Sorting Algorithms

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- Insertion Sort
- Selection Sort
- Heap Sort
- Merge Sort
- Quick Sort
- Radix Sort

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Overview

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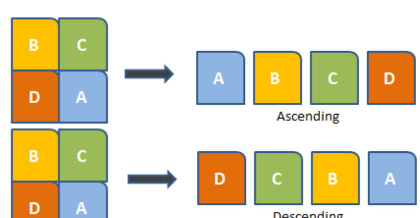
Sorting

- Sorting is:
  - A process organizing a collection of data into ascending/descending order



- Example:
  - List before sorting:

    {1, 25, 6, 5, 2, 37, 40}

  - List after sorting:

    {1, 2, 5, 6, 25, 37, 40}

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Sorting

- Sort key: data item which determines order
- Internal (sorting): data **fits in** memory
- External (sorting): data must reside on **secondary** storage
- In-place (algorithm): sorts the data **without using any additional** memory.
- Stable (algorithm): **preserves** the relative order of data elements.

Sorting

- We will analyze only **internal** sorting algorithms.
- Sorting also has indirect uses. An initial sort of the data can significantly enhance the performance of an algorithm.
- Majority of programming projects use a sort somewhere, and in many cases, the sorting cost determines the running time.
- A comparison-based sorting algorithm makes ordering decisions only based on comparisons.

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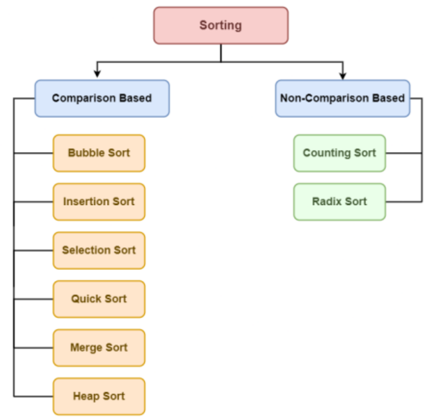
Sorting

- Some popular sorting algorithms:
  - Bubble Sort
  - Insertion Sort
  - Selection Sort
  - Heap Sort
  - Merge Sort
  - Quick Sort
  - Radix Sort
  - Counting Sort



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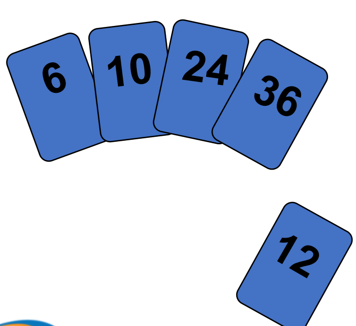
Insertion Sort

Insertion Sort

- Idea: like sorting a hand of playing cards
  - Start with an empty left hand and the cards facing down on the table.
  - Remove one card at a time from the table, and insert it into the correct position in the left hand
    - compare it with each of the cards already in the hand, from right to left
  - The cards held in the left hand are sorted
    - these cards were originally the top cards of the pile on the table

Insertion Sort



To insert 12, we need to make room for it by moving first 36 and then 24.

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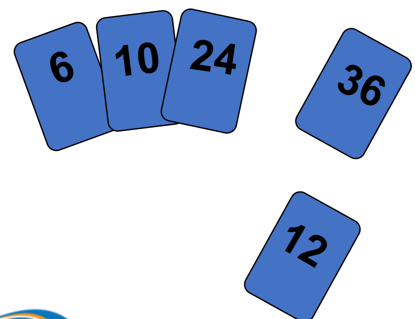
Insertion Sort



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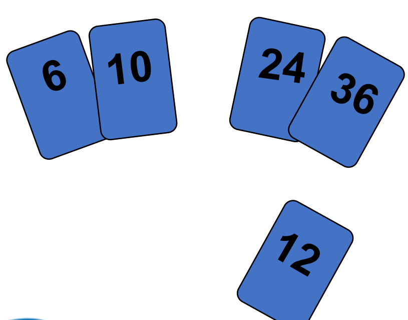
Insertion Sort



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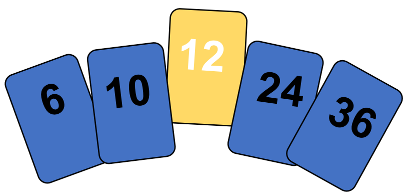
Insertion Sort



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Insertion Sort

- **Insertion Sort**: orders a list of values by repetitively inserting a particular value into a sorted subset of the list
- more specifically:
  - consider the first item to be a sorted sublist of length 1
  - insert second item into sorted sublist, shifting first item if needed
  - insert third item into sorted sublist, shifting items 1-2 as needed
  - ...
  - repeat until all values have been inserted into their proper positions

