Assignment 1

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Question 1

i. $S = \{x \in \mathbb{R} | x = \sqrt{2} + k, k \in \mathbb{N}\}$

ii. **proof:** For any arbitrary $k \in \mathbb{N}$, we define $S_k := (\sqrt{2}+k, \sqrt{2}+k+1)$. According to the definition, $S_k \subset \mathbb{R} \setminus S$. For arbitrary $x \in S_k$, let $\delta = min\{x - \sqrt{2} - k, \sqrt{2}+k+1-x\}$, $\exists B(x,\delta)$ is an open ball, thus S_k is an open set. Then the complement of S in $\mathbb{R}$ $\overline{S} = \bigcup_{k\in\mathbb{N}} S_k$ is an open set. Therefore, S is an closed set.

iii. **proof:** Since $\mathbb{N}$ is unbounded (according to the Archimedes character), S is an unbounded set.

iv. $B((0,0,4),1)$

v. **proof:** Obviously it is an open set since it is an oepn ball.

vi. $diam(B) = 2$.

proof: For arbitrary points $\vec{p}, \vec{q} \in B(\vec{x},1), d(\vec{x},\vec{p}) < 1, d(\vec{x},\vec{q}) < 1$, thus $d(\vec{x},\vec{y}) < 2$ according to the triangular inequality. Now consider two point sequences:

$$A_k = \{(0,0,3+\frac{1}{2^k}) | k \in \mathbb{N}\}, B_k = \{(0,0,5-\frac{1}{2^k}) | k \in \mathbb{N}\}$$

Obviously $A_k, B_k \subset B(\vec{x}, 1)$. Then we can tell

$$\lim_{k \to \infty} d(A_k, B_k) = 2$$

Which means $\forall \varepsilon > 0, \exists N > 0$, when $k > N, d(A_k, B_k) > 2 - \varepsilon$. Then we can tell $sup(\vec{p}, \vec{q}) = 2$, which means $diam(B) = 2$.

Question 2

i.

$$\lim_{(x,y) \to (3.5)} (\frac{\sin(y-x)}{y-x}, \sqrt{y^2 - x^2})$$

exists since when $(x,y) \to (3.5), y - x$ is nonzero. The limit is equal to $(\frac{\sin 2}{2}, 4)$

ii.

$$\lim_{(x,y) \to (0,0)} \frac{x^4 - y^3}{x^3 - y^4}$$

doesn't exist. **proof:** Replace y with $kx (k \in \mathbb{R})$, then the original formula is equal to

$$\lim_{x \to 0} \frac{x^4 - k^3 x^3}{x^3 - k^4 x^4}$$

=

$$\lim_{x \to 0} \frac{x - k^3}{1 - k^4 x}$$

= $-k^3$ which is uncertain. Therefore, the limit doesn't exist.

iii.

$$\lim_{\substack{x \to \infty \\ y \to \infty}} \frac{x^3 - y^3}{x^4 - y^4}$$