Assignment 2: Answer Key

There are two standard approaches to a question like this. One is to actually do all the partial derivatives and to assemble the Taylor polynomial...etc., and the other is to use Taylor expansions for the single variable exponential function and then to substitute $x - y$ in for the variable. We'll try to say a little on both.

1i) Let's first do it the "standard" way, we compute the following:

$f(1,1) = \exp(1-1) = 1,$
$f_x(1,1) = \exp(1-1) = 1,\ f_y(1,1) = -\exp(1-1) = -1,$
$f_{xx}(1,1) = \exp(1-1) = 1,\ f_{xy}(1,1) = f_{yx}(1,1) = -\exp(1-1) = -1\ f_{yy}(1,1) = \exp(1-1) = 1$
$f_{xxx}(1,1) = \exp(1-1) = 1,\ f_{xxy}(1,1) = f_{xyx}(1,1) = f_{yxx}(1,1) = -\exp(1-1) = -1$
$f_{xyy}(1,1) = f_{yxy} = (1,1) = f_{yyx}(1,1) = \exp(1-1) = 1,\ f_{yyy}(1,1) = \exp(1-1) = -1.$

Therefore, the degree 3 Taylor polynomial for f around $(x,y) = (1,1)$ is

$$\begin{aligned}&1 + (x-1) - (y-1) + \frac{(x-1)^2}{2} - (x-1)(y-1) + \frac{(y-1)^2}{2}\\ &+ \frac{(x-1)^3}{6} - \frac{(x-1)^2(y-1)}{2} + \frac{(x-1)(y-1)^2}{2} - \frac{(y-1)^3}{6}.\end{aligned}$$

You may also absorb al of the negative signs into $(y-1)$ and have

$$\begin{aligned}&1 + (x-1) + (1-y) + \frac{(x-1)^2}{2} + (x-1)(1-y) + \frac{(1-y)^2}{2}\\ &+ \frac{(x-1)^3}{6} + \frac{(x-1)^2(1-y)}{2} + \frac{(x-1)(1-y)^2}{2} + \frac{(1-y)^3}{6}.\end{aligned}$$

One may approach this question by utilizing the Taylor expansion of the exponential function $g(z) = \exp(z)$ directly. In this setting, we need to expand around $z = x - y = 1 - 1 = 0$, and we get:

$$g(0) = g_z(0) = \exp(0) = g_{zz}(0) = g_{zzz}(0) = 1,$$

and hence the degree Taylor polynomial is

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} = 1 + (x - 1 + 1 - y) + \frac{(x-1+1-y)^2}{2} + \frac{(x-1+1-y)^3}{6}.$$

Expanding each bracket then yields the standard expression for the Taylor polynomial.

Note: some of you may be wondering if it is necessary to expand, and strictly speaking it isn't. After all, whether you expand or not, it is the same polynomial. However, it is good practice to put it in the "standard form".

To finish off, the Peano form should be expressed as:

$$
\begin{aligned}
&1+(x-1)-(y-1)+\frac{(x-1)^2}{2}-(x-1)(y-1)+\frac{(y-1)^2}{2}\\
&\quad+\frac{(x-1)^3}{6}-\frac{(x-1)^2(y-1)}{2}+\frac{(x-1)(y-1)^2}{2}-\frac{(y-1)^3}{6}+o(\|(x-1,y-1)\|^3).
\end{aligned}
$$

1ii) We know from 1i) that the degree two Taylor polynomial is

$$1+(x-1)-(y-1)+\frac{(x-1)^2}{2}-(x-1)(y-1)+\frac{(y-1)^2}{2},$$

the error term is given by

$$\frac{e^{t(x-y)}(x-1)^3}{6}-\frac{e^{t(x-y)}(x-1)^2(y-1)}{2}+\frac{e^{t(x-y)}(x-1)(y-1)^2}{2}-\frac{e^{t(x-y)}(y-1)^3}{6}$$

where $t\in(0,1)$. To see this in the context of the first method, note that the $\vec{\xi}$ in $f(\vec{\xi}), f_x(\vec{\xi}), f_y(\vec{\xi}), f_{xx}(\vec{\xi}), f_{xy}(\vec{\xi})=f_{yx}(\vec{\xi}), f_{yy}(\vec{\xi})$ is necessarily a point between $(1,1)$ and (x,y) and hence takes the form $\vec{\xi}=(1,1)+t[(x,y)-(1,1)]=(1-t+tx,1-t+ty)$ for $t\in(0,1)$. Substituting this in then yields the desired Lagrange form error term.

