent
   - 4-node is left child:



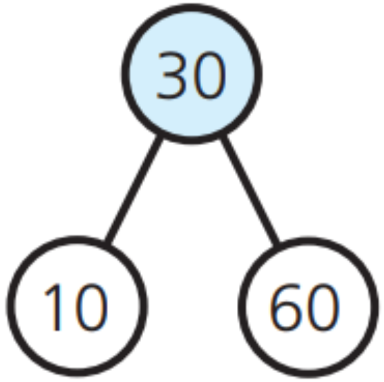
   - 4-node is middle child?
   - 4-node is right child?

Inserting to 2-3-4 tree

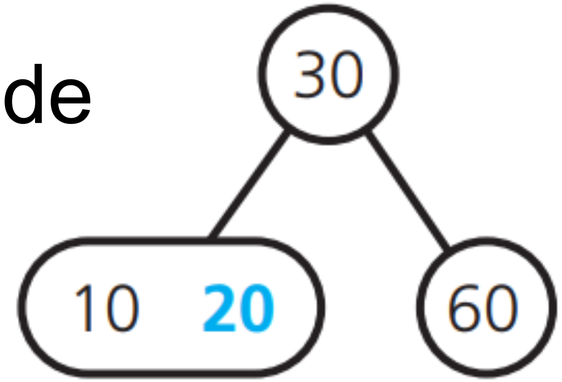
- Insert **20** into a one-node 2-3-4 tree



