Assignment 1: Answer Key

The first question is largely dependent on what you chose, we'll try to provide a sample set of solutions.

1i) We choose the set $\pi + \mathbb{Z} := \{x \in \mathbb{R} \mid x = \pi + n \text{ for } n \in \mathbb{Z}\}$.

1ii) The $\pi + \mathbb{Z}$ is closed because its complement is a union of open intervals

$$\mathbb{R} \setminus (\pi + \mathbb{Z}) = \bigcup_{n \in \mathbb{Z}} (\pi + n - 1, \pi + n).$$

1iii) This set is unbounded, because $d(\pi, \pi + n) = n$ may be arbitrarily large.

iv) We choose the set $B((0,0,4),1) \subset \mathbb{R}^3$.

v) $B((0,0,4),1)$ is open because it is an open ball (and we proved in class that open balls are open, during the proof of Theorem 1.1).

vi) $B((0,0,4),1)$ is bounded, because $\forall \vec{x}_1, \vec{x}_2 \in B((0,0,4),1)$, $d(\vec{x}_i, (0,0,4)) < 1$, and the triangle inequality ensures that

$$d(\vec{x}_1, \vec{x}_2) \leq d(\vec{x}_1, (0,0,4)) + d((0,0,4), \vec{x}_2) < 2.$$

vii) Our answer for part vi ensures us that 2 is an upper bound for the diameter, we now show that 2 is in fact the diameter by finding a sequence of pairs of points $\{(\vec{x}_i, \vec{y}_i)\}$ s.t. the distance between the $\vec{x}_i, \vec{y}_i$ tends to 2 as i tends to infinity. Specifically, we choose

$$\vec{x}_i = (0,0,3 + \tfrac{1}{i}) \text{ and } \vec{y}_i = (0,0,5 - \tfrac{1}{i}).$$

Then $d(\vec{x}_i, \vec{y}_i) = 2 - 2i^{-1}$, and

$$2 \geq \operatorname{diam}(B((0,0,4),1)) := \sup_{\vec{x},\vec{y} \in B((0,0,4),1)} d(\vec{x}, \vec{y}) \geq \lim_{i \to \infty} 2 - 2^{-i} = 2.$$

Some of you pointed out that there was some confusion with what the limit means with questions where the denominator could evaluate to 0. This is a matter of laxity on my part and I'm sorry. In order to define those limits properly, I should have specified the domains for each of the functions, and requires that the limit be taken whilst restricted to those domains.

Note for future reference that it is implicitly understood that if the domain is not explicitly written down, then one generally takes the maximum possible domain in Euclidean space.

2i) These are all well-defined elementary functions around $(x, y) = (3, 5)$, the limit exists (if you really want, you can use the fact that they're elementary to ensure that they're differentiable and hence continuous...) and we may directly evaluate to get $(\frac{1}{2}\sin(2), 4)$.

2ii) The limit does not exist. Consider approaching the origin along the y-axis, then $x = 0$ and the limit becomes $\lim_{y\to 0} \frac{1}{y}$, which does not converge (not even to $+\infty$ or $-\infty$ since it depends on the direction y approaches 0).

2iii) The limit exists. First observe that:

$$\frac{x^3 - y^3}{x^4 - y^4} = \frac{(x-y)(x^2 + xy + y^2)}{(x-y)(x+y)(x^2+y^2)} = \frac{(x^2 + xy + y^2)}{(x+y)(x^2+y^2)} \leq \frac{3}{2(x+y)},$$

where the last inequality is due to $xy \leq \frac{1}{2}(x^2 + y^2)$. Now, for every $\epsilon > 0$, consider $N = \epsilon^{-1}$, then for all $x, y > N$,

$$x + y > 2N = 2\epsilon^{-1} \Rightarrow 0 < \tfrac{2}{x+y} < \epsilon \Rightarrow \left| \frac{3}{2(x+y)} - 0 \right| < \epsilon,$$

and hence $\lim_{x\to+\infty, y\to+\infty} \frac{3}{2(x+y)} = 0$. The sandwich theorem ensures that

$$0 \leq \lim_{\substack{x\to+\infty \\ y\to+\infty}} \left| \frac{x^3 - y^3}{x^4 - y^4} \right| \leq \lim_{\substack{x\to+\infty \\ y\to+\infty}} \left| \frac{3}{2(x+y)} \right| = 0.$$

Which in turn tells us that

$$\lim_{\substack{x\to+\infty \\ y\to+\infty}} \frac{x^3 - y^3}{x^4 - y^4} \text{ exists and is equal to } 0.$$

2iv) The limit exists. This is an iterated limit, and we have to first consider the inner limit (i.e.: as $y \to -\infty$) whilst holding x constant. Then, for fixed $x \in \mathbb{R}$,

$$0 \leq \lim_{y\to-\infty} \left| \frac{x^3 - y^3}{x^4 - y^4} \right| = 0,$$

where I'm assuming that you guys know how to evaluate the above limit using Calculus A1 techniques (it's a limit in one variable). Therefore, the inner limit results in 0. We then take the outer limit

$$\lim_{x\to+\infty} 0 = 0,$$

to see that this iterated limit is 0.

