STAT 135 Lecture 9

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10 July 2025

Remark 0.1

Simple vs Simple Hypothesis Testing
Two possible values, $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$
Simple vs simple hypothesis test takes the form
$X_1, \dots, X_n \sim f$
$H_0 : f = f_0$
$H_1 : f = f_1$

Lemma 0.2 (Neyman-Pearson Lemma)

Likelihood ratio tests are good in simple vs simple hypothesis testing.

Method 0.3 (Likelihood Ratio Test playbook)

1. Write down likelihood ratios (LR, Λ)
2. Find c such that $P(LR < c|H_0) = \alpha$

Example 0.4

Normal with known variance $X_1, X_2, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 is known
$H_0 : \mu = \mu_0$
$H_1 : \mu = \mu_1$

Solution.

$$LR = \frac{\exp\left[\frac{1}{-2\sigma^2}\sum_{i=1}^n (X_i - \mu_0)^2\right]}{\exp\left[\frac{1}{-2\sigma^2}\sum_{i=1}^n (X_i - \mu_1)^2\right]} \qquad \text{Take log of LR}$$

$$\log(LR) = \sum_{i=1}^n (X_i - \mu_0)^2 - \sum_{i=1}^n (X_i - \mu_1)^2 = 2n\overline{X}(\mu_0 - \mu_1) + n\mu_1^2 - n\mu_0^2$$

Reject $\overline{X}$ large, find a c such that $P(\overline{X} > c|H_0) = \alpha$ ∎

Example 0.5

Laplace example

Have 1 observation $X \sim f(x|\sigma) = \frac{1}{2\sigma} e^{\frac{-|x|}{\sigma}}$

$H_0 : \sigma = 1$

$H_1 : \sigma = 2$

Solution.

$$\Lambda = \frac{\frac{1}{2} e^{-|x|}}{\frac{1}{4} e^{\frac{-|x|}{2}}}$$

Reject for small values of Λ ie $-|x| + \frac{|x|}{2}$, small values $-\frac{|x|}{2}$, ie large values of $|x|$
Need to find a critical value, ie c such that $P(|X| > c|H_0) = \alpha$
Under H_0 $X \sim f(x|1) = \frac{1}{2} e^{-|x|}$

$$2 \int_c^\infty \frac{1}{2} e^{-x} dx = \alpha, \qquad \alpha = 0.05 \text{normal value of } \alpha$$

$$-e^{-x}\Big|_c^\infty = 0.05 \rightarrow c = -\log 0.05 \approx 3$$

■

Example 0.6

p-values

$X_1, X_2, \ldots, X_{25} \overset{iid}{\sim} \mathcal{N}(\mu, 1)$

$H_0 : \mu = 1$ $H_1 : \mu = 2$

Solution. Reject for large values of $\overline{X}$. Under the null hypothesis $\overline{X} \sim N(1, \frac{1}{25})$

α	Reject for $\overline{X} > c$
0.10	$\overline{X} \geq 1.26$
0.05	$\overline{X} \geq 1.33$
0.03	$\overline{X} \geq 1.37$

Suppose we have that $\overline{X} = 1.37$, then we reject the null if we have $\alpha = 0.05$
To find the minimal α value where we would reject $\overline{X}$, we just take $1 - cdf(\overline{X})$ ■

Definition 0.7 (**p-value**)

Without setting α in advance, can look at test statistic and ask what is the smallest α for which we reject H_0
Reject if p-value $\leq \alpha$
Ways to interpret the p-value

(i) Smallest α for which we would reject H_0

(ii) P(Observing more extreme test statistics than we observed)

(iii) "Strength of evidence" against H_0

Method 0.8 (**Composite Tests**)

$\theta \in \Omega$
$H_0 : \theta \in \omega_0$
$H_1 : \theta \in \omega_1$
$\omega_0 \cup \omega_1 = \Omega$
$\omega_0 \cap \omega_1 = \emptyset$
$X_1, X_2, \ldots, X_n \sim f(x|\theta)$

Method 0.9 (**Generalized Likelihood Ratio Test**)

$$\Lambda^* = \frac{\max\limits_{\theta \in \omega_0} f(x_1, \ldots, x_n|\theta)}{\max\limits_{\theta \in \omega_1} f(x_1, \ldots, x_n|\theta)} = \frac{\text{Strongest case for } H_0}{\text{Strongest case for } H_1}$$

$$\Lambda = \frac{\max\limits_{\theta \in \omega_0} f(x_1, \ldots, x_n|\theta)}{\max\limits_{\theta \in \Omega} f(x_1, \ldots, x_n|\theta)} = \min(\Lambda^*, 1)$$

Example 0.10

$X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, 1)$
$H_0 : \mu = 5$
$H_1 : \mu \neq 5$

Solution.

$$\omega_0 = \{5\}, \omega_1 = \mathbb{R} - \{5\}$$

$$\text{Generalized LR Stat} : \Lambda = \frac{\exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i - 5)^2\right]}{\exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i - \overline{X})^2\right]}$$

Reject for small values of Λ
Take logs etc., simplifies to
"reject for large values of $n(\overline{x} - 5)^2$", $\sqrt{n}(\overline{X} - 5) \sim \mathcal{N}(0, 1)$, $n(\overline{X} - 5)^2 \sim \chi_1^2$
Find k such that $P(n(\overline{X} - 5)^2 > k|H_0) = \alpha$, reject for large values of $n(\overline{X} - 5)^2$, same as reject large values of $|\overline{X} - 5|$ ■

Theorem 0.11 (**Wilk's Theorem**)

Why am I mentioning χ^2, more generally,

$$-2 \log \Lambda \sim \chi_k^2, k = \dim \omega_1 - \dim \omega_0$$

Use the known distribution to get critical values
Reject for large values of $-2 \log \Lambda$ (quantiles of χ_k^2)