Assignment 2

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Question 1

i.

$$
\begin{aligned}
\boldsymbol{f}(x,y) = \boldsymbol{f}(1,1) + [(x-1)-(y-1)] + \frac{1}{2!}[(x-1)^2 - 2(x-1)(y-1) + (y-1)^2] + \\
\frac{1}{3!}[(x-1)^3 - 2(x-1)^2(y-1) - 2(x-1)^2(y-1) + (y-1)^3] + o(||(x,y)-(1,1)||^3) \\
= 1 + [x - y + \frac{1}{2}(x-y)^2 + \frac{1}{6}(x-y)^3] + o(||(x,y)-(1,1)||^3)
\end{aligned}
$$

ii.

$$
\begin{aligned}
\boldsymbol{f}(x,y) &= 1 + [x - y + \frac{1}{2}(x-y)^2] + R_k \\
&= 1 + [x - y + \frac{1}{2}(x-y)^2] + \frac{1}{6}(x-y)^3 \boldsymbol{f}(\vec{\xi})
\end{aligned}
$$

iii. Take point$(1, \frac{1}{2})$ into Taylor expansion in **i**:

$$
\begin{aligned}
&\boldsymbol{f}(1, \frac{1}{2}) = 1 + [\frac{1}{2} + \frac{1}{2}(\frac{1}{2})^2 + \frac{1}{6}(\frac{1}{2})^3] + o(||(x,y)-(1,1)||^3) \\
&\Rightarrow \sqrt{\mathrm{e}} \approx \frac{79}{48}
\end{aligned}
$$

iv. Take point $(1, \frac{1}{2})$ into Taylor expansion in **ii**:

$$
\begin{aligned}
&\boldsymbol{f}(x,y) = 1 + [\frac{1}{2} + \frac{1}{2}(\frac{1}{2})^2] + \frac{1}{6}(\frac{1}{2})^3 \boldsymbol{f}(\vec{\xi}) \\
&\Leftrightarrow \sqrt{\mathrm{e}} = \frac{13}{8} + \frac{1}{48}\boldsymbol{f}(\vec{\xi})
\end{aligned}
$$

Since $\vec{\xi}$ is in the open interval between point $(1,1)$ and $(1,\frac{1}{2})$, then $1 \leq \boldsymbol{f}(\vec{\xi}) \leq \sqrt{\mathrm{e}}$.

Therefore,

$$\begin{aligned}
\sqrt{e} &\geq \frac{13}{8} + \frac{1}{48} = \frac{79}{48} \\
\sqrt{e} &\leq \frac{13}{8} + \frac{1}{48}\sqrt{\mathrm{e}} \\
&\leq \frac{78}{47} \\
\Rightarrow \frac{79}{48} &\leq \sqrt{\mathrm{e}} \leq \frac{78}{47}
\end{aligned}$$

Question 2

i. Proof:

$$\nabla F = \begin{bmatrix} -2\mathrm{e}^{2z}x \\ 3y^2 \\ 8 - 2\mathrm{e}^{2z}x^2 \end{bmatrix}$$

Suppose that ∇F would vanish, then $x, y \equiv 0$. Consider $Z(F) = \{(x,y,z) | -x^2\mathrm{e}^{2z} + y^3 + 8z = 0\}$. For a point on $Z(F)$, if $x = y = 0$, then $z = 0$. However, ∇F at point $(0,0,0) = 8$ doesn't vanish. Therefore, ∇F never vanishes on $Z(F)$.

ii. Denote an arbitrary vector $\vec{x} = [x,y,z]^T$, $\vec{p} = [a,b,c]^T \in Z(F)$. At point $\vec{p}$, the normal vector of tangent plane of $Z(F)$ is $\nabla F|_{\vec{p}} = \begin{bmatrix} -2\mathrm{e}^{2c}a \\ 3b^2 \\ 8 - 2\mathrm{e}^{2c}a^2 \end{bmatrix}$.