N.B.: have a think about what the answer might have been if this was a double limit and not an iterated limit... (spoiler: it doesn't exist).

2v) The limit exists. First observe that

$$(1+2020y)^{\frac{1}{y-x^2y^2}} = (1+2020y)^{\frac{1}{2020y}\cdot\frac{2020}{1-x^2y}} = \left((1+2020y)^{\frac{1}{2020y}}\right)^{\frac{2020}{1-x^2y}}.$$

Next observe that we know the following limits exist:

$$\lim_{(x,y)\to(e,0)}(1+2020y)^{\frac{1}{2020y}} = \lim_{y\to 0}(1+2020y)^{\frac{1}{2020y}} = e$$

(note that we're not taking an iterated limit here, we're using the fact that the limit is independent of x to turn it into a one-variable limit) and

$$\lim_{(x,y)\to(e,0)}\frac{2020}{1-x^2y} = 2020.$$

Then, since $u^v = e^{v\log u}$ is continuous at $(u,v)=(e,2020)$, we obtain that

$$\lim_{(x,y)\to(e,0)}(1+2020y)^{\frac{1}{y-x^2y^2}} = \lim_{(u,v)\to(e,2020)} u^v = e^{2020}.$$

2vi) We'll demonstrate two approaches to this final limit for your reference.

METHOD 1: we use $x^2+y^2 \geq \frac{1}{2}(x^2+y^2)$ to show that

$$0 \leq \left|\frac{\log(x+y)}{x^2+y^2}\right| \leq \left|\frac{\log{(x+y)}}{\frac{1}{2}(x+y)^2}\right| = \left|\frac{-2\log(\frac{1}{x+y})}{(x+y)^2}\right| = \left|\frac{\log(\frac{1}{(x+y)^2})}{(x+y)^2}\right|.$$

Our goal is to show that the right-most expression converges to 0 as $x\to 3$ and $y\to+\infty$. The sandwich theorem then show that our desired limit exists, and is 0. So, in this vein of thought, let us consider functions $f:(0,+\infty)\times(0,+\infty)\to\mathbb{R}_{>0}$ defined by $f(x,y)=\frac{1}{(x+y)^2}$ and $g:(0,+\infty)\to\mathbb{R}$ defined by $g(u)=u\log(u)$. We know $f(x,y)$ is always positive, and hence the image of f lies in the domain of g, thereby allowing us to define the function $g\circ f$. Since x is positive,

$$0 \leq \lim_{\substack{x\to 3\\ y\to+\infty}}\frac{1}{(x+y)^2} \leq \lim_{\substack{x\to 3\\ y\to+\infty}}\frac{1}{y^2} = \lim_{y\to+\infty}\frac{1}{y^2} = 0.$$

Fortunately, 0 is a limit point of $(0,+\infty)$ (i.e.: the domain of g), and we may consider applying the composition rule for limits (lecture 3, slide 14). Specifically, f is not constant in an open neighborhood of $(3,+\infty)$ since f isn't constant on any open ball, and

$$\lim_{(0,+\infty)\ni u\to 0} u\log(u) =: \lim_{u\to 0^+} u\log(u) = 0 \quad \text{(L'Hopital's rule)}.$$

Therefore,

$$\lim_{\substack{x\to 3\\ y\to+\infty}}\left|\frac{\log(\frac{1}{(x+y)^2})}{(x+y)^2}\right| = 0, \text{ as desired.}$$

METHOD 2: Let's also prove this straight-up using the ϵ-"blah" definition of convergence. Given an arbitrary $\epsilon > 0$, consider $N = \max\{1, 2\epsilon^{-1}\}$. then for any x such that $|x-3| < \frac{1}{N}$ and $y > N$, we know that $x > 0$ hence $x + y > N \geq 1$. Therefore,

$$\log(x+y) < x+y-1 < x+y.$$

This in turn implies that

$$\left|\frac{\log(x+y)}{x^2+y^2} - 0\right| \leq \left|\frac{2\log(x+y)}{(x+y)^2}\right| < \left|\frac{2(x+y)}{(x+y)^2}\right| = \left|\frac{2}{x+y}\right| < \frac{2}{N} \leq \frac{2}{2\epsilon^{-1}} = \epsilon.$$

This definitionally means that the double limit converges to 0.