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Insertion Sort

- Input: (unsorted) a[] (*n* elements, 0..n-1)
- Output: (sorted) a[] (*n* elements)

```
for i = 1 to n-1 {
      v = a[i] //the ith element
      j = i
       // slide elements right to make room for a[i]
      while (j >= 1 and a[j-1] > v){
            a[j] = a[j-1] // Shift right one by one
            j = j - 1
      }
      a[j] = v
}
```


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Insertion Sort Example

Sort: 34 8 64 51 32 21

- 34 **8** 64 51 32 21
  - Pull out 8 into Temp
- 34 8 64 51 32 21
  - Compare 34 and 8 - move 34 up a spot
- 34 34 64 51 32 21
  - Spot is found for 8 - place it where it belongs
- 8 34 64 51 32 21

Insertion Sort Example

Sort: 34 8 64 51 32 21

- 8 34 **64** 51 32 21
  - Pull out 64 into Temp
- 8 34 64 51 32 21
  - Compare 64 and 34 - place 64 back into slot 2
- 8 34 64 51 32 21

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Insertion Sort Example

Sort: 34 8 64 51 32 21

- 8 34 64 **51** 32 21
  - Pull out 51 into Temp
- 8 34 64 51 32 21
  - Compare 51 and 64 - move 64 to the right
- 8 34 64 64 32 21
  - Compare 51 and 34 - place 51 into slot 2
- 8 34 51 64 32 21

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Insertion Sort Example

Sort: 34 8 64 51 32 21

- 8 34 51 64 **32** 21
  - Pull out 32 into Temp
- 8 34 51 64 32 21
  - Compare 32 and 64 - move 64 to the right
- 8 34 51 64 64 21
  - Compare 32 and 51 - move 51 to the right
- 8 34 51 51 64 21
  - Compare 32 and 34 - move 34 to the right
- 8 34 34 51 64 21
  - Compare 32 and 8 - place 32 in slot 1
- 8 32 34 51 64 21

One more element

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Insertion Sort Example

index	0	1	2	3	4	5	6	7
value	15	2	8	1	17	10	12	5
pass 1	**2**	**15**	8	1	17	10	12	5
pass 2	**2**	**8**	**15**	1	17	10	12	5
pass 3	**1**	**2**	**8**	**15**	17	10	12	5
pass 4	**1**	**2**	**8**	**15**	**17**	10	12	5
pass 5	**1**	**2**	**8**	**10**	**15**	**17**	12	5
pass 6	**1**	**2**	**8**	**10**	**12**	**15**	**17**	5
pass 7	**1**	**2**	**5**	**8**	**10**	**12**	**15**	**17**

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Analysis

- **worst case:** reverse-ordered elements in array.

$$\sum_{i=1}^{N-1} i = 1+2+3+\ldots+(N-1) = \frac{(N-1)N}{2}$$
$$= O(N^2)$$

- best case: array is in sorted ascending order. $\sum_{i=1}^{N-1} 1 = N-1 = O(N)$
- average case: each element is about halfway in order.

$$\sum_{i=1}^{N-1} \frac{i}{2} = \frac{1}{2}(1+2+3\ldots+(N-1)) = \frac{(N-1)N}{4}$$
$$= O(N^2)$$

Analysis

- The insertion sort is a good choice for sorting lists of a few thousand items or less.
- Some observations about insertion sort:
  - insertion sort runs fast if the input is almost sorted
  - insertion sort's weakness is that it swaps each element just one step at a time, taking many swaps to get the element into its correct position

An Extension for Insert Sort - Shell Sort

- For some sequence of gaps $g_1, g_2, g_3, \ldots, 1$:
  - Sort all elements that are g_1 indexes apart (using insertion sort)
  - Then sort all elements that are g_2 indexes apart, ...
  - Then sort all elements that are 1 index apart (using insertion sort)

- Some sequence of gaps can be used:
  - .., 121, 40, 13, 4, 1
  - .., 31, 15, 7, 3, 1

Shell Sort Example

- An example that sorts by gaps of **8**, then **4**, then **2**, then **1**:

index	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
start	27	88	92	-4	22	30	36	50	7	18	11	76	2	65	56	3	85
gap 8	7	18	11	-4	2	30	36	3	27	88	92	76	22	65	56	50	85
gap 4	2	18	11	-4	7	30	36	3	22	65	56	50	27	88	92	76	85
gap 2	2	-4	7	3	11	18	22	30	27	50	36	65	56	76	85	88	92
gap 1	-4	2	3	7	11	18	22	27	30	36	50	56	65	76	85	88	92

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Selection Sort

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Selection Sort - Idea

- Sort naturally the same as in real-life:
  - The list is divided into two sub-lists, *sorted* and *unsorted*, which are divided by an imaginary wall.
  - Find the **smallest element** from the unsorted sub-list and move to the correct position (swap it with the element at the beginning of the unsorted data.)
  - After each selection and swapping, increase the number of sorted elements and decrease the number of unsorted ones.
  - Loop those steps until the unsorted list has only 1 element.

