Assignment 4

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Question 1

i.

$$\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2, \boldsymbol{F}(x,y) = [x^2 - y^2 - 2x + 1, -2xy + 2y]^T$$

Proof:

$$\int (x^2 - y^2 - 2x + 1)\mathrm{d}x = \frac{1}{3}x^3 - xy^2 - x^2 + x + C(y)$$

$$\int (-2xy + 2y)\mathrm{d}y = -xy^2 + y^2 + C(x)$$

$$\Rightarrow \boldsymbol{F} = \nabla \cdot (\frac{1}{3}x^3 - xy^2 - x^2 + x + y^2)$$

This function does display path-independence.

ii.

$$\boldsymbol{G} : \mathbb{R}^2 \to \mathbb{R}^2, \boldsymbol{G}(x,y) = [x^3 - y^3 - 2x^2, -3xy^2 + 3y^2]^T$$

Proof:

$$\int (x^3 - y^3 - 2x^2, -3xy^2 + 3y^2)\mathrm{d}x = \frac{1}{4}x^4 - xy^3 - \frac{2}{3}x^3 + C(y)$$

$$\int (-3xy^2 + 3y^2)\mathrm{d}y = -xy^3 + y^3 + C(x)$$

$$\Rightarrow \boldsymbol{G} = \nabla \cdot (\frac{1}{4}x^4 - xy^3 - \frac{2}{3}x^3 + y^3)$$

This function does display path-independence.

iii.

$$\boldsymbol{H} : \{[0,0]^T\} \to \mathbb{R}^2, \boldsymbol{H}(x,y) = [x^3 - y^3 - 2x^2, -3xy^2 + 3y^2]$$

Proof:

Consider an arbitrary line L from $\vec{u}$ to $\vec{v}$. **Firstly**, if L doesn't go through $[0,0]^T$, then obviously $\int_L \boldsymbol{H} \cdot \mathrm{d}\vec{l} = 0$. **Secondly**, if L does go through $[0,0]^T$, then we can observe that $\boldsymbol{H}(0,0) = \vec{0}$, then $\int_L \boldsymbol{H} \cdot \mathrm{d}\vec{l} = 0$. Therefore, the function does display path-independence.

iv.

$$\boldsymbol{I} : \mathbb{R}^3 \to \mathbb{R}^3, \boldsymbol{I}(x,y,z) = [\sin(x + \mathrm{e}^x), \cos(\cos(y)), z + z^7]^T$$

Proof:

$$\int (\sin(x + \mathrm{e}^x))\mathrm{d}x = A(x) + C(y,z)$$

$$\int (\cos(\cos(y)))\mathrm{d}y = B(y) + C(x,z)$$

$$\int (z + z^7)\mathrm{d}z = D(z) + C(x,y)$$

$$\Rightarrow \boldsymbol{I} = \nabla \cdot (A(x) + B(y) + D(z) + C)$$

This function does display path-independence.

v.

$$\boldsymbol{J} : \mathbb{R}^3 \to \mathbb{R}^3, \boldsymbol{J}(x,y,z) = [\mathrm{e}^{xy}, \mathrm{e}x - y, x + y]^T$$

Proof:

$$\int (\mathrm{e}^{xy})\mathrm{d}x = \frac{1}{y}\mathrm{e}^{xy} + C(y,z)$$

$$\int (\mathrm{e}^{x-y})\mathrm{d}y = -\mathrm{e}^{x-y} + C(x,z)$$

$$\int (x + y)\mathrm{d}z = z(x + y) + C(x,y)$$

Therefore, $\boldsymbol{J}$ is not the tangent field for any function in $\mathbb{R}^3$. This function doesn't display path-independence.

vi.

$$\boldsymbol{K} : \mathbb{Z}^3 \to \mathbb{R}^3, \boldsymbol{K}(x,y,z) = [y,z,x+y+z]^T$$

Proof:

