Assignment 1

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Question 1

i. $S = \{x \in \mathbb{R} | x = \sqrt{2} + k, k \in \mathbb{N}\}$

ii. **proof:** For any arbitrary $k \in \mathbb{N}$, we define $S_k := (\sqrt{2} + k, \sqrt{2} + k + 1)$. According to the definition, $S_k \subset \mathbb{R} \setminus S$. For arbitrary $x \in S_k$, let $\delta = min\{x - \sqrt{2} - k, \sqrt{2} + k + 1 - x\}$, $\exists B(x, \delta)$ is an open ball, thus S_k is an open set. Then the complement of S in $\mathbb{R}$ $\overline{S} = \bigcup_{k \in \mathbb{N}} S_k$ is an open set. Therefore, S is an closed set.

iii. **proof:** Since $\mathbb{N}$ is unbounded (according to the Archimedes character), S is an unbounded set.

iv. $B((0,0,4),1)$

v. **proof:** Obviously it is an open set since it is an oepn ball.

vi. $diam(B) = 2$.

proof: For arbitrary points $\vec{p}, \vec{q} \in B(\vec{x}, 1), d(\vec{x}, \vec{p}) < 1, d(\vec{x}, \vec{q}) < 1$, thus $d(\vec{x}, \vec{y}) < 2$ according to the triangular inequality. Now consider two point sequences:

$$A_k = \{(0,0,3 + \frac{1}{2^k}) | k \in \mathbb{N}\}, B_k = \{(0,0,5 - \frac{1}{2^k}) | k \in \mathbb{N}\}$$

Obviously $A_k, B_k \subset B(\vec{x}, 1)$. Then we can tell

$$\lim_{k\to\infty} d(A_k, B_k) = 2$$

Which means $\forall \varepsilon > 0, \exists N > 0$, when $k > N, d(A_k, B_k) > 2 - \varepsilon$. Then we can tell $sup(\vec{p}, \vec{q}) = 2$, which means $diam(B) = 2$.

Question 2

i.

$$\lim_{(x,y)\to(3.5)} (\frac{\sin(y-x)}{y-x}, \sqrt{y^2-x^2})$$

exists since when $(x, y) \to (3.5), y - x$ is nonzero. The limit is equal to $(\frac{\sin 2}{2}, 4)$

ii.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^3}{x^3-y^4}$$

doesn't exist.

proof: Replace y with $kx(k \in \mathbb{R})$, then the original formula is equal to

$$\lim_{x\to 0} \frac{x^4-k^3x^3}{x^3-k^4x^4}$$

=

$$\lim_{x\to 0} \frac{x-k^3}{1-k^4x}$$

= $-k^3$ which is uncertain. Therefore, the limit doesn't exist.

iii.

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \frac{x^3-y^3}{x^4-y^4}$$

exists.

proof: Replace y with $kx(k \in \mathbb{R})$, then the original formula is equal to

$$\lim_{x \to \infty} \frac{x^3 - k^3x^3}{x^4 - k^4x^4}$$

=

$$\lim_{x \to \infty} \frac{1 - k^3}{(1 - k^4)x}$$

= 0

Therefore, the limit is equal to 0.