Selection Sort

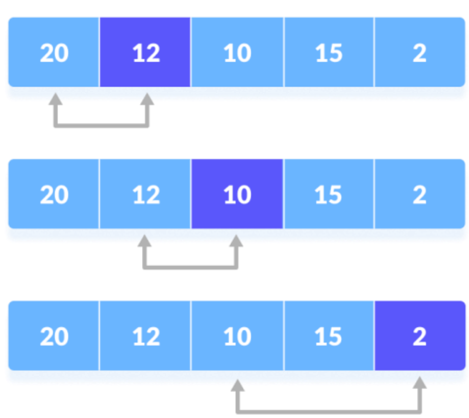
Input: (unsorted) a[] (*n* elements)

Output: (sorted) a[] (*n* elements)

- Step 1. Initialize *i* = 0.
- Step 2. Loop:
  - 2.1. Find the **smallest value** a[min] in the list with index from *i* to *n*-1 (a[i],.., a[n-1]).
  - 2.2. Swap a[min] and a[i]
- Step 3. Compare *i* with *n*:
  - If *i* < *n* then increase *i* by 1, back to step 2.
  - Otherwise, Stop.

Selection Sort

- Step 1: find the smallest value of the (unsorted) list



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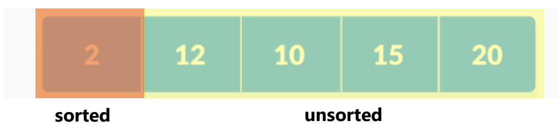
Selection Sort

- Step 2: the smallest value is placed in the front of the unsorted list

- Step 3: repeatedly step 1-2 for the unsorted parts



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Example

i = 0	15	2	8	7	3	6	9	17
i = 1	2	15	8	7	3	6	9	17
i = 2	2	3	8	7	15	6	9	17
i = 3	2	3	6	7	15	8	9	17
i = 4	2	3	6	7	15	8	9	17
i = 5	2	3	6	7	8	15	9	17
i = 6	2	3	6	7	8	9	15	17
i = 7	2	3	6	7	8	9	15	17

Analysis

- Which operation should be used for analysis?
- How many operations are there with size of the problem *n*?
- Best case? Worst case?

Analysis

- In general, we compare keys and move items (or exchange items) in a sorting algorithm (which uses key comparisons).
- To analyze a sorting algorithm we should count the number of key comparisons and the number of moves.
  - Ignoring other operations does not affect our result.
- The outer for loop executes *n-1* times. We invoke **swap** function once at each iteration.

  Total Swaps: n-1

  Total Moves: 3*(n-1) (Each swap has three moves)

Analysis

- The inner for loop executes the size of the unsorted part minus 1 (from 1 to *n-1*), and in each iteration we make one key comparison.

  number of key comparisons = 1+2+...+n-1 = n*(n-1)/2

Analysis

- The best case, the worst case, and the average case of the selection sort algorithm are same.
- Order of the algorithm: $O(n^2)$.

Analysis

- If sorting a very large array, selection sort algorithm probably too inefficient to use.
- What is the advantage of this algorithm?

Analysis

- The behavior of the selection sort algorithm does not depend on the initial organization of data.
- Although the selection sort algorithm requires $O(n^2)$ key comparisons, it only requires $O(n)$ moves.
- A selection sort could be a good choice if data moves are costly but key comparisons are not costly (short keys, long records).

Heap Sort

Heap Structure

- Definition (*array-based representation*):
  - Heap is a collection of *n* elements (a_0, a_1, … a_{n-1}) in which every element (at position ***i***) in the **first half** is greater than or equal to the elements at position **2*i*+1** and **2*i*+2**.
    (if 2i+2 ≥ n, just $a_i \geq a_{2i+1}$ satisfied).
  - i.e., for every *i* ($0 \leq i \leq n/2-1$)
    $a_i \geq a_{2i+1}$
    $a_i \geq a_{2i+2}$
  - Heap in above definition is called **max-heap**. *(We also have **min-heap** structure).*

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Heap Structure

- Examples:
  - A max-heap: 9, 5, 6, 4, 5, 2, 3, 3
  - A min-heap: 8, 15, 10, 20, 17, 12, 18, 21, 20
- Give some more examples of:
  - A max-heap with 8 elements.
  - A max-heap with 11 elements.
  - A min-heap with 7 elements.

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Heap Structure

- Property:
  - The first element of the max-heap is always the largest.