The error is slightly simpler to derive for Method 2, ξ in this case needs to be something between 0 and $z=x-y$, and hence is $t(x-y)$ for $t\in(0,1)$.

N.B.: Method 2 is partially contingent on the fact that $z=x-y$ is homogeneous. It is possible to still determine the Taylor polynomial using this method when z is inhomogeneous (e.g.: $\exp(z=x-y^2)$), however, the error term requires additional work.

1iii)

To approximate $\sqrt{e}=e^{\frac{1}{2}}$, set $(x,y)=(\frac{3}{2},1)$, then

$$
\begin{aligned}
\sqrt{e}\approx\ &1+(\tfrac{3}{2}-1)-(1-1)+\frac{(\frac{3}{2}-1)^2}{2}-(\tfrac{3}{2}-1)(1-1)+\frac{(1-1)^2}{2}\\
&+\frac{(\frac{3}{2}-1)^3}{6}-\frac{(\frac{3}{2}-1)^2(1-1)}{2}+\frac{(\frac{3}{2}-1)(1-1)^2}{2}-\frac{(1-1)^3}{6}=\frac{79}{48}.
\end{aligned}
$$

The choice to set $(x,y)=(\frac{3}{2},1)$ seems arbitrary, and it is. However, it makes no difference what one chooses, provided that $x-y=\frac{1}{2}$. The computation for Method 2 is similar.

iv) The degree 2 Taylor polynomial approximation for $\sqrt{e} = e^{\frac{1}{2}}$ is

$$1 + \tfrac{1}{2} - 0 + \frac{(\frac{1}{2})^2}{2} - (\tfrac{1}{2})(0) + \frac{(0)^2}{2} = \frac{13}{8}.$$

The error term is

$$\frac{e^{\frac{t}{2}}(\frac{1}{2})^3}{6} - \frac{e^{\frac{t}{2}}(\frac{1}{2})^2(0)}{2} + \frac{e^{\frac{t}{2}}(\frac{1}{2})(0)^2}{2} - \frac{e^{\frac{t}{2}}(0)^3}{6} = \frac{e^{\frac{t}{2}}}{48}, \text{ for } t \in (0,1).$$

Since the exponential function is monotonically increasing, the error term is therefore necessarily bounded above by $\frac{\sqrt{e}}{48}$. Thus,

$$\begin{aligned} \frac{13}{8} - \frac{\sqrt{e}}{48} < \sqrt{e} < \frac{13}{8} + \frac{\sqrt{e}}{48} &\Rightarrow \frac{47\sqrt{e}}{48} < \frac{13}{8} < \frac{49\sqrt{e}}{48} \\ &\Rightarrow \frac{78}{49} < \sqrt{e} < \frac{78}{47}. \end{aligned}$$

Note that the assignment has non-strict equality signs, this is not meant to be confusing.

2i) The gradient of F is:

$$\nabla F = [-2xe^{2z}, 3y^2, 8 - 2x^2e^{2z}]^T.$$

We need to show that $\nabla F \neq \vec{0}$ at points in $Z(F)$. In fact, we shall see that $\nabla F \neq \vec{0}$ for all points in $\mathbb{R}^3$.

In order for $\nabla F = \vec{0}$, $-2xe^{2z} = 0 \Rightarrow x = 0$. However, then $8 - 2x^2e^{2z} = 8 \neq 0$.

2ii) The point-normal form for the tangent plane is given by $0 = \nabla F|_{[a,b,c]^T} \cdot \big([x,y,z]^T - [a,b,c]^T\big)$, and hence:

$$0 = [-2ae^{2c}, 3b^2, 8 - 2a^2e^{2c}]^T \cdot [x-a, y-b, z-c]^T.$$

There's no need to expand the dot product to

$$0 = -2ae^{2c}(x-a) + 3b^2(y-b) + 8 - 2a^2e^{2c}(z-c).$$

2iii) We require two orthogonal vectors to the gradient vector $\nabla F|_{[a,b,c]^T}$. This choice is not canonical, we give the following possibilities:

$$\vec{v} = [8 - 2a^2e^{2c}, 0, 2ae^{2c}]^T \text{ and } \vec{w} = [-3ab^2, -8, 3b^2]^T.$$

