STAT 151A Lecture 6

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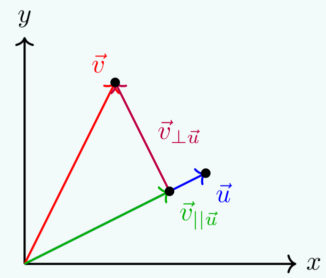
Remark 0.1 (Recap)

Project $\vec{v} \in \mathbb{R}^n$ onto $\vec{u} \in \mathbb{R}^n$
$\vec{v}_{||\vec{u}} = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$



$\vec{v}_{\perp\vec{u}} = \vec{v} - \vec{v}_{||\vec{u}}$
Project $\vec{v} \in \mathbb{R}^n$ onto subspace $U \subseteq \mathbb{R}^n$ with orthogonal basis $\{\vec{u}_1, \ldots, \vec{u}_p\}$
$\vec{v}_{||\vec{u}} = \sum_{i=1}^{p} \vec{v}_{||\vec{u}} = \sum_{i=1}^{p} \frac{\vec{v}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}\vec{u}_i$
Key applications $\vec{y}_{||\vec{1}} = \bar{y}\cdot\vec{1}$
$\vec{v}_{||\{\vec{1},\vec{x}\}} = \vec{y}_{||\{\vec{1},\vec{x}_{\perp\vec{1}}\}} = \bar{y}\vec{1} + \frac{(\vec{y}-\bar{y}\vec{1})\cdot(\vec{x}-\bar{x}\vec{1})}{(\vec{x}-\bar{x}\vec{1})\cdot(\vec{x}-\bar{x}\vec{1})}(\vec{x}-\bar{x}\vec{1})$
$\vec{y} = \vec{y}_{||\{\vec{1},\vec{x}\}} + \vec{y}_{\perp\{\vec{1},\vec{x}\}} = \hat{y} + \vec{e} =$ fitted values + residuals
Properties

- $\bar{e} = \frac{1}{n}\sum_{e_i}^{n} = 0$
- $\vec{e} \perp \vec{x}$
- $\vec{e} \perp \hat{y}$

$\sqrt{n-1}SD = \left\|\vec{y}_{\perp\vec{1}}\right\|$
$(n-1)\widehat{\mathrm{Var}} = \left\|\vec{y}_{\perp\vec{1}}\right\|^2$
$= \left\|\vec{y} - \bar{y}\vec{1}\right\|^2 = \|\vec{y} - \hat{y}\|^2 + \left\|\hat{y} - \bar{y}\vec{1}\right\|^2$
Total sum of squares = error sum of squares + regression sum of squares

Remark 0.2 (Why does it decompose nicely)

$\left\|\vec{y} - \bar{y}\vec{1}\right\|^2 = (\vec{y} - \bar{y}\vec{1})\cdot(\vec{y} - \bar{y}\vec{1})$
$(\vec{y} - \hat{y} + \hat{y} - \bar{y}\vec{1})\cdot(\vec{y} - \hat{y} + \hat{y} - \bar{y}\vec{1}) = \|\vec{y} - \hat{y}\|^2 + \left\|\hat{y} + \bar{y}\vec{1}\right\| + 2(\vec{y} - \hat{y})(\hat{y} - \bar{y}\vec{1})$
$2(\vec{y} - \hat{y})(\hat{y} - \bar{y}\vec{1}) = 0$ properties of least squares

Remark 0.3 (Sum of Squares Decomposition)

$$\frac{\mathrm{Reg}_{SS}}{\mathrm{Tot}_{SS}} = \frac{\left\|\hat{y} - \bar{y}\vec{1}\right\|^2}{\left\|\vec{y} - \bar{y}\vec{1}\right\|^2} = \frac{\sum \hat{b}^2 (x_i - \bar{x})^2}{\sum (y_i - \bar{y})^2} = r^2 \frac{\sum (y_i - \bar{y})^2}{\sum (x_i - \bar{x})^2} \cdot \frac{\sum (x_i - \bar{x})^2}{\sum (y_i - \bar{y})^2} = r^2$$

Method 0.4 (Multiple Regression)

$\vec{y} \approx a + b\vec{x}$
$\vec{y} \approx a + b_1\vec{x}_1 + b_2\vec{x}_2 + \cdots + b_p\vec{x}_p$
Multiple regression $\rightarrow$ as projection
Rewrite this in matrix form

$$\vec{Y} = \begin{pmatrix} | & | & | & & | \\ \vec{1} & \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_p \\ | & | & | & & | \end{pmatrix} \begin{pmatrix} a \\ b_1 \\ \vdots \\ b_p \end{pmatrix} = \mathbf{X}\hat{\beta}$$

$\mathbf{X} \in \mathbb{R}^{n \times p+1}$ is the design matrix and $\hat{\beta} \in \mathbb{R}^{p+1}$ is the coefficient vector
How to calculate $\hat{\beta}$? Choose it such that $\mathbf{X}\hat{\beta} = Y_{||\{\vec{1}, \vec{x}_1, \ldots, \vec{x}_p\}}$
$\mathbf{Y} = \mathbf{X}\hat{\beta} + \vec{e} = \mathbf{Y}_{||\,\mathrm{span}(\mathbf{X})} + \mathbf{Y}_{\perp \mathrm{span}(\mathbf{X})}$
We have that $\vec{e} \perp \vec{1}, \vec{e} \perp \vec{x}_i \forall i, \hat{Y} = \mathbf{X}\hat{\beta} \perp \vec{e}$
$\mathbf{X}^\intercal\mathbf{Y} = \mathbf{X}^\intercal\mathbf{X}\hat{\beta} + \mathbf{X}^\intercal\vec{e}$
$\mathbf{X}^\intercal\mathbf{Y} = \mathbf{X}^\intercal\mathbf{X}\hat{\beta} \Rightarrow \hat{\beta} = (\mathbf{X}^\intercal\mathbf{X})^{-1}\mathbf{X}^\intercal\mathbf{Y}$, only true if $\mathbf{X}^\intercal\mathbf{X}$ is invertible, otherwise, $\mathbf{X}\hat{\beta} = \hat{Y}$ is still well defined, but $\hat{\beta}$ is not unique