

Useful Formulas for the Analysis of Algorithms

This appendix contains a list of useful formulas and rules that are helpful in the mathematical analysis of algorithms. More advanced material can be found in [Gra94], [Gre07], [Pur04], and [Sed96].

Properties of Logarithms

All logarithm bases are assumed to be greater than 1 in the formulas below; $\lg x$ denotes the logarithm base 2, $\ln x$ denotes the logarithm base $e = 2.71828 \dots$; x, y are arbitrary positive numbers.

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a x^y = y \log_a x$
4. $\log_a xy = \log_a x + \log_a y$
5. $\log_a \frac{x}{y} = \log_a x - \log_a y$
6. $a^{\log_b x} = x^{\log_b a}$
7. $\log_a x = \frac{\log_b x}{\log_b a} = \log_a b \log_b x$

Combinatorics

1. Number of permutations of an n -element set: $P(n) = n!$
2. Number of k -combinations of an n -element set: $C(n, k) = \frac{n!}{k!(n-k)!}$
3. Number of subsets of an n -element set: 2^n

Important Summation Formulas

1. $\sum_{i=l}^u 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$ (l, u are integer limits, $l \leq u$); $\sum_{i=1}^n 1 = n$
2. $\sum_{i=1}^n i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2$
3. $\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3}n^3$
4. $\sum_{i=1}^n i^k = 1^k + 2^k + \cdots + n^k \approx \frac{1}{k+1}n^{k+1}$
5. $\sum_{i=0}^n a^i = 1 + a + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}$ ($a \neq 1$); $\sum_{i=0}^n 2^i = 2^{n+1} - 1$
6. $\sum_{i=1}^n i2^i = 1 \cdot 2 + 2 \cdot 2^2 + \cdots + n2^n = (n-1)2^{n+1} + 2$
7. $\sum_{i=1}^n \frac{1}{i} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n + \gamma$, where $\gamma \approx 0.5772 \dots$ (Euler's constant)
8. $\sum_{i=1}^n \lg i \approx n \lg n$

Sum Manipulation Rules

1. $\sum_{i=l}^u ca_i = c \sum_{i=l}^u a_i$
2. $\sum_{i=l}^u (a_i \pm b_i) = \sum_{i=l}^u a_i \pm \sum_{i=l}^u b_i$
3. $\sum_{i=l}^u a_i = \sum_{i=l}^m a_i + \sum_{i=m+1}^u a_i$, where $l \leq m < u$
4. $\sum_{i=l}^u (a_i - a_{i-1}) = a_u - a_{l-1}$

Approximation of a Sum by a Definite Integral

$$\int_{l-1}^u f(x)dx \leq \sum_{i=l}^u f(i) \leq \int_l^{u+1} f(x)dx \quad \text{for a nondecreasing } f(x)$$

$$\int_l^{u+1} f(x)dx \leq \sum_{i=l}^u f(i) \leq \int_{l-1}^u f(x)dx \quad \text{for a nonincreasing } f(x)$$

Floor and Ceiling Formulas

The *floor* of a real number x , denoted $\lfloor x \rfloor$, is defined as the greatest integer not larger than x (e.g., $\lfloor 3.8 \rfloor = 3$, $\lfloor -3.8 \rfloor = -4$, $\lfloor 3 \rfloor = 3$). The *ceiling* of a real number x , denoted $\lceil x \rceil$, is defined as the smallest integer not smaller than x (e.g., $\lceil 3.8 \rceil = 4$, $\lceil -3.8 \rceil = -3$, $\lceil 3 \rceil = 3$).

1. $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
2. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ and $\lceil x + n \rceil = \lceil x \rceil + n$ for real x and integer n
3. $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$
4. $\lceil \lg(n+1) \rceil = \lfloor \lg n \rfloor + 1$

Miscellaneous

1. $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ as $n \rightarrow \infty$ (Stirling's formula)
2. Modular arithmetic (n, m are integers, p is a positive integer)

$$(n + m) \bmod p = (n \bmod p + m \bmod p) \bmod p$$

$$(nm) \bmod p = ((n \bmod p)(m \bmod p)) \bmod p$$