It is straight-forward to verify that $\vec{v}$ and $\vec{w}$ are orthgonal to $\nabla F|_{[a,b,c]^T}$ and that they are non-zero vectors, however we require that they be linearly independent. Due to the 0-entry in the middle of $\vec{v}$ versus the 8 in the middle of $\vec{w}$. the only possibility of linear dependent requires that $\vec{v} = \vec{v} + 0\vec{w} = \vec{0}$. This is impossible. Thus, the parametrized form may be given by:

$$[a, b, c]^T + s\vec{v} + t\vec{w}, \quad s, t \in \mathbb{R}.$$

N.B.: this is a slightly trickier version of the usual "find the parametrized form" question, since a, b, c are not explicitly given and this non-explicitness makes it harder to justify that the vectors $\vec{v}$ and $\vec{w}$ are linearly independent (and you may need to use the fact that $F(a, b, c) = 0$, or possibly provide different $\vec{v}$ and $\vec{w}$ for different choices of a, b and c). You are unlikely to see something without explicit numbers in exams.

2iv) The Jacobian of F is:

$$[dF] = [-2xe^{2z}, 3y^2, 8 - 2x^2e^{2z}],$$

and since the determinant of the 1×1 matrix $[8 - 2x^2e^{2z}]$ is non-zero (which is due to the given assumption that $x \neq 2e^{-z}$), it is an invertible matrix and we may apply the implicit function theorem to assert that there exists a $\mathcal{C}^1$-map $\zeta : B([\begin{smallmatrix} a \\ b \end{smallmatrix}], \delta) \to B(c, \epsilon) \subset \mathbb{R}$ such that $\forall [\begin{smallmatrix} x \\ y \end{smallmatrix}] \in B([\begin{smallmatrix} a \\ b \end{smallmatrix}], \delta)$, $F(x, y, \zeta(x, y)) = 0$. Finally, the Jacobian for ζ is given by

$$[d\zeta] = -[8 - 2x^2e^{2z}]^{-1}[-2xe^{2z}, 3y^2] = \left[\frac{xe^{2z}}{4 - x^2e^{2z}}, \frac{3y^2}{2x^2e^{2z} - 8}\right].$$

2v) There are various small variations to how one might approach this question, those who understand the essence of the implicit function theorem may regard parts of this approach as being trivial or unnecessary.

Define a new function $G : \mathbb{R} \to \mathbb{R}$ which permutes the x, y, z coordinates of F as follows: $G(y, z, x) = F(x, y, z)$. The Jacobian of G is:

$$[dG] = [\partial_1 G, \partial_2 G, \partial_3 G] = [\partial_y G, \partial_z G, \partial_x G] = [3y^2, 8 - 2x^2e^{2z}, -2xe^{2z}],$$

and provided that $x \neq 0$, the 1×1 matrix $[-2xe^{2z}]$ is invertible. The implicit function theorem then tells us that there exists a $\mathcal{C}^1$ map $\xi : B([\begin{smallmatrix} b \\ c \end{smallmatrix}], \delta) \to B(a, \epsilon) \subset \mathbb{R}$ such that $\forall [\begin{smallmatrix} y \\ z \end{smallmatrix}] \in B([\begin{smallmatrix} b \\ c \end{smallmatrix}], \delta)$, $G(y, z, \xi(y, z)) = F(\xi(y, z), y, z) = 0$. The Jacobian for ξ is given by

$$[d\xi] = -[-2xe^{2z}]^{-1}[3y^2, 8 - 2x^2e^{2z}] = \left[\frac{3y^2}{2xe^{2z}}, \frac{4 - x^2e^{2z}}{xe^{2z}}\right].$$

2vi) In order to invoke the inverse function theorem, we need to show that the Jacobian of g at $[\begin{smallmatrix} 0 \\ -2 \end{smallmatrix}]$ is invertible.

$$[dg] = \begin{bmatrix} \partial_x(x + \zeta(x, y)) & \partial_y(x + \zeta(x, y)) \\ \partial_x y & \partial_y y \end{bmatrix} = \begin{bmatrix} 1 + \partial_x\zeta(x, y) & \partial_y\zeta(x, y) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{xe^{2z}}{4 - x^2e^{2z}} & \frac{3y^2}{2x^2e^{2z} - 8} \\ 0 & 1 \end{bmatrix}.$$

Evaluated at $\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]$ yields:

$$[dg]|_{\left[\begin{smallmatrix}1\\-2\end{smallmatrix}\right]} = \begin{bmatrix} 1 & \frac{-3}{2} \\ 0 & 1 \end{bmatrix}, \text{ which is invertible.}$$

