Assignment 1

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Question 1

i. $S = \{x \in \mathbb{R} | x = \sqrt{2} + k, k \in \mathbb{N}\}$

ii. Proof: For any arbitrary $k \in \mathbb{N}$, we define $S_k := (\sqrt{2}+k, \sqrt{2}+k+1)$. According to the definition, $S_k \subset \mathbb{R} \setminus S$. For arbitrary $x \in S_k$, let $\delta = min\{x - \sqrt{2} - k, \sqrt{2}+k+1-x\}$, $\exists B(x, \delta)$ is an open ball, thus S_k is an open set. Then the complement of S in $\mathbb{R}$ $\overline{S} = \bigcup_{k \in \mathbb{N}} S_k$ is an open set. Therefore, S is an closed set.

iii. Proof: Since $\mathbb{N}$ is unbounded (according to the Archimedes character), S is an unbounded set.

iv. $B((0,0,4),1)$

v. Proof: Obviously it is an open set since it is an oepn ball.

vi. $diam(B) = 2$.

Proof: For arbitrary points $\vec{p}, \vec{q} \in B(\vec{x}, 1), d(\vec{x}, \vec{p}) < 1, d(\vec{x}, \vec{q}) < 1$, thus $d(\vec{x}, \vec{y}) < 2$ according to the triangular inequality. Now consider two point sequences:

$$A_k = \{(0,0,3+\frac{1}{2^k}) | k \in \mathbb{N}\}, B_k = \{(0,0,5-\frac{1}{2^k}) | k \in \mathbb{N}\}$$

Obviously $A_k, B_k \subset B(\vec{x}, 1)$. Then we can tell

$$\lim_{k\to\infty} d(A_k, B_k) = 2$$

Which means $\forall \varepsilon > 0, \exists N > 0$, when $k > N, d(A_k, B_k) > 2 - \varepsilon$. Then we can tell $sup(\vec{p}, \vec{q}) = 2$, which means $diam(B) = 2$.

Question 2

i.

$$\lim_{(x,y)\to(3.5)} (\frac{\sin(y-x)}{y-x}, \sqrt{y^2-x^2})$$

exists

Proof: Both $\frac{\sin(y-x)}{y-x}$ and $\sqrt{y^2-x^2}$ are continuous at point $(3,5)$, thus the limit is equal to $(\frac{\sin 2}{2}, 4)$.

ii.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^3}{x^3-y^4}$$

doesn't exist.

Proof: Replace y with $kx(k \in \mathbb{R})$, then the original formula is equal to

$$\lim_{x\to 0} \frac{x^4-k^3x^3}{x^3-k^4x^4}$$

=

$$\lim_{x\to 0} \frac{x-k^3}{1-k^4x}$$

= $-k^3$ which is uncertain. Therefore, the limit doesn't exist.

iii.

$$\lim_{\substack{x\to+\infty \\ y\to+\infty}} \frac{x^3-y^3}{x^4-y^4}$$

exists.

Proof: The original formula is equal to

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{x^2+xy+y^2}{x^3+x^2y+xy^2+y^3}$$

which is obviously no less than zero. Then

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{x^2+xy+y^2}{x^3+x^2y+xy^2+y^3}$$

=

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{x^2+xy+y^2+\frac{y^3}{x}-\frac{y^3}{x}}{x^3+x^2y+xy^2+y^3}$$

=

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} (\frac{1}{x}-\frac{y^3}{x^4+x^3y+x^2y^2+xy^3})$$

$\leq$

$$\lim_{x\to+\infty} \frac{1}{x}=0$$

As a result,

$$0 \leq \lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{x^3-y^3}{x^4-y^4} \leq 0$$

Therefore, the limit is equal to 0.

iv.

$$\lim_{x\to+\infty} \lim_{y\to-\infty} \frac{x^3-y^3}{x^4-y^4}$$

v.

$$\lim_{(x,y)\to(\mathrm{e},0)} (1+2020y)^{\frac{1}{y-x^2y^2}}$$

exists.

Proof: Consider

$$\boldsymbol{f}:\mathbb{R}^2\to\mathbb{R}^2, \boldsymbol{f}(x,y)=((1+2020y)^{\frac{1}{2020y}}, \frac{2020y}{y-x^2y^2})$$

=

$$\boldsymbol{f}:\mathbb{R}^2\to\mathbb{R}^2, \boldsymbol{f}(x,y)=((1+2020y)^{\frac{1}{2020y}},\frac{2020}{1-x^2y})$$

, Therefore,

$$\lim_{(x,y)\to(\mathrm{e},0)}\boldsymbol{f}(x,y)=(e,2020)$$

$$\boldsymbol{g}:\mathbb{R}^2\to\mathbb{R}:\boldsymbol{g}(u,v)=u^v$$

Then invoke the composition rule:

$$\lim_{(u,v)\to(\mathrm{e},2020)}\boldsymbol{g}(u,v)=\mathrm{e}^{2020}$$

vi.

$$\lim_{(x,y)\to(3,+\infty)}\frac{\log(x+y)}{x^2+y^2}$$

exists.

Proof: