Tsinghua University

Autumn/Fall 2020

**Calculus A2 (English) Mid-semester Examination**

**Objective:** This exam aims to provide you with a comprehensive assessment of your competency/mastery on concepts encountered thus far in the Calculus A2 (English) course.

**General Instructions:** This is an open-book examination, and you are permitted to use whatever resources you have at your disposal (e.g.: subject slides, online reference websites, textbooks), but not anything which would consitute computational aid (e.g.: calculator, mathematics software, Wolfram Alpha, asking a fellow sentient being). Please write up your solutions on paper in clear and legible font, and upload the answers onto Tsinghua Web Learning with your student number and name in the file name.

- This examination is designed to be completed within 120 minutes, however, you may go beyond that time limit. Nevertheless, I encourage you to time yourself and to clearly mark on your exam how much time you need to complete this examination.
- You may have 15 minutes of reading time prior to the commencement of the test (these 15 minutes are not counted as a part of the 120 minutes of test time).
- Your reading time commences as soon as you take a look at the next page.
- Unclear handwriting resulting in ambiguous answers (e.g.: T/F written to both resemble T and F) will be marked as incorrect. Also, please clearly mark what your intended answer/s is/are for the short answer questions (e.g.: by highlighting or by drawing a box around the answer).
- I trust you to conduct yourself with academic integrity. However, should any instances of cheating/collusion become apparent, all parties involved will be disqualified from the examination.
- This examination is not a hurdle for passing the subject. However, I strongly encourage you to attempt it.
- This examination is out of a total of 60 marks. There are bonus questions provided, but they cannot take you beyond 60 marks in total.

**Part I.** TRUE OR FALSE QUESTIONS. (1 mark each)

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You may write either A/B or T/F to designate your answer.

1. This is a mid-semester examination for Calculus A1 (English) and *not* for Linear Algebra.

   A. True  B. False

2. $\mathbb{Z} \cup (-4, 5] \subset \mathbb{R}$ is a closed set.

   A. True  B. False

3. $\mathbb{Z} \cup (-4, 5] \subset \mathbb{R}$ is a bounded set.

   A. True  B. False

4. Consider $f : \Omega := \mathbb{R}^2 \setminus \{\vec{0}\} \to \mathbb{R}$ defined by $f(x) = \frac{\sin(x^2+y^4)}{x^2+y^4}$. Then the limit $\lim_{\Omega \ni \vec{x} \to \vec{0}} f(\vec{x})$ exists.

   A. True  B. False

5. For every polynomial $P(x, y)$, $\partial_x \partial_y \partial_x P = \partial_y \partial_x^2 P$.

   A. True  B. False

6. Given a differentiable function $\mathbf{f} : B(\vec{0}, 1) \subset \mathbb{R}^n \to \mathbb{R}^m$, its derivative at $\vec{0}$ is a linear function $d\mathbf{f}|_{\vec{0}} : B(\vec{0}, 1) \to \mathbb{R}^m$.

   A. True  B. False

7. If the Hessian of a multivariate $\mathcal{C}^\infty$ function $f : \mathbb{R}^n \to \mathbb{R}$ has all (strictly) positive eigenvalues at a point $\vec{x} \in \mathbb{R}^n$, then $\vec{x}$ is a local minimum of $f$.

   A. True  B. False

8. Every closed and bounded set in $\mathbb{R}^n$ is Jordan measurable.

   A. True  B. False

9. The function $f : [-20, 20) \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous.

   A. True  B. False

10. The integral $\int_{[0,1]\times[-1,1]} \tan(\sin(x)y^3)\; dxdy$ evaluates to a strictly positive constant.

    A. True  B. False

**Part II.** Short Answer Questions. (2 marks each)

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Neither proof nor rough working is required in this section (although you're free to include rough working).

11. Compute diameter of the set $\left\{\left[\begin{smallmatrix}-1\\3\end{smallmatrix}\right], \left[\begin{smallmatrix}-1\\-3\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]\right\} \subset \mathbb{R}^2$.

12. Determine the following limit:

$$\lim_{\substack{x\to+\infty\\ y\to\pi}} \left(x + \frac{y}{\pi}\right)^{\frac{\cos(y)}{x}}.$$

13. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]) = \left[\begin{smallmatrix}x^3\\x^2y\end{smallmatrix}\right]$. If the Jacobian of a function $g : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$[dg]|_{\left[\begin{smallmatrix}8\\-4\end{smallmatrix}\right]} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix},$$

calculate the Jacobian of $g \circ f$ at $\left[\begin{smallmatrix}2\\-1\end{smallmatrix}\right]$.

14. Consider the function $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x, y) = \exp(\sin(xy)) + \cos(x^2 + y^2)$. What is the direction of steepest increase/ascent of $h$ at $\left[\begin{smallmatrix}3\\-1\end{smallmatrix}\right]$?

