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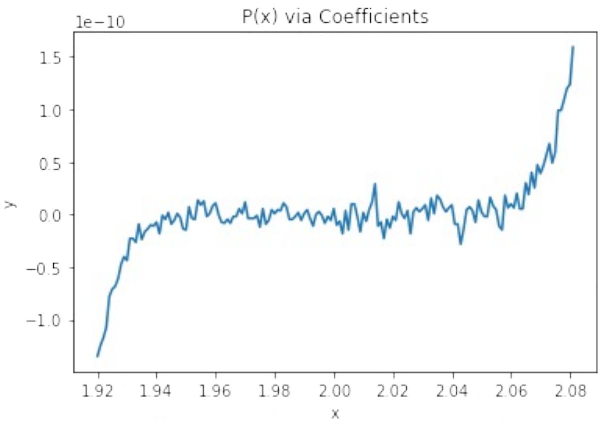
4600 HW1

1. Consider the polynomial
$p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512.$

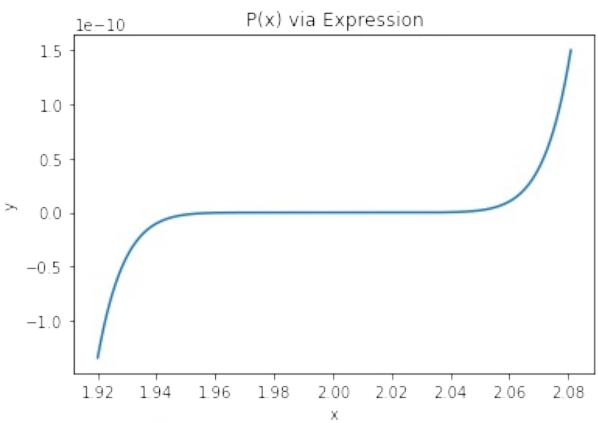
   i. Plot $p(x)$ for $x = 1.920, 1.921, 1.922, \ldots, 2.080$ (i.e. $x = [1.920 : 0.001 : 2.080];$) evaluating $p$ via its coefficients.
   ii. Produce the same plot again, now evaluating $p$ via the expression $(x-2)^9$.
   iii. What is the difference? What is causing the discrepancy? Which plot is correct?

i.




* All code at end of document

ii.




iii. Clearly, the two forms of p(x) are producing different graphs. In i, there are many more operations needed to produce the values for p(x) leading to increased error. Neither graph is totally correct but ii is more correct since only two operations (1 subtraction vs 5) are needed to calculate p(x).

2. How would you perform the following calculations to avoid cancellation? Justify your answers.

   i. Evaluate $\sqrt{x+1} - 1$ for $x \simeq 0$.
   ii. Evaluate $\sin(x) - \sin(y)$ for $x \simeq y$.
   iii. Evaluate $\frac{1-\cos(x)}{\sin(x)}$ for $x \simeq 0$.

Goal is to eliminate subtractions as this leads to loss of precision most commonly.

i. $\sqrt{x+1} - 1 = \left(\sqrt{x+1} - 1\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) = \boxed{\frac{x}{\sqrt{x+1}+1}}$

ii. $\sin x - \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \boxed{2\sin\left(\frac{x+y}{2}\right)\left(\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) + \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\right)}$

iii. $\frac{1-\cos(x)}{\sin(x)} = \frac{1-\cos(x)}{\sin(x)} \cdot \left(\frac{1+\cos(x)}{1+\cos(x)}\right) = \frac{1-\cos^2(x)}{\sin(x)(1+\cos(x))} = \frac{\sin^2 x}{\sin(x)(1+\cos(x))} = \boxed{\frac{\sin(x)}{1+\cos(x)}}$

3. Find the second degree Taylor polynomial $P_2(x)$ for $f(x) = (1 + x + x^3)\cos(x)$ about $x_0 = 0$.

(a) Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for the error $|f(0.5) - P_2(0.5)|$ using the error formula and compare it to the actual error.

(b) Find a bound for the error $|f(x) - P_2(x)|$ when $P_2(x)$ is used to approximate $f(x)$. This will be a function of $x$.

(c) Approximate $\int_0^1 f(x)dx$ using $\int_0^1 P_2(x)dx$.

