Henry Dyer, I agree to the honor pledge

1A)

$$M \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 1 & 0 \\ 1 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -2 \\ 0 \\ 2 \\ 6 \\ 3 \end{pmatrix} \leftarrow Y$$

(1B) Normal Equations : $M^T M c = M^T Y$

$\Rightarrow \quad c = (M^T M)^{-1} M^T Y$

1C) $M^T M$ can be ill conditioned and hard to invert, instead using the singular value decomposition of M can be make this process easier.

2A)

x_0	-1	0	1
e^{3x_0}	e^{-3}	1	e^{3}

$$p(x) = \begin{cases} (1-e^{-3})x+1 : & x \in [-1,0) \\ (e^{3}-1)x+1 : & x \in [0,1] \end{cases}$$

2B) $[x_0, x_1] = [-1, 0]$

error: $e(x) = |e^{3x} - e^{-3}x - 2| \leq 1$

since function contains no roots $e^{3x} - e^{-3}x < 2 \ \forall x$ on interval and

$x=0 : |e^{0} - 0 - 2| = 1 > |e^{-3} + e^{-3} - 2| : x = -1$

$[x_1, x_2]$

error: $e(x) = |e^{3x} - e^{-3}x - 2| \leq e^{3} - 2$

since $e' = 3e^{3x} - e^{-3}$ so $e' > 0 \ \forall x$ on $[0,1]$

hence $e(x)$ is monotonically increasing and so right end point $(x=1)$ is functions max.

This same logic for $[x_0, x_1]$ as well actually

$2 \cdot 1 \cdot -2$ $4 \cdot 3 \cdot 2$ $1(-1)(-3)$

3A)

$$L_0(x) = \frac{(x-1)(x-2)(x-4)}{-8}, \quad L_1(x) = \frac{x(x-2)(x-4)}{4}$$

$$L_2(x) = \frac{x(x-1)(x-4)}{-4}, \quad L_3(x) = \frac{x(x-1)(x-2)}{24}$$

Then $p(x) = -16L_0(x) - 2L_1(x) + 0L_2(x) + 16L_3(x)$

$\frac{-2+16}{1}$ $\frac{0+2}{1}$ $\frac{16}{4-2}$ $\frac{2-14}{2-0}$ $\frac{8-2}{4-1}$

3B)

0	-16			
		14		
1	-2		-6	
		2		2
2	0		2	
		8		
4	16			

So $p(x) = -16 + 14(x) - 6(x)(x-1) + 2(x)(x-1)(x-2)$

3c) It is more stable as it requires fewer steps to evaluate when a new term is added as it has already been accounted for in previous calculations.

$1 = \tan(x)$

4A) $P_0(x) = 1, \; P_1(x) = x$

$$P_2(x) = P_{\underset{n}{1}+1}(x) = \frac{(2(1)+1)\,x\,P_1(x) - (1)\,P_0(x)}{2}$$

So $P_2(x) = \dfrac{3x^2 - 1}{2}$

some inverse trig function

4B)
$$c_0 = \frac{\int_{-1}^{1} (1)\frac{1}{1+x^2}\,dx}{\int_{-1}^{1} 1^2\,dx} = \frac{\arctan(x)\Big|_{-1}^{1}}{2} = \frac{\frac{\pi}{4} - \frac{7\pi}{4}}{2} = \frac{3\pi}{2}$$

$\frac{1}{2}(\ln(2) - \ln(2)) = 0$

$$c_1 = \frac{\int_{-1}^{1} x\frac{1}{1+x^2}\,dx}{\int_{-1}^{1} x^2\,dx} = \frac{\frac{1}{2}(\ln(1+x^2))\Big|_{-1}^{1}}{\frac{2}{3}} = \frac{0}{\frac{2}{3}}$$

So degree one approx is $P(x) = \frac{3\pi}{2} + 0x$

4C)
$$c_3 = \frac{\int_{-1}^{1} (x^3 - \frac{3}{5}x)\left(\frac{1}{1+x^2}\right)dx}{\int_{-1}^{1} (x^3 - \frac{3}{5}x)^2\,dx}$$

4A Continued)

$$\langle P_0, P_1 \rangle = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0 \checkmark$$

$$\langle P_0, P_2 \rangle = \frac{1}{2}\int_{-1}^{1} 3x^2 - 1\,dx = \frac{1}{2}(x^3 - x)\Big|_{-1}^{1} = \frac{1}{2}(0 - 0) = 0 \checkmark$$

$$\langle P_1, P_2 \rangle = \frac{1}{2}\int_{-1}^{1} 3x^3 - x\,dx = \frac{1}{2}\left(\frac{3}{4}x^4 - \frac{x^2}{2}\right) = \frac{1}{2}\left(\left(\frac{3}{4} - \frac{1}{2}\right) - \left(\frac{3}{4} - \frac{1}{2}\right)\right)$$

$$= 0 \checkmark$$

5A) $\nabla g(x,y) = -2(1-x) + 8(y-x^2)(-2x)$
$+ 8(y-x^2)$

$= 8(y-x^2)(1-2x) - 2 + 2x$

$\vec{x}_1 = x_0 - \alpha \nabla g(x,y) \therefore \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \alpha \nabla g(0,0)$

We choose α often by line search so that the step size is not to large causing us to over shoot the minimum.

5B) Lazy Newton: $x_{n+1} = x_n - J_F^{-1}(x_0) F(x_n)$, inverse Jacobian only calculated once so less computationally expensive.

Broyden: $x_{n+1} = x_n - (J_F(x_0) + uv^T) F(x_n)$ For uv^T a rank 1 update, super linear convergence while updating inverse w/ low rank update only $O(n^2)$ vs $O(n^3)$ to compute inverse like in Newton.

5C) $\text{Error}(x_i) = \frac{f^{(n+1)}(\sigma)}{(n+1)!}(x_i - \sigma)$, $\sigma \in [-1, 1]$

Chebyshev nodes are found from projecting equispaced nodes on unit circle to x axis and help tame the approximation (particularly n+1 deriv) near end points to avoid Runge Phenomenon.

5D) By definition of orthogonality,

$\forall x \in [0, 8]$, $\langle P_J(x), P_I(x) \rangle = 0$ so for any x, $\left(\frac{x-4}{4}\right)$ is a linear factor which can be factored out of inner product by linearity (of inner product operator) thus giving us $\left(\frac{x-4}{4}\right)^2 \langle P_J(x), P_I(x) \rangle = 0$, so orthogonality is preserved.