Calculus Note

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1 Euclidean n-space

1.1 Concepts:

$$\mathbb{R}^n := \{(x_1, ..., x_n) | x_1, ..., x_n \in \mathbb{R}\}$$

$$A \times B := (a \in A, b \in B), \mathbb{R}^n := \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}$$

$$|| \cdot || := \mathbb{R}^n \to \mathbb{R} \text{ (Euclidean n-norm)}$$

$$||\vec{x}|| := \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

$$d := \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \text{ (Distance)}$$

$$d(\vec{x}, \vec{y}) := ||\vec{x} - \vec{y}|| = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$$

Properties of distance:

$$\text{Symmetry:} \forall \vec{x}, \vec{y} \in \mathbb{R}^n, d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$$

$$\text{Triangle inequality:} \forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n, d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}) \geq d(\vec{x}, \vec{z})$$

$$\text{Positivity:} \forall \vec{x}, \vec{y} \in \mathbb{R}^n, d(\vec{x}, \vec{y}) \geq 0$$

$$\Rightarrow \forall \vec{x}, \vec{y} \in \mathbb{R}^n, d(\vec{x}, \vec{y}) = 0 \Leftrightarrow \vec{x} = \vec{y}$$

1.2 Open n-ball

1.2.1 Concept:

For $\delta > 0, B(\vec{x}, \delta) := \{\vec{y} \in \mathbb{R}^n | d(\vec{x}, \vec{y}) < \delta\}$

1.2.2 Interior points:

$\vec{x} \in \mathbb{R}^n$ is an interior point of S iff. $\exists \delta > 0 s.t. B(\vec{x}, \delta) \subset S$

1.2.3 Exterior points:

$\vec{x} \in \mathbb{R}^n$ is an exterior point of S iff. $\exists \delta > 0 s.t. B(\vec{x}, \delta) \subset \mathbb{R}^n \setminus S$

1.2.4 Boundary points:

$\vec{x} \in \mathbb{R}^n$ is an boundary point of S iff. $\forall \delta > 0, B(\vec{x}, \delta) \cap S \neq \varnothing$ and$B(\vec{x}, \delta) \cap (\mathbb{R}^n \setminus S) \neq \varnothing$

1.2.5 Adherent points:

$\vec{x} \in \mathbb{R}^n$ is an adherent point of S iff. $\forall \delta > 0, B(\vec{x}, \delta) \cap S \neq \varnothing$

1.2.6 Limit points:

$\vec{x} \in \mathbb{R}^n$ is an limit point of S iff. $\forall \delta > 0, (B(\vec{x}, \delta) \setminus \{\vec{x}\}) \cap S \neq \varnothing$

1.3 Type of sets

1.3.1 Open sets:

A set S is open iff. every point of S is an interior point.

1.3.2 Closed sets:

A set S is closed iff. $\mathbb{R}^n \setminus S$ is open.

1.3.3 Interior of a set:

All interior points of S.

1.3.4 Exterior of a set:

All exterior points of S.

1.3.5 Boundary of a set

All boundary points of S $\rightarrow \partial S$

1.3.6 Closure of a set:

$\bar{S}$ of S is $S \cup \partial S$

1.4 Theorem 1.1

A set $S \subset \mathbb{R}^n$ is open iff. it is equal to the union of a collection of open balls.

Proof: 1. S = union of open balls $\Rightarrow S =$ open

Consider arbitrary $\vec{x} \in S$, then $\exists$ an open ball $B(\vec{y}, \delta_y) \subset S$ which contains $\vec{x}$. Let $\delta_{\vec{x}} := \delta_{\vec{y}} - d(\vec{x}, \vec{y})$, then $\delta_{\vec{x}} > 0$ and $B(\vec{x}, \delta - \vec{x}) \subset B(\vec{y}, \delta_{\vec{y}}) \subset S$

$\Rightarrow \vec{x}$ is an interior point of S.

Since $\vec{x}$ was arbitrary chosen, every point of S is an interior point. Therefore, S is open.

2. $S =$ open $\Rightarrow S =$ union of open balls

Assume S is open, $\Rightarrow \forall \vec{x} \in S, \exists \delta_{\vec{x}} > 0 s.t. B(\vec{x}, \delta_{\vec{x}}) \subset S$

$$U := \bigcup_{\vec{x} \in \mathrm{Int}(S)} B(\vec{x}, \delta_{\vec{x}}) \subset S$$

Conversely, $S \subset U \Rightarrow S = U$ Therefore, S is the union of open balls.

1.5 Theorem 1.2

$\mathrm{Int}(S)$ of any set $S \subset \mathbb{R}^n$ is an open set. Likewise, $\mathrm{Ext}(S)$ of S is also open.

Proof: Every $\vec{x} \in \mathrm{Int}(S)$ is an interior point of S

$\Rightarrow \exists \delta_{\vec{x}} > 0, s.t. B(\vec{x}, \delta_{\vec{x}}) \subset S$

$\Rightarrow U := \bigcup_{\vec{x} \in \mathrm{Int}(S)} B(\vec{x}, \delta_{\vec{x}})$ is open.

Thus, $\mathrm{Int}(S) \subset U$, because every $\vec{x} \in \mathrm{Int}(S)$ is contained in some open ball $B(\vec{x}, \delta_{\vec{x}}) \subset U$. For every $\vec{x} \in \mathrm{Int}(S), B(\vec{x}, \delta_{\vec{x}}) \subset \mathrm{Int}(S)$.

Since $B(\vec{x}, \delta_{\vec{x}})$ is open, every point in $B(\vec{x}, \delta_{\vec{x}})$ is an interior point of $B(\vec{x}, \delta_{\vec{x}}) \Rightarrow$ is an interior point of $S \supset B(\vec{x}, \delta_{\vec{x}})$.

1.6 Proposition 1.3

The union of any collection of (arbitrarily many) open sets is open.

1.7 Proposition 1.4

Finite intersections of open sets is open.

1.8 Theorem 1.5

Finite unions of closed sets is closed, and intersections of closed sets is closed.

2 Further properties

2.1 Proposition 2.1

Finite intersections of open sets is open.

2.2 Theorem 2.2

Finite unions of closed set is closed, and intersections of closed set is closed.

Proof:

$$
\begin{aligned}
&\mathbb{R}^n \setminus (\bigcup_{\alpha \in \mathscr{A}} S_\alpha) = \bigcup_{\alpha \in \mathscr{A}} (\mathbb{R}^n \setminus S_\alpha) \\
\Rightarrow &\vec{x} \in \mathbb{R}^n \setminus (\bigcup_{\alpha \in \mathscr{A}} S_\alpha) \\
\Leftrightarrow &\vec{x} \notin \bigcup_{\alpha \in \mathscr{A}} S_\alpha \\
\Leftrightarrow &\forall \alpha \in \mathscr{A}, \vec{x} \notin S_\alpha \\
\Leftrightarrow &\forall \alpha \in \mathscr{A}, \vec{x} \in \bigcap_{\alpha \in \mathscr{A}} (\mathbb{R}^N \setminus S_\alpha)
\end{aligned}
$$

2.3 Connectedness

2.3.1 Path-connectedness:

$\forall \vec{x}, \vec{y} \in S, \exists$ a continuous function $f : [0,1] \to S, s.t. f(0) = \vec{x}$ and $f(1) = \vec{y}$

2.3.2 Connectedness:

$S \subset \mathbb{R}, s.t. \exists$ open sets $U, V \subset \mathbb{R}^n, s.t. U \cap V = \varnothing, U, V \neq \varnothing$

$S \subset U \cup V$ and $S \cap V$ and $S \cup V$ are both non-empty.