Answer Sheet

Pan Hao, 2019012632

Part I

1-5 F T F T T

6-10 T T T T F

Part II

11. 6

12. e^{-1}

13. $\begin{bmatrix} 24 & 0 \\ 8 & 4 \end{bmatrix}$

14. $\begin{bmatrix} -\cos(3)e^{-\sin 3} - 6\sin 10 \\ 3\cos(3)e^{-\sin 3} + 2\sin 10 \end{bmatrix}$

15. $(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -\sqrt[3]{4} \\ \sqrt[3]{2} \end{bmatrix}) \cdot \begin{bmatrix} 1 \\ 2\sqrt[3]{2} \end{bmatrix} = 0$

16. $1 + y + y^2$

17. $(0,0), (-\frac{1}{3}, -1)$

18. $(0,0)$: neither; $(-\frac{1}{3},-1)$: local maximum

19. $\frac{\mathrm{d}y}{\mathrm{d}x}=-12, \frac{\mathrm{d}z}{\mathrm{d}x}=4$

20. $\frac{17}{12}$

Part III

21.

$$\lim_{\vec{x}\to[3,1,-4]^T} \frac{1-xy}{x^2+y^2+z^2+3}$$

Proof: The limit exists. This is a well-defined elementary function around $(x,y,z)=(3,1,-4)$. The limit exists and is equal to $-\frac{2}{29}$.

$$\Omega_1 := \{[x,y]^T \in \mathbb{R} | x \neq 0\}, \lim_{\Omega_1 \ni \vec{x}\to[0,1]^T} (y+x)^{\frac{1}{x}}$$

Proof: The limit exists. First observe that we know the following limits exist:

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \mathrm{e}$$

Therefore,

$$\lim_{\Omega_1 \ni \vec{x}\to[0,1]^T} (y+x)^{\frac{1}{x}} = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \mathrm{e}$$

$$\Omega_2 := \mathbb{R}^2 \setminus \{\vec{0}\}, \lim_{\Omega_2 \ni \vec{x}\to\vec{0}} \frac{\tan(xy)\log(1+|xy|)}{x^2+y^2}$$

Proof: The limit exists. First we observe that

$$\lim_{\Omega_2 \ni \vec{x}\to\vec{0}} \frac{\tan(xy)\log(1+|xy|)}{x^2+y^2} \geq 0$$

Since $x^2+y^2 \geq 2xy$, we observe that

$$
\begin{aligned}
\lim_{\Omega_2 \ni \vec{x} \to \vec{0}} \frac{\tan(xy)\log(1+|xy|)}{x^2+y^2} &\leq \lim_{\Omega_2 \ni \vec{x} \to \vec{0}} \frac{\tan(xy)\log(1+|xy|)}{2xy} \\
&= \lim_{\Omega_2 \ni \vec{x} \to \vec{0}} \frac{(xy)^2+o((xy)^2)}{2xy} \\
&= 0
\end{aligned}
$$

According to the sandwich theorem, the limits is equal to 0.

22.

$$
\begin{bmatrix} \mathrm{d}H \end{bmatrix} = \begin{bmatrix} 2x & 6y & \sin(2z) \end{bmatrix}
$$

$$
\boldsymbol{H} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2\cos(2z) \end{bmatrix}
$$

Let $[\mathrm{d}H] = \vec{0}$, we get stationary points $\vec{p} = [0,0,\frac{k\pi}{2}]^T$

First consider $k = 2n, n \in \mathbb{N}^*$:

$$
\boldsymbol{H}_{\vec{p}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Since this is a diagonal matrix, its eigenvalues are $\lambda_1 = 6 > \lambda_2 = 2 > \lambda_3 = 1 > 0$.

Therefore, Hessian Matrixes at $\vec{p}$ are positive definite, $\vec{p}$ are local minima.

Then consider $k = 2n+1, n \in \mathbb{N}^*$:

$$
\boldsymbol{H}_{\vec{p}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -1 \end{bmatrix}
$$

Since this is a diagonal matrix, its eigenvalues are $\lambda_1 = 6 > \lambda_2 = 2 > 0, \lambda_3 = -1 < 0$. Therefore, the Hessian Matrix are neither positive definite nor semi-positive definite, these stationary points are not extrema.

