Assignment 3

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Question 1

$$
\begin{aligned}
\int_0^1 \int_y^1 y\sqrt{1-x^6}\mathrm{d}x\mathrm{d}y &= \int_0^1 \int_0^x y\sqrt{1-x^6}\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \frac{1}{2}x^2\sqrt{1-x^6}\mathrm{d}x \\
&= \frac{1}{6}\int_0^1 \sqrt{1-x^6}\mathrm{d}(x^3) \\
&= \frac{1}{6}\int_0^1 \sqrt{1-u^2}\mathrm{d}u
\end{aligned}
$$

Substitute u with $\sin v$, then $\mathrm{d}u = \cos v$, $\sqrt{1-u^2} = \cos v$

$$
\begin{aligned}
I = \int_0^1 \sqrt{1-u^2}\mathrm{d}u &= \int_0^{\frac{\pi}{2}} \cos^2 v\mathrm{d}v \\
&= \cos v \sin v|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \sin^2 v\mathrm{d}v \\
&= \int_0^{\frac{\pi}{2}} \mathrm{d}v - I \\
\Rightarrow I &= \frac{1}{2}v|_0^{\frac{\pi}{2}} = \frac{\pi}{4} \\
\Rightarrow \int_0^1 \int_y^1 y\sqrt{1-x^6}\mathrm{d}x\mathrm{d}y &= \frac{1}{6}I = \frac{\pi}{24}
\end{aligned}
$$

Question 2

$$
\int_1^\infty (\int_2^3 \frac{y}{x^3}\exp(\frac{y}{x^2})\mathrm{d}y)\mathrm{d}x
$$

Consider $F(x) = \frac{3}{x^3}\exp(\frac{3}{x^2}), \forall y \in [2,3], \frac{y}{x^3}\exp(\frac{y}{x^2}) \leq F(x)$

Since $\int_1^\infty F(x)\mathrm{d}x = -\frac{1}{2}\int_1^\infty \exp(\frac{3}{x^2})\mathrm{d}(\frac{3}{x^2}) = -\frac{1}{2}(1-\mathrm{e}^3)$ is convergent, the original formula is uniformly convergent wrt $y \in [2,3]$

$$
\begin{aligned}
\int_1^\infty \Big(\int_2^3 \frac{y}{x^3}\exp(\frac{y}{x^2})\mathrm{d}y\Big)\mathrm{d}x &= \int_2^3 \int_1^\infty \frac{y}{x^3}\exp(\frac{y}{x^2})\mathrm{d}x\mathrm{d}y \\
&= \int_2^3 \mathrm{d}y \int_1^\infty -\frac{1}{2}\exp(\frac{y}{x^2})\mathrm{d}(\frac{y}{x^2}) \\
&= \frac{1}{2}\int_2^3 (\mathrm{e}^y - 1)\mathrm{d}y \\
&= \frac{1}{2}(\mathrm{e}^3 - \mathrm{e}^2 - 1)
\end{aligned}
$$

Question 3

$$\int_0^\infty \frac{\cos x}{x}(\mathrm{e}^{-x} - \mathrm{e}^{-2x})\mathrm{d}x$$

Since

$$\lim_{x\to 0} \frac{\cos x(\mathrm{e}^{-x} - \mathrm{e}^{-2x})}{x} = \lim_{x\to 0} \cos x(2\mathrm{e}^{-2x} - \mathrm{e}^{-x}) = 0$$

The integrand extends continuously to a well-defined function on $[0,1)$.

Since

$$\int\int \mathrm{e}^{-xy}\mathrm{d}y\mathrm{d}x = \int \frac{1}{x}\mathrm{e}^{-xy}\mathrm{d}x$$

The integrand is equal to

$$\int_0^\infty \int_1^2 (\cos x)\mathrm{e}^{-xy}\mathrm{d}y\mathrm{d}x$$

Consider $F(y) = \mathrm{e}^{-xy}, |(\cos x)\mathrm{e}^{-xy}| \le F(y), \int_0^\infty F(y)\mathrm{d}y = \frac{1}{y}$ is convergent, the integrand is uniformly convergent wrt $y \in [1,2]$

$$\begin{aligned}
\int_0^\infty \frac{\cos x}{x}(\mathrm{e}^{-x} - \mathrm{e}^{-2x})\mathrm{d}x &= \int_1^2 \int_0^\infty (\cos x)\mathrm{e}^{-xy}\mathrm{d}x\mathrm{d}y \\
I = \int_0^\infty (\cos x)\mathrm{e}^{-xy}\mathrm{d}x &= -\frac{1}{y}\int_0^\infty (\cos x)\mathrm{d}(\mathrm{e}^{-xy}) \\
&= -\frac{1}{y}(\int_0^\infty (\sin x)\mathrm{e}^{-xy}\mathrm{d}x - 1) \\
&= \frac{1}{y^2}I + \frac{1}{y} \\
\Rightarrow I &= \frac{y}{y^2+1} \\
\Rightarrow \int_1^2 \frac{y}{y^2+1}\mathrm{d}y &= \frac{1}{2}\int_1^2 \frac{1}{y^2+1}\mathrm{d}(y^2+1) \\
&= \frac{1}{2}(\ln 5 - \ln 2)
\end{aligned}$$

Question 4

$$\int_1^\infty (\int_2^3 \frac{y}{x^3}\exp(\frac{y}{x^2})\mathrm{d}y)\mathrm{d}x$$

Consider $f(\cdot, y) = \frac{y}{x^3}, g(\cdot, y) = \exp(\frac{y}{x^2})$,

$$\int_1^\infty f(x,y)\mathrm{d}x = \int_1^\infty \frac{y}{x^3}\mathrm{d}x \text{ is uniformly convergent wrt y since } 3 > 1$$

$$1 < g(x,y) \le \mathrm{e}^y \text{ is bounded and monotonic wrt } x$$

Therefore, the integrand is uniformly convergent wrt y according to Albel test.