Homework 1 Answer Key

1a) Y N Y Y N and **1b)** Y Y N N Y. [Yi: I'm just providing rough guides to how you can think about these sets, this is not meant to be a formal proof of openness or closedness. Although in some cases, the proof is more-or-less given here. For 1, $\mathbb{R}^2$ is a clopen (closed and open) set when regarded as a subset of itself. Just take an open ball around any point, it has to be contained in $\mathbb{R}^2$ so it's open. The complement of $\mathbb{R}^2$ is $\varnothing$, which is trivially open. However, when you regard $\mathbb{R}^2$ as a plane in $\mathbb{R}^3$ as specified in 2, try taking an open ball around any point in the plane, it will contain points outside the plane and hence the plane cannot be open. It is closed, however, because its complement is open. In particular, the complement consists of two copies of upper half-space (just translated and rotated), which is open because $U \supset B(\vec{x}, x_n)$. So parts 2 and 3 are closely related. The proof for openness of the rectangle in 4 is more-or-less the same as for $C(\vec{v}, r)$, and 5 is the closure of 4 and is closed.]

2a) N Y Y Y N Y N and **2b)** Y N N N N N Y. [Yi: 1 is a "ring" which contains both its inner and outer concentric circle boundaries and is closed. 2 is an arbitrary union of open balls and is open. Note that I originally had $\frac{1}{|k|}$ as the radius, which is a typo and should have been $\frac{1}{|k|+1}$. 3 is the plane with infinitely many disjoint closed balls dug out of it (it is possible for infinite unions of closed sets to be closed, it's just not always true, as is highlighted by the next example). 4 is actually the open unit ball $B(\vec{x}, 1)$. 5 if you draw this one properly, you'll see that the closed ball $\overline{B}((0,0), \sqrt{3})$ is not fully contained in the two open balls, and instead has two of its points $(0, \sqrt{3})$ and $(0, -\sqrt{3})$ poking out. Thus this is neither an open nor a closed set.For 5, this time the two open balls are large enough to contain all of $\overline{B}((0,0), \sqrt{3})$. For 6, notice that there are only finitely many points between $[n, n+1]$, so the complement of this set is just a whole bunch of open intervals.]

3)
i. [Yi: strictly speaking, I only asked you to find the diameter and didn't ask you prove/establish that what you got is the diameter. Nevertheless, you'll find below explicit arguments detailing how one might formally show that the diameter of a set is BLAH.]

Consider the following sequences of points

$$\{\vec{x}_k = (r - \tfrac{1}{k}, \ldots, r - \tfrac{1}{k})\} \text{ and } \{\vec{y}_k = (-r + \tfrac{1}{k}, \ldots, -r + \tfrac{1}{k})\}.$$

We see that

$$\lim_{k \to \infty} d(\vec{x}_k, \vec{y}_k) = \lim_{k \to \infty} 2(r - \tfrac{1}{k})\sqrt{n} = 2r\sqrt{n}.$$

Therefore, the diameter is at least $2r\sqrt{n}$. On the other hand, given arbitrary $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} =$

$(y_1, \ldots, y_n) \in (-r, r)^n$,

$$d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < \sqrt{\sum_{i=1}^{n} (2r)^2} = 2r\sqrt{n}.$$

Therefore, the diameter (which the supremum of the distance between two points in $(-r, r)^n$) cannot be greater than $2r\sqrt{n}$ and hence has to be $2r\sqrt{n}$ precisely.

ii. For any $\vec{x} \in (-r, r)^n$, take

$$R_{\vec{x}} = \min_{1 \leqslant i \leqslant n} \{|x_i - r|, |x_i + r|\},$$

then $B(\vec{x}, R_{\vec{x}}) \subseteq (-r, r)^n$.

iii. [Yi: Let us introduce a little notation (in fact, this is standard notation): given a vector $\vec{v} \in \mathbb{R}^n$ and a set $U \subset \mathbb{R}^n$, we define

$$\vec{v} + U := \{\vec{p} \in \mathbb{R}^n \mid \vec{p} = \vec{v} + \vec{u}, \text{ where } \vec{u} \in U\}.$$

Geometrically speaking, one can think of $\vec{v} + U$ as taking the set U and translating it by adding $\vec{v}$ to every point in U. Now let's return to the problem at hand.]

For any $x \in C(\vec{v}, r)$, $\vec{x} - \vec{v} \in (-r, r)^n$, then from part ii,

$$B(\vec{x}, R_{\vec{x}-\vec{v}}) = \vec{v} + B(\vec{x} - \vec{v}, R_{\vec{x}-\vec{v}}) \subseteq \vec{v} + (-r, r)^n = C(\vec{v}, r).$$

[Yi note: this is a very clean proof[1], and demonstrates mathematical maturity. With practice, I'm sure that we can all learn to write equally polished proofs!]

iv. For any $\vec{y} \in B(\vec{x}, R)$, take $r_{\vec{y}} > 0$ such that

$$r_{\vec{y}} < \frac{R - d(\vec{y}, \vec{x})}{2\sqrt{n}}.$$

[Yi: note that this is always possible because $d(\vec{y}, \vec{x}) < R$ and hence $R - d(\vec{y}, \vec{x})$ is positive.]