- This is a 4-node → split it

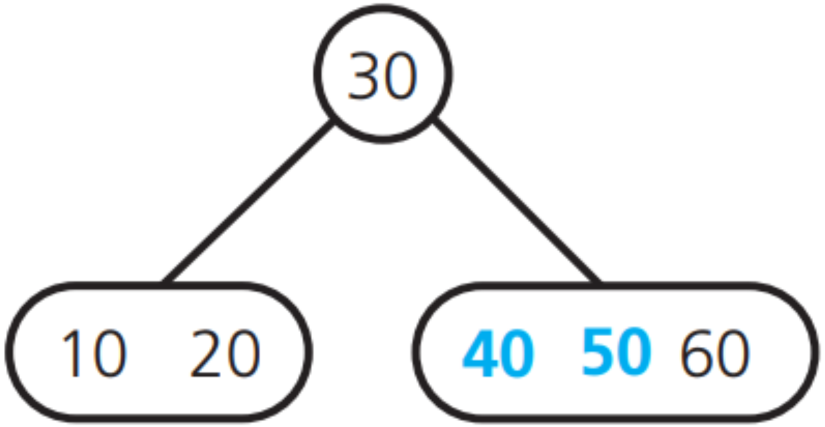


- Then, insert to the leaf node

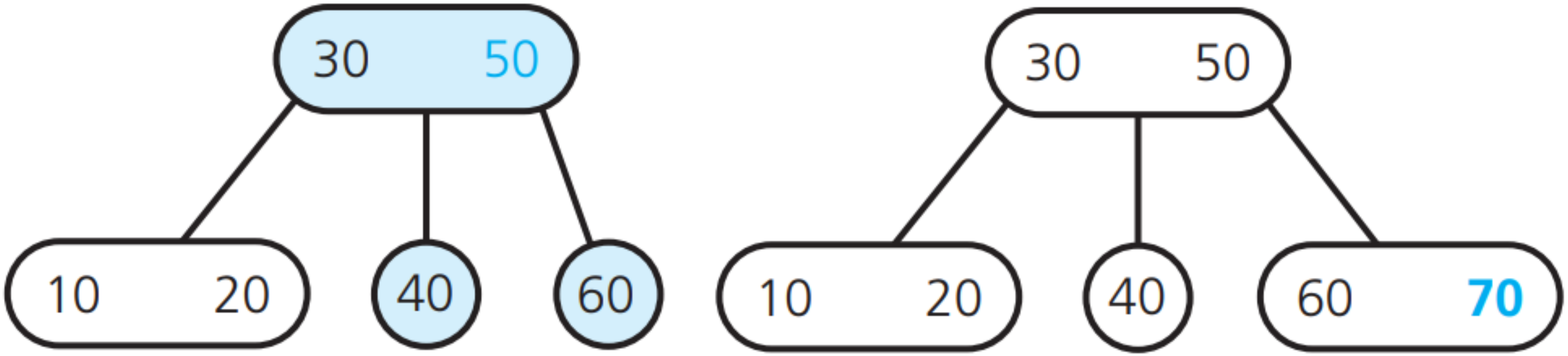


Inserting to 2-3-4 tree

- Insert **50, 40** do not require split nodes

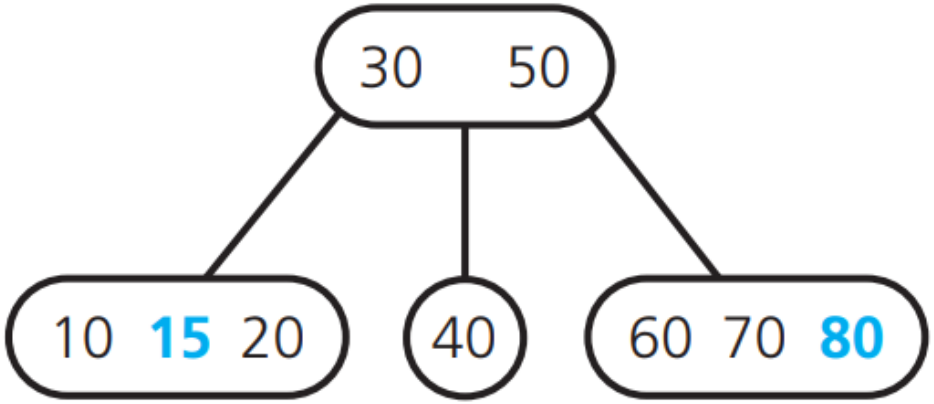


- Insert **70** → split the <40 50 60> node, then insert

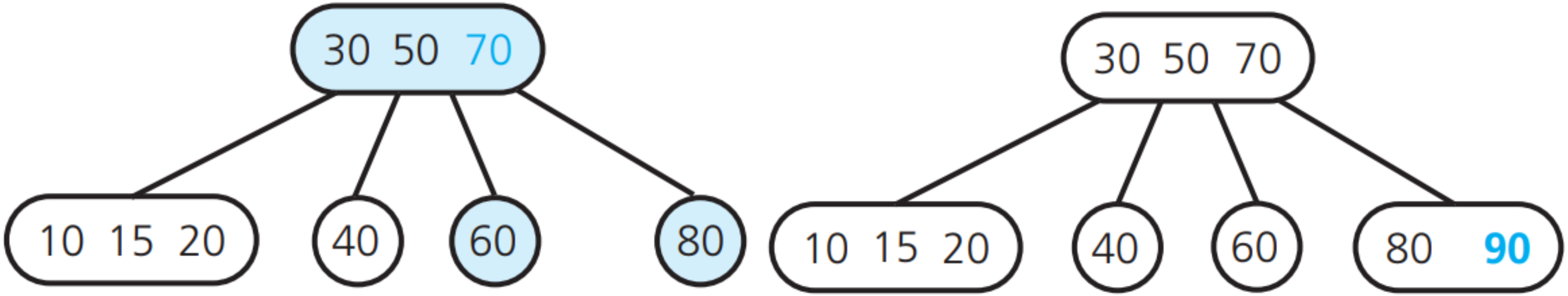


Inserting to 2-3-4 tree

- Insert **80, 15** do not require split nodes

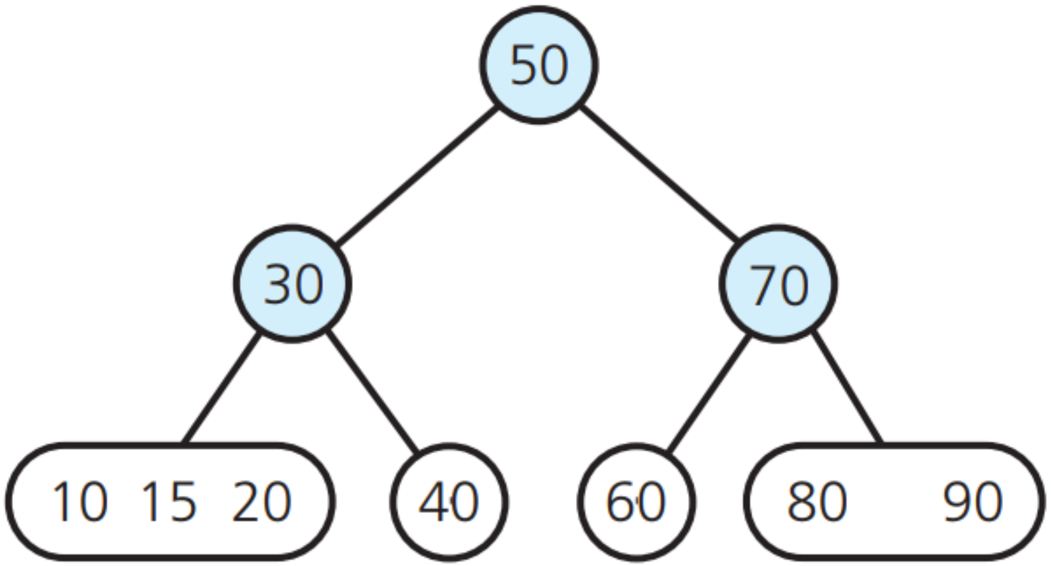


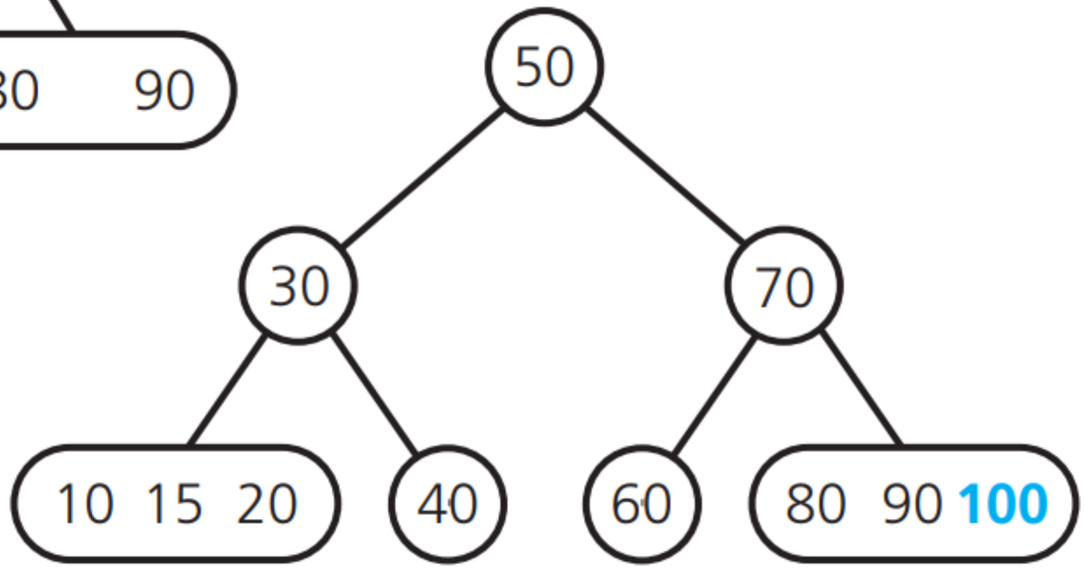
- Insert **90** → split the <60 70 80> node, then insert



Inserting to 2-3-4 tree

- Insert **100** → split the <30 50 70> node, then insert





Deleting from a 2-3-4 tree

- Similar to deleting a key from a 2-3 tree:
  - Locate the node *i* that contains the key *k* to remove
  - Find *k*'s successor (at leaves) and swap with *k*
  - Ensure that *k* doesn't occur in a 2-node so that we can perform one pass removal (unlike removal from a 2-3 tree)

→ *Transform each 2-node that we encounter during the search for k into a 3-node or 4-node*

→ Detail of deleting from a 2-3-4 are left as exercises

Conclusion

- Pros:
  - A 2-3-4 is balanced