Thus, the inverse function theorem tells us that there are open sets U and V which respectively contain $\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0+\zeta(0,-2)\\-2\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\-2\end{smallmatrix}\right]$ such that there is an inverse function $f : V \to U$ to $g : U \to V$. In particular, one may restrict U to a sufficiently small open ball $B(\left[\begin{smallmatrix}1\\-2\end{smallmatrix}\right], \epsilon)$ to get the desired map and its inverse. Finally, the Jacobian of f at $\left[\begin{smallmatrix}1\\-2\end{smallmatrix}\right]$ is given by the inverse matrix

$$[dg]|^{-1}_{\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]} = \begin{bmatrix} 1 & \frac{-3}{2} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix} = [df]|_{\left[\begin{smallmatrix}1\\-2\end{smallmatrix}\right]}.$$

Note: I believe that none of you used ϕ, for which I'm grateful, because that was a terribly careless "fix" by me. I'll only provide answers for f, as the solution is far worse for ϕ.

3i) To find stationary points, we solve for $[df] = \vec{0}$. Firstly, we compute the Jacobian:

$$[df] = [2(x+y-1)+2(x+y+2), 2(x+y-1)-2(x+y+2), 4z+\tfrac{1}{2}z^2] = [4x+2, 4y-6, 4z+\tfrac{1}{2}z^2].$$

Solving for $[df] = \vec{0}$ yields $x = -\frac{1}{2}$, $y = \frac{3}{2}$ and $z = 0$ or -8, thus giving us two stationary points: $[\frac{-1}{2}, \frac{3}{2}, 0]^T$ and $[\frac{-1}{2}, \frac{3}{2}, -8]^T$. In particular, both of these vectors have norm less than 10 and lie in $B(\vec{0}, 10)$.

3ii) To determine if a stationary point is a local extremum, we (first) check its Hessian:

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4+z \end{bmatrix}.$$

Since the Hessian is already diagonalized, its eigenvalues are $4, 4$ and $4+z$. Therefore, the stationary point $[\frac{-1}{2}, \frac{3}{2}, 0]^T$ has Hessian with all positive eigenvalues. This in turn means that the Hessian is positive definite, and that $[\frac{-1}{2}, \frac{3}{2}, 0]^T$ is a local minimum. The stationary point $[\frac{-1}{2}, \frac{3}{2}, -8]^T$, on the other had, has Hessian with both positive and negative eigenvalues. Thus, the Hessian is indefinite, and is neither a local maximum nor a local minimum. To clarify, we're combining two statements: the Hessian is not positive semi-definite and hence is not a local minimum, and the Hessian is not negative semi-definite and hence is not a local maximum.

3iii) Define $C : \mathbb{R}^3 \to \mathbb{R}$, $C(x, y, z) = x^2 + y^2 + z^2 - 100$. The zero-set $Z(C)$ is closed (it's the complement of the interior and exterior of $B(\vec{0}, 10)$) and bounded (it's contained in

$\overline{B}(\vec{0}, 10)$), and therefore must exhibit maxima and minima. Let us set up the Lagrangian and try to compute these points (as well as any other local extrema):

$$\mathcal{L}_f(x,y,z,\lambda) := f - \lambda C = (x+y-1)^2 + (x-y+2)^2 + 2z^2 + \tfrac{1}{6}z^3 - \lambda(x^2+y^2+z^2-100).$$

N.B.: it is possible also at this stage to expand f and replace $x^2+y^2+z^2$ with 100. doing this can sometimes simplify the computation, but generally not by much.

For $\nabla\mathcal{L}_f = \vec{0}$, we have

$$\nabla\mathcal{L}_f = \begin{bmatrix} 4x+2-2\lambda x \\ 4y-6-2\lambda y \\ 4z+\frac{1}{2}z^2-2\lambda z \\ x^2+y^2+z^2-100 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

N.B.: you may regard λ as a fixed constant and compute the gradient $\nabla\mathcal{L}_f(\cdot,\cdot,\lambda)$ without differentiating wrt λ since the $C=0$ condition is already given.

From the top row, we see that $\lambda \neq 2$ (since $2 \neq 0$), and hence the two two rows tell us that $y = -3x = \frac{-3}{\lambda-2}$. The third row then tells us either that $z = 0$ or $z = 4(\lambda-2) = \frac{4}{x}$ (these two conditions are actually incompatible because $\lambda \neq 2$, not that this is important).

