APPM 4600 — HOMEWORK # 3

1. Consider the equation $2x - 1 = \sin x$.

   (a) Find a closed interval $[a, b]$ on which the equation has a root r, and use the Intermediate Value Theorem to prove that r exists.

   (b) Prove that r from (a) is the only root of the equation (on all of $\mathbb{R}$).

   (c) Use the bisestion code from class (or your own) to approximate r to eight correct decimal places. Include the calling script, the resulting final approximation, and the total number of iterations used.

2. The function $f(x) = (x-5)^9$ has a root (with multiplicity 9) at $x = 5$ and is monotonically increasing (decreasing) for $x > 5$ ($x < 5$) and should thus be a suitable candidate for your function above. Use `a=4.82` and `b=5.2` and `tol = 1e-4` and use `bisection` with:

   (a) $f(x) = (x-5)^9$.

   (b) The expanded expanded version of $(x-5)^9$, that is, $f(x) = x^9 - 45x^8 + \ldots - 1953125$.

   (c) Explain what is happening.

3. (a) Use a theorem from class (Theorem 2.1 from text) to find an upper bound on the number of iterations in the bissection needed to approximate the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$ with an accuracy of 10^{-3}.

   (b) Find an approximation of the root using the bisection code from class to this degree of accuracy. How does the number of iterations compare with the upper bound you found in part (a)?

4. **Definition 1** *Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p with $p_n \neq p$ for all n. If there exists postive constants λ and α such that*

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

   then $\{p_n\}_{n=1}^{\infty}$ converges to p with an order α and asympotic error constant λ. If $\alpha = 1$ and $\lambda < 1$ then the sequence converges linearly*. If $\alpha = 2$, the sequence is* quadratically *convergent.*

   Which of the following iterations will converge to the indicated fixed point x_* (provided x_0 is sufficiently close to x_*)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

   (a) (10 points) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}$, $x_* = 2$

   (b) (10 points) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $x_* = 3^{1/3}$

   (c) (10 points) $x_{n+1} = \frac{12}{1+x_n}$, $x_* = 3$

5. All the roots of the scalar equation

$$x - 4\sin(2x) - 3 = 0,$$

are to be determined with at least 10 accurate digits[1].

(a) Plot $f(x) = x - 4\sin(2x) - 3$ (using your `Python` toolbox). All the zero crossings should be in the plot. How many are there?

(b) Write a program or use the code from class to compute the roots using the fixed point iteration

$$x_{n+1} = -\sin(2x_n) + 5x_n/4 - 3/4.$$

Use a stopping criterium that gives an answer with ten correct digits. (*Hint: you may have to change the error used in determing the stopping criterion.*) Find, empirically which of the roots that can be found with the above iteration. Give a theoretical explanation.

[1] n accurate digits is equivalent to a relative error smaller than 0.5×10^{-n}.