STAT 151A Lecture 5

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Definition 0.1 (Dimension)
Number of vectors in a basis for V of V

Definition 0.2 (Subspace)
If a vector space U contains vectors of dimension n then $U \subseteq \mathbb{R}^n$

Definition 0.3 (Orthogonal Basis)
Suppose we have a subspace V with a basis $\{\vec{v}_1, \dots, \vec{v}_p\}$
If $\vec{v}_i \perp \vec{v}_j$, $\forall i \neq j$, we call this an orthogonal basis
If $\|\vec{v}_i\| = 1, \forall i$, this is an orthonormal basis

Remark 0.4
If $\vec{v}_i, \dots, \vec{v}_p$ is an orthogonal basis for V then $\forall \vec{w} \in V$, we can write $\vec{w}$ uniquely as a linear combination of $\vec{v}_i$'s

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p, \text{ where } c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

Proof.

$$\vec{w} \cdot \vec{v}_i = (c_1\vec{v}_1 + \cdots + c_p\vec{v}_p) \cdot \vec{v}_i$$
$$= c_1\vec{v}_1 \cdot \vec{v}_i + c_2\vec{v}_2 \cdot \vec{v}_i + \cdots + c_p v_p \cdot \vec{v}_i$$
$$= c_i\vec{v}_i \cdot \vec{v}_i \Rightarrow c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

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Definition 0.5 (Projection)
$\vec{v}, \vec{u} \in V$
Find $c\vec{u}$ that is closest to $\vec{v}$
Answer $c^*\vec{u}$, where $c^* = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$

Proof.

$$\|\vec{v} - c\vec{u}\|^2 = \|v - c^*\vec{u} + c^*\vec{u} - c\vec{u}\|^2$$

$$= (\vec{v} - c^*\vec{u} + c^*\vec{u} - c\vec{u}) \cdot (\vec{v} - c^*\vec{u} + c^*\vec{u} - c\vec{u})$$
$$= \|\vec{v} - c^*\vec{u}\|^2 + \|c^*\vec{u} - c\vec{u}\|^2 + 2(\vec{v} - c^*\vec{u}) \cdot (c^*\vec{u} - c\vec{u})$$

First term does not depend on c, second term is smallest when $c = c^*$

$$(\vec{v} - \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \cdot u) \cdot (c^*\vec{u} - c\vec{u}) = 2(\vec{v} - \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}) \cdot \vec{u}(c^* - c) = 2(\vec{v} \cdot \vec{u} - \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} \cdot \vec{u})(c^* - c) = 2(c^* - c)(\vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{u}) = 0$$

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Remark 0.6 (Notation)
$c^*\vec{u} = \vec{v}_{||\vec{u}} = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}$
$\vec{v} - c^*\vec{u} = \vec{v}_{\perp\vec{u}}$ "Orthogonal complement (of $\vec{v}$ projected onto $\vec{u}$)"

Example 0.7 (Mean and SD)
$y_1, \ldots, y_n \Rightarrow \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \vec{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \subset \mathbb{R}^n$
Project $\vec{y}$ onto $\vec{1}$:

$$c^* = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \Rightarrow c = \frac{\vec{y} \cdot \vec{1}}{\vec{1} \cdot \vec{1}}$$
$$\vec{y}_{||\vec{1}} = \frac{\vec{y} \cdot \vec{1}}{\vec{1} \cdot \vec{1}}\vec{1} = \frac{\sum y_i}{n}\vec{1} = \bar{y} \cdot \vec{1}$$

SD: $\|\vec{y}_{\perp\vec{1}}\| = \left\|\vec{y} - \vec{y}_{||\vec{1}}\right\| = \|\vec{y} - \bar{y}\vec{1}\| = \sqrt{\sum(y_i - \bar{y})^2} = \sqrt{n}SD$

Example 0.8 (Simple Regression as Projection)
$(x_1, y_1), \ldots (x_n, y_n)$
Project $\vec{y}$ onto span$\{\vec{1}, \vec{x}\}$

Solution. Start by finding an orthogonal basis (assume not all x_i are the same)
Gram-Schmidt Process: Orthogonalize a set of vectors

1) Put $\vec{1}$ into basis
2) Find orthogonal complement of $\vec{x}$ with respect to $\vec{1}$: $\vec{x} - \vec{x}_{||\vec{1}} = \vec{x}_{\perp\vec{1}}$
3) Put this into an orthogonal basis

Now project $\vec{y}$ onto span$\{\vec{1}, \vec{x}_{\perp\vec{1}}\}$

$$\vec{y}_{||\{\vec{1},\vec{x}\}} = \vec{y}_{||\{\vec{1},\vec{x}_{\perp\vec{1}}\}} = \vec{y}_{||\vec{1}} + \vec{y}_{||\vec{x}_{\perp\vec{1}}} = \bar{y}\vec{1} + \vec{y}_{||(\vec{x}-\bar{x}\vec{1})} = \bar{y}\vec{1} + (\vec{y} - \bar{y}\vec{1})_{||(\vec{x}-\bar{x}\vec{1})}$$
$$= \bar{y}\vec{1} + \frac{(\vec{y} - \bar{y}\vec{1}) \cdot (\vec{x} - \bar{x}\vec{1})}{(\vec{x} - \bar{x}\vec{1})(\vec{x} - \bar{x}\vec{1})}(\vec{x} - \bar{x}\vec{1}) = \frac{1}{n}\sum y_i + \frac{\sum(y_i - \bar{y})(x_i - \bar{x})}{(\sum(x_i - \bar{x}))}$$

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