STAT 135 Lecture 8

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Example 0.1 (Rao-Blackwellization Example)

Have $X_1, \ldots, X_n \overset{iid}{\sim} Bernoulli(p)$
Estimate p

Solution. $T = \sum_{i=1}^n X_i$ is sufficient

$$\hat{\theta} = X_1$$

Use Rao-Blackwell to improve

$$\tilde{\theta} = E(\hat{\theta}|T)$$

$$E\left[\sum_{i=1}^n X_i|T = t\right] = t \to \sum_{i=1}^n E\left[X_i|T = t\right] = t$$

$$E(X_1|\sum_{i=n}^n X_i = t) = \frac{t}{n} = \frac{\sum_{i=1}^n x_i}{n} \qquad \text{sample fraction}$$

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Method 0.2 (Rao-Blackwell Playbook)

1. Start with an estimator $\hat{\theta}$, could be bad
2. Find a sufficient statistic T
3. Rao-Blackwell says make a new estimator $\tilde{\theta} = E(\hat{\theta}|T)$
4. $\tilde{\theta}$ is better than $\hat{\theta}$

Remark 0.3 (Statistics Big Picture, Coin Flipping)

1. What is a good estimate for p? (Estimation)
2. Is the coin fair? (Hypothesis testing)
3. What is a reasonable range for p? (Interval esimation/confidence intervals)

Definition 0.4 (Hypothesis Testing)

- Statements about a parameter and whether data provides evidence for/against
- Decisions based on statements

Example 0.5 (Analogy: Criminal Trial)

H_0 : Defendent is not guilty
H_1 : Defendent is guilty

True state of the world	Don't reject H_0 (Find not Guilty)	Reject H_0 (Find Guilty)
H_0 : Defendent Not Guilty	No Error	Type II Error
H_1 : Defendent Is Guilty	Type I Error	No Error

Definition 0.6 (Simple Hypotheses)

Two possible values for unknown θ, write the problem as:
Null $H_0 : \theta = \theta_0$ vs.
Alternative $H_1 : \theta = \theta_1$
Sample: One possible value

Method 0.7 (Setup)

Data $X_1, \ldots, X_n \sim f(x|\theta)$
$H_0 : \theta = \theta_0$
$H_1 : \theta = \theta_1$
Binary decisions $d(X_1, \ldots, X_n) = 0$ or 1
$P(d(X_1, \ldots, X_n) = 1|H_0) = P(\text{Type I error}) = \alpha$ (Significance level)
$P(d(X_1, \ldots, X_n) = 0|H_1) = P(\text{Type II error}) = \beta, 1 - \beta$ (Power)

Example 0.8

Two types of coin
$p = P(\text{heads})$
$H_0 : p = 0.5$
$H_1 : p = 0.7$
Three flips of the coin
X = number of heads, $X \sim bin(3, p)$

Solution.

x	$P(X = x\|H_0)$	$P(X = x\|H_1)$	$\frac{P(X=x\|H_0)}{P(X=x\|H_1)}$
0	0.125	0.027	4.63
1	0.375	0.189	1.98
2	0.375	0.441	0.85
3	0.125	0.343	0.36

Suppose we want $\alpha = \frac{1}{2}$, we would make the cutoff at $X \geq 2$ or $LR < 1.98$ (LR = Likelihood Ratio) What is β if we do that? $\beta = 0.216$ and power of test $= 1 - 0.216$

$$LR = \frac{P(X = x|H_0)}{P(X = x|H_1)}$$

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Definition 0.9 (Test Statistic)

Function of data we use to make the decision

Definition 0.10 (Rejection Region)

Sample values for which we reject H_0

Remark 0.11 (Problem Framing)

Require a hard limit on α. Try to minimize β subject to that or maximize power $1 - \beta$. Start with a limit on α

Lemma 0.12 (Neyman-Pearson Lemma)

Suppose H_0 and H_1 are simple hypotheses, if there is a likelihood ratio test with level α. No other test has higher power

Definition 0.13 (Likelihood Ratio Test)

$$\frac{f(X_1, \ldots, X_n|\theta_0)}{f(X_1, \ldots, X_n|\theta_1)}$$

Rejects for H_0 for some LR ¡ C