Heap Structure - Heap Construction

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- Input: An array a[], *n* elements
- Output: A heap a[], *n* elements

```
Step 1. Start from the middle of the array (first
half). Initialize index = n /2 - 1
Step 2. while (index >= 0)
{
   heapRebuild at position index //heapRebuild(index, a, n)
   index = index - 1
}
```


Heap Structure – heapRebuild (pos, A, n)

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```
○ Step 1. Initialize k = pos, v = A[k], isHeap =
  false
○ Step 2. while not isHeap and 2*k+1 < n do
  j = 2*k + 1 //first element
  if j < n – 1 //has enough 2 elements
     if A[j] < A[j + 1] then j = j + 1 //position of the larger
     between A[2*k+1] and A[2*k+2]
  if A[k] >= A[j] then isHeap = true
  else
     swap between A[k] and A[j]
     k = j
```

Heap Construction - An Example

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- Construct a heap from the following list:

2, 9, 7, 6, 5, 8

Heap Sort

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- An interesting sorting algorithm discovered by J.W.J. Williams (in 1964).
- Idea is same as Selection Sort.
- It has two stages:
  - Stage 1: **(heap construction)**. Construct a heap for a given array.
  - Stage 2: **(maximum deletion)**. Apply the maximum key deletion n-1 times to the remaining heap
    - Exchange the first and the last element of the heap.
    - Decrease the heap size by 1.
    - Rebuild the heap at the first position.

Heap Sort

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```
HeapSort(a[], n)
{
	heapConstruct(a, n);
	r = n - 1;
	while (r > 0)
	{
		swap(a[0], a[r]);
		heapRebuild(0, a, r); //heapConstruct(a, r);
		r = r - 1;
	}
}
```


Heap Sort - Analysis

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- Best case, Worst case, Average case are the same.
- The order of this algorithm: $O(n\log_2 n)$

Merge Sort

Divide-and-Conquer

- This technique can be divided into the following three parts:
  - **Divide**: This involves dividing the problem into smaller sub-problems.
  - **Conquer**: Solve sub-problems by calling **recursively** until solved.
  - **Combine**: **Combine** the sub-problems to get the final solution of the whole problem

Merge Sort

- Merge Sort algorithm is one of two important **divide-and-conquer** sorting algorithms.

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Merge Sort - Idea

- It is a recursive algorithm.
  - Divides the list into halves,
  - Sort each halve separately, and
  - Then merge the sorted halves into one sorted array.

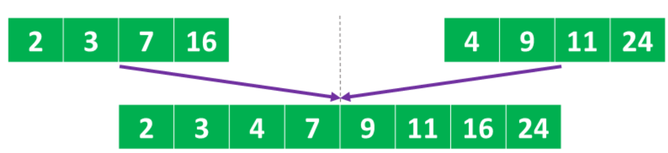
- Note:
  - A list with 0 or 1 element is a sorted list.

Merge Sort - Idea

- **Merge** procedure:
  - Goal: Merge two ordered lists into an order list.
  - Input: two ordered lists A[] (*n* elements), B[] (*m* elements)
  - Output: a new ordered list C[] (*n* + *m* elements) (containing all elements of A and B).
  - Example:
    - A = {1, 5, 7, 9}, B = {2, 9, 10, 12, 17, 26}; C = {1, 2, 5, 7, 9, 9, 10, 12, 17, 26}
  - Propose the efficient algorithm.

Merge Sort - Idea

- Merge procedure:
  - **Input**:
    - A = {2, 3, 7, 16},
    - B = {4, 9, 11, 24};
  - **Output**:
    - C = {2, 3, 4, 7, 9, 11, 16, 24}



Merge Sort

- Input: A[], left, right (list A from index left to right).
- Output: (Ordered) A[] (from left, to right)

```
MergeSort(A[], left, right)
{
   if (left < right) {
      mid = (left + right)/2;
      MergeSort(A, left, mid);
      MergeSort(A, mid+1, right);
      Merge(A, left, mid, right);
   }
}
```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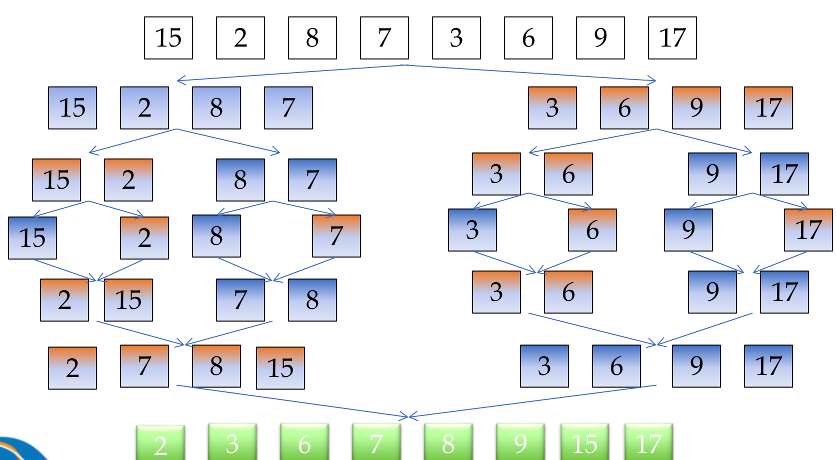
Merge Sort - An Example



