

# STAT 151A Lecture 13

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## Remark 0.1 (Multivariate Normal Distribution (MVN))

Scalar (usual) normal:  $y \sim \mathcal{N}(\mu, \sigma^2)$

Multivariate normal:  $\vec{Y} \sim \mathcal{N}_n(\vec{\mu}, \Sigma)$ ,  $\vec{\mu} \in \mathbb{R}^n$ ,  $\Sigma \in \mathbb{R}^{n \times n}$ ,  $\Sigma \succ 0$

Properties:

- If  $\vec{Y} \sim \mathcal{N}_n(\vec{\mu}, \Sigma)$ , linear functions of  $\vec{Y}$  are also MVN
- If  $\vec{Y} \sim \mathcal{N}_n(\vec{\mu}, \Sigma)$ , then  $Y_j(\mu_j, \sigma_{jj}^2)$
- If  $\vec{Y}$  and  $\vec{Z}$  are both MVN and independent,  $\vec{Y} + \vec{Z}$  is also MVN,  $\vec{Y} + \vec{Z} \sim \mathcal{N}_n(\vec{\mu}_Y + \vec{\mu}_Z, \Sigma_{YY} + \Sigma_{ZZ} + \Sigma_{YY})$
- If  $\vec{Z} \in \mathbb{R}^m$ ,  $\vec{Y} \in \mathbb{R}^n$ ,  $\begin{pmatrix} \vec{Z} \\ \vec{Y} \end{pmatrix} \sim MVN_{n+m}$  and  $\text{Cov}(\vec{Z}, \vec{Y}) \Leftrightarrow \vec{Z} \perp \vec{Y}$

Counterexample:  $X \sim \mathcal{N}(0, 1)$  :  $X$  and  $X^2$ ,  $\text{Cov}(X, X^2) = 0$

## Method 0.2 (Normal Linear Model)

$\vec{Y}_{|X} \sim \mathcal{N}_n(\mathbf{X}\beta, \sigma^2 \mathbb{I}_n)$

$\vec{Y} = \mathbf{X}\beta + \vec{\varepsilon}$ ,  $\vec{\varepsilon} \sim \mathcal{N}_n(\vec{0}, \sigma^2 \mathbb{I}_n)$  Stronger than Gauss-Markov model due to multivariate normal (No covariance)

Why? Back to  $\hat{\beta}_j$

Test whether null hypothesis  $H_0 : \beta_j = 0$

Use NLM:  $\hat{\beta}_j$

$\hat{\beta} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{Y}$

Distribution of  $\hat{\beta}$  under NLM we get another Multivariate normal

$\hat{\beta} \sim \mathcal{N}_p((\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{X} \beta, (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \sigma^2 \mathbb{I}_n \mathbf{X} (\mathbf{X}^\top \mathbf{X})^{-1}) \Rightarrow \hat{\beta} \sim \mathcal{N}_p(\beta, \sigma^2 (\mathbf{X}^\top \mathbf{X})^{-1})$

Under  $H_0$ :

$\hat{\beta}_j \stackrel{H_0}{\sim} \mathcal{N}(0, \sigma^2 (\mathbf{X}^\top \mathbf{X})_{jj})$

$\frac{\hat{\beta}_j}{\hat{\sigma} \sqrt{(\mathbf{X}^\top \mathbf{X})_{jj}}} \stackrel{H_0}{\sim} \mathcal{N}(0, 1)$

Since we have to estimate the standard deviation, we should instead use the  $t$  distribution

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**Example 0.3** (Derive the  $t$ -test for  $\hat{\beta}_j$ )

What is the  $t$ -distribution?

Take a  $Z \sim \mathcal{N}(0, 1)$

Take a  $W \sim \chi_k^2 \rightarrow$  sum of  $k$  independent standard normals, each squared

$$T = \frac{Z}{\sqrt{\frac{W}{k}}} \rightarrow T \sim t_k$$

Goal: Show that under Normal Linear Model and  $H_0$ ,  $\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{(\mathbf{X}^\top \mathbf{X})_{jj}^{-1}}} \sim t_{n-p-1}$ ,  $\hat{\sigma}$  is a function of  $\vec{e}$

Steps:

- (a)  $\hat{\beta}_j \sim \mathcal{N}(\mu = 0)$   
 $\vec{e} \sim \mathcal{N}_n(\mu = 0)$
- (b)  $\hat{\beta}$  and  $\vec{e}$  are uncorrelated  $\xrightarrow{NLM}$  independent
- (c)  $\vec{e} \sim \mathcal{N}_n \rightarrow \frac{\vec{e}^\top \vec{e}}{\sigma^2} \sim \chi_{n-p-1}^2$
- (d) Put it all together

**Proof.**

- (a)  $\hat{\beta} \sim \mathcal{N}_{p+1}(\beta, \sigma^2(\mathbf{X}^\top \mathbf{X})^{-1})$  under NLM

Under  $H_0 \rightarrow \hat{\beta}_j \sim \mathcal{N}(0, \sigma^2(\mathbf{X}^\top \mathbf{X})_{jj}^{-1})$

$$\vec{e} = (\mathbb{I} - \mathbb{H})\vec{Y}$$

Under NLM:

$$\vec{e} \sim \mathcal{N}_n((\mathbb{I} - \mathbb{H})\mathbf{X}\beta, (\mathbb{I} - \mathbb{H})\sigma^2\mathbb{I}(\mathbb{I} - \mathbb{H})) = \mathcal{N}_n(\vec{0}, \sigma^2(\mathbb{I} - \mathbb{H}))$$

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