Therefore, the point-normal form for the tangent plane is

$$(\vec{x} - \vec{p}) \cdot \begin{bmatrix} -2\mathrm{e}^{2c}a \\ 3b^2 \\ 8 - 2\mathrm{e}^{2c}a^2 \end{bmatrix} = 0$$

iii. Take two non-zero vectors which are orthonormal to $\nabla F|_{\vec{p}}$:

$$\vec{u} = \begin{bmatrix} -\frac{1}{2e^{2c}a} \\ \frac{1}{3b^2} \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 0 \\ \frac{1}{3b^2} \\ \frac{1}{8-2e^{2c}a^2} \end{bmatrix}$$

Then the parametric form for the tangent place is

$$\{\vec{x} = \vec{p} + s\vec{u} + t\vec{v} | s, t \in \mathbb{R}\}$$

iv. Proof:

$$[\mathrm{d}\boldsymbol{F}] = \begin{bmatrix} -2e^{2z}x & 3y^2 & 8-2e^{2z}x^2 \end{bmatrix}$$

When $x \neq \pm 2e^{-z}, 2e^{2z}x^2 \neq 0$. According to the implicit function theorem, $\forall \vec{p} \in Z(F), \exists C^1$ map $\boldsymbol{\zeta} : B([a,b]^T, \delta) \to \mathbb{R}$ satisfying $F(x, y, \zeta(x,y)) = 0$.

$$\begin{aligned} \begin{bmatrix} \mathrm{d}\boldsymbol{\zeta} \end{bmatrix} &= -\begin{bmatrix} 8-2e^{2c}a^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -2e^{2c}a & 3b^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{2c}a}{4-e^{2c}a^2} & -\frac{3b^2}{8-2e^{2c}a^2} \end{bmatrix} \end{aligned}$$

v. Proof: Consider $G(y,z,x) = -x^2e^{2z} + y^3 + 8z$, then $F(x,y,z) = G(y,z,x)$

$$[\mathrm{d}\boldsymbol{G}] = \begin{bmatrix} 3y^2 & 8-2e^{2z}x^2 & -2e^{2z}x \end{bmatrix}$$

When $x \neq 0, -2e^{2z}x \neq 0$. According to the implicit function theorem, $\forall \vec{p} \in Z(F), \exists C^1$ map $\boldsymbol{\xi} : B([b,c]^T, \delta) \to \mathbb{R}$ satisfying $G(z, y, \xi(y,z)) = F(\xi(y,z), y, z) = 0$.

$$\begin{aligned} \begin{bmatrix} \mathrm{d}\boldsymbol{\xi} \end{bmatrix} &= -\begin{bmatrix} -2e^{2c}a \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3b^2 & 8-2e^{2c}a \end{bmatrix} \\ &= \begin{bmatrix} \frac{3b^2}{2e^{2c}a} & \frac{4-e^{2c}a^2}{e^{2c}a} \end{bmatrix} \end{aligned}$$

vi. Proof: At point $[0,-2,1]^T$, according to **iv.**,

$$\begin{bmatrix} \mathrm{d}\boldsymbol{\zeta}|_{[0,-2,1]^T} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{2} \end{bmatrix}$$

Thus,

$$\begin{aligned}
\left[\mathrm{d}\boldsymbol{g}\right] &= \begin{bmatrix} 1+\partial_x\boldsymbol{\zeta} & \partial_y\boldsymbol{\zeta} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Its columns are independent, so the matrix is invertible. According to the inverse function theorem, $\exists \boldsymbol{f} : B([1,-2]^T, \varepsilon) \to f(B([1,-2]^T, \varepsilon))$ which is inverse to $\boldsymbol{g}$: $f(B([1,-2]^T, \varepsilon)) \to B([1,-2]^T, \varepsilon)$.

$$\begin{aligned}
\left[\mathrm{d}\boldsymbol{f}\right] &= \left[\mathrm{d}\boldsymbol{g}\right]^{-1} \\
&= \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

1 Question 3

$$f(x,y,z) = (x+y-1)^2 + (x-y+2)^2 + 2z^2 + \frac{1}{6}z^3$$

i.

$$\begin{aligned}
\left[\mathrm{d}\boldsymbol{f}\right] &= \left[2(x+y-1)+2(x-y+2) \quad 2(x+y-1)-2(x-y+2) \quad 4z+\tfrac{1}{2}z^2\right] \\
&= \left[4x+2 \quad 4y-6 \quad 4z+\tfrac{1}{2}z^2\right]
\end{aligned}$$

In $B(\vec{0},10)$, let $[\mathrm{d}\boldsymbol{f}] = \vec{0}$, we get stationary points $\vec{u} = [-\frac{1}{2}, \frac{3}{2}, 0]^T$, $\vec{v} = [-\frac{1}{2}, \frac{3}{2}, -8]^T$.

ii.

$$H_{\boldsymbol{f}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4+z \end{bmatrix}$$

Consider two stationary points $\vec{u}, \vec{v}$:

$$H_{\boldsymbol{f}}(\vec{u}) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I$$

Thus the Hessian at this point is positive definite and therefore is a local minimum.

