# Homework 3

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## Problem 1

(a) Find a closed interval $[a, b]$ on which the equation $2x - 1 = \sin x$ has a root r.

Let $f(x) = 2x - \sin x - 1$, $a = -1$, and $b = \pi/2$. $f(a) = f(0) = -1$ and $f(b) = f(\pi/2) = \pi - 2$. $f(x)$ is continuous $\forall x \in [a, b]$ so the Intermediate Value Theorem guarantees $\exists x_0 \in (a, b)$ such that $f(x_0) = 0$ since $f(a) < 0, f(b) > 0$, hence a root $r$ to our initial equation exists.

(b) Prove that $r$ from $(a)$ is the only root of the equation.

Again consider $f(x) = 2x - \sin x - 1$, $f^{'}(x) = 2 - \sin x$. $\forall x, \sin x \geq 1$, so $\forall x, f^{'}(x) \geq 1$. Since $f^{'}(x)$ is always positive, $f(x)$ is a strictly monotone increasing function so $r$ must be its only root.

(c) Use the bisection method to approximate $r$ to eight correct decimal places.

```
f = lambda x: 2 * x - 1 -np.sin(x)
a = 0
b = np.pi / 2
Nmax = 100
tol = 0.5 * 10**-8
[astar, ier] = bisection(f, a, b, tol, Nmax)
print('the approximate root is', astar)
print('the error message reads:', ier)
```

Using the bisection script from class, $r$ was approximated to eight correct decimal places in 28 iterations to a value of $r = 0.8878622096305406$.

## Problem 2

(a) Apply the bisection method to $f(x) = (x-5)^9$ with $a = 4.82, b = 5.2$, and tol$= 10^{-4}$.

```
f = lambda x: (x - 5) ** 9
a = 4.82
b = 5.2
Nmax = 100
tol = 10 ** -4
[astar, ier] = bisection(f, a, b, tol, Nmax)
print('the approximate root is', astar)
print('the error message reads:', ier)
```

This returns a root $r = 4.9999804687500005$ in 11 iterations.

(b) Apply the bisection method to the expanded version of $(x-5)^9$.

```
f = lambda x: x**9 - 45*x**8 + 900*x**7 - 10500*x**6
+ 78750*x**5 - 393750*x**4 + 1312500*x**3
- 2812500*x**2 + 3515625*x - 1953125
a = 4.82
b = 5.2
Nmax = 100
tol = 10 ** -4
[astar, ier] = bisection(f, a, b, tol, Nmax)
print('the approximate root is', astar)
print('the error message reads:', ier)
```

Here the bisection script returns a root of $r = 5.12875$ and does so on the first iteration because it evaluates $f(c) = 0$ for $c = (4.82 + 5.2)/2$.

(c) Explain what is happening.

This happens because of the round-off errors caused by the repeated subtractions when evaluating the expanded version of $f(x)$.

## Problem 3

(a) Find an upper bound on the number of iterations in the bisection method needed to approximate the root of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$ with accuracy of $10^{-3}$.

From Theorem 2.1, $|p_n - p| \leq 2^{-n}(b - a)$. For the above function:

$$|p_n - p| \leq 2^{-n}(4 - 1) = 10^{-3}$$

We can solve for n

$$2^{-n} = \frac{1}{3}10^{-3}$$
$$-n = \log_2(\frac{1}{3}10^{-3})$$
$$n \approx 11.55$$

Hence we can expect it to take 12 or fewer iterations to converge with an accuracy of $10^{-3}$.

(b) Find an approximation of the bisection code from class. How does this number of iterations compare with the upper bound in part (a)?

The code (pushed to my GitHub) converged to accuracy under $10^{-3}$ in 11 iterations, which is in line with the upper bound found above.

## Problem 4

(a) $x_{n+1} = -16 + 6x_n + \frac{12}{x_n}, x_* = 2$

Let $x_{n+1} = f(x_n) = -16 + 6x_n + \frac{12}{x_n}$, so $f'(x_n) = 6 - \frac{12}{x_n^2}$, at $x_* = 2, f'(x_*) = 6 - 4 = 2$. $|2| > 1$ so this iteration will not converge to $x_* = 2$.

