Answer Sheet

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Part I

1-5 F T F T T

6-10 T T T T F

Part II

11. 6

12. e^{-1}

13. $\begin{bmatrix} 24 & 0 \\ 8 & 4 \end{bmatrix}$

14. $\begin{bmatrix} -\cos(3)e^{-\sin 3} - 6\sin 10 \\ 3\cos(3)e^{-\sin 3} + 2\sin 10 \end{bmatrix}$

15. $(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -\sqrt[3]{4} \\ \sqrt[3]{2} \end{bmatrix}) \cdot \begin{bmatrix} 1 \\ 2\sqrt[3]{2} \end{bmatrix} = 0$

16. $1 + y + y^2$

17. $(0,0), (-\frac{1}{3}, -1)$

18. $(0,0)$: neither; $\left(-\frac{1}{3},-1\right)$: local maximum

19. $\frac{dy}{dx} = -12, \frac{dz}{dx} = 4$

20. $\frac{17}{12}$

Part III

21.

$$\lim_{\vec{x} \to [3,1,-4]^T} \frac{1-xy}{x^2+y^2+z^2+3}$$

Proof: The limit exists. This is a well-defined elementary function around $(x,y,z) = (3,1,-4)$. The limit exists and is equal to $-\frac{2}{29}$.

$$\Omega_1 := \{[x,y]^T \in \mathbb{R} | x \neq 0\}, \lim_{\Omega_1 \ni \vec{x} \to [0,1]^T} (y+x)^{\frac{1}{x}}$$

Proof: The limit exists. First observe that we know the following limits exist:

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$$

Therefore,

$$\lim_{\Omega_1 \ni \vec{x} \to [0,1]^T} (y+x)^{\frac{1}{x}} = \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$$

$$\Omega_2 := \mathbb{R}^2 \setminus \{\vec{0}\}, \lim_{\Omega_2 \ni \vec{x} \to \vec{0}} \frac{\tan(xy) \log(1+|xy|)}{x^2+y^2}$$

Proof: The limit exists. First we observe that

$$\lim_{\Omega_2 \ni \vec{x} \to \vec{0}} \frac{\tan(xy) \log(1+|xy|)}{x^2+y^2} \geq 0$$

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Since $x^2 + y^2 \geq 2xy$, we observe that

$$\lim_{\Omega_2 \ni \vec{x} \to \vec{0}} \frac{\tan(xy)\log(1+|xy|)}{x^2+y^2} \leq \lim_{\Omega_2 \ni \vec{x} \to \vec{0}} \frac{\tan(xy)\log(1+|xy|)}{2xy}$$

$$= \lim_{\Omega_2 \ni \vec{x} \to \vec{0}} \frac{(xy)^2 + o((xy)^2)}{2xy}$$

$$= 0$$

According to the sandwich theorem, the limits is equal to 0.

22.

$$\left[dH \right] = \begin{bmatrix} 2x & 6y & \sin(2z) \end{bmatrix}$$

$$H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2\cos(2z) \end{bmatrix}$$

Let $[dH] = \vec{0}$, we get stationary points $\vec{p} = [0,0,\frac{k\pi}{2}]^T$

First consider $k = 2n, n \in \mathbb{N}^*$:

$$H_{\vec{p}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since this is a diagonal matrix, its eigenvalues are $\lambda_1 = 6 > \lambda_2 = 2 > \lambda_3 = 1 > 0$.

Therefore, Hessian Matrixes at $\vec{p}$ are positive definite, $\vec{p}$ are local minima.

Then consider $k = 2n+1, n \in \mathbb{N}^*$:

$$H_{\vec{p}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Since this is a diagonal matrix, its eigenvalues are $\lambda_1 = 6 > \lambda_2 = 2 > 0, \lambda_3 = -1 < 0$. Therefore, the Hessian Matrix are neither positive definite nor semi-positive

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definite, these stationary points are not extrema.

