

Calculus Note

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March 26, 2020

1 Euclidean n-space

1.1 Concepts:

$$\mathbb{R}^n := \{(x_1, \dots, x_n) | x_1, \dots, x_n \in \mathbb{R}\}$$

$$A \times B := (a \in A, b \in B), \mathbb{R}^n := \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$$

$$\|\cdot\| := \mathbb{R}^n \rightarrow \mathbb{R} \text{ (Euclidean n-norm)}$$

$$\|\vec{x}\| := \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$$d := \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \text{ (Distance)}$$

$$d(\vec{x}, \vec{y}) := \|\vec{x} - \vec{y}\| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

Properties of distance:

$$\text{Symmetry: } \forall \vec{x}, \vec{y} \in \mathbb{R}^n, d(\vec{x}, \vec{y}) = d(\vec{y}, \vec{x})$$

$$\text{Triangle inequality: } \forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n, d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}) \geq d(\vec{x}, \vec{z})$$

$$\text{Positivity: } \forall \vec{x}, \vec{y} \in \mathbb{R}^n, d(\vec{x}, \vec{y}) \geq 0$$

$$\Rightarrow \forall \vec{x}, \vec{y} \in \mathbb{R}^n, d(\vec{x}, \vec{y}) = 0 \Leftrightarrow \vec{x} = \vec{y}$$

1.2 Open n-ball

1.2.1 Definition:

$$\text{For } \delta > 0, B(\vec{x}, \delta) := \{\vec{y} \in \mathbb{R}^n | d(\vec{x}, \vec{y}) < \delta\}$$

1.2.2 Interior points:

$\vec{x} \in \mathbb{R}^n$ is an interior point of S iff. $\exists \delta > 0$ s.t. $B(\vec{x}, \delta) \subset S$

1.2.3 Exterior points:

$\vec{x} \in \mathbb{R}^n$ is an exterior point of S iff. $\exists \delta > 0$ s.t. $B(\vec{x}, \delta) \subset \mathbb{R}^n \setminus S$

1.2.4 Boundary points:

$\vec{x} \in \mathbb{R}^n$ is a boundary point of S iff. $\forall \delta > 0, B(\vec{x}, \delta) \cap S \neq \emptyset$ and $B(\vec{x}, \delta) \cap (\mathbb{R}^n \setminus S) \neq \emptyset$

1.2.5 Adherent points:

$\vec{x} \in \mathbb{R}^n$ is an adherent point of S iff. $\forall \delta > 0, B(\vec{x}, \delta) \cap S \neq \emptyset$

1.2.6 Limit points:

$\vec{x} \in \mathbb{R}^n$ is a limit point of S iff. $\forall \delta > 0, (B(\vec{x}, \delta) \setminus \{\vec{x}\}) \cap S \neq \emptyset$

1.3 Type of sets

1.3.1 Open sets:

A set S is open iff. every point of S is an interior point.

1.3.2 Closed sets:

A set S is closed iff. $\mathbb{R}^n \setminus S$ is open.

1.3.3 Interior of a set:

All interior points of S .

1.3.4 Exterior of a set:

All exterior points of S .

1.3.5 Boundary of a set

All boundary points of $S \rightarrow \partial S$

1.3.6 Closure of a set:

\bar{S} of S is $S \cup \partial S$

1.4 Theorem 1.1

A set $S \subset \mathbb{R}^n$ is open iff. it is equal to the union of a collection of open balls.

Proof: 1. $S = \text{union of open balls} \Rightarrow S = \text{open}$

Consider arbitrary $\vec{x} \in S$, then \exists an open ball $B(\vec{y}, \delta_y) \subset S$ which contains \vec{x} . Let $\delta_x := \delta_y - d(\vec{x}, \vec{y})$, then $\delta_x > 0$ and $B(\vec{x}, \delta_x) \subset B(\vec{y}, \delta_y) \subset S$
 $\Rightarrow \vec{x}$ is an interior point of S .

Since \vec{x} was arbitrary chosen, every point of S is an interior point. Therefore, S is open.

2. $S = \text{open} \Rightarrow S = \text{union of open balls}$

Assume S is open, $\Rightarrow \forall \vec{x} \in S, \exists \delta_x > 0$ s.t. $B(\vec{x}, \delta_x) \subset S$

$$U := \bigcup_{\vec{x} \in \text{Int}(S)} B(\vec{x}, \delta_x) \subset S$$

Conversely, $S \subset U \Rightarrow S = U$ Therefore, S is the union of open balls.

1.5 Theorem 1.2

$\text{Int}(S)$ of any set $S \subset \mathbb{R}^n$ is an open set. Likewise, $\text{Ext}(S)$ of S is also open.