(b) $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}, x_* = 3^{1/3}$

Let $x_{n+1} = f(x_n) = \frac{2}{3}x_n + \frac{1}{x_n^2}$, so $f'(x_n) = \frac{2}{3} - \frac{2}{x_n^3}$, at $x_* = 3^{1/3}, f'(x_*) = 0.0 < 1$ so this iteration will converge to $x_* = 3^{1/3}$. $f'(x_*) = 0$, but $f''(x) = \frac{6}{x_n^4}$ evaluated at $x_* \approx 1.39 \neq 0$. Therefore, this converges quadratically at $x_* = 3^{1/3}$.

(c) $x_{n+1} = \frac{12}{1+x_n}, x_* = 3$

Let $x_{n+1} = f(x_n) = \frac{12}{1+x_n}$, so $f'(x) = \frac{-12}{(1+x_n)^2}$, at $x_* = 3, f'(x_*) = \frac{-3}{4}$. $\left|\frac{-3}{4}\right| < 1$ so this iteration will converge to $x_* = 3$. Our uncertainty $\zeta$ term will converge to $x_*$ so our rate of converge is equal to $\left|f'(x_*)\right| = \frac{3}{4}$. The above calculation is formulated from the following formula:

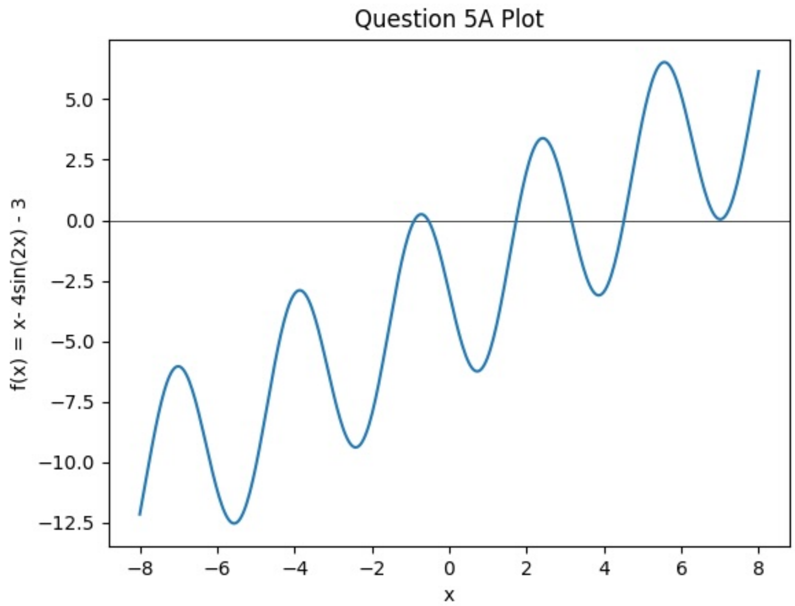
$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^q} = \mu$$

where $q$ is the order of convergence.

## Problem 5

(a) Plot $f(x) = x - 4\sin(2x) - 3$. All the zero crossings should be in the plot. How many are there?

This function crosses the x-axis five times.




(b) Write a program or use the code from class to compute the roots using the fixed point iteration

$$x_{n+1} = -\sin(2x_n) + 5x_n/4 - 3/4$$

Use a stopping criterion that gives an answer with ten correct digits. Find, empirically which of the roots can be found with the above iteration. Give a theoretical explanation.

The fixed point method finds two roots of this function to ten correct digits, one at $r = 3.1618264865$, and the other at $r = -0.54444240068$. It does not converge to the other three roots because the derivative of the fixed point function, $x_{n+1} = -\sin(2x_n) + 5x_n/4 - 3/4$, which is equal to $-2\cos(2x_n) + 5/4$, is greater than 1 at the other roots, so by Theorem 2.3, the fixed point method will not converge.