Assignment 1

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Question 1

i. $S = \{x \in \mathbb{R} | x = \sqrt{2} + k, k \in \mathbb{N}\}$

ii. **Proof:** For any arbitrary $k \in \mathbb{N}$, we define $S_k := (\sqrt{2}+k, \sqrt{2}+k+1)$. According to the definition, $S_k \subset \mathbb{R} \setminus S$. For arbitrary $x \in S_k$, let $\delta = min\{x - \sqrt{2} - k, \sqrt{2}+k+1-x\}$, $\exists B(x,\delta)$ is an open ball, thus S_k is an open set. Then the complement of S in $\mathbb{R}$ $\overline{S} = \bigcup_{k\in\mathbb{N}} S_k$ is an open set. Therefore, S is an closed set.

iii. **Proof:** Since $\mathbb{N}$ is unbounded (according to the Archimedes character), S is an unbounded set.

iv. $B((0,0,4),1)$

v. **Proof:** Obviously it is an open set since it is an oepn ball.

vi. **Proof:** $\forall \vec{x_1}, \vec{x_2} \in B(0,0,4), d(\vec{x_i},(0,0,4)) < 1$. Therefore, according to triangular inequality, $d(\vec{x_1},\vec{x_2}) \leq d(\vec{x_1},(0,0,4)) + d(\vec{x_2},(0,0,4)) < 2$. Thus, this open ball is bounded.

vii. $diam(B) = 2$.

Proof: For arbitrary points $\vec{p}, \vec{q} \in B(\vec{x},1), d(\vec{x},\vec{p}) < 1, d(\vec{x},\vec{q}) < 1$, thus $d(\vec{x},\vec{y}) < 2$

according to the triangular inequality. Now consider two point sequences:

$$A_k = \{(0,0,3+\frac{1}{2^k})|k \in \mathbb{N}\}, B_k = \{(0,0,5-\frac{1}{2^k})|k \in \mathbb{N}\}$$

Obviously $A_k, B_k \subset B(\vec{x}, 1)$. Then we can tell

$$\lim_{k\to\infty} d(A_k, B_k) = 2$$

Which means $\forall \varepsilon > 0, \exists N > 0$, when $k > N, d(A_k, B_k) > 2 - \varepsilon$. Then we can tell $sup(\vec{p}, \vec{q}) = 2$, which means $diam(B) = 2$.

Question 2

i.

$$\lim_{(x,y)\to(3.5)} (\frac{\sin(y-x)}{y-x}, \sqrt{y^2-x^2})$$

exists

Proof: Both $\frac{\sin(y-x)}{y-x}$ and $\sqrt{y^2-x^2}$ are continuous at point $(3,5)$, thus the limit is equal to $(\frac{\sin 2}{2}, 4)$.

ii.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^3}{x^3-y^4}$$

doesn't exist.

Proof: Replace y with $kx(k \in \mathbb{R})$, then the original formula is equal to

$$\begin{aligned}\lim_{x\to 0} \frac{x^4-k^3x^3}{x^3-k^4x^4} &= \lim_{x\to 0} \frac{x-k^3}{1-k^4x} \\ &= -k^3\end{aligned}$$

which is uncertain. Therefore, the limit doesn't exist.

iii.

$$\lim_{\substack{x\to+\infty\\y\to+\infty}} \frac{x^3-y^3}{x^4-y^4}$$

exists.

Proof: The original formula is equal to

$$\lim_{\substack{x\to+\infty\\y\to+\infty}} \frac{x^2+xy+y^2}{x^3+x^2y+xy^2+y^3}$$

which is obviously no less than zero. Then

$$\begin{aligned}
\lim_{\substack{x\to+\infty\\y\to+\infty}} \frac{x^2+xy+y^2}{x^3+x^2y+xy^2+y^3} &= \lim_{\substack{x\to+\infty\\y\to+\infty}} \frac{x^2+xy+y^2+\frac{y^3}{x}-\frac{y^3}{x}}{x^3+x^2y+xy^2+y^3} \\
&= \lim_{\substack{x\to+\infty\\y\to+\infty}} (\frac{1}{x}-\frac{y^3}{x^4+x^3y+x^2y^2+xy^3}) \\
&\leq \lim_{x\to+\infty} \frac{1}{x} = 0
\end{aligned}$$

As a result,

$$0 \leq \lim_{\substack{x\to+\infty\\y\to+\infty}} \frac{x^3-y^3}{x^4-y^4} \leq 0$$

Therefore, the limit is equal to 0.

iv.

$$\lim_{x\to+\infty} \lim_{y\to-\infty} \frac{x^3-y^3}{x^4-y^4}$$

exists.

Proof: Replace y with $u = \frac{1}{y}$, then

$$\lim_{y\to-\infty} \frac{x^3-y^3}{x^4-y^4} = \lim_{u\to0^-} \frac{x^3-\frac{1}{u^3}}{x^4-\frac{1}{u^4}} = \lim_{u\to0^-} \frac{(u^3x^3-1)u}{u^4x^4-1} = 0$$

Therefore,

$$\lim_{x\to+\infty} \lim_{y\to-\infty} \frac{x^3-y^3}{x^4-y^4} = \lim_{x\to+\infty} 0 = 0$$

v.

$$\lim_{(x,y)\to(\mathrm{e},0)} (1+2020y)^{\frac{1}{y-x^2y^2}}$$

exists.