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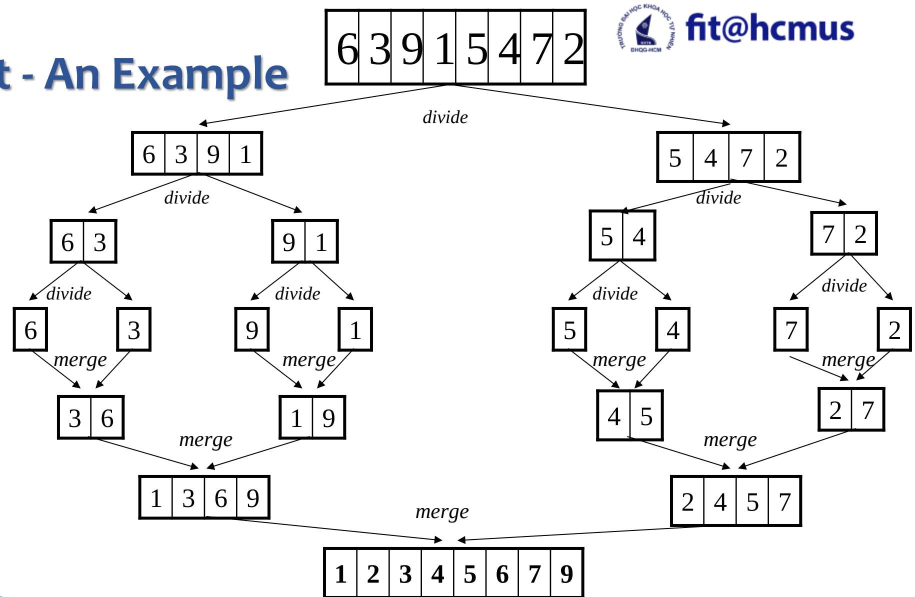
Merge Sort - An Example



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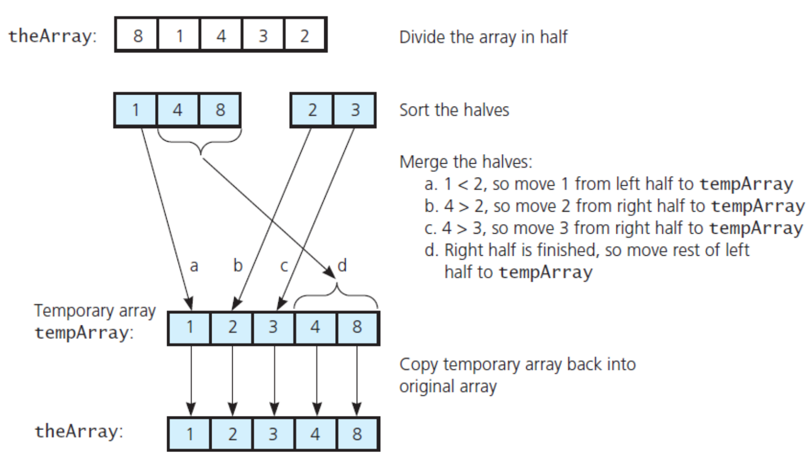
Merge Sort - An Example



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Analysis

- Merge Sort is extremely efficient algorithm with respect to time.
  - Both worst case and average case are O (n * $\log_2 n$)
- Merge Sort requires an extra array whose size equals to the size of the original array.
- If we use a linked list, we do not need an extra array
  - But we need space for the links
  - And, it will be difficult to divide the list into half (O(n))

Quick Sort

Quick Sort - Idea

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- Like Merge Sort, Quick Sort is also based on the **divide-and-conquer** paradigm.
- It works as follows:
  - First, it **partitions** an array into two parts,
  - Then, it **sorts** the parts **independently**,
  - Finally, it **combines** the sorted subsequences by a simple concatenation.

Quick Sort - Idea

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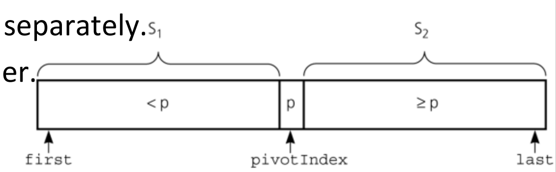
- The algorithm consists of the following three steps:
  - **Divide:** ***Partition*** the list.
    - To partition the list, we first choose some element from the list for which we hope about half the elements will come before and half after. Call this element the ***pivot***.
    - Then we partition the elements so that all those with values less than the pivot come in one sub-list and all those with greater values come in another.
  - **Recursion:** Recursively sort the sub-lists separately.
  - **Conquer**: Put the sorted sub-lists together.



