STAT 151A Lecture 13

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Remark 0.1 (Multivariate Normal Distribution (MVN))

Scalar (usual) normal: $y \sim \mathcal{N}(\mu, \sigma^2)$
Multivariate normal: $\vec{Y} \sim \mathcal{N}_n(\vec{\mu}, \Sigma), \vec{\mu} \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n \times n}, \Sigma \succ 0$
Properties:

- If $\vec{Y} \sim \mathcal{N}_n(\vec{\mu}, \Sigma)$, linear functions of $\vec{Y}$ are also MVN
- If $\vec{Y} \sim \mathcal{N}_n(\vec{\mu}, \Sigma)$, then $Y_j(\mu_j, \sigma_{jj}^2)$
- If $\vec{Y}$ and $\vec{Z}$ are both MVN and independent, $\vec{Y} + \vec{Z}$ is also MVN, $\vec{Y} + \vec{Z} \sim \mathcal{N}_n(\vec{m}u_Z + \vec{\mu}_Y, \Sigma_{ZZ} + \Sigma_{YY})$
- If $\vec{Z} \in \mathbb{R}^m, \vec{Y} \in \mathbb{R}^n$, $\begin{pmatrix} \vec{Z} \\ \vec{Y} \end{pmatrix} \sim MVN_{n+m}$ and $\text{Cov}(\vec{Z}, \vec{Y}) \Leftrightarrow \vec{Z} \perp\!\!\!\perp \vec{Y}$

  Counterexample: $X \sim \mathcal{N}(0,1) : X$ and X^2, $\text{Cov}(X, X^2) = 0$

Method 0.2 (Normal Linear Model)

$\vec{Y}_{|X} \sim \mathcal{N}_n(\mathbf{X}\beta, \sigma^2 \mathbb{I}_n)$
$\vec{Y} = \mathbf{X}\beta + \vec{\varepsilon}, \vec{\varepsilon} \sim \mathcal{N}_n(\vec{0}, \sigma^2 \mathbb{I}_n)$ Stronger than Gauss-Markov model due to multivariate normal (No covariance)
Why? Back to $\hat{\beta}_j$
Test whether null hypothesis $H_0 : \beta_j = 0$
Use NLM: $\hat{\beta}_j$
$\hat{\beta} = (\mathbf{X}^\intercal \mathbf{X})^{-1} \mathbf{X}^\intercal \mathbf{Y}$
Distribution of $\hat{\beta}$ under NLM we get another Multivariate normal
$\hat{\beta} \sim \mathcal{N}_p\left((\mathbf{X}^\intercal \mathbf{X})^{-1} \mathbf{X}^\intercal \mathbf{X}\beta, (\mathbf{X}^\intercal \mathbf{X})^{-1} \mathbf{X}^\intercal \sigma^2 \mathbb{I}_n \mathbf{X} (\mathbf{X}^\intercal \mathbf{X})^{-1}\right) \Rightarrow \hat{\beta} \sim \mathcal{N}_p(\beta, \sigma^2 (\mathbf{X}^\intercal \mathbf{X})^{-1})$
Under H_0:
$\hat{\beta}_j \overset{H_0}{\sim} \mathcal{N}(0, \sigma^2 (\mathbf{X}^\intercal \mathbf{X})_{jj})$
$\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{(\mathbf{X}^\intercal \mathbf{X})_{jj}}} \overset{H_0}{\sim} \mathcal{N}(0,1)$
Since we have to estimate the standard deviation, we should instead use the t distribution

Example 0.3 (Derive the t-test for $\hat{\beta}_j$)

What is the t-distribution?
Take a $Z \sim \mathcal{N}(0,1)$
Take a $W \sim \chi^2_k \rightarrow$ sum of k independent standard normals, each squared
$T = \frac{Z}{\sqrt{\frac{W}{k}}} \rightarrow T \sim t_k$
Goal: Show that under Normal Linear Model and H_0, $\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{(\mathbf{X}^\intercal\mathbf{X})^{-1}_{jj}}} \sim t_{n-p-1}, \hat{\sigma}$ is a function of $\vec{e}$
Steps:

(a) $\hat{\beta}_j \sim \mathcal{N}(\mu = 0)$
$\vec{e} \sim \mathcal{N}_n(\mu = 0)$

(b) $\hat{\beta}$ and $\vec{e}$ are uncorrelated $\overset{NLM}{\rightarrow}$ independent

(c) $\vec{e} \sim \mathcal{N}_n \rightarrow \frac{\vec{e}^\intercal\vec{e}}{\sigma^2} \sim \chi^2_{n-p-1}$

(d) Put it all together

Proof.

(a) $\hat{\beta} \sim \mathcal{N}_{p+1}(\beta, \sigma^2(\mathbf{X}^\intercal\mathbf{X})^{-1})$ under NLM
Under $H_0 \rightarrow \hat{\beta}_j \sim \mathcal{N}(0, \sigma^2(\mathbf{X}^\intercal\mathbf{X})^{-1}_{jj})$
$\vec{e} = (\mathbb{I} - \mathbb{H})\vec{Y}$
Under NLM:
$\vec{e} \sim \mathcal{N}_n((\mathbb{I} - \mathbb{H})\mathbf{X}\beta, (\mathbb{I} - \mathbb{H})\sigma^2\mathbb{I}(\mathbb{I} - \mathbb{H})) = \mathcal{N}_n(\vec{0}, \sigma^2(\mathbb{I} - \mathbb{H}))$

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