Proof: Consider

$$\boldsymbol{f}:\mathbb{R}^2\to\mathbb{R}^2, \boldsymbol{f}(x,y) = ((1+2020y)^{\frac{1}{2020y}}, \frac{2020y}{y-x^2y^2})$$

=

$$\boldsymbol{f}:\mathbb{R}^2\to\mathbb{R}^2, \boldsymbol{f}(x,y) = ((1+2020y)^{\frac{1}{2020y}}, \frac{2020}{1-x^2y})$$

, Therefore,

$$\lim_{(x,y)\to(\mathrm{e},0)} \boldsymbol{f}(x,y) = (e, 2020)$$

$$\boldsymbol{g}:\mathbb{R}^2\to\mathbb{R}: \boldsymbol{g}(u,v) = u^v$$

Then invoke the composition rule:

$$\lim_{(u,v)\to(\mathrm{e},2020)} \boldsymbol{g}(u,v) = \mathrm{e}^{2020}$$

vi.

$$\lim_{(x,y)\to(3,+\infty)} \frac{\log(x+y)}{x^2+y^2}$$

exists.

Proof: According to the fundamental inequality, $x^2+y^2 \geq \frac{x^2+y^2}{2}$, then

$$\frac{\log(x+y)}{x^2+y^2} \leq \frac{\log(x+y)}{\frac{(x+y)^2}{2}}$$

Replace x^2+y^2 with u. When $(x^2+y^2)\to(3,\infty), u\to\infty$. Then

$$\lim_{(x,y)\to(3,\infty)} \frac{\log(x+y)}{\frac{(x+y)^2}{2}} = \lim_{u\to\infty} \frac{2\log u}{u^2} = 0$$

Meanwhile, the original formula is no less than zero, which means

$$0 \leq \lim_{(x,y)\to(3,+\infty)} \frac{\log(x+y)}{x^2+y^2} \leq 0$$

Therefore, the limit is equal to 0.

Question 3

i. Proof: Denote $\vec{h}$ as $\begin{bmatrix} h_x \\ h_y \end{bmatrix}$, $\vec{t}$ as $\begin{bmatrix} x \\ y \end{bmatrix}$. Take the linear function $\boldsymbol{L}(\vec{h})$ as $\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} h_x \\ h_y \end{bmatrix}$.

Consider

$$\boldsymbol{f}(\vec{t}+\vec{h}) - \boldsymbol{f}(\vec{t}) - \boldsymbol{L}(\vec{h}) = \begin{bmatrix} 2(x+h_x)+3-(2x+3)-2h_x \\ (x+h_x)^2+(y+h_y)^2-(x^2+y^2)-2h_y \\ (y+h_y)\mathrm{e}^{2(x+h_x)} - y\mathrm{e}^{2x} - (2h_x+h_y) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ h_x^2+2xh_x+2h_y(y-1)+h_y^2 \\ y\mathrm{e}^{2x}(\mathrm{e}^{2h_x}-1)+h_y(\mathrm{e}^{2(x+h_x)}-1)-2h_x \end{bmatrix}$$

When $\vec{t}=(0,1)$,

$$\boldsymbol{f}(\vec{t}+\vec{h}) - \boldsymbol{f}(\vec{t}) - \boldsymbol{L}(\vec{h}) = \begin{bmatrix} 0 \\ h_x^2+2xh_x+2h_y(y-1)+h_y^2 \\ \mathrm{e}^{2h_x}-1+h_y(\mathrm{e}^{2h_x}-1)-2h_x \end{bmatrix}$$

$$\lim_{\vec{h}\to 0} \frac{||\boldsymbol{f}(\vec{t}+\vec{h}) - \boldsymbol{f}(\vec{t}) - \boldsymbol{L}(\vec{h})||}{||\vec{h}||}$$

=

$$\lim_{\vec{h}\to 0} \sqrt{\frac{(h_x^2+2xh_x+2h_y(y-1)+h_y^2)^2+(\mathrm{e}^{2h_x}-1+h_y(\mathrm{e}^{2h_x}-1)-2h_x)^2}{h_x^2+h_y^2}}$$

=

$$\lim_{\vec{h}\to 0}\sqrt{\frac{(h_x^2+h_y^2)^2+(2h_xh_y)^2}{h_x^2+h_y^2}}(\vec{h}\to 0, 2h_x = \mathrm{e}^{2h_x}-1+o(2h_x))$$

=

$$\lim_{\vec{h}\to 0}\sqrt{h_x^2+h_y^2+\frac{4}{\frac{1}{h_y^2}+\frac{1}{h_x^2}}}=0$$

Therefore, $\boldsymbol{f}$ is differentiable at $(0,1)$.

ii. At $(x,y)=(0,1)$

$$\frac{\partial f_1}{\partial x}=2, \frac{\partial f_1}{\partial y}=0$$
$$\frac{\partial f_2}{\partial x}=2x=0, \frac{\partial f_2}{\partial y}=2y=2$$
$$\frac{\partial f_3}{\partial x}=2y\mathrm{e}^{2x}=2, \frac{\partial f_2}{\partial y}=\mathrm{e}^{2x}=1$$

iii.

$$[\mathrm{d}\boldsymbol{f}|_{(0,1)}]=\begin{bmatrix}2 & 0\\ 0 & 2\\ 2 & 1\end{bmatrix}$$