STAT 151A Lecture 8

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Remark 0.1 (Recap)

$$\mathbf{Y} = \mathbf{X}\hat{\beta} + \vec{e} = \mathbf{Y}_{||\,\text{span}(\mathbf{X})} + \mathbf{Y}_{\perp \text{span}(\mathbf{X})}$$

$$\hat{\beta} = (\mathbf{X}^\intercal \mathbf{X})^{-1}\mathbf{X}^\intercal \mathbf{Y}$$

Remark 0.2 (Enter Probability)

$$\vec{Y} = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}, \vec{Y} \text{ a random vector}$$

$$\mathbb{E}(\vec{Y}) = \begin{pmatrix} \mathbb{E}(Y_1) \\ \vdots \\ \mathbb{E}(Y_n) \end{pmatrix}$$

$$\text{Var}(\vec{Y}) = \Sigma_{\vec{Y}\vec{Y}} = \begin{pmatrix} \text{Var}(Y_1) & \text{Cov}(Y_1, Y_2) & \text{Cov}(Y_1, Y_3) & \cdots & \text{Cov}(Y_1, Y_n) \\ \text{Cov}(Y_2, Y_1) & \text{Var}(Y_2) & \text{Cov}(Y_2, Y_3) & \cdots & \text{Cov}(Y_2, Y_n) \\ \text{Cov}(Y_3, Y_1) & \text{Cov}(Y_3, Y_2) & \text{Var}(Y_3) & \cdots & \text{Cov}(Y_3, Y_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(Y_n, Y_1) & \text{Cov}(Y_n, Y_2) & \text{Cov}(Y_n, Y_3) & \cdots & \text{Var}(Y_n) \end{pmatrix}$$

$\Sigma_{\vec{Y}\vec{Y}} \in \mathbb{R}^{n \times n}$

Remark 0.3 (Linear Transformations)

Random Vectors: $\vec{V} \in \mathbb{R}^m, \vec{U} \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^m$

$$\vec{V} = \vec{c} + A\vec{U}$$

$$\mathbb{E}(\vec{V}) = \vec{c} + A\mathbb{E}(\vec{U})$$

$$\Sigma_{\vec{V}\vec{V}} = A\Sigma_{\vec{U}\vec{U}}A^\intercal \to \Sigma_{\vec{V}\vec{V}(i,j)} = \sum_{k=1}^{n}\sum_{l=1}^{n} a_{ik}a_{jl}\,\text{Cov}(U_k, U_l)$$

Remark 0.4 (Quadratic Form)

Random Vector $\vec{V} \in \mathbb{R}^n$, Constant Matrix $A \in R^{n\times n}$
Quadratic Form: $\vec{v}^\intercal A \vec{v}$

$$\mathbb{E}(\vec{v}^\intercal A \vec{v}) = \mathbb{E}(\vec{v})^\intercal A \mathbb{E}(\vec{V}) + \mathrm{Tr}(A\Sigma_{\vec{V}\vec{V}})$$

Random Vectors $\vec{V} \in \mathbb{R}^p, \vec{W} \in \mathbb{R}^m, \vec{U} \in \mathbb{R}^n$
$\vec{V} = A\vec{U}, \vec{W} = B\vec{U}$

$$\Sigma_{\vec{V}\vec{W}} = A\Sigma_{\vec{U}\vec{U}} B^\intercal$$

Remark 0.5 (Classical Linear Probabiltiy Model, Gauss-Markov Model)

Reminder: Simple Regression
$y_i = \alpha + \beta x_i + \varepsilon_i$
$\mathbb{E}(\varepsilon_i|x_i) = 0, \mathrm{Var}(\varepsilon_i|x_i) = \sigma^2, \Sigma_{\vec{e}ps\vec{e}ps|X} = \sigma^2 \mathbb{I}_n$

$$Y = X\hat{\beta} + \vec{e}, \hat{\beta} = (X^\intercal X)^{-1} X^\intercal Y, \vec{e} = Y - X(X^\intercal X)^{-1} X^\intercal Y$$

$$\begin{aligned}
\mathbb{E}(\hat{\beta}|X) &= \mathbb{E}[(X^\intercal X)^{-1} X^\intercal Y|X] \\
&= (X^\intercal X)^{-1} X^\intercal \mathbb{E}[Y|X] \\
&= (X^\intercal X)^{-1} X^\intercal \mathbb{E}[X\hat{\beta} + \vec{e}|X] \\
&= (X^\intercal X)^{-1} X^\intercal X \mathbb{E}[\beta|X] + (X^\intercal X)^{-1} X^\intercal \mathbb{E}(\vec{e}|X) \\
&= \beta \\
\mathrm{Var}(\hat{\beta}|X) = \Sigma_{\hat{\beta}\hat{\beta}|X} &= (X^\intercal X)^{-1} X^\intercal \Sigma_{\vec{Y}\vec{Y}|X} \left[(X^\intercal X)^{-1} X^\intercal\right]^\intercal \\
&= \left[(X^\intercal X)^{-1} X^\intercal\right] \sigma^2 \mathbb{I} \left[X(X^\intercal X)^{-1}\right] \\
&= \sigma^2 (X^\intercal X)^{-1} X^\intercal X (X^\intercal X)^{-1} \\
&= \sigma^2 (X^\intercal X)^{-1}
\end{aligned}$$