5.25 a/

$$G(x) = \sum_{r=0}^{\infty} \frac{1}{(r+1)} \cdot \frac{x^r}{r!} = \sum_{r=0}^{\infty} \frac{1}{(r+1)!} x^r$$

$$= 1 + \frac{1}{2!} \cdot x + \frac{1}{3!} \cdot x^2 + \dots\dots$$

$$= \frac{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots}{x} = \frac{e^x - 1}{x}$$

b/

$$G(x) = \sum_{r=0}^{\infty} r! \cdot \frac{x^r}{r!} = \sum_{r=0}^{\infty} x^r = \frac{1}{1-x}$$

5.27 a/ $0 + 1 + 2 + \dots + n = \sum_{r=0}^{n} r$

Tìm hàm sinh $H(x)$ có hệ số r

Ta có : $\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{r=0}^{\infty} r \cdot x^{r-1}$$

$$\Rightarrow x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{r=0}^{\infty} r \cdot x^r = H(x)$$

$$\Rightarrow H(x) = \frac{x}{(1-x)^2} \Rightarrow H^*(x) = \frac{x}{(1-x)^3}$$

Tìm hệ số x^n trong $H^*(x)$, tức là tìm hs x^{n-1} trong $\frac{1}{(1-x)^3}$

$$= K_3^{n-1} = \binom{n-1+3-1}{n-1} = \binom{n+1}{n-1} = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

b/ $\displaystyle\sum_{r=1}^{n} r^2$

Tìm hàm sinh hệ số r^2:

$$x\, d\left(\frac{1}{1-x}\right) = \sum_{r=0}^{\infty} r x^r$$

$$x\,\frac{d\left(x \cdot d\left(\frac{1}{1-x}\right)\right)}{dx} = \sum_{r=0}^{\infty} r^2 x^r = H(x)$$

$$\Rightarrow H(x) = x \cdot \frac{d\left(\frac{x}{(1-x)^2}\right)}{dx}$$

$$= x \cdot \left[\frac{(1-x)^2 - 2(1-x)\cdot(-1)\cdot x}{(1-x)^4}\right]$$

$$= x\left[\frac{(1-x)(1-x+2x)}{(1-x)^4}\right] = \frac{x^2+x}{(1-x)^3}$$

$$\Rightarrow H^*(x) = \frac{x^2+x}{(1-x)^4}$$, tìm hệ số x^n:

$$= K_4^{n-2} + K_4^{n-1} = \binom{n+1}{n-2} + \binom{n+2}{n-1}$$

$$= \binom{n+1}{3} + \binom{n+2}{3}$$

$$\frac{(n+1)!}{3!\,(n-2)!}$$

$$= \frac{(n-1)n(n+1)}{6} + \frac{n(n+1)(n+2)}{6}$$

$$= \frac{n(n+1)(n-1+n+2)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}$$

c/ $\sum_{r=1}^{n} 13$

Tìm hàm sinh hệ số 13

$$13\left(\frac{1}{1-x}\right) = \sum_{r=0}^{\infty} 13x^r = H(x)$$

$H^*(x) = \frac{13}{(1-x)^2}$, ta tìm hệ số x^n - hệ số x^0

$$= 13(K_2^n - K_2^0) = 13\left[\binom{n+1}{n} - 1\right]$$

$$= 13\left[\binom{n+1}{1} - 1\right]$$

$$= 13n$$

f/ $4.3.2.1 + 5.4.3.2 + \dots + n(n-1)(n-2)(n-3)$

$$= \sum_{r=0}^{n} r(r-1)(r-2)(r-3)$$

Tìm hsinh hệ số $r(r-1)(r-2)(r-3)$

Ta có: $\frac{1}{(1-x)^5} = \sum_{r=0}^{\infty} K_5^r x^r$

$$= \sum_{r=0}^{\infty} \binom{r+4}{r} x^r = \sum_{r=0}^{\infty} \binom{r+4}{4} x^r$$

$$= \sum_{r=0}^{\infty} \frac{1}{4!} (r+1)(r+2)(r+3)(r+4)\, x^r$$

$$\Rightarrow \underbrace{4! \cdot \frac{1}{(1-x)^5} \cdot x^4}_{H(x)} = \sum_{r=0}^{\infty} r(r-1)(r-2)(r-3)\, x^r$$

$$\Rightarrow H^*(x) = 4! \cdot \frac{x^4}{(1-x)^6}$$, tìm hệ số x^n

$$= 4! \cdot K_6^{n-4} = 4! \cdot \binom{n+1}{n-4} = 4! \binom{n+1}{5}$$

$$= 4! \cdot C_{n+1}^5$$

5.28. b/

$$\sum_{r=0}^{\infty} (r-1)^2 \cdot x^r$$

$$\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r$$

$$\Rightarrow \frac{1}{x(1-x)} = \sum_{r=0}^{\infty} x^{r-1}$$

$$x \cdot \frac{d\left(\frac{1}{x(1-x)}\right)}{dx} = \sum_{r=0}^{\infty} (r-1)\, x^{r-1}$$

$$x^2 \cdot \frac{d\left[x \cdot \frac{d\left(\frac{1}{x(1-x)}\right)}{dx}\right]}{dx} = \sum_{r=0}^{\infty} (r-1)^2 \cdot x^r = H(x)$$

$$\Rightarrow H(x) = x^2 \cdot \frac{d\left[\cdot \frac{(2x-1)}{x.(1-x)^2}\right]}{dx}$$

$$= \cancel{x^2} \cdot \frac{2.x(1-x)^2 - (2x-1).\left[(1-x)^2 - 2x(1-x)\right]}{\cancel{x^2}(1-x)^4}$$

$$= \frac{2x\cancel{(1-x)}^2 - 2x\cancel{(1-x)}^2 + (1-x)^2 + (2x-1).2x(1-x)}{(1-x)^4}$$

$$= \frac{(1-x)\left[1-x + (2x-1) \cdot 2x\right]}{(1-x)^4} = \frac{4x^2-3x+1}{(1-x)^3}$$

5.29.f/

$$\sum_{r=0}^{\infty} \frac{1}{(r+1)(r+2)} \cdot \frac{x^r}{r!}$$