

Assignment 1

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Question 1

i. $S = \{x \in \mathbb{R} | x = \sqrt{2} + k, k \in \mathbb{N}\}$

ii. **Proof:** For any arbitrary $k \in \mathbb{N}$, we define $S_k := (\sqrt{2} + k, \sqrt{2} + k + 1)$.

According to the definition, $S_k \subset \mathbb{R} \setminus S$. For arbitrary $x \in S_k$, let $\delta = \min\{x - \sqrt{2} - k, \sqrt{2} + k + 1 - x\}$, $\exists B(x, \delta)$ is an open ball, thus S_k is an open set. Then the complement of S in \mathbb{R} $\bar{S} = \bigcup_{k \in \mathbb{N}} S_k$ is an open set. Therefore, S is a closed set.

iii. **Proof:** Since \mathbb{N} is unbounded (according to the Archimedes character), S is an unbounded set.

iv. $B((0, 0, 4), 1)$

v. **Proof:** Obviously it is an open set since it is an open ball.

vi. $\text{diam}(B) = 2$.

Proof: For arbitrary points $\vec{p}, \vec{q} \in B(\vec{x}, 1)$, $d(\vec{x}, \vec{p}) < 1, d(\vec{x}, \vec{q}) < 1$, thus $d(\vec{p}, \vec{q}) < 2$ according to the triangular inequality. Now consider two point sequences:

$$A_k = \{(0, 0, 3 + \frac{1}{2^k}) | k \in \mathbb{N}\}, B_k = \{(0, 0, 5 - \frac{1}{2^k}) | k \in \mathbb{N}\}$$

Obviously $A_k, B_k \subset B(\vec{x}, 1)$. Then we can tell

$$\lim_{k \rightarrow \infty} d(A_k, B_k) = 2$$

Which means $\forall \varepsilon > 0, \exists N > 0$, when $k > N, d(A_k, B_k) > 2 - \varepsilon$. Then we can tell $\sup(\vec{p}, \vec{q}) = 2$, which means $\text{diam}(B) = 2$.

Question 2

i.

$$\lim_{(x,y) \rightarrow (3,5)} \left(\frac{\sin(y-x)}{y-x}, \sqrt{y^2-x^2} \right)$$

exists

Proof: Both $\frac{\sin(y-x)}{y-x}$ and $\sqrt{y^2-x^2}$ are continuous at point $(3,5)$, thus the limit is equal to $(\frac{\sin 2}{2}, 4)$.

ii.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - y^3}{x^3 - y^4}$$

doesn't exist.

Proof: Replace y with $kx (k \in \mathbb{R})$, then the original formula is equal to

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x^4 - k^3 x^3}{x^3 - k^4 x^4} &= \lim_{x \rightarrow 0} \frac{x - k^3}{1 - k^4 x} \\ &= -k^3 \end{aligned}$$

which is uncertain. Therefore, the limit doesn't exist.

iii.

$$\lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \frac{x^3 - y^3}{x^4 - y^4}$$

exists.

Proof: The original formula is equal to

$$\lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \frac{x^2 + xy + y^2}{x^3 + x^2y + xy^2 + y^3}$$

which is obviously no less than zero. Then

$$\begin{aligned} \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \frac{x^2 + xy + y^2}{x^3 + x^2y + xy^2 + y^3} &= \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \frac{x^2 + xy + y^2 + \frac{y^3}{x} - \frac{y^3}{x}}{x^3 + x^2y + xy^2 + y^3} \\ &= \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \left(\frac{1}{x} - \frac{y^3}{x^4 + x^3y + x^2y^2 + xy^3} \right) \\ &\leq \lim_{x \rightarrow +\infty} \frac{1}{x} = 0 \end{aligned}$$

As a result,

$$0 \leq \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \frac{x^3 - y^3}{x^4 - y^4} \leq 0$$

Therefore, the limit is equal to 0.

iv.

$$\lim_{x \rightarrow +\infty} \lim_{y \rightarrow -\infty} \frac{x^3 - y^3}{x^4 - y^4}$$

exists.

Proof: Replace y with $u = \frac{1}{y}$, then

$$\lim_{y \rightarrow -\infty} \frac{x^3 - y^3}{x^4 - y^4} = \lim_{u \rightarrow 0^-} \frac{x^3 - \frac{1}{u^3}}{x^4 - \frac{1}{u^4}} = \lim_{u \rightarrow 0^-} \frac{(u^3x^3 - 1)u}{u^4x^4 - 1} = 0$$

Therefore,

$$\lim_{x \rightarrow +\infty} \lim_{y \rightarrow -\infty} \frac{x^3 - y^3}{x^4 - y^4} = \lim_{x \rightarrow +\infty} 0 = 0$$

v.

$$\lim_{(x,y) \rightarrow (e,0)} (1 + 2020y)^{\frac{1}{y - x^2y^2}}$$

exists.

