

Assignment 2

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Question 1

i.

$$\begin{aligned}f(x, y) &= f(1, 1) + [(x-1) - (y-1)] + \frac{1}{2!}[(x-1)^2 - 2(x-1)(y-1) + (y-1)^2] + \\&\quad \frac{1}{3!}[(x-1)^3 - 2(x-1)^2(y-1) - 2(x-1)^2(y-1) + (y-1)^3] + o(\|(x, y) - (1, 1)\|^3) \\&= 1 + [x - y + \frac{1}{2}(x - y)^2 + \frac{1}{6}(x - y)^3] + o(\|(x, y) - (1, 1)\|^3)\end{aligned}$$

ii.

$$\begin{aligned}f(x, y) &= 1 + [x - y + \frac{1}{2}(x - y)^2] + R_k \\&= 1 + [x - y + \frac{1}{2}(x - y)^2] + \frac{1}{6}(x - y)^3 f(\vec{\xi})\end{aligned}$$

iii. Take point $(1, \frac{1}{2})$ into Taylor expansion in i:

$$\begin{aligned}f(1, \frac{1}{2}) &= 1 + [\frac{1}{2} + \frac{1}{2}(\frac{1}{2})^2 + \frac{1}{6}(\frac{1}{2})^3] + o(\|(x, y) - (1, 1)\|^3) \\&\Rightarrow \sqrt{e} \approx \frac{79}{48}\end{aligned}$$

iv. Take point $(1, \frac{1}{2})$ into Taylor expansion in ii:

$$\begin{aligned}f(x, y) &= 1 + [\frac{1}{2} + \frac{1}{2}(\frac{1}{2})^2] + \frac{1}{6}(\frac{1}{2})^3 f(\vec{\xi}) \\&\Leftrightarrow \sqrt{e} = \frac{13}{8} + \frac{1}{48} f(\vec{\xi})\end{aligned}$$

Since $\bar{\xi}$ is in the open interval between point $(1, 1)$ and $(1, \frac{1}{2})$, then $1 \leq f(\bar{\xi}) \leq \sqrt{e}$.

Therefore,

$$\begin{aligned}\sqrt{e} &\geq \frac{13}{8} + \frac{1}{48} = \frac{79}{48} \\ \sqrt{e} &\leq \frac{13}{8} + \frac{1}{48}\sqrt{e} \\ &\leq \frac{78}{47} \\ \Rightarrow \frac{79}{48} &\leq \sqrt{e} \leq \frac{78}{47}\end{aligned}$$

Question 2

i. Proof:

$$\nabla F = \begin{bmatrix} -2e^{2z}x \\ 3y^2 \\ 8 - 2e^{2z}x^2 \end{bmatrix}$$

Suppose that ∇F would vanish, then $x, y \equiv 0$. Consider $Z(F) = \{(x, y, z) \mid -x^2e^{2z} + y^3 + 8z = 0\}$. For a point on $Z(F)$, if $x = y = 0$, then $z = 0$. However, ∇F at point $(0, 0, 0) = 8$ doesn't vanish. Therefore, ∇F never vanishes on $Z(F)$.

ii. Denote an arbitrary vector $\vec{x} = [x, y, z]^T$, $\vec{p} = [a, b, c]^T \in Z(F)$. At point \vec{p} , the

normal vector of tangent plane of $Z(F)$ is $\nabla F|_{\vec{p}} = \begin{bmatrix} -2e^{2c}a \\ 3b^2 \\ 8 - 2e^{2c}a^2 \end{bmatrix}$.

Therefore, the point-normal form for the tangent plane is

$$(\vec{x} - \vec{p}) \cdot \begin{bmatrix} -2e^{2c}a \\ 3b^2 \\ 8 - 2e^{2c}a^2 \end{bmatrix} = 0$$

iii. Take two non-zero vectors which are orthonormal to $\nabla F|_{\vec{p}}$:

$$\vec{u} = \begin{bmatrix} -\frac{1}{2e^{2c}a} \\ \frac{1}{3b^2} \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ \frac{1}{3b^2} \\ \frac{1}{8-2e^{2c}a^2} \end{bmatrix}$$

Then the parametric form for the tangent plane is

$$\{\vec{x} = \vec{p} + s\vec{u} + t\vec{v} | s, t \in \mathbb{R}\}$$

iv. **Proof:**

$$[d\mathbf{F}] = \begin{bmatrix} -2e^{2z}x & 3y^2 & 8-2e^{2z}x^2 \end{bmatrix}$$

When $x \neq \pm 2e^{-z}, 2e^{2z}x^2 \neq 0$. According to the implicit function theorem, $\forall \vec{p} \in Z(F), \exists C^1$ map $\zeta : B([a, b]^T, \delta) \rightarrow \mathbb{R}$ satisfying $F(x, y, \zeta(x, y)) = 0$.

$$\begin{aligned} [d\zeta] &= - \begin{bmatrix} 8-2e^{2c}a^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -2e^{2c}a & 3b^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{2c}a}{4-e^{2c}a^2} & -\frac{3b^2}{8-2e^{2c}a^2} \end{bmatrix} \end{aligned}$$

v. **Proof:** Consider $G(y, z, x) = -x^2e^{2z} + y^3 + 8z$, then $F(x, y, z) = G(y, z, x)$

$$[d\mathbf{G}] = \begin{bmatrix} 3y^2 & 8-2e^{2z}x^2 & -2e^{2z}x \end{bmatrix}$$

