Calculus A2 (English) — Assignment 1.

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Submission instructions: please submit your solutions to Tsinghua Web Learning as a PDF file[1]. Please name your solution file in the following format: "id number + surname + first name".pdf. (e.g.: 20200202zhuyidan.pdf'). The assignment is out of a total of 20 marks, and the marks are not proportional to the difficulty of the problem. Bonus questions provide bonus marks which only count if you lose marks in the main portion of the assignment, and do not take your total over 20. Please feel free to discuss the assignment in your allocated study groups, but write up your solutions by yourselves. Also feel free to look at and/or provide feedback on each other's solutions.

Question 1: (10 marks)

i. Provide an example of a non-empty *unbounded* (i.e.: it is not bounded), *closed* set in $\mathbb{R}$ which does not intersect $\mathbb{Q}$. (2 marks)

ii. Prove that the set you gave in i. is closed. (1 mark)

iii. Prove that the set you gave in i. is unbounded. (1 mark)

iv. Provide an example of a non-empty *bounded open* set in $\mathbb{R}^3$ which is in the complement of $B((0,0,0),2)$. (2 marks)

v. Prove that the set you gave in iv. is open. (1 mark)

vi. Prove that the set you gave in iv. is bounded. (1 mark)

vii. Compute the diameter for the set you gave in iv., and prove that this is indeed the diameter. (2 mark)

Question 2: (6 marks)

Determine if the following limits exist, and prove it. Moreover, for limits which exist, compute them. You may use any limit/continuity arguments you learned in Calculus A1.

i.
$$\lim_{(x,y)\to(3,5)} \left(\frac{\sin(y-x)}{y-x}, \sqrt{y^2-x^2}\right). \quad \text{(1 mark)}$$

ii.
$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^3}{x^3-y^4}. \quad \text{(1 mark)}$$

iii.
$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{x^3-y^3}{x^4-y^4}. \quad \text{(1 mark)}$$

[1] I'm sure that you're all tech-savvy enough to figure out how to do this!

iv.

$$\lim_{x\to+\infty}\lim_{y\to-\infty}\frac{x^3-y^3}{x^4-y^4}. \quad \text{(1 mark)}$$

v.

$$\lim_{(x,y)\to(e,0)}(1+2020y)^{\frac{1}{y-x^2y^2}}. \quad \text{(1 mark)}$$

vi.

$$\lim_{(x,y)\to(3,+\infty)}\frac{\log(x+y)}{x^2+y^2}. \quad \text{(1 mark)}$$

Question 3: (4 marks)

Consider the following vector-valued function

$$\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^3,\ \mathbf{f}(x,y)=(2x+3,x^2+y^2,ye^{2x}).$$

i Prove that $\mathbf{f}$ is (totally) differentiable at $(x,y)=(0,1)$ from first principles, i.e.: using the definition of differentiability given in class. (1 mark) (*Hint: do parts ii. and iii. first to work out what the total derivative of of* $\mathbf{f}$ *at* $(x,y)=(0,1)$ *should be, then substitute this into the expression in the definition of total differentiability and show that the condition of differentiability is satisfied.*)

ii Compute all the partial derivatives of $\mathbf{f}$ at $(x,y)=(0,1)$. (1 mark)

iii Compute the Jacobian (i.e.: the total derivative matrix) of $\mathbf{f}$ at $(x,y)=(0,1)$. (2 marks)

Bonus questions:

i Prove that clopen (i.e.:closed and open) sets in $\mathbb{R}^n$ must have empty boundary. (1 mark)

ii The following limit does not converge. Determine all possible "values" (in $\mathbb{R}^2$) that the following expression may take as (x,y) approaches $(0,0)$:

$$\lim_{(x,y)\to(0,0)}\left(e^{\frac{x}{x^2+y^2}}\cos\left(\frac{y}{x^2+y^2}\right),e^{\frac{x}{x^2+y^2}}\sin\left(\frac{y}{x^2+y^2}\right)\right). \quad \text{(1 mark)}$$