

STAT 151A Lecture 14

Henry Liev

29 September 2025

Example 0.1 (Derive the t -test for $\hat{\beta}_j$)

What is the t -distribution?

Take a $Z \sim \mathcal{N}(0, 1)$

Take a $W \sim \chi_k^2 \rightarrow$ sum of k independent standard normals, each squared

$$T = \frac{Z}{\sqrt{\frac{W}{k}}} \rightarrow T \sim t_k$$

Goal: Show that under Normal Linear Model and H_0 , $\frac{\hat{\beta}_j}{\hat{\sigma} \sqrt{(\mathbf{X}^\top \mathbf{X})_{jj}^{-1}}} \sim t_{n-p-1}$, $\hat{\sigma}$ is a function of \vec{e}

Steps:

- (a) $\hat{\beta}_j \sim \mathcal{N}(\mu = 0)$
 $\vec{e} \sim \mathcal{N}_n(\mu = 0)$
- (b) $\hat{\beta}$ and \vec{e} are uncorrelated \xrightarrow{NLM} independent
- (c) $\vec{e} \sim \mathcal{N}_n \rightarrow \frac{\vec{e}^\top \vec{e}}{\sigma^2} \sim \chi_{n-p-1}^2$
- (d) Put it all together

Proof.

- (a) $\hat{\beta} \sim \mathcal{N}_{p+1}(\beta, \sigma^2(\mathbf{X}^\top \mathbf{X})^{-1})$ under NLM

Under $H_0 \rightarrow \hat{\beta}_j \sim \mathcal{N}(0, \sigma^2(\mathbf{X}^\top \mathbf{X})_{jj}^{-1})$

$$\vec{e} = (\mathbb{I} - \mathbb{H})\vec{Y}$$

Under NLM:

$$\vec{e} \sim \mathcal{N}_n((\mathbb{I} - \mathbb{H})\mathbf{X}\beta, (\mathbb{I} - \mathbb{H})\sigma^2\mathbb{I}(\mathbb{I} - \mathbb{H})) = \mathcal{N}_n(\vec{0}, \sigma^2(\mathbb{I} - \mathbb{H}))$$

- (b) $\vec{Y}|\mathbf{X} \sim \mathcal{N}_n(\mathbf{X}\beta, \sigma^2\mathbb{I}_n)$
 $\hat{\beta}|\mathbf{X} \sim \mathcal{N}_{p+1}(\beta, \sigma^2(\mathbf{X}^\top \mathbf{X})^{-1})$
 $\vec{e}|\mathbf{X} \sim \mathcal{N}_n(0, \sigma^2(\mathbb{I} - \mathbb{H}))$

$$\begin{aligned} \text{Cov}(\vec{e}, \hat{\beta}|\mathbf{X}) &= \text{Cov}((\mathbb{I} - \mathbb{H})\vec{y}, (\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \vec{y}) \\ &= (\mathbb{I} - \mathbb{H})\Sigma_{\vec{y}\vec{y}}\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \\ &= (\mathbb{I} - \mathbb{H})\sigma^2\mathbb{I}_n\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \\ &= \sigma^2(\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} - \mathbb{H}\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}) \\ &= \sigma^2[\mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} - \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}] = 0, \text{Cov}(\vec{e}, \hat{y}) = 0 \end{aligned}$$

$$\begin{pmatrix} \vec{e} \\ \hat{\beta} \end{pmatrix} = \begin{pmatrix} (\mathbb{I} - \mathbb{H})\vec{y} \\ (\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \vec{y} \end{pmatrix} = \begin{pmatrix} \mathbb{I} - \mathbb{H} \\ (\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \end{pmatrix} \vec{y}|\mathbf{X} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ \beta \end{pmatrix}, \sigma^2 \begin{pmatrix} (\mathbb{I} - \mathbb{H}) & 0 \\ 0 & (\mathbf{X}^\top \mathbf{X})^{-1} \end{pmatrix}\right)$$

$\vec{e}, \hat{\beta}$ are independent under normal linear model

Remark. $\vec{e} \cdot \hat{y} \rightarrow \vec{e} \perp \vec{y} \rightarrow \text{Cov}(\vec{e}, \hat{y}) = 0$ BUT if $\vec{y} \sim MVN$ and \vec{e} and \hat{y} are orthogonal projections of the same \vec{y} , then both statements hold

(c) $\vec{e} \sim \mathcal{N}_n(0, \sigma^2(\mathbb{I} - \mathbb{H}))$

$\frac{\vec{e}^\top \vec{e}}{\sigma^2} \overset{\text{Claim:}}{\underset{\text{Under NLM}}{\sim}} \chi_{n-p-1}^2$

Recall: if $Z_i \sim N(0, 1^2)$ and Z_1, \dots, Z_k are all independent then $\sum Z_i^2 \sim \chi_k^2$

$\frac{\vec{e}}{\sigma} \sim N(0, \mathbb{I}_n)$ i.e. $e_i \sim N(0, 1^2) \perp e_j \rightarrow \frac{\vec{e}^\top \vec{e}}{\sigma^2} \sim \chi_n^2$

How to deal with dependence between elements of \vec{e} ?

Intuition: vector \vec{e} belongs to an $n - p - 1$ dimensional subspace $\rightarrow \vec{e} \in \mathbb{R}_{\perp \text{Col}\{\mathbb{X}\}}^n$

So $\sum_{i=1}^n \frac{e_i^2}{\sigma^2}$ is akin to $\sum_{i=1}^{n-p-1} Z_i^2$ independent standard normals

2 ways to make this more precise

Method 1:

Theorem. If $\vec{Z} \sim \mathcal{N}_n(0, E)$ where E satisfies $\text{Rank}(E) = k \leq n, E^2 = E, E^\top = E$
Then $Z^\top E Z \sim \chi_k^2$

Look for a transformation $G \in \mathbb{R}^{m \times n}$ so that $\Sigma_{(G\vec{e}, G\vec{e})} = \mathbb{I}_m$ and $(G\vec{e})^\top G\vec{e} = \vec{e}^\top \vec{e}$, turns out this only works if $m = n - p - 1$ (and there always exists some G like this)

(d) $\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_{jj}}}, v_{jj} = [(\mathbf{X}^\top \mathbf{X})^{-1}]_{jj}$
 $\frac{\hat{\beta}_j}{\sigma\sqrt{v_{jj}}} \sim N(0, 1) \perp \frac{\vec{e}^\top \vec{e}}{\sigma^2} \sim \chi_{n-p-1}^2$

$$\frac{\frac{\hat{\beta}_j}{\sigma\sqrt{v_{jj}}}}{\sqrt{\frac{\vec{e}^\top \vec{e}}{\sigma^2} \cdot \frac{1}{n-p-1}}} = \frac{\hat{\beta}_j}{\sqrt{\frac{\vec{e}^\top \vec{e}}{n-p-1} \cdot v_{jj}}} \sim t_{n-p-1}$$

$$\sqrt{\hat{\sigma}^2} = \hat{\sigma}$$

■