Proof: Every $\vec{x} \in \text{Int}(S)$ is an interior point of S

$\Rightarrow \exists \delta_x > 0$, s.t. $B(\vec{x}, \delta_x) \subset S$

$\Rightarrow U := \bigcup_{\vec{x} \in \text{Int}(S)} B(\vec{x}, \delta_x)$ is open.

Thus, $\text{Int}(S) \subset U$, because every $\vec{x} \in \text{Int}(S)$ is contained in some open ball

$B(\vec{x}, \delta_x) \subset U$. For every $\vec{x} \in \text{Int}(S), B(\vec{x}, \delta_x) \subset \text{Int}(S)$.

Since $B(\vec{x}, \delta_{\vec{x}})$ is open, every point in $B(\vec{x}, \delta_{\vec{x}})$ is an interior point of $B(\vec{x}, \delta_{\vec{x}}) \Rightarrow$ is an interior point of $S \supset B(\vec{x}, \delta_{\vec{x}})$.

1.6 Proposition 1.3

The union of any collection of (arbitrarily many) open sets is open.

1.7 Proposition 1.4

Finite intersections of open sets is open.

1.8 Theorem 1.5

Finite unions of closed sets is closed, and intersections of closed sets is closed.

2 Further properties

2.1 Proposition 2.1

Finite intersections of open sets is open.

2.2 Theorem 2.2

Finite unions of closed set is closed, and intersections of closed set is closed.

Proof:

$$\begin{aligned}
 \mathbb{R}^n \setminus \left(\bigcup_{\alpha \in \mathcal{A}} S_{\alpha} \right) &= \bigcup_{\alpha \in \mathcal{A}} (\mathbb{R}^n \setminus S_{\alpha}) \\
 \Rightarrow \vec{x} \in \mathbb{R}^n \setminus \left(\bigcup_{\alpha \in \mathcal{A}} S_{\alpha} \right) \\
 \Leftrightarrow \vec{x} \notin \bigcup_{\alpha \in \mathcal{A}} S_{\alpha} \\
 \Leftrightarrow \forall \alpha \in \mathcal{A}, \vec{x} \notin S_{\alpha} \\
 \Leftrightarrow \forall \alpha \in \mathcal{A}, \vec{x} \in \bigcap_{\alpha \in \mathcal{A}} (\mathbb{R}^n \setminus S_{\alpha})
 \end{aligned}$$

2.3 Connectedness

2.3.1 Path-connectedness:

$\forall \vec{x}, \vec{y} \in S, \exists$ a continuous function $f : [0, 1] \rightarrow S, s.t. f(0) = \vec{x}$ and $f(1) = \vec{y}$

2.3.2 Connectedness:

$S \subset \mathbb{R}^n, s.t. \exists$ open sets $U, V \subset \mathbb{R}^n, s.t. U \cap V = \emptyset, U, V \neq \emptyset$

$S \subset U \cup V$ and $S \cap U$ and $S \cap V$ are both non-empty.

2.4 Convergence

2.4.1 Definition

Let $\{\vec{x}_k\}_{k \in \mathbb{N}} = \vec{x}_1, \vec{x}_2, \dots$ be a sequence in \mathbb{R}^n , we say that this sequence converges to $\vec{a} \in \mathbb{R}^n$ iff. $\forall \epsilon > 0, \exists N \in \mathbb{N}, s.t. \forall k > N, \|\vec{x}_k - \vec{a}\| < \epsilon$

$$\Rightarrow \lim_{k \rightarrow \infty} \vec{x}_k = \vec{a} \Leftrightarrow d(\vec{x}_k, \vec{a}) = 0$$

2.4.2 Cauchy convergent sequences

A sequence $\{\vec{x}_k\}$ in \mathbb{R}^n is called Cauchy convergent iff. $\forall \epsilon > 0, \exists N, s.t. \forall j, k > N, d(\vec{x}_j, \vec{x}_k) < \epsilon$

2.5 Theorem 2.4

Let $\{\vec{x}_k\}$ be a sequence in \mathbb{R}^n , we adopt the following notion:

$$\vec{x}_k = (x_{k,1}, x_{k,2}, \dots, x_{k,n})$$

Then $\lim_{k \rightarrow \infty} \vec{x}_k = \vec{a} \Leftrightarrow \forall j = 1, \dots, n, \lim_{k \rightarrow \infty} x_{k,j} = a_j$ **Similar to Cauchy convergence**

N.B. In \mathbb{R}^n , Cauchy convergent **doesn't mean** convergent in S (∂S).