$$H_{\boldsymbol{f}}(\vec{v}) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

For this diagonal matrix, its eigenvalues are $\lambda_1 = \lambda_2 = 4 > 0, \lambda_3 = -4 < 0$. Thus the Hessian at this point is not positive semi-definite and therefore is not a local extrema.

iii. Denote $\varphi(x,y,z) = x^2 + y^2 + z^2 - 100 = 0$. Consider:

$$\begin{aligned}
F(x,y,z,\lambda) &= f(x,y,z) + \lambda\varphi(x,y,z) \\
&= (x+y-1)^2 + (x-y+2)^2 + 2z^2 + \frac{1}{6}z^3 + \lambda(x^2+y^2+z^2-100) \\
\Rightarrow \partial_x \boldsymbol{F} &= 4x + 2 + 2\lambda x = 0 \\
\partial_y \boldsymbol{F} &= 4y - 6 + 2\lambda y = 0 \\
\partial_z \boldsymbol{F} &= 4z + \frac{1}{2}z^2 + 2\lambda z = 0 \\
\Rightarrow \lambda &= -\frac{2x+1}{x} \\
\lambda &= \frac{3-2y}{y} \\
\lambda &= \frac{4z+\frac{1}{2}z^2}{2z} (z \neq 0) \\
&\Rightarrow \begin{cases} y = -3x \\ z = \dfrac{4}{x} \end{cases}
\end{aligned}$$

Take $y=-3x, z=\frac{4}{x}$ into $x^2+y^2+z^2=100$:

$$\begin{aligned}
x^2+(-3x)^2+(\frac{4}{x})^2 &= 100 \\
\Leftrightarrow 5x^4-50x^2+8 &= 0 \\
\Rightarrow x_1^2 = \frac{25+3\sqrt{65}}{5}, x_2^2 &= \frac{25-3\sqrt{65}}{5} \\
\Rightarrow x_1 = \sqrt{\frac{25+3\sqrt{65}}{5}}, x_2 &= -\sqrt{\frac{25+3\sqrt{65}}{5}} \\
x_3 = \sqrt{\frac{25-3\sqrt{65}}{5}}, x_4 &= -\sqrt{\frac{25-3\sqrt{65}}{5}}
\end{aligned}$$

if $z=0$, take $y=-3x$ into $x^2+y^2+z^2=100$:

$$\begin{aligned}
10x^2 &= 100 \\
\Rightarrow x_5 = \sqrt{10}, x_6 &= -\sqrt{10}
\end{aligned}$$

iv.

$$\boldsymbol{g}: \mathbb{R}\times(0,+\infty)\times[0,\frac{\pi}{2})\to\mathbb{R}$$

$$g(u,v,w) = u^6+u^3+(\log v)^2-3\log v+\tan^2 w+\frac{1}{12}\tan^3 w$$

Obviously, $\boldsymbol{g}$ goes to infinity when u,v,w go to their boundaries. Thus, the minima would only occur at the stationary point.

$$\begin{aligned}
\left[\mathrm{d}\boldsymbol{g}\right] &= \left[6u^5+3u^2 \quad \tfrac{2\log v-3}{v} \quad 2\sec^2 w\tan w+\tfrac{1}{4}\sec^2 w\tan^2 w\right] \\
&\text{Let } [\mathrm{d}\boldsymbol{g}]=0 \\
&\Rightarrow 3u^2(2u^3+1)=0 \\
&\frac{2\log v-3}{v}=0 \\
&2\sec^2 w\tan w+\frac{1}{4}\sec^2 w\tan^2 w=0
\end{aligned}$$

$$\Rightarrow u_1 = 0, u_2 = (-\frac{1}{2})^{\frac{1}{3}}$$

$$v = \mathrm{e}^{\frac{3}{2}}$$

$$w = 0$$

$$g(0, \mathrm{e}^{\frac{3}{2}}, 0) = -\frac{9}{4}$$

$$g((-\frac{1}{2})^{\frac{1}{3}}, \mathrm{e}^{\frac{3}{2}}, 0) = -\frac{9}{4}$$

$$\Rightarrow g(0, \mathrm{e}^{\frac{3}{2}}, 0) = g(0, \mathrm{e}^{\frac{3}{2}}, 0) = -\frac{9}{4}$$

Therefore, there are two minima for this function $[0, \mathrm{e}^{\frac{3}{2}}, 0]^T, [(-\frac{1}{2})^{\frac{1}{3}}, \mathrm{e}^{\frac{3}{2}}, 0]^T$