Assignment 5

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Question 1

i.

$$\sum_{n=1}^{\infty} 3^{-2n+\frac{1}{n}}$$

is converged.

Proof:

$$\begin{aligned}
\lim_{n\to\infty} (3^{-2n+\frac{1}{n}})^{\frac{1}{n}} &= \lim_{n\to\infty} 3^{-2+\frac{1}{n^2}} \\
&= \frac{1}{9} < 1
\end{aligned}$$

According to the root test, $\sum\limits_{n=1}^{\infty} 3^{-2n+\frac{1}{n}}$ is converged.

ii.

$$\sum_{n=2020}^{\infty} \sin(\pi(n^2 - n + \frac{2}{n}))$$

is diverged.

Proof:

$$\begin{aligned}
\sin(\pi(n^2 - n + \frac{2}{n})) &= \sin(n(n-1)\pi + \frac{2}{n}\pi) \\
&= \sin(\frac{2\pi}{n}) \\
\lim_{n\to\infty} \frac{\sin(\frac{2\pi}{n})}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{\frac{2\pi}{n} + o(\frac{1}{n})}{\frac{1}{n}} \\
&= 2\pi + o(1)
\end{aligned}$$

Since $\sum\limits_{n=2020}^{\infty} \frac{1}{n}$ is diverged, $\sum\limits_{n=2020}^{\infty} \sin(\pi(n^2 - n + \frac{2}{n}))$ is diverged as well.

iii.

$$\sum_{n=2020}^{\infty} \frac{\tanh(n)}{n-\cos(n)}$$

is diverged.

Proof:

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{\tanh(n)}{n-\cos(n)}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{\tanh(n)}{1-\frac{\cos(n)}{n}} \\ &= 1\end{aligned}$$

Since $\sum\limits_{n=2020}^{\infty} \frac{1}{n}$ is diverged, $\sum\limits_{n=2020}^{\infty} \frac{\tanh(n)}{n-\cos(n)}$ is diverged as well.

iv.

$$\sum_{n=1}^{\infty} (1-\frac{1}{n})^{n^2}$$

is converged.

Proof:

$$\begin{aligned}((1-\frac{1}{n})^{n^2})^{\frac{1}{n}} &= (1-\frac{1}{n})^n \\ \lim_{n\to\infty}(1-\frac{1}{n})^n &= \lim_{n\to\infty}((1+(-\frac{1}{n}))^{-n})^{-1} \\ &= \frac{1}{\mathrm{e}} < 1\end{aligned}$$

According to the root test, $\sum\limits_{n=1}^{\infty} (1-\frac{1}{n})^{n^2}$ is converged.

v.

$$\sum_{n=0}^{\infty} \frac{n!}{n^n}$$

is converged.

Proof:

$$\begin{aligned}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} &= \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= (\frac{n}{n+1})^n < 1\end{aligned}$$

According to the ratio test, $\sum\limits_{n=0}^{\infty} \frac{n!}{n^n}$ is converged.

vi.

$$\sum_{n=1}^{\infty} \frac{n\cos(n)}{n^2+1}$$

is converged.

Proof: First consider $\sum\limits_{n=1}^{\infty} \cos(n)$:

$$\begin{aligned}
2\cos(1)\sin(\frac{1}{2}) &= \sin(\frac{3}{2}) - \sin(\frac{1}{2}) \\
2\cos(2)\sin(\frac{1}{2}) &= \sin(\frac{5}{2}) - \sin(\frac{3}{2}) \\
&\dots \\
2\cos(n)\sin(\frac{1}{2}) &= \sin(n+\frac{1}{2}) - \sin(n-\frac{1}{2}) \\
\Rightarrow \sum_{n=1}^{\infty}\cos(n) &= \frac{\sin(n+\frac{1}{2}) - \sin(\frac{1}{2})}{2\sin(\frac{1}{2})} \\
&< \frac{1+1}{2\sin(\frac{1}{2})} = \csc(\frac{1}{2})
\end{aligned}$$

Thus $\sum\limits_{n=1}^{\infty} \cos(n)$ is bounded. Then consider sequence $\{\frac{n}{n^2+1}\}(n \geq 1)$ which is monotonic.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{n^2+1} &= \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} \\
&= 0
\end{aligned}$$

Thus $\{\frac{n}{n^2+1}\}(n \geq 1)$ monotonically goes to zero. Therefore, according to Dirichlet test, $\sum\limits_{n=1}^{\infty} \frac{n\cos(n)}{n^2+1}$ is converged.

vii.

$$\sum_{n=2020}^{\infty} \frac{\log(\log(n))}{n(\log(n))^2}$$

is converged.