(d) Estimate the error in the integral.

a) $f'(x) = (1+3x^2)\cos x - (1+x+x^3)\sin x$

$f''(x) = 6x\cos x - (1+3x^2)\sin x - (1+3x^2)\sin x - (1+x+x^3)\cos x = (-1+5x-x^3)\cos x - 2(1+3x^2)\sin x$

$f'''(x) = (5-3x^2)\cos x - (-1+5x-x^3)\sin x - 12\sin x - 2(1+3x^2)\cos x = (x^3-17x+1)\sin x + (-9x^2+3)\cos x$

$P_2(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2}(x)^2 = 1 + x - \frac{x^2}{2}$ so $P_2(0.5) = 1 + 0.5 - \frac{(0.5)^2}{2} = 1.375$

$R_2(x) = \frac{f'''(c)}{6}(x)^3$ for $c \in (0, x)$, $R_3(x) = \frac{1}{6}((c^3-17c+1)\sin c + (-9c^2+3)\cos c)x^3$

$(c^3-17c+1)\sin c + (-9c^2+3)\cos c$ is maximized on $c \in (0, \frac{1}{2})$ at $c = 0 \Rightarrow$ a max of $3$

So $R_2(x) \le \frac{3}{6}x^3 = \frac{1}{2}x^3$, for $x = 0.5$, $\frac{1}{2}(0.5)^3 = .0625$

Actual Value $\approx 1.426$, $0.0625 >$ actual error of $(1.426 - 1.375) = 0.0512$

b) In general $R_2(x) \le \left|\frac{1}{6}(\max\{f'''(c) : c \in (0,x)\}) \cdot x^3\right|$

c) $\int_0^1 P_2(x)dx = \int_0^1 1 + x - \frac{x^2}{2}\,dx = \left(x + \frac{x^2}{2} - \frac{x^3}{6}\right)\Big|_0^1 = 1 + \frac{1}{2} - \frac{1}{6} = \frac{4}{3}$

d) $\int_0^1 \frac{1}{2}x^3 dx = \frac{x^4}{8}\Big|_0^1 = \frac{1}{8}$

4. Consider the quadratic equation $ax^2 + bx + c = 0$ with $a = 1, b = -56, c = 1$.

   (a) Assume you can calculate the square root with 3 correct decimals (e.g. $\sqrt(2) \approx 1.414 \pm \frac{1}{2}10^{-3}$) and compute the relative errors for the two roots to the quadratic when computed using the standard formula

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

   (b) A better approximation for the "bad" root can be found by manipulating $(x - r_1)(x - r_2) = 0$ so that $r_1$ and $r_2$ can be related to $a, b, c$. Find such relations (there are two) and see if either can be used to compute the "bad" root more accurately.



a) $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} = \dfrac{56 + 55.964}{2} = 55.982$

w/ Relative Error: $\left| \dfrac{55.982137 - 55.982}{55.982137} \right| \approx 2.45 \times 10^{-6}$

$r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} = \dfrac{56 - 55.964}{2} = 0.018$

w/ relative error: $\left| \dfrac{0.0178628 - 0.018}{0.0178628} \right| \approx .008$

b) Let $ax^2 + bx + c = 0 = a(x - r_1)(x - r_2) = a(x^2 - (r_1 + r_2)x + r_1 r_2) \Rightarrow b = -a(r_1 + r_2),\ c = a r_1 r_2$

From above, the "good root" is $r_1 = 55.982$

$b = -a(r_1 + r_2) \Rightarrow -56 = -1(55.982 + r_2) \Rightarrow r_2 = 56 - 55.982 = 0.018$ which is what we computed in the previous part so it is not an improvement.

$c = a r_1 r_2 \Rightarrow 1 = 55.982 r_2 \Rightarrow \underbrace{r_2 \approx 0.01786}_{\text{even better if you keep more decimals}}$ which has rel error $\dfrac{|0.017826 - 0.01781|}{0.017826} \approx 0.0015$

which is better than our approximation in part a.

5. **Cancellation of terms**. Consider computing $y = x_1 - x_2$ with $\tilde{x}_1 = x_1 + \Delta x_1$ and $\tilde{x}_2 = x_2 + \Delta x_2$ being approximations to the exact values. If the operation $x_1 - x_2$ is carried out exactly we have $\tilde{y} = y + \underbrace{(\Delta x_1 - \Delta x_2)}_{\Delta y}$.