Proof: Consider

$$\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \mathbf{f}(x, y) = ((1 + 2020y)^{\frac{1}{2020y}}, \frac{2020y}{y - x^2y^2})$$

=

$$\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \mathbf{f}(x, y) = ((1 + 2020y)^{\frac{1}{2020y}}, \frac{2020}{1 - x^2y})$$

, Therefore,

$$\lim_{(x,y) \rightarrow (e,0)} \mathbf{f}(x, y) = (e, 2020)$$

$$\mathbf{g} : \mathbb{R}^2 \rightarrow \mathbb{R} : \mathbf{g}(u, v) = u^v$$

Then invoke the composition rule:

$$\lim_{(u,v) \rightarrow (e,2020)} \mathbf{g}(u, v) = e^{2020}$$

vi.

$$\lim_{(x,y) \rightarrow (3,+\infty)} \frac{\log(x+y)}{x^2 + y^2}$$

exists.

Proof: According to the fundamental inequality, $x^2 + y^2 \geq \frac{x^2+y^2}{2}$, then

$$\frac{\log(x+y)}{x^2 + y^2} \leq \frac{\log(x+y)}{\frac{(x+y)^2}{2}}$$

Replace $x^2 + y^2$ with u . When $(x^2 + y^2) \rightarrow (3, \infty), u \rightarrow \infty$. Then

$$\lim_{(x,y) \rightarrow (3,\infty)} \frac{\log(x+y)}{\frac{(x+y)^2}{2}} = \lim_{u \rightarrow \infty} \frac{2 \log u}{u^2} = 0$$

Meanwhile, the original formula is no less than zero, which means

$$0 \leq \lim_{(x,y) \rightarrow (3,+\infty)} \frac{\log(x+y)}{x^2 + y^2} \leq 0$$

Therefore, the limit is equal to 0.

Question 3

i. **Proof:** Denote \vec{h} as $\begin{bmatrix} h_x \\ h_y \end{bmatrix}$, \vec{t} as $\begin{bmatrix} x \\ y \end{bmatrix}$. Take the linear function $\mathbf{L}(\vec{h})$ as $\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \end{bmatrix}$.

Consider

$$\begin{aligned} \mathbf{f}(\vec{t} + \vec{h}) - \mathbf{f}(\vec{t}) - \mathbf{L}(\vec{h}) &= \begin{bmatrix} 2(x + h_x) + 3 - (2x + 3) - 2h_x \\ (x + h_x)^2 + (y + h_y)^2 - (x^2 + y^2) - 2h_y \\ (y + h_y)e^{2(x+h_x)} - ye^{2x} - (2h_x + h_y) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ h_x^2 + 2xh_x + 2h_y(y - 1) + h_y^2 \\ ye^{2x}(e^{2h_x} - 1) + h_y(e^{2(x+h_x)} - 1) - 2h_x \end{bmatrix} \end{aligned}$$

When $\vec{t} = (0, 1)$,

$$\mathbf{f}(\vec{t} + \vec{h}) - \mathbf{f}(\vec{t}) - \mathbf{L}(\vec{h}) = \begin{bmatrix} 0 \\ h_x^2 + 2xh_x + 2h_y(y - 1) + h_y^2 \\ e^{2h_x} - 1 + h_y(e^{2h_x} - 1) - 2h_x \end{bmatrix}$$

$$\lim_{\vec{h} \rightarrow 0} \frac{\|\mathbf{f}(\vec{t} + \vec{h}) - \mathbf{f}(\vec{t}) - \mathbf{L}(\vec{h})\|}{\|\vec{h}\|}$$

=

$$\lim_{\vec{h} \rightarrow 0} \sqrt{\frac{(h_x^2 + 2xh_x + 2h_y(y - 1) + h_y^2)^2 + (e^{2h_x} - 1 + h_y(e^{2h_x} - 1) - 2h_x)^2}{h_x^2 + h_y^2}}$$

=

$$\lim_{\vec{h} \rightarrow 0} \sqrt{\frac{(h_x^2 + h_y^2)^2 + (2h_xh_y)^2}{h_x^2 + h_y^2}}$$

=

$$\lim_{\vec{h} \rightarrow 0} \sqrt{h_x^2 + h_y^2 + \frac{4}{\frac{1}{h_y^2} + \frac{1}{h_x^2}}} = 0$$

Therefore, \mathbf{f} is differentiable at $(0, 1)$.

ii. At $(x, y) = (0, 1)$

$$\frac{\partial f_1}{\partial x} = 2, \frac{\partial f_1}{\partial y} = 0$$

$$\frac{\partial f_2}{\partial x} = 2x = 0, \frac{\partial f_2}{\partial y} = 2y = 2$$

$$\frac{\partial f_3}{\partial x} = 2ye^{2x} = 2, \frac{\partial f_3}{\partial y} = e^{2x} = 1$$

iii.

$$[d\mathbf{f}]_{(0,1)} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$$