STAT 151A Lecture 32

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Remark 0.1 (Linear Probability Model)
$y_i = \alpha + \beta x_i + \varepsilon_i$, where y_i is binary
NLM: $y_i = \alpha + \beta x_i + \varepsilon_i, \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$
LPM: is $\varepsilon_i \sim N(0, \sigma^2)$, no
$\varepsilon_i = y_i - \mathbb{E}(y_i)$
$\varepsilon_i = \begin{cases} 1 - (\alpha + \beta x_i) = 1 - \pi_i \\ -(\alpha + \beta x_i) = \pi_i \end{cases}$
Not normally distributed and not homoskedastic
Bottom line: Inference is messy (HLM or bootstrap can help a bit)

Remark 0.2 (Logistic Regression)

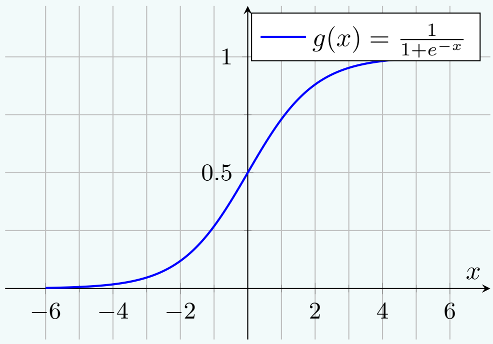
What we need: a (nonlinear) mapping from the linear predictor to $[0, 1]$
$P(y_i = 1|x_i) = \pi_i = g(\alpha + \beta x_i)$
Choose logistic function:

$$g(z) = \frac{1}{1 + \exp(-z)}$$

Why?

(1) It is nearly linear for $\pi \in (0.2, 0.8)$

(2) Nice interpretation for β:
$\pi_i = \frac{1}{1+\exp(-\alpha-\beta x_i)} = \frac{\exp(\alpha+\beta x_i)}{\exp(\alpha+\beta x_i)+1}$
$1 - \pi_i = \frac{\exp(\alpha+\beta x_i)+1}{\exp(\alpha+\beta x_i)+1} - \frac{\exp(\alpha+\beta x_i)}{\exp(\alpha+\beta x_i)+1} = \frac{1}{\exp(\alpha+\beta x_i)+1}$
$\frac{\pi_i}{1-\pi_i} = \exp(\alpha + \beta x_i)$
$\log\left(\frac{\pi_i}{1-\pi_i}\right) = \alpha + \beta x_i$
A 1-unit difference in x_j si associated with a β_j difference in log odds $\leftrightarrow e^{\beta_j}$ multiplicative difference in odds



Remark 0.3 (How do we fit Logistic Regression?)

Maximum Likelihood

$y_1, \dots, y_n \overset{iid}{\sim} \text{Bern}(\pi_i), \pi_i = \frac{1}{1+\exp(-\beta x_i - \alpha)}$

Likelihood

$$\begin{aligned}
\mathcal{L}(y_1, \dots, y_n | \beta, \alpha, x_1, \dots, x_n) &= \prod_{i=1}^{n} f(y_i) \\
&= \prod_{i=1}^{n} \pi_i^{y_i}(1-\pi_i)^{1-y_i} \\
&= \prod_{i=1}^{n} \left(\frac{\pi_i}{1-\pi_i}\right)^{y_i}(1-\pi_i) \\
&= \prod_{i=1}^{n} [\exp(\alpha + \beta x_i)]^{y_i} \frac{1}{1+\exp(\alpha + \beta x_i)} \\
\log(L(y_1, \dots, y_n | \beta, \alpha, x_1, \dots, x_n)) &= \sum [y_i(\alpha + \beta x_i) - \log(1 + \exp(\alpha + \beta x_i))]
\end{aligned}$$

Take derivatives wrt β, α and set to 0

$$\begin{aligned}
\frac{\partial}{\partial \alpha} &= \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \frac{\exp(\alpha + \beta x_i)}{1 + \exp(\alpha + \beta x_i)} \\
\frac{\partial}{\partial \beta} &= \sum y_i x_i - \sum \frac{\exp(\alpha + \beta x_i)}{1 + \exp(\alpha + \beta x_i)} x_i
\end{aligned}$$

Need to solve

$$\begin{aligned}
\sum y_i &= \sum \frac{1}{1 + \exp(-\hat{\alpha} - \hat{\beta} x_i)} \\
\sum y_i x_i &= \sum \frac{1}{1 + \exp(-\hat{\alpha} - \hat{\beta} x_i)} x_i
\end{aligned}$$