Global minima of H is $min(\sin(\frac{k\pi}{2})) = 0(k = 2n, n \in \mathbb{N}^*)$

24.

$$\int_0^\infty \int_1^7 \frac{x^2}{x^6 + y^2} dy dx = \int_1^7 \int_0^\infty \frac{x^2}{x^6 + y^2} dx dy$$

$$\frac{x^2}{x^6 + y^2} = \frac{1}{x^4 + \frac{y^2}{x^2}} < \frac{1}{x^4}$$

Since $\int_1^\infty \frac{1}{x^4} dx$ is convergent, $\int_1^\infty \frac{x^2}{x^6 + y^2}$ is also convergent. Therefore,

$$\int_0^\infty \frac{x^2}{x^6 + y^2} dx = \frac{1}{3} \int_0^\infty \frac{1}{x^6 + y^2} d(x^3)$$

$$= \frac{1}{3y^2} \int_0^\infty \frac{1}{(\frac{x^3}{y})^2 + 1} d(x^3)$$

$$= \frac{1}{3y} \int_0^\infty \frac{1}{(\frac{x^3}{y})^2 + 1} d(\frac{x^3}{y})$$

$$= \frac{1}{3y} \arctan(\frac{x^3}{y}) \Big|_0^\infty = \frac{\frac{\pi}{2}}{3y}$$

$$\int_1^7 \int_0^\infty \frac{x^2}{x^6 + y^2} dx dy = \int_1^7 \frac{\frac{\pi}{2}}{3y} dy$$

$$= \frac{\pi}{6} \log y \Big|_1^7$$

$$= \frac{\pi}{6} \log 7$$

25.

$$M = \int_0^1 \int_0^1 \int_0^1 \exp(x - y + 2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \int_0^1 \int_0^1 (\exp(1 - y + 2z) - \exp(-y + 2z))\mathrm{d}y\mathrm{d}z$$

$$= \int_0^1 (\exp(1 + 2z) + \exp(2z - 1) + \exp(2z))\mathrm{d}z$$

$$= \frac{1}{2}e^3 - e^2 - \frac{1}{2}e^{-1} + 1$$

$$\hat{x} = \frac{1}{M} \int_0^1 \int_0^1 \int_0^1 x\exp(x - y + 2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \frac{1}{M} \int_0^1 \int_0^1 -\exp(-y + 2z)\mathrm{d}y\mathrm{d}z$$

$$= \frac{1}{M} \int_0^1 \exp(2z - 1) - \exp(2z)\mathrm{d}z$$

$$= \frac{1}{2M}(e - e^{-1} - e^2 + 1)$$

$$\hat{y} = \frac{1}{M} \int_0^1 \int_0^1 \int_0^1 y\exp(x - y + 2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \frac{1}{M} \int_0^1 \int_0^1 y\exp(1 - y + 2z) - \exp(-y + 2z)\mathrm{d}y\mathrm{d}z$$

$$= \frac{1}{M} \int_0^1 (3\exp(2z) - 2\exp(2z - 1) - \exp(2z + 1))\mathrm{d}z$$

$$= \frac{1}{2M}(3e^2 - e^3 + 2e^{-1} - 3)$$

$$\hat{z} = \frac{1}{M} \int_0^1 \int_0^1 \int_0^1 z\exp(x - y + 2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \frac{1}{M} \int_0^1 z(\exp(1 + 2z) + \exp(2z - 1) - 2\exp(2z))\mathrm{d}z$$

$$= \frac{1}{2M}(e^3 + 2e - 2e^2 + e^{-1} - 2)$$

Therefore, the center of mass $\vec{x} =$

$$\begin{bmatrix} \frac{e - e^{-1} - e^2 + 1}{e^3 - 2e^2 - e^{-1} + 2} \\ \frac{3e^2 - e^3 + 2e^{-1} - 3}{e^3 - 2e^2 - e^{-1} + 2} \\ \frac{e^3 + 2e - 2e^2 + e^{-1} - 2}{e^3 - 2e^2 - e^{-1} + 2} \end{bmatrix}$$

The above are not correct, but I am too tired to reconsider the new question...