STAT 135 Lecture 4

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Remark 0.1

Quiz:

- 30 minutes
- Closed everything
- No Electronics
- 2 Questions

Remark 0.2

Last time

- Method of moments
  - First strategy for estimation
  - One good property: Consistency

Example 0.3

Independent observations
$P(X = 0) = \frac{2}{3}\theta$
$P(X = 1) = \frac{1}{3}\theta$
$P(X = 2) = \frac{2}{3}(1 - \theta)$
$P(X = 3) = \frac{1}{3}(1 - \theta)$
Can estimate the mean μ write $\theta = f(\mu) = f(\overline{X})$

Example 0.4

$X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} exp(\lambda) = f(x) = \lambda e^{-\lambda x}, x > 0$
$\mu = E(X) = \int_0^\infty x\lambda e^{-\lambda x}dx, u = x, dv = \lambda e^{-\lambda}xdx$
$= xe^{-\lambda x}\big|_0^\infty + \int_0^\infty e^{-\lambda x}dx = \frac{1}{\lambda}$
$\lambda = \frac{1}{\mu_1} \to \hat{\lambda} = \frac{1}{\overline{X}}$

Method 0.5

Delta Method
Applies when our estimator is a smooth function of sample means
Helps us understand sampling distribution
Estimator $f(\overline{x})$
$f(\overline{X}) - f(\mu) \approx f'(\mu)(X - \mu)$
$Var(f(\overline{X}) - f(\mu)) \approx Var(f'(\mu)(X - \mu))$
$Var(f(\overline{X})) \approx (f'(\mu))^2 Var(\overline{X}) = f'(\mu)^2 \frac{\sigma^2}{n}$
$SE(f(\overline{X})) = \sqrt{Var(f(\overline{X}))}$
Now we have standard error of $f(\overline{X})$
Smooth:

- f differentiable on a neighborhood of μ
- f' continuous
- $f'(\mu) \neq 0$

Remark 0.6

$Var(X) = E[(X - E(X))^2]$
$SD(X) = \sqrt{Var(X)}$
When discussing Estimators
T is an estimator we call $\sqrt{Var(T)}$ the standard error

Example 0.7

Exponential Example:
$X_1, X_2, \ldots, X_n \overset{iid}{\sim} f(x) = \lambda e^{-\lambda x}, x > 0$
Estimate λ
$T = \frac{1}{\overline{X}}$
$Var(T) = Var(\frac{1}{\overline{X}})$
Use delta method
$f(x) = \frac{1}{x}, f'(x) = \frac{-1}{x^2}$
$Var(\frac{1}{\overline{X}}) = \left(\frac{-1}{\mu^2}\right)^2 \frac{\sigma^2}{n} = \left(\frac{1}{(\frac{1}{\lambda})^2}\right)^2 \frac{(\frac{1}{\lambda^2})}{n} = \frac{\lambda^2}{n}$
Also true:
$\sqrt{n}(f(\overline{X}) - f(\mu)) \xrightarrow{d} \mathcal{N}(0, (f'(\mu))^2\sigma^2)$
So we can form approximate confidence intervals

Definition 0.8

Maximum likelihood Estimation
Flip a possibly biased coin 10 times and get 6 heads lands head with unknown probability p
Guess p
P(6 heads in 10 flips) $= \binom{10}{6} p^6(1-p)^4$
Pick a value of p that maximizes the probability
$\arg\max_p 6\log(p) + 4\log(1-p)$
Take derivative $\frac{6}{p} - \frac{4}{1-p} = 0 \rightarrow p = \frac{3}{5}$
General setting
$X_1, X_2, \ldots, X_n \overset{iid}{\sim} f(x|\theta)$
MLE approach is to find the maximizing θ
Likelihood function $= l(\theta) = \prod_{i=1}^n f(x_i|\theta)$
Log Likelihood Function: $L(\theta) = \sum_{i=1}^n \log f(x_i|\theta)$
$\hat{\theta}_{MLE} = \arg\max_\theta l(\theta)$
Probability: $f(x|\theta)$
View as a function of x given θ
Likelihood: $f(x|\theta)$
View it as a function of θ given x (data)

Example 0.9

Exponential Example (MLE)
$X_1, \ldots, X_n \overset{iid}{\sim} f(x) = \lambda e^{-\lambda x}$
$L(\lambda) = \sum_{i=1}^n \log \lambda + \lambda x_i$
$L'(\lambda) = \sum_{i=1}^n \frac{1}{\lambda} - x_i = 0 \rightarrow \frac{n}{\lambda} = \sum_{i=1}^n x_i \rightarrow \lambda = \frac{n}{\sum_{i=1}^n x_i} = \frac{1}{\bar{X}}$

Remark 0.10

Takeaways:

- Delta Method:
  Helps when estimators are smooth functions of sample means
- Maximum Likelihood:
  New strategy for estimation