2.6 Lemma 2.5

$S \subset \mathbb{R}^n, \vec{a} \in \mathbb{R}^n, \vec{a}$ is adherent point of $S \Leftrightarrow \{\vec{x}_k\}$ in S converge to \vec{a}

2.7 Lemma 2.6

\vec{a} is a limit point $\Leftrightarrow \{\vec{x}_k\}$ in $S \setminus \{\vec{a}\}$ converges to \vec{a}

2.8 Theorem 2.7

$S \subset \mathbb{R}^n$ is closed $\Leftrightarrow S$ contains all of its adherent points.

2.9 Diameter of a set

2.9.1 Definition

$$\text{diam}(S) := \sup_{\vec{x}, \vec{y} \in S} d(\vec{x}, \vec{y}) \rightarrow R \in (0, +\infty) \cup \{+\infty\}$$

2.9.2 Bounded sets and sequences

$$\text{diam}(S) < \infty$$

2.10 Cantor's intersection theorem

$\{F_k\}_{k \in \mathbb{N}}$ of non-empty, closed (*without this is not true*) subsets of \mathbb{R}^n nested in the following way:

$$\mathbb{R}^n \supset F_1 \supset F_2 \supset \dots \supset F_k \supset \dots$$

If $\lim_{k \rightarrow \infty} \text{diam}(F_k) = 0$, then the $\bigcup_{k=1}^{\infty} F_k$ is a set consisting a single element.

3 Vector function

3.1 Function

f is a subset of $A \times B$:

$$\forall a \in A, \exists! b \in B, \text{s.t. } (a, b) \in f \Leftrightarrow f(a) = b$$

3.2 Scalar and vector valued functions

3.2.1 Definition

$f : \Omega_1 \rightarrow \Omega_2$ between a set $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ is called an **n-variable, m-dimensional vector valued** function. Especially when $m = 1$, it is a scalar function.

3.2.2 Algebraic constructions

For $\Omega \subset \mathbb{R}^n, \mu : \Omega \rightarrow \mathbb{R}, \mathbf{f}, \mathbf{g} : \Omega \rightarrow \mathbb{R}^n$:

$$\mu \mathbf{f} : \Omega \rightarrow \mathbb{R}^n, \vec{x} \mapsto \mu(\vec{x}) \mathbf{f}(\vec{x})$$

$$\mathbf{f} + \mathbf{g} : \Omega \rightarrow \mathbb{R}^n, \vec{x} \mapsto \mathbf{f}(\vec{x}) + \mathbf{g}(\vec{x})$$

3.2.3 Composition

$$\mathbf{f} : \Omega_1 \subset \mathbb{R}^n \rightarrow \Omega_2 \subset \mathbb{R}^m, \mathbf{g} : \Omega_2 \rightarrow \mathbb{R}^l :=$$

$$\mathbf{g} \circ \mathbf{f} : \Omega_1 \rightarrow \mathbb{R}^l, \vec{x} \mapsto \mathbf{g}(\mathbf{f}(\vec{x}))$$

Practically requires $\mathbf{f}(\Omega_1) = \mathbf{f}(\vec{x}) | \vec{x} \in \Omega_1$ is a subset of the domain of \mathbf{g}

3.3 Taking limits

Let $\Omega \subset \mathbb{R}^n$ be non-empty. Given a function $\mathbf{f} : \Omega \rightarrow \mathbb{R}^n$, a limit point \vec{x}_0 of Ω and $\vec{a} \in \mathbb{R}^m$. If

$$\forall \varepsilon > 0, \exists \delta > 0, s.t. \forall \vec{x} \in \Omega$$

$$0 < \|\vec{x} - \vec{x}_0\|_n < \delta \Rightarrow \|\mathbf{f}(\vec{x}) - \vec{a}\|_m < \varepsilon$$

$$\Leftrightarrow \vec{x} \in B(\vec{x}_0, \delta) \setminus \{\vec{x}_0\} \Rightarrow \mathbf{f}(\vec{x}) \in B(\vec{a}, \varepsilon) \Rightarrow \vec{x} \text{ limits to } \vec{x}_0 \text{ in } \Omega. \mathbf{f}(\vec{x}) \text{ limits to } \vec{a} :$$

$$\lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} \mathbf{f}(\vec{x}) = \vec{a}$$

When limit point \vec{x}_0 is an interior point of $\Omega \cup \{\vec{x}_0\}$, then we can denote

$$\lim_{\vec{x} \rightarrow \vec{x}_0} \mathbf{f}(\vec{x})$$

Coordinate-wise denotation:

$$\lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} \mathbf{f}(\vec{x}) = \vec{a} \Leftrightarrow \text{For } j = 1, \dots, m, \lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} f_j(\vec{x}) = a_j, \mathbf{f}(\vec{x}) = (f_1\vec{x}, \dots, f_n(\vec{x}))$$