When $z = 0$, the fourth row tells us that $10x^2 = 100$ and hence $x = \pm\sqrt{10}$. Therefore, we obtain two stationary points

$$[\sqrt{10}, -3\sqrt{10}, 0]^T \text{ and } [-\sqrt{10}, 3\sqrt{10}, 0]^T.$$

On the other hand, if $z = \frac{-4}{x}$, then the fourth row gives us:

$$10x^2 + \frac{16}{x^2} = 100.$$

Solving this yields four possible solutions:

$$x = \sqrt{5 \pm 3\sqrt{\tfrac{13}{5}}} \text{ and } x = -\sqrt{5 \pm 3\sqrt{\tfrac{13}{5}}}$$
$$x \approx \pm\, 0.40,\ \pm 3.14.$$

Numerically substituting this into y and z then yields:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \pm \begin{bmatrix} \sqrt{5-3\sqrt{\frac{13}{5}}} \\ -3\sqrt{5-3\sqrt{\frac{13}{5}}} \\ \frac{4}{\sqrt{5-3\sqrt{\frac{13}{5}}}} \end{bmatrix} \approx \pm \begin{bmatrix} -0.40 \\ 1.21 \\ -9.92 \end{bmatrix}, = \pm \begin{bmatrix} \sqrt{5+3\sqrt{\frac{13}{5}}} \\ -3\sqrt{5+3\sqrt{\frac{13}{5}}} \\ \frac{4}{\sqrt{5+3\sqrt{\frac{13}{5}}}} \end{bmatrix} \approx \pm \begin{bmatrix} -3.17 \\ 9.41 \\ -1.28 \end{bmatrix}.$$

Computing the value of f for all six of these points, we get that the minimum on the boundary is at approx. $[-0.40, 1.21, -9.92]^T$ (with a minimum value of approximately 34.32) and the maximum on the boundary is at approx. $[0.40, 1.21, 9.92]^T$ (with a maximum value of approximately 375.68). However, the local minimum found in part 3ii) actually realizes a lower value of 0. Thus, the global minimum is at $[\frac{-1}{2}, \frac{3}{2}, 0]$ and the global maximum is at

$$\left[\sqrt{5+3\sqrt{\tfrac{13}{5}}}, -3\sqrt{5+3\sqrt{\tfrac{13}{5}}}, \frac{4}{\sqrt{5+3\sqrt{\frac{13}{5}}}}\right]^T \approx [0.40, -1.21, 9.92]^T.$$

3iv) First observe that $f(x,y,z) = 2x^2+2y^2+2x-6y+2z^2+\frac{1}{6}z^3+5$, and hence $g(u,v,w) = \frac{1}{2}(f(u^3, \log v, \tan w) - 5)$. Since $u \in \mathbb{R} \Leftrightarrow u^3 \in \mathbb{X}$, $v \in (0, +\infty) \Leftrightarrow \log v \in \mathbb{R}$ and $w \in [0, \frac{\pi}{2}) \Leftrightarrow \tan w \in [0, +\infty)$, we see that the extremization of $g(u,v,w)$ is tantamount to extremizing $f(x,y,z)$ subject to $(x,y,z) \in \mathbb{R}^2 \times [0, +\infty)$. Since $z \geq 0$, $f(x,y,z) \geq (x+y-1)^2 + (x-y+2)^2 \geq 0$. At the same time, $f = 0$ is realized for $[\frac{-1}{2}, \frac{3}{2}, 0]^T$, and hence the global minimum of $f : \mathbb{R}^2 \times [0, +\infty) \to \mathbb{R}$ is $[\frac{-1}{2}, \frac{3}{2}, 0]^T$. This then implies that the global minimum for g is attained at

$$\begin{bmatrix} \sqrt[3]{\frac{-1}{2}} \\ \exp(\frac{3}{2}) \\ \text{Arctan}(0) \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt[3]{2}} \\ \exp(\frac{3}{2}) \\ 0 \end{bmatrix},$$

and takes the value of $\frac{1}{2}(0-5) = \frac{-5}{2}$. On the other hand, there can be no maximum since $g(0, 1, w)$ blows up to $+\infty$ as w approaches $\frac{\pi}{2}$.

We will not provide solutions to **Bonus i)** and **Bonus ii)**. You can think of them as the challenge questions of the assignment — if you want hints, talk to me! (I'll admit that Bonus question ii this time was lackluster. It's my fault – I miscomputed something when I was writing up the question).