Then for any $\vec{z} \in C(\vec{y}, r_{\vec{y}})$,

$$d(\vec{z}, \vec{x}) \leqslant d(\vec{z}, \vec{y}) + d(\vec{y}, \vec{x}) \leqslant 2r_{\vec{y}}\sqrt{n} + d(\vec{y}, \vec{x}) < R,$$

i.e. $\vec{z} \in B(\vec{x}, R)$, which shows that

$$C(\vec{y}, r_{\vec{y}}) \subseteq B(\vec{x}, R).$$

[1] Everybody say "Thanks, Jirui!"

[Yi: note also that instead of bounding $d(\vec{z}, \vec{y})$ by $r_{\vec{y}}\sqrt{n}$ (which many of you would naturally be tempted to do), we used $2r_{\vec{y}}\sqrt{n}$ here. This is actually a pretty good idea because we know this inequality purely from the definition of diameter and also because if you wanted to use $r_{\vec{y}}\sqrt{n}$ you should probably first prove that that's correct..., so it's extra work with no substantial payoff.]

Therefore

$$B(\vec{x}, R) = \bigcup_{\vec{y} \in B(\vec{x}, R)} C(\vec{y}, r_{\vec{y}}).$$

4)
The last two sets in question 1 are bounded with diameter $\sqrt{16 + (\pi - e)^2}$.

The 1st, 4th, 5th and 6th sets in question 2 are bounded with diameters 8, 2, 6 and 7 respectively.

5)
i. $\text{Int}(C(\vec{x}, 1)) = C(\vec{x}, 1)$ because $C(\vec{x}, 1)$ is open.
[Yi: sorry, I mistyped the hint. It should have been use 3iii]

ii. $\partial C(\vec{x}, 1) = \{\vec{y} \in \mathbb{R}^n | -1 \leqslant y_i - x_i \leqslant 1 \text{ for all } i \text{ and } \exists j \text{ s.t. } y_j = -1 \text{ or } y_j = 1\}$.
[Yi: again, I didn't ask you to prove that this is indeed the boundary. But if you had to, you could do so by showing that any δ-ball around one of these points necessarily contains a point in $C(\vec{x}, 1)$ as well as $\mathbb{R}^n \setminus C(\vec{x}, 1)$.]

iii. $\overline{C(\vec{x}, 1)} = \{\vec{y} \in \mathbb{R}^n | -1 \leqslant y_i - x_i \leqslant 1 \text{ for all } i\}$.

iv. $\text{Ext}(C(\vec{x}, 1)) = \{\vec{y} \in \mathbb{R}^n | \exists j, \text{s.t. } y_j < -1 \text{ or } y_j > 1\}$. [Yi: you can see from the definition of exterior points that adherent points cannot be exterior points. So, from Theorem 2.7 in class the exterior has to be a subset of $\mathbb{R}^n \setminus \overline{C(\vec{x}, 1)}$ (it's clear from definition that both interior and boundary points are adherent points). So then you would show that every point in $\mathbb{R}^n \setminus \overline{C(\vec{x}, 1)}$ is an exterior point, and this is equivalent to what Jirui wrote down.]

6)
[Yi: we're using the strategy suggested in the homework of showing that every adherent point of $\overline{S}$ is contained in $\overline{S}$. I added references to some lemmas we use along the way, you won't need to do this in an exam/test situation]

If a is an adherent point of $\overline{S}$, then by Lemma 2.5, there exists a sequence $\{x_n\}$ in $\overline{S}$, such that

$$\lim_{n \to \infty} x_n = a. \tag{1}$$

First, let's show that for any $\delta > 0$,

$$B(a, \delta) \cap S \neq \varnothing \tag{2}$$

Suppose, on the contrary, there exists $\delta > 0$ such that $B(a, \delta) \cap S = \varnothing$. (1) says that there exists $N \in \mathbb{N}$, such that for all $n > N$,

$$x_n \in B(a, \delta).$$

Thus we can take $r > 0$ such that

$$B(x_n, r) \subset B(a, \delta),$$

[Yi: this is using the fact that $B(a, \delta)$ is open and hence x_n is an interior point.] which implies

$$B(x_n, r) \cap S = \varnothing$$

contradicting $x_n \in \overline{S}$. Therefore (2) holds. If a is not in S [Yi: the implication here is that if $a \in S$, then we're already done because a would then be an element of $\overline{S}$], then $\forall \delta > 0$,

$$a \in B(a, \delta) \cap (\mathbb{R}^n \backslash S) \neq \varnothing,$$

i.e. $a \in \partial S$. We conclude that

$$a \in \partial S \cup S = \overline{S}.$$

7) and **8)**

No official solutions will be provided for questions 7 and 8, but talk to me if you want hints. It was correctly pointed out to me that question 8 seemed too easy, and that was because I mistyped it. The original form of question 8 that I had was for

$$\mathcal{E} := \left\{ \tfrac{p^m}{3^n} \mid p \in \mathbb{Z},\ m, n \in \mathbb{N},\ m \geq 2 \right\}.$$

I think that this will turn it into a proper challenge.