Henry Dyer, I agree to the honor pledge.

1A) $x^2 + 500x + 2 = 0$

$$x = \frac{-500 \pm \sqrt{500^2 - 4(2)}}{2} = -250 \pm \sqrt{62498}$$

$\sqrt{62498}$ is close to 250 so the root $-250 + \sqrt{62498}$ could incur a loss of precision due to the round off error from subtracting to close numbers.

1B) $a(x-r_1)(x-r_2) = ax^2 + ax(+r_1 + r_2) + ar_1 r_2 = ax^2 + bx + c$

In our case, $a = 1$

so $500 = -r_1 - r_2$, $2 = r_1 r_2$

From part A, we do not incur round off error when solving for $-250 - \sqrt{62498} = r_1$ root so set $r_2 = \frac{2}{r_1}$ which will find the other root to full precision since division is stable.

2A) f is continuous on $[15,25]$ with $f(15)=-8.5$, $f(25)=16.7$ so IVT guarantees $\exists c$ s.t. $f(c)=0$, $c\in(15,25)$ since $0\in$ image of f on this interval.

Bisection guarantees convergence since f is continuous and $f(a)f(b)=(-8.5)(16.7)<0$.

2B) For absolute error: $\frac{b-a}{2^n}=\frac{10}{2^n}\leq 10^{-6}$

but relative error should not consider length of interval since $b-a<10^{-6}$ would indicate this precision without iteration so

we use $\frac{1}{2^n}\leq 10^{-6}$ for relative

error $\Rightarrow$ $n=\lceil \log_2(10^6)\rceil = 20$ iterations.

3A) $g_b'(x) = b(1 - 8\cos(2x)) + 1 = -8b\cos(2x) + b + 1$

Convergence guaranteed if $|f'(x_i^*)| < 1$

$|-8b\cos(2x) + b + 1| < 1$

$\Rightarrow -1 < -8b\cos(2x) + b + 1 < 1$

$\Rightarrow -b < -8b\cos(2x) < b$

$\Rightarrow -1 < -8\cos(2x) < 1$ (If $b=0$ FPI would be useless)

$\Rightarrow \frac{-1}{8} < \cos(2x) < \frac{1}{8}$

If $|\cos(2x)| < \frac{1}{8}$ at x_i^*, FPI guarantees convergence.

3B) x_1^* will not converge, $|g_b'(x_1^*)| > 1$

x_2^* will converge, $|g_b'(x_2^*)| < 1$

x_3^* will not conv, $|g_b'(x_3^*)| > 1$

x_4^* will conv, $|g_b'(x_4^*)| < 1$

x_5^* will not conv, $|g_b'(x_5^*)| > 1$

3C) x_1^*, x_3^*, x_5^* will conv since $|g_b'| < 1$ at these points.

x_2^*, x_4^* will not since $|g_b'| > 1$

3D) For FPI, rate of conv $\lambda = g_b'(x^*)$ evaluated at x^*, λ closer to zero, ie $|g_b'|$ smaller means faster convergence.

3B, 3C achieve conv for all points so we ranked magnitude of g_b' $\forall x_i^*$ from small (fast) to large (slow)

Fast: x_5^*, x_3^*, x_1^* x_2^*, x_4^* slow

4A) $F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} (x-3)^2 + 2(y-1)^2 - 25 \\ xy - 3y - 9 \end{bmatrix}$ (set equations = to zero)

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x-6 & 4y-4 \\ y & x-3 \end{pmatrix}$$

4B) $$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0+2-25 \\ 6-6-9 \end{pmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -23 \\ -9 \end{pmatrix}$$

4C) No since $J(3,0) = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}$

which is not invertible so would be unable to compute step for Newtons method.

5A) Condition # = $\frac{\text{rel error in output}}{\text{rel error input}}$ so if we have $\varepsilon_{mach} \approx 10^{-16}$ (in double) with condition # $\kappa \approx 10^{8}$ we could trust roughly 8 significant digits (from $10^{8} \cdot 10^{-16}$) before relative error in output is to large for useful approximations.

5B) False, if $f'(x_0)$ (or x close to x_0) is close to zero, you newton step could shoot you off far away from root and would be difficult to get back, if you ever do.

5C) Method 1: linear, Δ log error from it_{i+1} to i is roughly 0.48 each step so convergence is linear.

Method 2: super linear (likely quadratic) since log error roughly halves each step, (so accuracy doubles) which is like with Newtons Method.

Method 3: of consecutive error terms grows with iterations so accuracy also grows (since negative exponent $10^{\log_{10}|en|}$) so this is super linear, likely FPI for double root since this is super linear but not quadratic like Newtons.