STAT 151A Lecture 14

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Example 0.1 (Derive the t-test for $\hat{\beta}_j$)
What is the t-distribution?
Take a $Z \sim \mathcal{N}(0,1)$
Take a $W \sim \chi^2_k \to$ sum of k independent standard normals, each squared
$T = \frac{Z}{\sqrt{\frac{W}{k}}} \to T \sim t_k$
Goal: Show that under Normal Linear Model and H_0, $\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{(\mathbf{X}^\intercal\mathbf{X})^{-1}_{jj}}} \sim t_{n-p-1}, \hat{\sigma}$ is a function of $\vec{e}$
Steps:

(a) $\hat{\beta}_j \sim \mathcal{N}(\mu = 0)$
$\vec{e} \sim \mathcal{N}_n(\mu = 0)$

(b) $\hat{\beta}$ and $\vec{e}$ are uncorrelated $\overset{NLM}{\to}$ independent

(c) $\vec{e} \sim \mathcal{N}_n \to \frac{\vec{e}^\intercal\vec{e}}{\sigma^2} \sim \chi^2_{n-p-1}$

(d) Put it all together

Proof.

(a) $\hat{\beta} \sim \mathcal{N}_{p+1}(\beta, \sigma^2(\mathbf{X}^\intercal\mathbf{X})^{-1})$ under NLM
Under $H_0 \to \hat{\beta}_j \sim \mathcal{N}(0, \sigma^2(\mathbf{X}^\intercal\mathbf{X})^{-1}_{jj})$
$\vec{e} = (\mathbb{I} - \mathbb{H})\vec{Y}$
Under NLM:
$\vec{e} \sim \mathcal{N}_n((\mathbb{I} - \mathbb{H})\mathbf{X}\beta, (\mathbb{I} - \mathbb{H})\sigma^2\mathbb{I}(\mathbb{I} - \mathbb{H})) = \mathcal{N}_n(\vec{0}, \sigma^2(\mathbb{I} - \mathbb{H}))$

(b) $\vec{Y}_|\mathbf{X} \sim \mathcal{N}_n(\mathbf{X}\beta, \sigma^2\mathbb{I}_n)$
$\hat{\beta}|\mathbf{X} \sim \mathcal{N}_{p+1}(\beta, \sigma^2(\mathbf{X}^\intercal\mathbf{X})^{-1})$
$\vec{e}|\mathbf{X} \sim \mathcal{N}_n(0, \sigma^2(\mathbb{I} - \mathbb{H}))$

$$\begin{aligned}
\text{Cov}(\vec{e}, \hat{\beta}|\mathbf{X}) &= \text{Cov}((\mathbb{I} - \mathbb{H})\vec{y}, (\mathbf{X}^\intercal\mathbf{X})^{-1}\mathbf{X}^\intercal\vec{y}) \\
&= (\mathbb{I} - \mathbb{H})\Sigma_{\vec{y}\vec{y}}\mathbf{X}(\mathbf{X}^\intercal\mathbf{X})^{-1} \\
&= (\mathbb{I} - \mathbb{H})\sigma^2\mathbb{I}_n\mathbf{X}(\mathbf{X}^\intercal\mathbf{X})^{-1} \\
&= \sigma^2(\mathbf{X}(\mathbf{X}^\intercal\mathbf{X})^{-1} - \mathbb{H}\mathbf{X}(\mathbf{X}^\intercal\mathbf{X})^{-1}) \\
&= \sigma^2\left[\mathbf{X}(\mathbf{X}^\intercal\mathbf{X})^{-1} - \mathbf{X}(\mathbf{X}^\intercal\mathbf{X})^{-1}\mathbf{X}^\intercal\mathbf{X}(\mathbf{X}^\intercal\mathbf{X})^{-1}\right] = 0, \text{Cov}(\vec{e}, \hat{y}) = 0
\end{aligned}$$