Quick Sort

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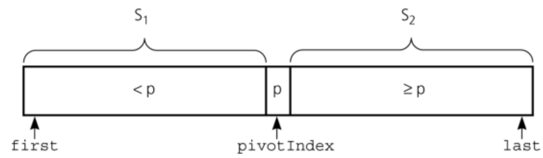
- Input: A[], first, last (Sort the list A[] from index *first* to *last*)
- Output: Ordered list A[first..last]




```
QuickSort(A[], first, last)
   if (first < last) {
      Select a pivot p from A[].
      pivotIndex = Partition(A, first, last) //Partition A[] into 2 sub-lists S1 (first … pivotIndex-1), S2 (pivotIndex+1…last)
      QuickSort (A, first, pivotIndex-1) //Sort S1
      QuickSort (A, pivotIndex + 1, last) //Sort S2
   }
```


Quick Sort - Partition

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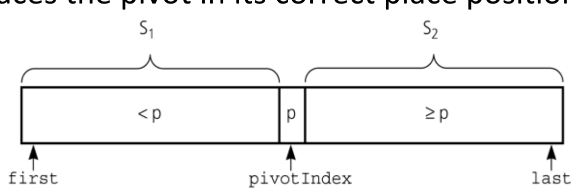
- Partitioning places the pivot in its correct place position within the array.



- Arranging the array elements around the pivot *p* generates two smaller sorting problems.
  - sort the **left section** of the array and sort the **right section** of the array.
  - when these two smaller sorting problems are solved recursively, our bigger sorting problem is solved.

Quick Sort - Partition

- Selecting the pivot
  - Select a pivot element among the elements of the given array
  - We put this pivot into the first location of the array before partitioning.
- Which array item should be selected as pivot?
  - We hope that we will get a good partitioning.

Quick Sort - Partition

- Selecting the pivot
  - Select a pivot element among the elements of the given array
  - We put this pivot into the first location of the array before partitioning.
- Which array item should be selected as pivot?
  - If the items in the array arranged randomly, we choose a pivot randomly.
  - We can choose the first or last element as a pivot (it may not give a good partitioning).
  - We can use different techniques to select the pivot.

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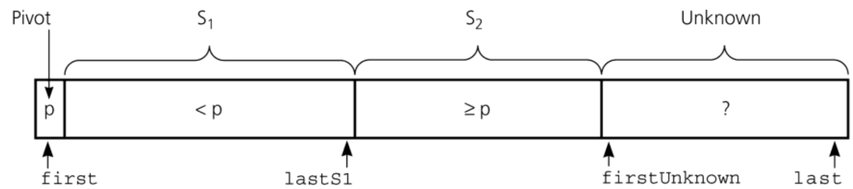
Quick Sort - Partition

- Partitioning uses two more variables:
  - `lastS1`: the last index of `S1` (the elements in A less than p).
  - `firstUnknown`: the first index of `Unknown`.
- Partitioning takes place when **`firstUnknown <= last`**.



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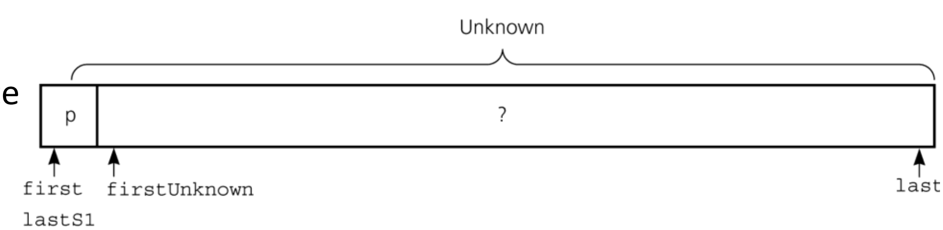
Quick Sort - Partition