3.4 Scalar function properties

3.4.1 Inequality

If $\forall \vec{x} \in \Omega \setminus \{\vec{x}_0\}, f(\vec{x}) \leq g(\vec{x})$, then

$$\lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} f(\vec{x}) \leq \lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} g(\vec{x})$$

3.4.2 Non-negativity and Sandwich Theorem

3.4.3 Calculation

If $\lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} \mu(\vec{x}), \mathbf{f}(\vec{x}), \mathbf{g}(\vec{x})$ all exist, then

$$\begin{aligned} \lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} (\mu \mathbf{f})(\vec{x}) &= \lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} \mu(\vec{x}) \cdot \mathbf{f}(\vec{x}) \\ \lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} (\mathbf{f} + \mathbf{g})(\vec{x}) &= \lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} \mathbf{f}(\vec{x}) + \lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} \mathbf{g}(\vec{x}) \\ \lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} \frac{1}{\mu(\vec{x})} &= \frac{1}{\lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} \mu(\vec{x})} (\mu \neq 0) \end{aligned}$$

3.5 Uniqueness

If a limit exists, then it must be unique.

Proof: Given $\lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} \mathbf{f}(\vec{x}) = \vec{a}, \vec{b}$,

\Rightarrow A sequence $\{\vec{x}_k\}_{k \in \mathbb{N}}$ has its tail $\rightarrow \vec{a}, \vec{b}$

$\Rightarrow \vec{a}, \vec{b}$ are arbitrary close (via triangle inequality)

$\Rightarrow \vec{a} = \vec{b}$

3.6 Iterated limits / Repeated limits

Taking limits of a function one variable at a time.

$$\lim_{x \rightarrow x_0} \lim_{y \rightarrow y_0} f(x, y)$$

1. Compute $\lim_{y \rightarrow y_0} f(x, y)$ for all x in a small punctured ball $B(x_0, \delta) \setminus \{x_0\}$ around x_0
2. This produces a function $\varphi(x) := \lim_{y \rightarrow y_0} f(x, y)$ on $B(x_0, \delta) \setminus \{x_0\} \Rightarrow \lim_{x \rightarrow x_0} \varphi(x)$

$$\Rightarrow \lim_{x \rightarrow x_0} \lim_{y \rightarrow y_0} f(x, y) = \lim_{x \rightarrow x_0} \varphi(x) = \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$$

If

$$\lim_{x \rightarrow x_0} \lim_{y \rightarrow y_0} f(x, y) \neq \lim_{y \rightarrow y_0} \lim_{x \rightarrow x_0} f(x, y)$$

then $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$ doesn't exist.

Conclusion: Iterated limits \nRightarrow double limits

4 Continuity

4.1 Continuity

4.1.1 Definition

$f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, \vec{x}_0 \in \Omega, f$ is continuous at $\vec{x}_0 \Leftrightarrow$

$$\forall \varepsilon > 0, \exists \delta > 0, s.t. \forall \vec{x} \in \Omega, \|\vec{x} - \vec{x}_0\| < \delta \Rightarrow \|f(\vec{x}) - f(\vec{x}_0)\| < \varepsilon$$

$$(\vec{x} \in B(\vec{x}_0, \delta) \cap \Omega \Rightarrow f(\vec{x}) \in B(f(\vec{x}_0), \varepsilon))$$

If $\vec{x}_0 \in \Omega$ is isolated, then f is always continuous at \vec{x}_0 .

if $\vec{x}_0 \in \Omega$ is a limit point, then Continuity at \vec{x}_0 is equal to the following limit being

true: $\lim_{\Omega \ni \vec{x} \rightarrow \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$

4.1.2 Continuous function

$\mathbf{f} : \Omega \rightarrow \mathbb{R}^m$ is continuous at every $\vec{x} \in \Omega := C^0$

Scalar function: $C^0(\Omega) := C^0(\Omega; \mathbb{R})$

$\mathbf{f}(\vec{x})$ is continuous $\Leftrightarrow f_1(\vec{x}), f_2(\vec{x}), \dots, f_m(\vec{x})$ are continuous.

4.2 Theorem 4.1

Given continuous functions $\mathbf{f} : \Omega_1 \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, \mathbf{g} : \Omega_2 \subset \mathbb{R}^m \rightarrow \mathbb{R}^l$

such that $\mathbf{f}(\Omega_1) \subset \Omega_2$, then the function $\mathbf{g} \circ \mathbf{f} : \Omega_1 \rightarrow \mathbb{R}^l$ is continuous.