$$\begin{pmatrix}\vec{e} \\ \hat{\beta}\end{pmatrix} = \begin{pmatrix}(\mathbb{I} - \mathbb{H})\vec{y} \\ (\mathbf{X}^\intercal\mathbf{X})^{-1}\mathbf{X}^\intercal\vec{y}\end{pmatrix} = \begin{pmatrix}\mathbb{I} - \mathbb{H} \\ (\mathbf{X}^\intercal\mathbf{X})^{-1}\mathbf{X}^\intercal\end{pmatrix}\vec{y}|\mathbf{X} \sim \mathcal{N}\left(\begin{pmatrix}0 \\ \beta\end{pmatrix}, \sigma^2\begin{pmatrix}(\mathbb{I} - \mathbb{H})0 \\ 0(\mathbf{X}^\intercal\mathbf{X})^{-1}\end{pmatrix}\right)$$

$\vec{e}, \hat{\beta}$ are independent under normal linear model

Remark. $\vec{e}\cdot\hat{y} \to \vec{e} \perp\!\!\!\perp \vec{y} \to \mathrm{Cov}(\vec{e},\hat{y}) = 0$ BUT if $\vec{y} \sim MVN$ and $\vec{e}$ and $\hat{y}$ are orthogonal projections of the same $\vec{y}$, then both statements hold

(c) $\vec{e} \sim \mathcal{N}_n(0, \sigma^2(\mathbb{I}-\mathbb{H}))$
$\frac{\vec{e}^\intercal\vec{e}}{\sigma|X} \underset{\text{Under NLM}}{\overset{\text{Claim:}}{\sim}} \chi^2_{n-p-1}$
Recall: if $Z_i \sim N(0,1^2)$ and $Z_1,\ldots,Z_k$ are all independent then $\sum \mathbb{Z}_i^2 \sim \chi^2_k$
$\frac{\vec{e}}{\sigma} \sim N(0,\mathbb{I}_n)$ i.e. $e_i \sim N(0,1^2) \perp\!\!\!\perp e_j \to \frac{\vec{e}^\intercal\vec{e}}{\sigma^2} \sim \chi^2_n$
How to deal with dependence between elements of $\vec{e}$?
Intuition: vector $\vec{e}$ belongs to an $n-p-1$ dimensional subspace $\to \vec{e} \in \mathbb{R}^n_{\perp\mathrm{Col}\{\mathbb{X}\}}$
So $\sum_{i=1}^n \frac{e_i^2}{\sigma^2}$ is akin to $\sum_{i=1}^{n-p-1} Z_i^2$ independent standard normals
2 ways to make this more precise
Method 1:

Theorem. If $\vec{Z} \sim \mathcal{N}_n(0,E)$ where E satisfies $\mathrm{Rank}(E) = k \le n, E^2 = E, E^\intercal = E$
Then $Z^\intercal E Z \sim \chi^2_k$

Look for a transformation $G \in \mathbb{R}^{m\times n}$ so that $\Sigma_{(G\vec{e},G\vec{e})} = \mathbb{I}_m$ and $(G\vec{e})^\intercal G\vec{e} = \vec{e}^\intercal\vec{e}$, turns out this only works if $m-n-p-1$ (and there always exists some G like this)

(d) $\frac{\hat{\beta}_j}{\hat{\sigma}\sqrt{v_{jj}}}, v_{jj} = \left[(\mathbf{X}^\intercal\mathbf{X})^{-1}\right]_{jj}$
$\frac{\hat{\beta}_j}{\sigma\sqrt{v_{jj}}} \sim N(0,1) \perp\!\!\!\perp \frac{\vec{e}^\intercal\vec{e}}{\sigma^2} \sim \chi^2_{n-p-1}$

$$\frac{\frac{\hat{\beta}_j}{\sigma\sqrt{v_{jj}}}}{\sqrt{\frac{\vec{e}^\intercal\vec{e}}{\sigma^2}\cdot\frac{1}{n-p-1}}} = \frac{\hat{\beta}_j}{\sqrt{\frac{\vec{e}^\intercal\vec{e}}{n-p-1}}\cdot\sqrt{v_{jj}}} \sim t_{n-p-1}$$

$\sqrt{\hat{\sigma}^2} = \hat{\sigma}$

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