15. Find the point-normal form of tangent plane of the surface $x + \sin(xy\pi) + y^2 = 0$ in $\mathbb{R}^3$ at the point $[x, y, z]^T = [-\sqrt[3]{4}, \sqrt[3]{2}, 0]^T$

16. Determine the Peano form of the degree 3 Taylor expansion of $\phi : \mathbb{R}^2 \to \mathbb{R}$, $\phi(x, y) = \exp(x^2 + \sin(y))$ around the origin (to clarify, the degree refers to the degree of the Taylor polynomial).

17. Find all the stationary points of the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x, y) = 3xy - 3x^2 + y^3$.

18. Classify each stationary point found in the previous question as either a local maximum, minimum or neither.

19. You're given a function $\mathbf{J} : \mathbb{R}^3 \to \mathbb{R}^2$ and told that at a given point $\vec{v} \in Z(\mathbf{J})$, the Jacobian of $\mathbf{J}$ is given by:

$$[d\mathbf{J}]|_{\vec{v}} = \begin{bmatrix} 3 & 4 & 3 \\ 2 & 0 & 1 \end{bmatrix}.$$

The implicit function theorem tells us that around $\vec{v}$, restricted to the zero-set $Z(\mathbf{J})$, the $y$ and $z$ coordinates for $Z(\mathbf{J})$ may be expressed as functions of the $x$-coordinate. What are $\frac{dy}{dx}$ and $\frac{dz}{dx}$ at $\vec{v}$?

20. Let $\triangle := \{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] \in \mathbb{R}^2 \mid x, y \in [-1, 1],\ y \leq x\}$, calculate the following double integral:

$$\int_{\triangle} 2x + 6x^2y \; dxdy.$$

## **Part III.** In-depth Questions. (6 marks each)

You need to show your working in this section.

21. For each of the following expressions, decide if the limit exists. If yes, determine the limit with proof that the limit exists. If not, prove that the limit does not exist.

    - Consider the limit $\lim_{\vec{x}\to[3,1,-4]^T} \frac{1-xy}{x^2+y^4+z^2+3}$.
    - Let $\Omega_1 := \{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathbb{R}^2 \mid x \neq 0\}$, consider the limit $\lim_{\Omega_1 \ni \vec{x} \to \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} (y+x)^{\frac{1}{x}}$.
    - Let $\Omega_2 := \mathbb{R}^2 \setminus \{\vec{0}\}$, consider the limit $\lim_{\Omega \ni \vec{x} \to \vec{0}} \frac{\tan(xy)\log(1+|xy|)}{x^2+y^2}$.

22. Consider the function $H : \mathbb{R}^3 \to \mathbb{R}$ defined by $H(x,y,z) = x^2 + 3y^2 + \sin^2(z)$. Compute the Jacobian of $H$ as well as its Hessian. Use the former to determine all of the stationary points of $H$, and use the latter to classify them as either local maxima, minima or neither. Hence determine all of the global extrema of $H$.

23. Use Lagrange multipliers to determine the global extrema of $8x - 3y + 2z$ constrained to the zero-set $Z(C)$, where $C(x,y,z) = 2x^2 + 3y^2 + 4z^2 - 16$ is defined over all of $\mathbb{R}^3$. Make sure to explain why the global extrema exist/do not exist.

24. Calculate the following iterated integral, making sure to justify that the following integral exists using uniform convergence.

$$\int_0^{+\infty} \left( \int_1^7 \frac{x^2}{x^6+y^2}\, dy \right) dx.$$

25. A radius 1 ball has mass density function $\rho : \overline{B}(\vec{0},1) \subset \mathbb{R}^3 \to \mathbb{R}$ given by:

$$\rho(x,y,z) = \exp(z).$$

    Find the center of mass of this ball. A little extra info: this is actually (more-or-less) equivalent to the original function on offer, provided that one assumes/establishes something like Question 27. But the calculation should be much much cleaner.

**Part IV.** Bonus Questions. (2 marks each)

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26. Consider the function $Q : \mathbb{R}^3 \to \mathbb{R}$ defined by $Q(x, y, z) = \cos(xy)\sin(z)$. Find all of the stationary points of $Q$, and use the Hessian or otherwise to classify them as either local maxima, minima or neither.

27. Given a Jordan measurable set $\Omega \subset \mathbb{R}^n$ and a Riemann integrable mass density function $\rho : \Omega \to \mathbb{R}$ with center of mass $\vec{0} \in \mathbb{R}^n$. Let $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^n$ be an arbitrary bijective linear transformation on $\mathbb{R}^n$ (i.e.: $\mathbf{L}$ is defined by an invertible matrix $M_{\mathbf{L}}$ acting on $\mathbb{R}^n$ by left multiplication), prove that the center of mass for $\rho \circ \mathbf{L}^{-1} : \mathbf{L}(\Omega) \to \mathbb{R}$ is still $\vec{0} \in \mathbb{R}^n$.