STAT 151A Lecture 9

Henry Liev

17 September 2025

Remark 0.1 (More Linear Algebra and Probability)

$\hat{\beta} \to \hat{y} = \mathbf{X}\hat{\beta} = \mathbf{X}(\mathbf{X^\intercal X})^{-1}\mathbf{X^\intercal Y} = \mathbf{H}\vec{Y}$

$\hat{\beta} \to \vec{e} = \mathbf{Y} - \mathbf{X}\hat{\beta} = Y - \mathbf{X}(\mathbf{X^\intercal X})^{-1}\mathbf{X^\intercal Y} = (\mathbb{I} - \mathbf{H})\vec{Y}$

What are projection matrices in general?

Any square matrix $\mathbf{A}$ such that $\mathbf{A^\intercal} = \mathbf{A}$ and $\mathbf{A}^2 = \mathbf{A} = \mathbf{AA}$

Remark 0.2 (Back to Sum of Squares)

$\left\| \vec{y} - \bar{y}\vec{1} \right\|^2 = \left\| \hat{y} - \bar{y}\vec{1} \right\|^2 + \left\| \hat{y} - \hat{y} \right\|^2$

TotSS = RegSS + ErrSS

$R^2 = \frac{RegSS}{TotSS} \in [0, 1]$

For simple regression $R^2 = r_{x,y}^2$

For multiple regression, not true

If you add more variables to your model, R^2 wil walys go up, not necessarily a good thing, how to fix?

$R^2 = 1 - \frac{ErrSS}{TotSS}$

TotSS $= \sum(y_i - \bar{y})^2 \Rightarrow \frac{1}{n-1} \cdot TotSS \to \hat{\sigma}_y^2$

ErrSS $= \sum(y_i - \hat{y}_i)^2$

$\frac{1}{n-p-1}ErrSS = \frac{1}{n-p-1}\sum(y_i - \hat{y}_i)^2 = \hat{\sigma}$, an estimate of σ^2 from the classical (mean probability model, ie at $\text{Var}(Y|X)$)

$R^2 = 1 - \frac{(n-p-1)\hat{\sigma}^2}{(n-1)\hat{\sigma}_Y}$

Adjusted $R^2 = 1 - \frac{\widehat{\text{Var}}(Y|X)}{\widehat{\text{Var}}(Y)} = 1 - \frac{n-1}{n-p-1}\frac{ErrSS}{TotSS}$

Remark 0.3 (Dig into elements of $\hat{\beta}$)

$\hat{\beta} \rightarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_p \end{pmatrix}, \hat{\beta}_0 \rightarrow$ intercept

How do we interpret?

Start with simple regression: (ignore intercept)

$$\hat{b} = \frac{\sum(y_i - \bar{y})(x_i - \bar{x})}{\sum(x_i - \bar{x})^2} = \frac{(\vec{y} - \bar{y}\vec{1})(\vec{x} - \bar{x}\vec{1})}{(\vec{x} - \bar{x}\vec{1})(\vec{x} - \bar{x}\vec{1})} = r_{x,y}\frac{\hat{\sigma}_Y}{\sigma_X}$$

What about for multiple regression? Start with 2 variable model

$\vec{y} = a\vec{1} + b\vec{x} + c\vec{z} + \vec{e} = \vec{y}_{||\,\text{span}\{\vec{1},\vec{x},\vec{z}\}} + \vec{e}$

Have a formula for the coefficients when the space we're projecting into has orthogonal basis

$y_{||u} = \sum_{i=1}^{n} \frac{y \cdot \vec{u}_i}{\vec{u}_i \cdot \vec{u}_i} \cdot \vec{u}_i$

Problem: $\vec{1}, \vec{x}, \vec{z}$ not orthogonal

Solution: Orthogonalize using Gram-Schmidt Process or repeated projectoins

1. Start with $\vec{1}$ in the basis

2. Take $\vec{x}_{\perp\vec{1}}$ and add it to the basis $= \vec{x} - \vec{x}_{||\vec{1}} = \vec{x} - \bar{x}\vec{1}$

3. Take $\vec{z}$ and project onto $\{\vec{1}, \vec{x} - \bar{x}\vec{1}\}$, subtract projection from $\vec{z}$ and add it to the basis

$$z_{||\{\vec{1},\vec{x}-\bar{x}\vec{1}\}} = \frac{\vec{z} \cdot \vec{1}}{\vec{1} \cdot \vec{1}}\vec{1} + \frac{\vec{z} \cdot (\vec{x} - \bar{x}\vec{1})}{(\vec{x} - \bar{x}\vec{1})(\vec{x} - \bar{x}\vec{1})}(\vec{x} - \bar{x}\vec{1})$$

Bottom line: $\hat{\beta}_i = \frac{\vec{y} \cdot \text{residuals from regressing on all other columns}}{\vec{e}_i \cdot \vec{e}_i}$