When $x \neq 0, -2e^{2z}x \neq 0$. According to the implicit function theorem, $\forall \vec{p} \in Z(F), \exists C^1$ map $\xi : B([b, c]^T, \delta) \rightarrow \mathbb{R}$ satisfying $G(z, y, \xi(y, z)) = F(\xi(y, z), y, z) = 0$.

$$\begin{aligned} [d\xi] &= - \begin{bmatrix} -2e^{2c}a \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3b^2 & 8-2e^{2c}a^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3b^2}{2e^{2c}a} & \frac{8-2e^{2c}a^2}{2e^{2c}a} \end{bmatrix} \end{aligned}$$

vi. **Proof:** At point $[0, -2, 1]^T$, according to iv.,

$$[d\zeta]_{[0, -2, 1]^T} = \begin{bmatrix} 0 & -\frac{3}{2} \end{bmatrix}$$

Thus,

$$\begin{aligned} [\mathbf{d}\mathbf{g}] &= \begin{bmatrix} 1 + \partial_x \boldsymbol{\zeta} & \partial_y \boldsymbol{\zeta} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Its columns are independent, so the matrix is invertible. According to the inverse function theorem, $\exists \mathbf{f} : B([1, -2]^T, \varepsilon) \rightarrow f(B([1, -2]^T, \varepsilon))$ which is inverse to $\mathbf{g} : f(B([1, -2]^T, \varepsilon)) \rightarrow B([1, -2]^T, \varepsilon)$.

$$\begin{aligned} [\mathbf{d}\mathbf{f}] &= [\mathbf{d}\mathbf{g}]^{-1} \\ &= \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} \end{aligned}$$

1 Question 3

$$f(x, y, z) = (x + y - 1)^2 + (x - y + 2)^2 + 2z^2 + \frac{1}{6}z^3$$

i.

$$\begin{aligned} [\mathbf{d}\mathbf{f}] &= \begin{bmatrix} 2(x + y - 1) + 2(x - y + 2) & 2(x + y - 1) - 2(x - y + 2) & 4z + \frac{1}{2}z^2 \\ 4x + 2 & 4y - 6 & 4z + \frac{1}{2}z^2 \end{bmatrix} \\ &= \begin{bmatrix} 4x + 2 & 4y - 6 & 4z + \frac{1}{2}z^2 \end{bmatrix} \end{aligned}$$

In $B(\vec{0}, 10)$, let $[\mathbf{d}\mathbf{f}] = \vec{0}$, we get stationary points $\vec{u} = [-\frac{1}{2}, \frac{3}{2}, 0]^T$, $\vec{v} = [-\frac{1}{2}, \frac{3}{2}, -8]^T$.

ii.

$$H_{\mathbf{f}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 + z \end{bmatrix}$$

Consider two stationary points \vec{u}, \vec{v} :

$$H_f(\vec{u}) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I$$

Thus the Hessian at this point is positive definite and therefore is a local minimum.

$$H_f(\vec{v}) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

For this diagonal matrix, its eigenvalues are $\lambda_1 = \lambda_2 = 4 > 0, \lambda_3 = -4 < 0$. Thus the Hessian at this point is not positive semi-definite and therefore is not a local extrema.

iii. Denote $\varphi(x, y, z) = x^2 + y^2 + z^2 - 100 = 0$. Consider:

$$\begin{aligned} F(x, y, z, \lambda) &= f(x, y, z) + \lambda \varphi(x, y, z) \\ &= (x + y - 1)^2 + (x - y + 2)^2 + 2z^2 + \frac{1}{6}z^3 + \lambda(x^2 + y^2 + z^2 - 100) \\ \Rightarrow \partial_x \mathbf{F} &= 4x + 2 + 2\lambda x = 0 \\ \partial_y \mathbf{F} &= 4y - 2 + 2\lambda y = 0 \\ \partial_z \mathbf{F} &= 4z + \frac{1}{2}z^2 + 2\lambda z = 0 \\ \Rightarrow \lambda &= -\frac{2x+1}{x} \\ \lambda &= \frac{3-2y}{y} \\ \lambda &= \frac{4z + \frac{1}{2}z^2}{2z} \\ \Rightarrow \begin{cases} y = -3x \\ z = \frac{4}{x} \end{cases} \end{aligned}$$

Take $y = -3x, z = \frac{4}{x}$ into $x^2 + y^2 + z^2 = 100$:

$$x^2 + (-3x)^2 + \left(\frac{4}{x}\right)^2 = 100$$

$$\Leftrightarrow 5x^4 - 50x^2 + 8 = 0$$

$$\Rightarrow x_1^2 = \frac{25 + 3\sqrt{65}}{5}, x_2^2 = \frac{25 - 3\sqrt{65}}{5}$$

$$\Rightarrow x_1 = \sqrt{\frac{25 + 3\sqrt{65}}{5}} > 1, x_2 = -\sqrt{\frac{25 + 3\sqrt{65}}{5}} < -1$$

$$x_3 = \sqrt{\frac{25 - 3\sqrt{65}}{5}} < 1, x_4 = -\sqrt{\frac{25 - 3\sqrt{65}}{5}} > -1$$