Calculus A2 (English) — Assignment 4.

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Submission instructions: same instructions as for assignments 1, 2 and 3.

Question 1: (7 marks)

Determine which of the following vector fields has path-independent integrals (ie.: the line integral of the function between any two points in the domain of the function is path independent). If the function does display path-independence, prove it (possibly by using results done in class). If not, disprove it.

i. $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{F}(x, y) = [x^2 - y^2 - 2x + 1, -2xy + 2y]^\mathsf{T}$;

ii. $\mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{G}(x, y) = [x^3 - y^3 - 2x^2, -3xy^2 + 3y^2]^\mathsf{T}$;

iii. $\mathbf{H} : \{[0, 0]^\mathsf{T}\} \to \mathbb{R}^2$, $\mathbf{H}(x, y) = [x^3 - y^3 - 2x^2, -3xy^2 + 3y^2]^\mathsf{T}$;

iv. $\mathbf{I} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{I}(x, y, z) = [\sin(x + e^x), \cos(\cos(y)), z + z^7]^\mathsf{T}$;

v. $\mathbf{J} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{J}(x, y, z) = [e^{xy}, e^{x-y}, x + y]^\mathsf{T}$.

vi. $\mathbf{K} : \mathbb{Z}^3 \to \mathbb{R}^3$, $\mathbf{K}(x, y, z) = [y, z, x + y + z]^\mathsf{T}$;

vii. $\mathbf{L} : \{[0, 0, z]^\mathsf{T} \in \mathbb{R}^3\} \to \mathbb{R}^3$, $\mathbf{L}(x, y, z) = [x + y + z, x^2 + y^2 - z^2, x^3 + 2y^3 + 3z^3]^\mathsf{T}$.

Question 2: (7 marks)

Consider the region Ω in $\mathbb{R}^3$ bounded by:

- the xz-plane (i.e.: $y = 0$),
- $z = \pm 1$ and
- $x^2 + y^2 = 1$.

Informally speaking, Ω resembles a "half-chopped-up-log", and is expressible as $D \times [-1, 1]$, where $D := \left\{ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1,\ y \geqslant 0 \right\}$. Let S_1, S_2, S_3 respectively denote the subsets of the boundary surface $\partial\Omega$ which lie on:

- the xz-plane (i.e.: $y = 0$),
- $z = \pm 1$ and
- $x^2 + y^2 = 1$.

We endow $\partial\Omega$ and each of these subsets with the exterior pointing orientation $\mathbf{n} : \partial\Omega \to \mathbb{R}^3$ on $\partial\Omega$. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) = [y^2, x^2, z^2]^\mathsf{T}.$$

i. Compute the integral $\int_{\partial\Omega} \mathbf{F} \cdot \, d\vec{A}$ by computing the corresponding surface integrals over S_1, S_2 and S_3.

ii. Compute the integral $\int_{\partial\Omega} \mathbf{F} \cdot d\vec{A}$ using Gauss' theorem (the divergence theorem).

iii. Prove that there cannot exist a function $\mathbf{G} : A \to \mathbb{R}^3$ for any simply-connected open set $A \subset \mathbb{R}^3$ such that $\nabla \times \mathbf{G} = \mathbf{F}$. (Hint: you might find lecture 21 useful.) Note that this shows that one should not try to compute $\int_{S_i} \mathbf{F} \cdot \, d\vec{A}$ using the Kelvin-Stokes theorem.

iv. **Bonus question**: Consider a Jordan measurable open set $U \subset \mathbb{R}^2$ with smooth boundary, and define $\Omega := U \times [-1, 1] \subset \mathbb{R}^3$. Prove that

$$\int_{\partial\Omega} \mathbf{F} \cdot \, d\vec{A} = 0$$

(assume that we're taking the outward-pointing normal). — this is worth up to 2 *bonus* marks, and is not counted as a part of the designated 7 marks assigned to this question.

Question 3: (6 marks)

Let $\Gamma \subset \mathbb{R}^3$ denote the intersection of the surfaces $x^2 + y^2 = 1$ and $z = y^2 - x^2$, oriented anticlockwise when regarded from "above the xy-plane". Compute the line integral $\int_\Gamma \mathbf{F} \cdot \, d\vec{\ell}$ for $\mathbf{F} = [y^2, 2xy, xy]^T$ both:

i. directly, and

ii. using the Kelvin-Stokes theorem.