3i) In order to show that $\mathbf{f}(x,y)$ is differentiable at $(x,y) = (0,1,)$, we need to first "guess" what the derivative at $(0,1)$ should be. One can in fact do this by straight up computing $\mathbf{f}(0+h_1, 1+h_2) - \mathbf{f}(0,1)$, and seeing the coefficients of the linear h_i-terms, but let's just do it the pedestrian way of using partial derivatives and what-not. A quick computation shows that we should expect the derivative (in general) to be

$$d\mathbf{f} = (2dx, 2xdx + 2ydy, 2ye^{2x}dx + e^{2x}dy), \text{ and hence } d\mathbf{f}|_{(0,1)} = (2dx, 2dy, 2dx + dy).$$

Now that we've got our "guess" of what the derivative is, our goal is to show that the following limit exists and converges to 0:

$$\lim_{(h_1,h_2)\to\vec{0}} \frac{\|\mathbf{f}(0+h_1, 1+h_2) - \mathbf{f}(0,1) - (2dx, 2dy, 2dx+dy)(h_1,h_2)\|}{\|(h_1,h_2)\|} = 0. \tag{1}$$

Let's algebraically manipulate the numerator first:

$$\begin{aligned}
&\|\mathbf{f}(0+h_1, 1+h_2) - \mathbf{f}(0,1) - (2dx, 2dy, 2dx+dy)(h_1,h_2)\|\\
=&\|(2h_1+3, h_1^2 + (1+h_2)^2, (1+h_2)e^{2h_1}) - (3,1,1) - (2h_1, 2h_2, 2h_1+h_2)\|\\
=&\left\|\left(0, h_1^2+h_2^2, (1+h_2)(e^{2h_1}-1) - 2h_1\right)\right\|\\
\leq&\|(0, h_1^2+h_2^2, 0)\| + \|(0,0,(1+h_2)(e^{2h_1}-1) - 2h_1)\|\\
=&|h_1^2+h_2^2| + |(1+h_2)(e^{2h_1}-1) - 2h_1|,
\end{aligned}$$

with the second last inequality due to the triangle inequality. Note that

$$\lim_{(h_1,h_2)\to\vec{0}} \frac{|h_1^2+h_2^2|}{\|(h_1,h_2)\|} = \lim_{(h_1,h_2)\to\vec{0}} \frac{|h_1^2+h_2^2|}{\sqrt{h_1^2+h_2^2}} = \lim_{(h_1,h_2)\to\vec{0}} \sqrt{h_1^2+h_2^2} = 0. \tag{2}$$

Further observe that

$$\begin{aligned}
|(1+h_2)(e^{2h_1}-1)-2h_1| &= |(1+h_2)(e^{2h_1}-1-2h_1)-2h_1h_2| \\
&\le \big|(1+h_2)(e^{2h_1}-1-2h_1)\big| + |2h_1h_2| \\
&\le |1+h_2|\cdot\left|\frac{e^{2h_1}-1}{h_1}-2\right|\cdot|h_1| + |h_1^2+h_2^2| \\
&\le |1+h_2|\cdot\left|\frac{e^{2h_1}-1}{h_1}-2\right|\cdot\|(h_1,h_2)\| + \|(h_1,h_2)\|^2.
\end{aligned}$$

$$\text{Since} \qquad \lim_{(h_1,h_2)\to(0,0)} |1+h_2| = \lim_{h_2\to 0}|1+h_2| = 1, \text{ and}$$

$$\lim_{(h_1,h_2)\to(0,0)}\left|\frac{e^{2h_1}-1}{h_1}-2\right| = \lim_{h_1\to 0}\left|\frac{e^{2h_1}-1}{h_1}-2\right| = |2-2| = 0,$$

we may use the sanwich theorem to show that the following middle limit is equal to 0:

$$0 \le \lim_{(h_1,h_2)\to(0,0)} \frac{|(1+h_2)(e^{2h_1}-1)-2h_1|}{\|(h_1,h_2)\|} \le \lim_{(h_1,h_2)\to(0,0)} 1\cdot 0\cdot 1+0 = 0. \tag{3}$$

Combining (2) and (3) with the sandwich theorem then suffices to ensure that (1) holds.

3ii) The partial derivatives of $\mathbf{f}$ are

$$\partial_x\mathbf{f}|_{(0,1)} = (2, 2x, 2ye^{2x})|_{(0,1)} = (2,0,2) \text{ and } \partial_y\mathbf{f}|_{(0,1)} = (0, 2y, e^{2x})|_{(0,1)} = (0,2,1).$$

3iii) Let $\mathbf{f} = (f_1, f_2, f_3)$, then the Jacobian of $\mathbf{f}$ at $(0,1)$ is given by

$$[d\mathbf{F}]|_{(0,1)} = \begin{bmatrix} \frac{\partial f_1}{\partial x}\Big|_{(1,0)} & \frac{\partial f_1}{\partial y}\Big|_{(1,0)} \\ \frac{\partial f_2}{\partial x}\Big|_{(1,0)} & \frac{\partial f_2}{\partial y}\Big|_{(1,0)} \\ \frac{\partial f_3}{\partial x}\Big|_{(1,0)} & \frac{\partial f_3}{\partial y}\Big|_{(1,0)} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}.$$

We will not provide solutions to **Bonus i)** and **Bonus ii)**. You can think of them as the challenge questions of the assignment — if you want hints, talk to me!