

Calculus A2 (English) — Assignment 2.

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Submission instructions: same instructions as for assignment 1.

Question 1: (5 marks) Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, $f(x, y) = \exp(x - y)$, and

- i. compute the degree 3 Peano form for the Taylor expansion of f around $(x, y) = (1, 1)$; (2 marks)
- ii. compute the degree 2 Lagrange form for the Taylor expansion of f around $(x, y) = (1, 1)$; (1 mark)
- iii. use your answer to part i to show that $\sqrt{e} \approx \frac{79}{48}$, no error terms required; (1 mark)
- iv. use your answer to part ii to show that $\frac{78}{49} \leq \sqrt{e} \leq \frac{78}{47}$. (1 mark)

Question 2: (8 marks)

Consider the zero-set $Z(F) \subset \mathbb{R}^3$ of the function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$,

$$F(x, y, z) = -x^2 e^{2z} + y^3 + 8z, \text{ and}$$

- i. show that the gradient ∇F of F never vanishes on $Z(F)$; (2 mark)
- ii. compute the point-normal form for the tangent plane of $Z(F)$ at $[a, b, c]^T \in Z(F)$; (1 mark)
- iii. compute the parametric form for the tangent plane of $Z(F)$ at $[a, b, c]^T \in Z(F)$; (1 marks)
- iv. show, using the implicit function theorem, that provided that $x \neq \pm 2e^{-z}$, then for any $[a, b, c]^T \in Z(F)$, there is C^1 map $\zeta : B([\frac{a}{b}], \delta) \rightarrow \mathbb{R}$ satisfying $F(x, y, \zeta(x, y)) = 0$. Compute the Jacobian of ζ . (2 marks)
- v. show, using the implicit function theorem, that provided that $x \neq 0$, then for any $[a, b, c]^T \in Z(F)$, there is C^1 map $\xi : B([\frac{b}{c}], \delta) \rightarrow \mathbb{R}$ satisfying $F(\xi(y, z), y, z) = 0$. Compute the Jacobian of ξ . (1 marks)
- vi. consider the point $[0, -2, 1]^T \in Z(F)$ and construct $\zeta : B([\frac{0}{-2}], \delta) \rightarrow \mathbb{R}$ as per part iv. Define the following function $g : B([\frac{0}{-2}], \delta) \rightarrow \mathbb{R}^2$,

$$g(x, y) = \begin{bmatrix} x + \zeta(x, y) \\ y \end{bmatrix}.$$

Show, using the inverse function theorem, that there exists a C^1 function $f : B([\frac{1}{-2}], \epsilon) \rightarrow f(B([\frac{1}{-2}], \epsilon))$ which is inverse to $g : f(B([\frac{1}{-2}], \epsilon)) \rightarrow B([\frac{1}{-2}], \epsilon)$. Compute $[df]$ at $[\frac{1}{-2}]$. (1 mark)

Question 3: (7 marks)

Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$,

$$f(x, y, z) = (x + y - 1)^2 + (x - y + 2)^2 + 2z^2 + \frac{1}{6}z^3, \text{ and}$$

- i. find all the stationary points of f inside $B(\vec{0}, 10) \subset \mathbb{R}^3$; (1 mark)
- ii. classify all the stationary points you found as local maxima, local minima or neither, give formal justification in each case (N.B.: the statement “the Hessian at this point is not positive definite and therefore is not a local minimum” is generally incorrect, but “the Hessian at this point is not positive semi-definite and therefore is not a local minimum” is correct); (2 mark)
- iii. use Lagrange multipliers to find and classify all the local extrema on $\partial B(\vec{0}, 10)$ (you may assume that $\partial B(\vec{0}, 10)$ is given by $x^2 + y^2 + z^2 = 100$), justifying why each point is a local maximum/minimum on $\partial B(\vec{0}, 10)$. Hence find all the maxima and minima of f over the closed ball $\overline{B}(\vec{0}, 10)$; (3 marks)
- iv. what is/are the (global) minimum/minima for the function

$$g : \mathbb{R} \times (0, +\infty) \times [0, \frac{\pi}{2}) \rightarrow \mathbb{R}$$

$$g(u, v, w) = u^6 + u^3 + (\log v)^2 - 3 \log v + \tan^2 w + \frac{1}{12} \tan^3 w,$$

justify your answer. (1 mark)

Bonus questions:

- i. Consider a C^∞ function $f : \mathbb{R} \rightarrow \mathbb{R}$ and define

$$g : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad g(x_1, x_2) = f(x_1 + x_2).$$

For small h , one may use the degree k Taylor polynomial of f around 0 to approximate $f(h)$. One may also approximate $f(h)$ using the degree k Taylor polynomial of g around $\vec{0}$ by approximating $g(h_1, h_2)$ for $h = h_1 + h_2$. Prove that there's no difference in the two approximations. (1 mark)

- ii. Repeat questions 3i. and 3ii., but for

$$f(x, y, z) = (x + y - 1)^2 + (x - y + 2)^2 + 2z^2 + \frac{1}{3}z^3. \quad (\frac{1}{2} \text{ mark})$$

Also try 3iii. for this function! ($\frac{1}{2}$ mark)