APPM 4600 Lab 10
Building L^2 approximations

1 Overview

In this lab, you will build a code that allows you to build L^2 approximations. You will use quadrature algorithm that is built into `SCIPY`.

2 Before Lab

1. The Legendre polynomials can be evaluated via the following three term recursion:

$$\phi_0(x) = 1$$
$$\phi_1(x) = x$$
$$\phi_{n+1}(x) = \frac{1}{n+1}\left((2n+1)x\phi_n(x) - n\phi_{n-1}(x)\right)$$

   Write a subroutine named `eval_legendre` that takes in an order n and value x where the polynomials are to be evaluated at and returns a vector **p** of length $n+1$ whose entries are the values of the Legendre polynmials at x. You can compare this routine with the scipy function legendre.

3 Lab Day: Building the L^2 approximations

During lab, you will write a code that evaluates L^2 approximations of functions.

3.1 Creating the L^2 approximation

Recall from class that the polynomial of degree n ($p_n(x)$) that approximates a function $f(x)$ with respect to a weight function $w(x) \geq 0$ on an interval I is given by

$$p_n(x) = \sum_{j=0}^{n} a_j\phi_j(x)$$

where

$$a_j = \frac{\langle \phi_j, f\rangle_{L^2_w}}{\langle \phi_j, \phi_j\rangle_{L^2_w}} = \frac{\int_I \phi_j(x)f(x)w(x)dx}{\int_I \phi_j^2(x)w(x)dx}$$

and $\phi_j(x)$ are a set of polynomials orthogonal on I with respect to $w(x)$.

3.2 Exercises

1. Using `scipy.integrate.quad`, create a one line code that evaluates a coefficient a_j. Note: you have to create a subroutine that evaluates the $f(x)\phi_j(x)w(x)$ to feed into this, and also a subroutine $\phi_j^2(x)w(x)$ for evaluating the normalization. You should not use any symbolic packages.

   You may want to use the following to import the package:
   ```
   from scipy.integrate import quad
   ```

2. Take the method you developed in the prelab and the coefficient evaluator in problem 1 and insert them into the partially completed code below.

```
import matplotlib.pyplot as plt
import numpy as np
import numpy.linalg as la
import math
from scipy.integrate import quad

def driver():

#  function you want to approximate
    f = lambda x: math.exp(x)

# Interval of interest
    a = -1
    b = 1
# weight function
    w = lambda x: 1.

# order of approximation
    n = 2

#  Number of points you want to sample in [a,b]
    N = 1000
    xeval = np.linspace(a,b,N+1)
    pval = np.zeros(N+1)

    for kk in range(N+1):
      pval[kk] = eval_legendre_expansion(f,a,b,w,n,xeval[kk])

    ''' create vector with exact values'''
    fex = np.zeros(N+1)
    for kk in range(N+1):
        fex[kk] = f(xeval[kk])

    plt.figure()
    plt.plot(xeval,fex,'ro-', label= 'f(x)')
    plt.plot(xeval,pval,'bs--',label= 'Expansion')
    plt.legend()
    plt.show()

    err = abs(pval-fex)
    plt.semilogy(xeval,err_l,'ro--',label='error')
    plt.legend()
    plt.show()
```

```
def eval_legendre_expansion(f,a,b,w,n,x):

#   This subroutine evaluates the Legendre expansion

#  Evaluate all the Legendre polynomials at x that are needed
# by calling your code from prelab
  p = ...
  # initialize the sum to 0
  pval = 0.0
  for j in range(0,n+1):
      # make a function handle for evaluating phi_j(x)
      phi_j = lambda x: ...
      # make a function handle for evaluating phi_j^2(x)*w(x)
      phi_j_sq = lambda x: ...
      # use the quad function from scipy to evaluate normalizations
      norm_fac,err = ...
      # make a function handle for phi_j(x)*f(x)*w(x)/norm_fac
      func_j = lambda x: ...
      # use the quad function from scipy to evaluate coeffs
      aj,err = ...
      # accumulate into pval
      pval = pval+aj*p[j]

  return pval

if __name__ == '__main__':
  # run the drivers only if this is called from the command line
  driver()
```

3. Change the function being approximated to $f(x) = \frac{1}{1+x^2}$. Does the accuracy of the approximation change? If so, how so?

3.3 Additional Exercises

As an additional exercise write a new code that creates an L^2 approximation using the Chebychev polynomials. They are also defined on the interval $[-1, 1]$ but the weight function is different

$$w(x) = \frac{1}{\sqrt{1-x^2}}.$$

The three term recursion is defined as follows

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n - T_{n-1}(x) \end{aligned}$$

Note that the code is the same except for: 1- You need a T_n evaluator and 2- write a new function that gets called in the coefficient evaluator.

3.4 Deliverables

All codes should be pushed to git and your responses to the questions should be entered into Canvas.