Consider an arbitrary line L from $[0.5,0.5,0.5]^T$ to $[3.5,3.5,3.5]^T$. If L doesn't go through any point in $\mathbb{Z}^3$, then $\int_L \boldsymbol{K} \cdot \mathrm{d}\vec{l} = 0$. Otherwise, if L goes through point $[1,1,1]^T$, then we can observe $\boldsymbol{K}(1,1,1) = [1,1,3]^T$. Once if $\boldsymbol{K}(1,1,1) \cdot \mathrm{d}\vec{l} \neq 0$, $\int_L \boldsymbol{K} \cdot \mathrm{d}\vec{l} \neq 0$. Therefore, the function doesn't display path-independence.

vii.

$$\boldsymbol{L} : \{[0,0,z]^T \in \mathbb{R}^3\} \to \mathbb{R}^3, \boldsymbol{L}(x,y,z) = [x+y+z, x^2+y^2-z^2, x^3+2y^3+3z^3]$$

Proof:

Consider an arbitrary line L from $[-1,-1,-1]^T$ to $[1,1,1]^T$. If L doesn't go through z-axis, $\int_L \boldsymbol{K} \cdot \mathrm{d}\vec{l} = 0$. Otherwise, if L go through any point $\vec{u}$ on z-axis except the origin, $\boldsymbol{K}(\vec{u}) \neq 0$. Once if $\boldsymbol{K}(\vec{u}) \cdot \mathrm{d}\vec{l} \neq 0$, $\int_L \boldsymbol{K} \cdot \mathrm{d}\vec{l} \neq 0$. Therefore, this function doesn't display path-independence.

Question 2

$$\mathrm{D} := \{[x,y]^T \in \mathbb{R}^2 | x^2+y^2 \leq 1, y \geq 0\}$$

$$\boldsymbol{F}(x,y,z) = [y^2, x^2, z^2]^T$$

i.

$$
\begin{aligned}
\int_{\partial S_1} \boldsymbol{F} \cdot \mathrm{d}\vec{A} &= \int_{\partial S_1} \begin{bmatrix} y^2 \\ x^2 \\ z^2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \mathrm{d}A \\
&= \int_{-1}^{1} \int_{-1}^{1} -x^2 \mathrm{d}x\mathrm{d}z \\
&= \int_{-1}^{1} -\frac{2}{3} \mathrm{d}z \\
&= -\frac{4}{3} \\
\int_{\partial S_2} \boldsymbol{F} \cdot \mathrm{d}\vec{A} &= \int_{\partial S_2} \begin{bmatrix} y^2 \\ x^2 \\ z^2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix} \mathrm{d}A \\
&= \int z^2 \mathrm{d}A - \int z^2 \mathrm{d}A \\
&= 0 \\
\partial S_3 &= \begin{bmatrix} \cos\theta \\ \sin\theta \\ t \end{bmatrix} (\theta \in [0, \pi], t \in [-1, 1]) \\
\partial_\theta S \times \partial_t S &= \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \\
\int_{\partial S_3} \boldsymbol{F} \cdot \mathrm{d}\vec{A} &= \int_{\partial S_3} \begin{bmatrix} y^2 \\ x^2 \\ z^2 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \mathrm{d}A \\
&= \int_{-1}^{1} \int_{0}^{\pi} (\cos^3\theta + \sin^3\theta) \mathrm{d}\theta\mathrm{d}t \\
&= \int_{-1}^{1} \frac{2}{3} \mathrm{d}t \\
&= \frac{4}{3} \quad 4 \\
\Rightarrow \int_{\partial\Omega} \boldsymbol{F} \mathrm{d}\vec{A} &= -\frac{4}{3} + 0 + \frac{4}{3} = 0
\end{aligned}
$$

ii.

