APPM 4600 — HOMEWORK # 4

1. Suppose we want to find a solution located near $(x, y) = (1, 1)$ to the nonlinear set of equations

$$\begin{aligned} f(x, y) &= 3x^2 - y^2 = 0, \\ g(x, y) &= 3xy^2 - x^3 - 1 = 0 \end{aligned} \tag{0.1}$$

   (a) Iterate on this system numerically (in Python), using the iteration scheme

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}, \quad n = 0, 1, 2, \ldots$$

   starting with $x_0 = y_0 = 1$, and check how well it converges.

   (b) Provide some motivation for the particular choice of the numerical 2×2 matrix in the equation above.

   (c) Iterate on (0.1) using Newton's method, using the same starting approximation $x_0 = y_0 = 1$, and check how well this converges.

   (d) Spot from your numerical result what the exact solution is, and then verify that analytically.

2. The nonlinear system

$$\begin{cases} f(x, y) = x^2 + y^2 - 4 = 0 \\ g(x, y) = e^x + y - 1 = 0 \end{cases}$$

   has two real solutions. Use Newton and two quasi-Newton methods (Lazy Newton and Broyden) with the different initial guesses:

   (i) $x = 1$, $y = 1$

   (ii) $x = 1$, $y = -1$

   (iii) $x = 0$, $y = 0$

   Is the performance better or worse that of Newton's methods?

3. Consider the nonlinear system

$$\begin{aligned} x + \cos(xyz) - 1 &= 0, \\ (1 - x)^{1/4} + y + 0.05z^2 - 0.15z - 1 &= 0, \\ -x^2 - 0.1y^2 + 0.01y + z - 1 &= 0. \end{aligned}$$

   Test the following three techniques for approximating the solution to the nonlinear system to within 10^{-6}:

   - Newton's method

- Steepest descent method
- First Steepest descent method with a stopping tolerance of 5×10^{-2}. Use the result of this as the initial guess for Newton's method.

Using the same initial guess, which technique converges the fastest? Try to explain the performance.

4. We are given the following interpolation data with a degree 4 polynomial $p(x)$:

x_j	0	2	3	5	8
y_j	-125	-27	-8	0	27

- Write $p(x)$ using the Lagrange polynomial basis for this problem.
- Find the coefficients for Newton interpolation, and write $p(x)$ as a linear combination of Newton polynomials.
- This data was sampled from $f(x) = (x-5)^3$. Explain what this means in terms of differences of order 4 or higher (like the last coefficient above) using Newton interpolation.