Global minima of H is $min(\sin(\frac{k\pi}{2})) = 0(k = 2n, n \in \mathbb{N}^*)$

24.

$$\int_0^\infty \int_1^7 \frac{x^2}{x^6+y^2} \mathrm{d}y\mathrm{d}x = \int_1^7 \int_0^\infty \frac{x^2}{x^6+y^2} \mathrm{d}x\mathrm{d}y$$
$$\frac{x^2}{x^6+y^2} = \frac{1}{x^4+\frac{y^2}{x^2}} < \frac{1}{x^4}$$

Since $\int_1^\infty \frac{1}{x^4}\mathrm{d}x$ is convergent, $\int_1^\infty \frac{x^2}{x^6+y^2}$ is also convergent. Therefore,

$$\begin{aligned}
\int_0^\infty \frac{x^2}{x^6+y^2}\mathrm{d}x &= \frac{1}{3}\int_0^\infty \frac{1}{x^6+y^2}\mathrm{d}(x^3) \\
&= \frac{1}{3y^2}\int_0^\infty \frac{1}{(\frac{x^3}{y})^2+1}\mathrm{d}(x^3) \\
&= \frac{1}{3y}\int_0^\infty \frac{1}{(\frac{x^3}{y})^2+1}\mathrm{d}(\frac{x^3}{y}) \\
&= \frac{1}{3y}\arctan(\frac{x^3}{y})\Big|_0^\infty = \frac{\frac{\pi}{2}}{3y} \\
\int_1^7 \int_0^\infty \frac{x^2}{x^6+y^2}\mathrm{d}x\mathrm{d}y &= \int_1^7 \frac{\frac{\pi}{2}}{3y}\mathrm{d}y \\
&= \frac{\pi}{6}\log y\Big|_1^7 \\
&= \frac{\pi}{6}\log 7
\end{aligned}$$

25.

$$
\begin{aligned}
M &= \int_0^1 \int_0^1 \int_0^1 \exp(x-y+2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_0^1 \int_0^1 (\exp(1-y+2z)-\exp(-y+2z))\mathrm{d}y\mathrm{d}z \\
&= \int_0^1 (\exp(1+2z)+\exp(2z-1)+\exp(2z))\mathrm{d}z \\
&= \frac{1}{2}\mathrm{e}^3-\mathrm{e}^2-\frac{1}{2}\mathrm{e}^{-1}+1 \\
\hat{x} &= \frac{1}{M}\int_0^1 \int_0^1 \int_0^1 x\exp(x-y+2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{M}\int_0^1 \int_0^1 -\exp(-y+2z)\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{M}\int_0^1 \exp(2z-1)-\exp(2z)\mathrm{d}z \\
&= \frac{1}{2M}(\mathrm{e}-\mathrm{e}^{-1}-\mathrm{e}^2+1) \\
\hat{y} &= \frac{1}{M}\int_0^1 \int_0^1 \int_0^1 y\exp(x-y+2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{M}\int_0^1 \int_0^1 y\exp(1-y+2z)-\exp(-y+2z)\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{M}\int_0^1 (3\exp(2z)-2\exp(2z-1)-\exp(2z+1))\mathrm{d}z \\
&= \frac{1}{2M}(3\mathrm{e}^2-\mathrm{e}^3+2\mathrm{e}^{-1}-3) \\
\hat{z} &= \frac{1}{M}\int_0^1 \int_0^1 \int_0^1 z\exp(x-y+2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \frac{1}{M}\int_0^1 z(\exp(1+2z)+\exp(2z-1)-2\exp(2z))\mathrm{d}z \\
&= \frac{1}{2M}(\mathrm{e}^3+2\mathrm{e}-2\mathrm{e}^2+\mathrm{e}^{-1}-2)
\end{aligned}
$$

Therefore, the center of mass $\vec{x} =$

$$
\begin{bmatrix}
\frac{\mathrm{e}-\mathrm{e}^{-1}-\mathrm{e}^2+1}{\mathrm{e}^3-2\mathrm{e}^2-\mathrm{e}^{-1}+2} \\
\frac{3\mathrm{e}^2-\mathrm{e}^3+2\mathrm{e}^{-1}-3}{\mathrm{e}^3-2\mathrm{e}^2-\mathrm{e}^{-1}+2} \\
\frac{\mathrm{e}^3+2\mathrm{e}-2\mathrm{e}^2+\mathrm{e}^{-1}-2}{\mathrm{e}^3-2\mathrm{e}^2-\mathrm{e}^{-1}+2}
\end{bmatrix}
$$