- Initialize
  - `lastS1 = first`
  - `firstUnknown = first + 1`
- Initial state



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Quick Sort - Partition

Partition(A[], first, last, pivot) -> pivotIndex

Step 1. while (*firstUnknown* <= *last*) //not finish

1.1 If the element at position *firstUnknown* is **less than** *pivot* then move that element to *S1*

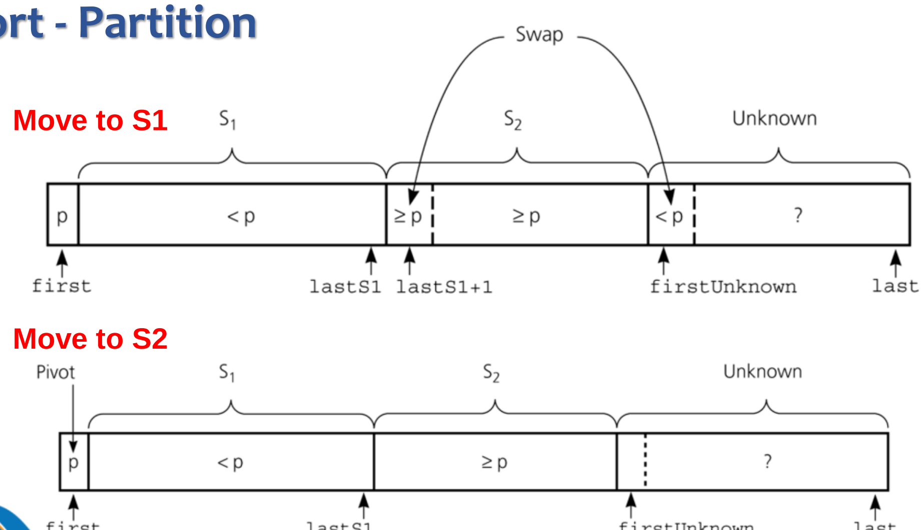
Otherwise, move that element to *S2*

1.2 *firstUnknown* = *firstUnknown* + 1 //next element

Step 2. Move *pivot* to the correct position (between *S1* and *S2*): Swap two elements at *lastS1* and *first*.

Step 3. pivotIndex = lastS1

Quick Sort - Partition



Quick Sort - Partition

- Partition this list: 27, 38, 12, 39, 27, 16

Pivot	Unknown				
27	38	12	39	27	16

Pivot	S2	Unknown			
27	38	12	39	27	16

Pivot	S1	S2	Unknown		
27	12	38	39	27	16

Quick Sort - Partition

- Partition this list: 27, 38, 12, 39, 27, 16

Pivot	S1	S2	Unknown		
27	12	38	39	27	16

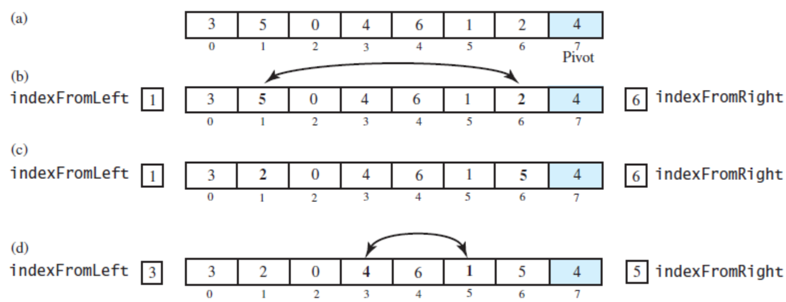
Pivot	S1	S2			U.K
27	12	38	39	27	16

Pivot	S1		S2		
27	12	16	39	27	38

S1		Pivot	S2		
16	12	27	39	27	38

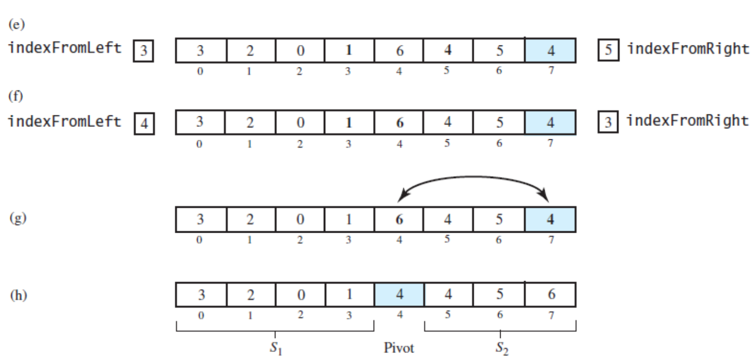
Quick Sort - Partition

- Another technique



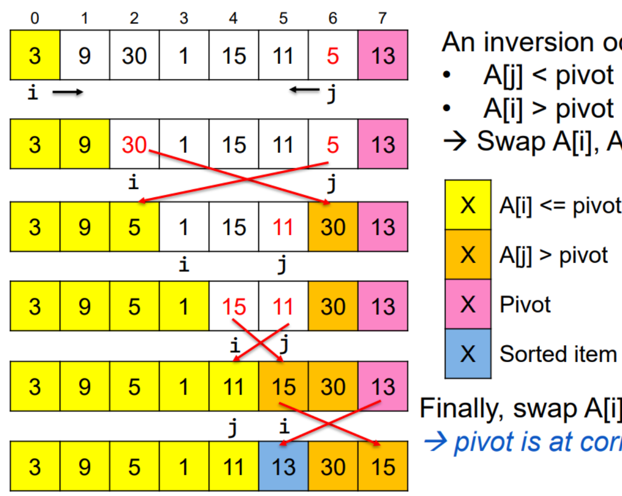
Quick Sort - Partition

- Another technique



Quick Sort - Partition

- Another technique



An inversion occurs:
- A[j] < pivot
- A[i] > pivot

→ Swap A[i], A[j]

Finally, swap A[i] and pivot

→ pivot is at correct position

Quick Sort - Partition

- Median-of-three pivot selection

Analysis

- Worst case: **$O(n^2)$**
- Quick Sort is **$O(n\log_2 n)$** in the best case and average case.

