*Calculus A2 (English) — Assignment 2.*

YI HUANG

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**Submission instructions:** same instructions as for assignment 1.

**Question 1:** (5 marks) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \exp(x - y)$, and

i. compute the degree 3 Peano form for the Taylor expansion of $f$ around $(x, y) = (1, 1)$ (to clarify, the degree 3 refers to the Taylor polynomial, not the error term); (2 marks)

ii. compute the degree 2 Lagrange form for the Taylor expansion of $f$ around $(x, y) = (1, 1)$; (1 mark)

iii. use your answer to part i to show that $\sqrt{e} \approx \frac{79}{48}$, no error terms equired; (1 mark)

iv. use your answer to part ii to show that $\frac{78}{49} \leqslant \sqrt{e} \leqslant \frac{78}{47}$. (1 mark)

**Question 2:** (8 marks)

Consider the zero-set $Z(F) \subset \mathbb{R}^3$ of the function $F : \mathbb{R}^3 \to \mathbb{R}$,

$$F(x, y, z) = -x^2 e^{2z} + y^3 + 8z, \text{ and}$$

i. show that the gradient $\nabla F$ of $F$ never vanishes on $Z(F)$; (2 mark)

ii. compute the point-normal form for the tangent plane of $Z(F)$ at $[a, b, c]^T \in Z(F)$; (1 mark)

iii. compute the parametric form for the tangent plane of $Z(F)$ at $[a, b, c]^T \in Z(F)$; (1 mark)

iv. show, using the implicit function theorem, that provided that $x \neq \pm 2e^{-z}$, then for any $[a, b, c]^T \in Z(F)$, there is a $C^1$ map $\zeta : B(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right], \delta) \to \mathbb{R}$ satisfying $F(x, y, \zeta(x, y)) = 0$. Compute the Jacobian of $\zeta$. (2 marks)

v. show, using the implicit function theorem, that provided that $x \neq 0$, then for any $[a, b, c]^T \in Z(F)$, there is a $C^1$ map $\xi : B(\left[\begin{smallmatrix} b \\ c \end{smallmatrix}\right], \delta) \to \mathbb{R}$ satisfying $F(\xi(y, z), y, z) = 0$. Compute the Jacobian of $\xi$. (1 mark)

vi. consider the point $[0, -2, 1]^T \in Z(F)$ and construct $\zeta : B(\left[\begin{smallmatrix} 0 \\ -2 \end{smallmatrix}\right], \delta) \to \mathbb{R}$ as per part iv. Define the following function $g : B(\left[\begin{smallmatrix} 0 \\ -2 \end{smallmatrix}\right], \delta) \to \mathbb{R}^2$,

$$g(x, y) = \begin{bmatrix} x + \zeta(x, y) \\ y \end{bmatrix}.$$

Show, using the inverse function theorem, that there exists a $C^1$ function $f : B(\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right], \epsilon) \to f(B(\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right], \epsilon))$ which is inverse to $g : f(B(\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right], \epsilon)) \to B(\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right], \epsilon)$. Compute $[df]$ at $\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]$. (1 mark)

**Question 3:** (7 marks)

Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$,

$$f(x, y, z) = (x + y - 1)^2 + (x - y + 2)^2 + 2z^2 + \tfrac{1}{6}z^3, \text{ and}$$

i. find all the stationary points of $f$ inside $B(\vec{0}, 10) \subset \mathbb{R}^3$; (1 mark)

ii. classify all the stationary points you found as local maxima, local minima or neither, give formal justification in each case (N.B.: the statement "*the Hessian at this point is not positive definite and therefore is not a local minimum*" is generally incorrect, but "*the Hessian at this point is not positive semi-definite and therefore is not a local minimum*" is correct); (2 marks)

iii. use Lagrange multipliers to find the maximum and minimum on $\partial B(\vec{0}, 10)$ (you may assume that $\partial B(\vec{0}, 10)$ is given by $x^2+y^2+z^2 = 100$), explain why maxima and minima of $f$ necessarily exist on the boundary. Thus find the maximum and minimum of $f$ over the closed ball $\overline{B}(\vec{0}, 10)$; (3 marks)

iv. what is/are the (global) minimum/minima for the function

$$g : \mathbb{R} \times (0, +\infty) \times [0, \tfrac{\pi}{2}) \to \mathbb{R}$$
$$g(u, v, w) = u^6 + u^3 + (\log v)^2 - 3 \log v + \tan^2 w + \tfrac{1}{12} \tan^3 w,$$

justify your answer. (1 mark)

**Bonus questions:**

i. Consider a $C^\infty$ function $f : \mathbb{R} \to \mathbb{R}$ and define

$$g : \mathbb{R}^2 \to \mathbb{R}, \; g(x_1, x_2) = f(x_1 + x_2).$$

For small $h$, one may use the degree $k$ Taylor polynomial of $f$ around 0 to approximate $f(h)$. One may also approximate $f(h)$ using the degree $k$ Taylor polynomial of $g$ around $\vec{0}$ by approximating $g(h_1, h_2)$ for $h = h_1 + h_2$. Prove that there's no difference in the two approximations. (1 mark)

ii. Repeat questions 3i. and 3ii., but for

$$f(x, y, z) = (x + y - 1)^2 + (x - y + 2)^2 + 2z^2 + \tfrac{1}{3}z^3. \text{ (0.5 marks)}$$

Also try 3iii. for this function! (0.5 marks)