*Play with different values of $x$. One really small value ($< 1$) and one large value $> 10^5$.*

(a) Find upper bounds on the absolute error $|\Delta y|$ and the relative error $|\Delta y|/|y|$, when is the relative error large?

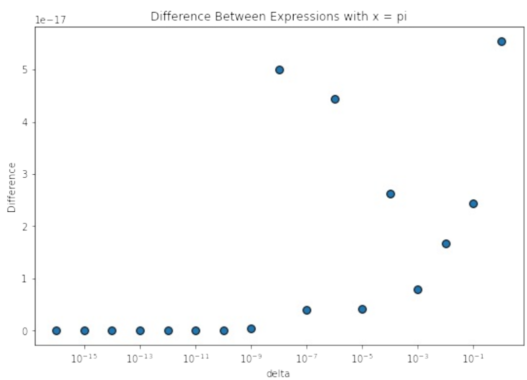
(b) First manipulate $\cos(x+\delta) - \cos(x)$ into an expression without subtraction. Pick two values of $x$; say $x = \pi$ and $x = 10^6$. Then for each $x$, tabulate or plot the difference between your expression and $\cos(x+\delta) - \cos(x)$ for $\delta = 10^{-16}, 10^{-15}, \cdots, 10^{-2}, 10^{-1}, 10^0$ (note that you can use your logx command to uniformly distribute $\delta$ on the x-axis).

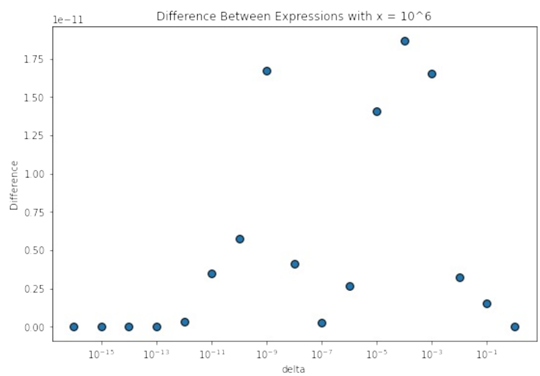
(c) Taylor expansion yields $f(x+\delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2!} f''(\xi), \quad \xi \in [x, x+\delta]$. Use this expression to create your own algorithm for approximating $\cos(x+\delta) - \cos(x)$. Explain why you chose the algorithm. Then compare the approximation from your algorithm with the techniques in part (b). Use the same values for $x$ and $\delta$.

a) Absolute error : $|\Delta y| = |\Delta x_1 + \Delta x_2| \le |\Delta x_1| + |\Delta x_2|$

Relative error $\dfrac{|\Delta y|}{|y|} \le \dfrac{|\Delta x_1| + |\Delta x_2|}{|x_1| + |x_2|}$

b) $\cos(x+\delta) - \cos(x) = -2\sin\left(\frac{x+\delta+x}{2}\right)\sin\left(\frac{x+\delta-x}{2}\right) = -2\sin\left(\frac{2x+\delta}{2}\right)\sin\left(\frac{\delta}{2}\right)$

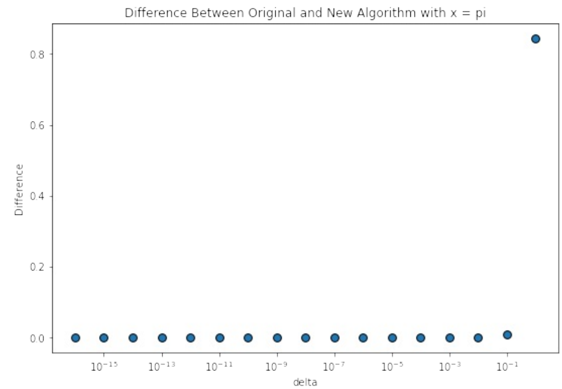


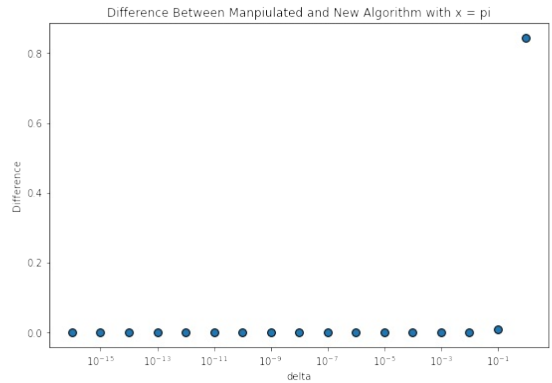



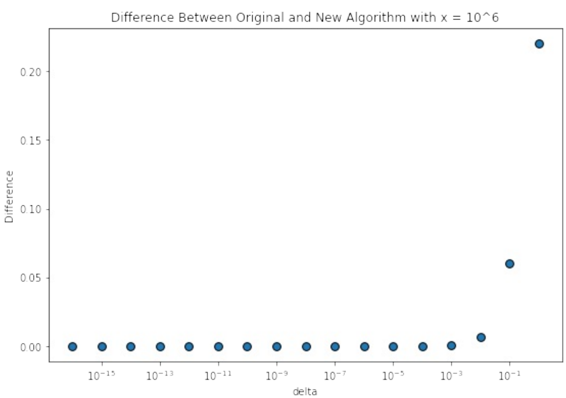
c) $f(x+\delta)-f(x)=\delta f'(x)+\frac{\delta^2}{2}f'(c) = -\delta\sin(x)-\frac{\delta^2}{2}\cos(c)=-1\left(\delta\sin(x)+\frac{\delta^2}{2}\cos(c)\right)$, $c\in[x,x+\delta]$

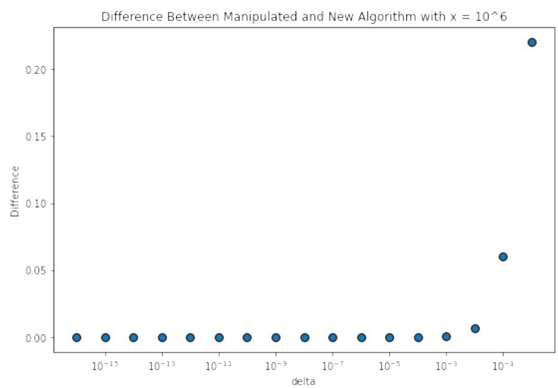
Assuming $\delta$ is small, $\delta^2$ will be even smaller so the $\frac{\delta^2}{2}\cos(c)$ term will have minimum impact on our result. Furthermore, we replace $\cos(c)$ with an average of the left and right input over the interval $[x, x+\delta]$.

$$\Rightarrow -1\left(\delta\sin(x)+\frac{\delta^2}{2}\cos(c)\right)\approx -1\left(\delta\sin x+\frac{\delta^2}{2}\left(\frac{\cos(x)+\cos(x+\delta)}{2}\right)\right)$$




* Notice more difference is visible in SB graphs since error axis is scaled down significantly.

```python
#HW1 Q1
def expanded(x):
     return 1*x**9 -18*x**8 +144*x**7 -672*x**6 +2016*x**5 -4032*x**4 +5376*x**3 -4608*x**2
+2304*x -512

def compact(x):
    return (x-2)**9

x_values = np.arange(1.92, 2.081, 0.001)
y_values = expanded(x_values)

plt.plot(x_values, y_values, label='expanded function')
plt.title('P(x) via Coefficients')
plt.xlabel('x')
plt.ylabel('y')
plt.show()

y_values = compact(x_values)

plt.plot(x_values, y_values, label='expanded function')
plt.title('P(x) via Expression')
plt.xlabel('x')
plt.ylabel('y')
plt.show()

#HW1 Q5
x = 10**6

delta = 10**(np.linspace(-16,0,17))

fig, ax = plt.subplots(figsize = (9, 6))
ax.scatter(delta, abs(np.cos(x + delta) - np.cos(x) - -2 * np.sin((2 * x + delta) / 2) *
np.sin(delta/ 2)), s=60, edgecolors="k")
ax.set_xscale("log")
plt.xlabel('delta')
plt.ylabel('Difference')
plt.title('Difference Between Expressions with x = 10^6')
plt.show()

#Implementation of new Algorithm
-1 * (delta * np.sin(x) + (.25 * delta ** 2 * (np.cos(x) + np.cos(x+delta))))
```