- **Notes:**
  - Quick Sort is slow when the array is sorted and we choose the first element as the pivot.
  - Although the worst case behavior is not so good, its average case behavior is much better than its worst case.
  - Quick Sort is one of best sorting algorithms using key comparisons.

Radix Sort

Radix Sort

- Radix Sort algorithm different than other sorting algorithms that we talked.
  - It DOES NOT use key comparisons to sort an array.

Radix Sort - Idea

- Treats each data item as a character string.
- Repeat (*for all character positions from the rightmost to the leftmost*)
  - Groups data items according to their rightmost character
  - Put these groups into order with respect to this rightmost character.
  - Combine all the groups.
  - Move to the next left position.
- At the end, the sort operation will be completed.

Radix Sort

```
RadixSort(A[], n, d) // sort n d-digit integers in the array A
   for (j = d down to 1) {
         Initialize 10 groups to empty
         Initialize a counter for each group to 0
         for (i = 0 through n-1) {
               k = jth digit of A[i]
               Place A[i] at the end of group k
               Increase kth counter by 1
         }
         Replace the items in A with all the items in group 0,
   followed by all the items in group 1, and so on.
   }
```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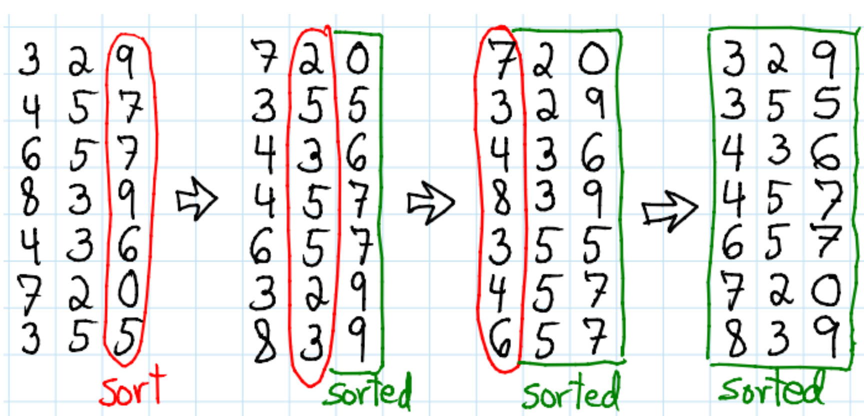
Radix Sort



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Radix Sort - An Example

- Sort the following list ascendingly using Radix Sort:

  27, 78, 52, 39, 17, 46
- Base: 10, Number of digits: 2
- First Pass. The rightmost digit

0	1	2	3	4	5	6	7	8	9
							17		
		52				46	27	78	39

Combine after first pass: **52, 46, 27, 17, 78, 39**

Radix Sort - An Example

- **Second Pass**. The second rightmost digit of : **52, 46, 27, 17, 78, 39**

0	1	2	3	4	5	6	7	8	9
	17	27	39	46	52		78		

Resulting list: **17, 27, 39, 46, 52, 78**

Analysis

- Radix Sort is O(n)

- What are the strength and weakness of this algorithm?

Analysis

- Although the radix sort is O(n), it is NOT appropriate as a general-purpose sorting algorithm.
  - Memory needed?

- The Radix Sort is more appropriate for a linked list than an array.

Comparison of Sorting Algorithms

	Worst case	Average case
Selection sort	n^2	n^2
Bubble sort	n^2	n^2
Insertion sort	n^2	n^2
Mergesort	n * log n	n * log n
Quicksort	n^2	n * log n
Radix sort	n	n
Treesort	n^2	n * log n
Heapsort	n * log n	n * log n

Summary

- Selection Sort is $O(n^2)$ algorithm. Good in some particular case but it is slow for large problems.
- Heap Sort converts an array into a heap to locate the array's largest items, enabling to sort more efficient.
- Quick Sort and Merge Sort are efficient recursive sorting algorithms.
- Quick Sort is $O(n^2)$ in worst case but rarely occurs.
- Merge Sort requires additional storage.
- Radix Sort is O(n) but not always applicable as not a general-purpose sorting algorithm.

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Questions and Answers