

# STAT 151A Lecture 5

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**Definition 0.1** (Dimension)

Number of vectors in a basis for  $V$  of  $V$

**Definition 0.2** (Subspace)

If a vector space  $U$  contains vectors of dimension  $n$  then  $U \subseteq \mathbb{R}^n$

**Definition 0.3** (Orthogonal Basis)

Suppose we have a subspace  $V$  with a basis  $\{\vec{v}_1, \dots, \vec{v}_p\}$

If  $\vec{v}_i \perp \vec{v}_j, \forall i \neq j$ , we call this an orthogonal basis

If  $\|\vec{v}_i\| = 1, \forall i$ , this is an orthonormal basis

**Remark 0.4**

If  $\vec{v}_1, \dots, \vec{v}_p$  is an orthogonal basis for  $V$  then  $\forall \vec{w} \in V$ , we can write  $\vec{w}$  uniquely as a linear combination of  $\vec{v}_i$ 's

$$\vec{w} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_p \vec{v}_p, \text{ where } c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i}$$

**Proof.**

$$\begin{aligned} \vec{w} \cdot \vec{v}_i &= (c_1 \vec{v}_1 + \dots + c_p \vec{v}_p) \cdot \vec{v}_i \\ &= c_1 \vec{v}_1 \cdot \vec{v}_i + c_2 \vec{v}_2 \cdot \vec{v}_i + \dots + c_p \vec{v}_p \cdot \vec{v}_i \\ &= c_i \vec{v}_i \cdot \vec{v}_i \Rightarrow c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} \end{aligned}$$

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**Definition 0.5** (Projection)

$\vec{v}, \vec{u} \in V$

Find  $c\vec{u}$  that is closest to  $\vec{v}$

Answer  $c^*\vec{u}$ , where  $c^* = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$

**Proof.**

$$\|\vec{v} - c\vec{u}\|^2 = \|\vec{v} - c^*\vec{u} + c^*\vec{u} - c\vec{u}\|^2$$

$$\begin{aligned}
&= (\vec{v} - c^* \vec{u} + c^* \vec{u} - c \vec{u}) \cdot (\vec{v} - c^* \vec{u} + c^* \vec{u} - c \vec{u}) \\
&= \|\vec{v} - c^* \vec{u}\|^2 + \|c^* \vec{u} - c \vec{u}\|^2 + 2(\vec{v} - c^* \vec{u}) \cdot (c^* \vec{u} - c \vec{u})
\end{aligned}$$

First term does not depend on  $c$ , second term is smallest when  $c = c^*$

$$(\vec{v} - \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \cdot \vec{u}) \cdot (c^* \vec{u} - c \vec{u}) = 2(\vec{v} - \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}) \cdot \vec{u}(c^* - c) = 2(\vec{v} \cdot \vec{u} - \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \cdot \vec{u})(c^* - c) = 2(c^* - c)(\vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{u}) = 0$$

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#### Remark 0.6 (Notation)

$$\begin{aligned}
c^* \vec{u} &= \vec{v}_{||\vec{u}} = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \\
\vec{v} - c^* \vec{u} &= \vec{v}_{\perp \vec{u}} \text{ "Orthogonal complement (of } \vec{v} \text{ projected onto } \vec{u}) \text{"}
\end{aligned}$$

#### Example 0.7 (Mean and SD)

$$y_1, \dots, y_n \Rightarrow \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \vec{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$$

Project  $\vec{y}$  onto  $\vec{1}$ :

$$c^* = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \Rightarrow c = \frac{\vec{y} \cdot \vec{1}}{\vec{1} \cdot \vec{1}}$$

$$\vec{y}_{||\vec{1}} = \frac{\vec{y} \cdot \vec{1}}{\vec{1} \cdot \vec{1}} \vec{1} = \frac{\sum y_i}{n} \vec{1} = \bar{y} \cdot \vec{1}$$

$$\text{SD: } \|\vec{y}_{\perp \vec{1}}\| = \|\vec{y} - \vec{y}_{||\vec{1}}\| = \|\vec{y} - \bar{y} \vec{1}\| = \sqrt{\sum (y_i - \bar{y})^2} = \sqrt{n} SD$$

#### Example 0.8 (Simple Regression as Projection)

$$\begin{aligned}
&(x_1, y_1), \dots, (x_n, y_n) \\
&\text{Project } \vec{y} \text{ onto } \text{span}\{\vec{1}, \vec{x}\}
\end{aligned}$$

**Solution.** Start by finding an orthogonal basis (assume not all  $x_i$  are the same)

Gram-Schmidt Process: Orthogonalize a set of vectors

- 1) Put  $\vec{1}$  into basis
- 2) Find orthogonal complement of  $\vec{x}$  with respect to  $\vec{1}$ :  $\vec{x} - \vec{x}_{||\vec{1}} = \vec{x}_{\perp \vec{1}}$
- 3) Put this into an orthogonal basis

Now project  $\vec{y}$  onto  $\text{span}\{\vec{1}, \vec{x}_{\perp \vec{1}}\}$

$$\begin{aligned}
\vec{y}_{||\{\vec{1}, \vec{x}\}} &= \vec{y}_{||\{\vec{1}, \vec{x}_{\perp \vec{1}}\}} = \vec{y}_{||\vec{1}} + \vec{y}_{||\vec{x}_{\perp \vec{1}}} = \bar{y} \vec{1} + \vec{y}_{||(\vec{x} - \bar{y} \vec{1})} = \bar{y} \vec{1} + (\vec{y} - \bar{y} \vec{1})_{||(\vec{x} - \bar{y} \vec{1})} \\
&= \bar{y} \vec{1} + \frac{(\vec{y} - \bar{y} \vec{1}) \cdot (\vec{x} - \bar{y} \vec{1})}{(\vec{x} - \bar{y} \vec{1}) \cdot (\vec{x} - \bar{y} \vec{1})} (\vec{x} - \bar{y} \vec{1}) = \frac{1}{n} \sum y_i + \frac{\sum (y_i - \bar{y})(x_i - \bar{x})}{(\sum (x_i - \bar{x}))}
\end{aligned}$$

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