STAT 151A Lecture 9

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Remark 0.1 (More Linear Algebra and Probability)
$\hat{\beta} \to \hat{y} = \mathbf{X}\hat{\beta} = \mathbf{X}(\mathbf{X}^\intercal\mathbf{X})^{-1}\mathbf{X}^\intercal\mathbf{Y} = \mathbf{H}\vec{Y}$
$\hat{\beta} \to \vec{e} = \mathbf{Y} - \mathbf{X}\hat{\beta} = Y - \mathbf{X}(\mathbf{X}^\intercal\mathbf{X})^{-1}\mathbf{X}^\intercal\mathbf{Y} = (\mathbb{I} - \mathbf{H})\vec{Y}$
What are projection matrices in general?
Any square matrix $\mathbf{A}$ such that $\mathbf{A}^\intercal = \mathbf{A}$ and $\mathbf{A}^2 = \mathbf{A} = \mathbf{A}\mathbf{A}$

Remark 0.2 (Back to Sum of Squares)
$\left\|\vec{y} - \bar{y}\vec{1}\right\|^2 = \left\|\hat{y} - \bar{y}\vec{1}\right\|^2 + \|\hat{y} - \hat{y}\|^2$
TotSS = RegSS + ErrSS
$R^2 = \frac{RegSS}{TotSS} \in [0,1]$
For simple regression $R^2 = r^2_{x,y}$
For multiple regression, not true
If you add more variables to your model, R^2 wil walys go up, not necessarily a good thing, how to fix?
$R^2 = 1 - \frac{ErrSS}{TotSS}$
TotSS $= \sum(y_i - \bar{y})^2 \Rightarrow \frac{1}{n-1} \cdot TotSS \to \hat{\sigma}^2_y$
ErrSS $= \sum(y_i - \hat{y}_i)^2$
$\frac{1}{n-p-1} ErrSS = \frac{1}{n-p-1}\sum(y_i - \hat{y}_i)^2 = \hat{\sigma}$, an estimate of σ^2 from the classical (mean probability model, ie at $\mathrm{Var}(Y|X)$)
$R^2 = 1 - \frac{(n-p-1)\hat{\sigma}^2}{(n-1)\hat{\sigma}_Y}$
Adjusted $R^2 = 1 - \frac{\widehat{\mathrm{Var}(Y|X)}}{\widehat{\mathrm{Var}(Y)}} = 1 - \frac{n-1}{n-p-1}\frac{ErrSS}{TotSS}$

Remark 0.3 (Dig into elements of $\hat{\beta}$)

$\hat{\beta} \to \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_p \end{pmatrix}, \hat{\beta}_0 \to$ intercept

How do we interpret?

Start with simple regression: (ignore intercept)

$$\hat{b} = \frac{\sum(y_i - \bar{y})(x_i - \bar{x})}{\sum(x_i - \bar{x})^2} = \frac{(\vec{y} - \bar{y}\vec{1})(\vec{x} - \bar{x}\vec{1})}{(\vec{x} - \bar{x}\vec{1})(\vec{x} - \bar{x}\vec{1})} = r_{x,y}\frac{\hat{\sigma}_Y}{\sigma_X}$$

What about for multiple regression? Start with 2 variable model

$\vec{y} = a\vec{1} + b\vec{x} + c\vec{z} + \vec{e} = \vec{y}_{||\,\text{span}\{\vec{1},\vec{x},\vec{z}\}} + \vec{e}$

Have a formula for the coefficients when the space we're projecting into has orthogonal basis

$y_{||u} = \sum_{i=1}^{n} \frac{y \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i$

Problem: $\vec{1}, \vec{x}, \vec{z}$ not orthogonal

Solution: Orthogonalize using Gram-Schmidt Process or repeated projectoins

1. Start with $\vec{1}$ in the basis
2. Take $\vec{x}_{\perp\vec{1}}$ and add it to the basis $= \vec{x} - \vec{x}_{||\vec{1}} = \vec{x} - \bar{x}\vec{1}$
3. Take $\vec{z}$ and project onto $\{\vec{1}, \vec{x} - \bar{x}\vec{1}\}$, subtract projection from $\vec{z}$ and add it to the basis

$$z_{||\{\vec{1},\vec{x}-\bar{x}\vec{1}\}} = \frac{\vec{z} \cdot \vec{1}}{\vec{1} \cdot \vec{1}}\vec{1} + \frac{\vec{z} \cdot (\vec{x} - \bar{x}\vec{1})}{(\vec{x} - \bar{x}\vec{1})(\vec{x} - \bar{x}\vec{1})}(\vec{x} - \bar{x}\vec{1})$$

Bottom line: $\hat{\beta}_i = \frac{\vec{y} \cdot \text{residuals from regressing on all other columns}}{\vec{e}_i \cdot \vec{e}_i}$