Proof: Since $\frac{\log(\log(n))}{n(\log(n))^2}$ is positive and monotonically decreasing, according to the integral comparison test, $\sum\limits_{n=2020}^{\infty} \frac{\log(\log(n))}{n(\log(n))^2}$ agrees in convergent/divergent with $\int\limits_{n=2020}^{\infty} \frac{\log(\log(n))}{n(\log(n))^2} \mathrm{d}n$.

$$
\begin{aligned}
\int\limits_{n=2020}^{\infty} \frac{\log(\log(n))}{n(\log(n))^2} \mathrm{d}n &= \int\limits_{n=2020}^{\infty} \frac{\log(\log(n))}{(\log(n))^2} \mathrm{d}(\log(n)) \\
&= \int\limits_{u=\log(2020)}^{\infty} \frac{\log(u)}{u^2} \mathrm{d}u \\
&= \int\limits_{u=\log(2020)}^{\infty} \frac{\log(u)}{u^{\frac{1}{2}}} \cdot \frac{1}{u^{\frac{3}{2}}} \mathrm{d}u
\end{aligned}
$$

First consider $\frac{\log(u)}{u^{\frac{1}{2}}}$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u} \frac{\log(u)}{u^{\frac{1}{2}}} &= \frac{u^{-\frac{1}{2}} - \frac{1}{2}\log(u)u^{-\frac{1}{2}}}{u} \\
&= (1 - \frac{1}{2}\log(u))u^{-\frac{3}{2}} < 0 \\
\lim_{n\to\infty} \frac{\log(u)}{u^{\frac{1}{2}}} &= \lim_{n\to\infty} \frac{\frac{1}{u}}{\frac{1}{2}u^{-\frac{1}{2}}} = 0
\end{aligned}
$$

Thus $\frac{\log(u)}{u^{\frac{1}{2}}}$ monotonically bounded. Since $\int\limits_{u=\log(2020)}^{\infty} \frac{1}{u^{\frac{3}{2}}} \mathrm{d}u$ converges, according to the Abel test, $\int\limits_{n=2020}^{\infty} \frac{\log(\log(n))}{n(\log(n))^2} \mathrm{d}n$ is converged. Therefore, $\sum\limits_{n=2020}^{\infty} \frac{\log(\log(n))}{n(\log(n))^2}$ is converged as well.

Question 2

i.

$$
\sum_{n=2}^{\infty} \frac{n^x}{(\log(n))^n}
$$

Consider the ratio test:

$$
\begin{aligned}
\frac{\frac{(n+1)^x}{(\log(n+1))^{n+1}}}{\frac{n^x}{(\log(n))^n}} &= (\frac{n+1}{n})^x \cdot \frac{(\log(n))^n}{(\log(n+1))^{n+1}} \\
\frac{(\log(n))^n}{(\log(n+1))^{n+1}} &= \exp(n\log(\log(n)) - (n+1)\log(\log(n+1))) \\
&= \exp(n\log\frac{\log(n)}{\log(n+1)} - \log(\log(n+1))) \\
\text{Since } \log(n) < \log(n+1), \ \frac{\log(n)}{\log(n+1)} &< 0
\end{aligned}
$$

$$\lim_{n\to\infty} n\log\frac{\log(n)}{\log(n+1)} - \log(\log(n+1)) = -\infty$$

$$\lim_{n\to\infty} \exp(n\log\frac{\log(n)}{\log(n+1)} - \log(\log(n+1))) = 0$$

Since $(\frac{n+1}{n})^x$ goes to 1 when n goes to ∞, $(\frac{n+1}{n})^x \cdot \frac{(\log(n))^n}{(\log(n+1))^{n+1}}$ goes to 0. Therefore, $\sum\limits_{n=2}^{\infty} \frac{n^x}{(\log(n))^n}$ is converged for all $x \in \mathbb{R}$.

ii.

$$\sum_{n=1}^{\infty} x^{-1-\frac{1}{2}-\cdots-\frac{1}{n}}$$

According to the logarithmic test:

$$
\begin{aligned}
\log(\frac{1}{a_n}) &= \log(x^{1+\frac{1}{2}+\cdots+\frac{1}{n}}) \\
&= (1+\frac{1}{2}+\cdots+\frac{1}{n})\log(x) \\
&= (\log(n)+\gamma)\log(x) \\
L_n &= \frac{\log(\frac{1}{a_n})}{\log(n)} \\
&= \frac{(\log(n)+\gamma)\log(x)}{\log(n)} \\
&= \log(x) + \frac{\gamma}{\log(n)}\log(x) > \log(x)
\end{aligned}
$$

Therefore, if $\sum\limits_{n=1}^{\infty} x^{-1-\frac{1}{2}-\cdots-\frac{1}{n}}$ converges, $\log(x) \geq 1$. Thus the domain of convergence is $x > \mathrm{e}$.