  - Its insertion and removal operations use only one pass from root to leaf.
- Cons:
  - Requires more storage than a binary search tree that contains the same data

→ Red-black tree can represent a 2-3-4 tree that retains the advantages of a 2-3-4 tree without the storage overhead.

Comparing Trees

- Binary search tree can become *unbalanced* and *lose* their good time complexity
- A 2-3 tree and a 2-3-4 tree are variants of a BST that keeps the tree balanced easily.
- A Red-black tree is a binary tree representation of a 2-3-4 tree that requires less storage than a 2-3-4 tree. Insertions and removals in RB tree are more efficient than 2-3-4 tree.
- AVL tree is strict binary trees that *overcome the balance problem*
- B-tree is balanced search tree designed to work well on disk drivers or other direct-access secondary storage devices

What's next?

- After today:
  - Reading
    - ✓ B-tree: Textbook 1 chap 18 (page 655~)
    - ✓ 2-3 tree: Textbook 2 sec 19.2 (page 569~)
    - ✓ 2-3-4 tree: Textbook 2 sec 19.3 (page 585~)
    - ✓ 2-3-4 tree to Red-black tree: Textbook 2 sec 19.4 (page 592~)
  - Do homework 6.2
- Next class:
  - Individual Assignment 4
  - Lecture 7: Graphs

Q&A