$$\begin{aligned}\int_{\partial\Omega} \boldsymbol{F} \mathrm{d}\vec{A} &= \int_{\partial\Omega} \nabla\cdot\boldsymbol{F} \mathrm{d}A \\ &= \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{-1}^{1}(2z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= z^2|_{-1}^{1} \\ &= 0\end{aligned}$$

iii. Proof:

$$\begin{aligned}\nabla\cdot\boldsymbol{F}(x,y,z) &= \nabla\cdot[y^2,x^2,z^2]^T \\ &= 2z \\ z &\in [-1,1]\end{aligned}$$

Therefore, $\nabla\cdot\boldsymbol{F}$ is not identically equal to 0, which means that the vector field $\boldsymbol{F}$ is not solenoidal. There cannot exist a function $\boldsymbol{G} : A \to \mathbb{R}^3$ for any simply-connected open set $A \subset \mathbb{R}^3$ such that $\nabla\times\boldsymbol{G} = \boldsymbol{F}$.

Question 3

$$\text{Intersection of surfaces } x^2 + y^2 = 1, z = y^2 - x^2$$

$$\boldsymbol{F} = [y^2, 2xy, xy]^T$$

i.

$$x = \cos\theta$$

$$y = \sin\theta$$

$$z = y^2 - x^2 = -\cos 2\theta$$

$$\mathrm{d}\vec{l} = l'\mathrm{d}\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 2\sin 2\theta \end{bmatrix} \mathrm{d}\theta (0 \le \theta \le 2\pi)$$

$$\oint_\Gamma \boldsymbol{F} \cdot \mathrm{d}\vec{l} = \int_0^{2\pi} \begin{bmatrix} \sin^2\theta \\ 2\sin\theta\cos\theta \\ \sin\theta\cos\theta \end{bmatrix} \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 2\sin 2\theta \end{bmatrix} \mathrm{d}\theta$$

$$= \int_0^{2\pi} (-\sin^3\theta + 2\sin\theta\cos^2\theta + 4\sin^2\theta\cos^2\theta)\mathrm{d}\theta$$

$$= \int_0^{2\pi} \sin^2 2\theta \mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1 - \cos 4\theta)\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \mathrm{d}\theta = \pi$$

ii.

$$\oint_{\Gamma} \boldsymbol{F} \cdot \mathrm{d}\vec{l} = \int \int_S \nabla \times \boldsymbol{F} \cdot \mathrm{d}\vec{A}$$

$$= \int \int_S \begin{bmatrix} x \\ -y \\ 0 \end{bmatrix} \cdot \mathrm{d}\vec{A}$$

$$S = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ r^2\sin^2\theta - r^2\cos^2\theta \end{bmatrix} (0 \le r \le 1, 0 \le \theta \le 2\pi)$$

$$\partial_r S \times \partial_\theta S = \begin{bmatrix} \cos\theta \\ \sin\theta \\ -2r\cos 2\theta \end{bmatrix} \times \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 2r^2\sin 2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 2r^2\sin\theta + 2r^2\cos\theta\cos 2\theta \\ 2r^2\sin\theta\cos 2\theta - 2r^2\cos\theta\sin 2\theta \\ r\cos^2\theta + r\sin^2\theta \end{bmatrix}$$

$$\int \int_S \begin{bmatrix} x \\ -y \\ 0 \end{bmatrix} \cdot \mathrm{d}\vec{A} = \int \int_S \begin{bmatrix} r\cos\theta \\ -r\sin\theta \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2r^2\sin\theta + 2r^2\cos\theta\cos 2\theta \\ 2r^2\sin\theta\cos 2\theta - 2r^2\cos\theta\sin 2\theta \\ r\cos^2\theta + r\sin^2\theta \end{bmatrix} \mathrm{d}A$$

$$= \int_0^{2\pi} \int_0^1 (r^3\sin^2 2\theta + 2r^3\cos^2\theta +$$

$$r^3\sin^2 2\theta - 2r^3\sin^2\theta\cos 2\theta)\mathrm{d}r\mathrm{d}\theta$$

$$= \int_0^{2\pi} \int_0^1 (2r^3)\mathrm{d}r\mathrm{d}\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\mathrm{d}\theta = \pi$$