Calculus Note

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1 Euclidean n-space

1.1 Concepts:

$$\mathbb{R}^n := \{(x_1,...,x_n)|x_1,...,x_n \in \mathbb{R}\}$$

$$A \times B := (a \in A, b \in B), \mathbb{R}^n := \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}$$

$$||\cdot|| := \mathbb{R}^n \to \mathbb{R} \text{ (Euclidean n-norm)}$$

$$||\vec{x}|| := \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

$$d := \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \text{ (Distance)}$$

$$d(\vec{x},\vec{y}) := ||\vec{x}-\vec{y}|| = \sqrt{(x_1-y_1)^2 + ... + (x_n-y_n)^2}$$

$$\text{Properties of distance:}$$

$$\text{Symmetry:} \forall \vec{x},\vec{y} \in \mathbb{R}^n, d(\vec{x},\vec{y}) = d(\vec{y},\vec{x})$$

$$\text{Triangle inequality:} \forall \vec{x},\vec{y},\vec{z} \in \mathbb{R}^n, d(\vec{x},\vec{y}) + d(\vec{y},\vec{z}) \geq d(\vec{x},\vec{z})$$

$$\text{Positivity:} \forall \vec{x},\vec{y} \in \mathbb{R}^n, d(\vec{x},\vec{y}) \geq 0$$

$$\Rightarrow \forall \vec{x},\vec{y} \in \mathbb{R}^n, d(\vec{x},\vec{y}) = 0 \Leftrightarrow \vec{x} = \vec{y}$$

1.2 Open n-ball

1.2.1 Definition:

For $\delta > 0, B(\vec{x},\delta) := \{\vec{y} \in \mathbb{R}^n | d(\vec{x},\vec{y}) < \delta\}$

1.2.2 Interior points:

$\vec{x} \in \mathbb{R}^n$ is an interior point of S iff. $\exists \delta > 0 s.t. B(\vec{x}, \delta) \subset S$

1.2.3 Exterior points:

$\vec{x} \in \mathbb{R}^n$ is an exterior point of S iff. $\exists \delta > 0 s.t. B(\vec{x}, \delta) \subset \mathbb{R}^n \setminus S$

1.2.4 Boundary points:

$\vec{x} \in \mathbb{R}^n$ is an boundary point of S iff. $\forall \delta > 0, B(\vec{x}, \delta) \cap S \neq \varnothing$ and$B(\vec{x}, \delta) \cap (\mathbb{R}^n \setminus S) \neq \varnothing$

1.2.5 Adherent points:

$\vec{x} \in \mathbb{R}^n$ is an adherent point of S iff. $\forall \delta > 0, B(\vec{x}, \delta) \cap S \neq \varnothing$

1.2.6 Limit points:

$\vec{x} \in \mathbb{R}^n$ is an limit point of S iff. $\forall \delta > 0, (B(\vec{x}, \delta) \setminus \{\vec{x}\}) \cap S \neq \varnothing$

1.3 Type of sets

1.3.1 Open sets:

A set S is open iff. every point of S is an interior point.

1.3.2 Closed sets:

A set S is closed iff. $\mathbb{R}^n \setminus S$ is open.

1.3.3 Interior of a set:

All interior points of S.

1.3.4 Exterior of a set:

All exterior points of S.

1.3.5 Boundary of a set

All boundary points of S $\to \partial S$

1.3.6 Closure of a set:

$\bar{S}$ of S is $S \cup \partial S$

1.4 Theorem 1.1

A set $S \subset \mathbb{R}^n$ is open iff. it is equal to the union of a collection of open balls.

Proof: 1. S = union of open balls $\Rightarrow S =$ open

Consider arbitrary $\vec{x} \in S$, then $\exists$ an open ball $B(\vec{y}, \delta_y) \subset S$ which contains $\vec{x}$. Let $\delta_{\vec{x}} := \delta_{\vec{y}} - d(\vec{x}, \vec{y})$, then $\delta_{\vec{x}} > 0$ and $B(\vec{x}, \delta - \vec{x}) \subset B(\vec{y}, \delta_{\vec{y}}) \subset S$

$\Rightarrow \vec{x}$ is an interior point of S.

Since $\vec{x}$ was arbitrary chosen, every point of S is an interior point. Therefore, S is open.

2. $S =$ open $\Rightarrow S =$ union of open balls

Assume S is open, $\Rightarrow \forall \vec{x} \in S, \exists \delta_{\vec{x}} > 0 s.t. B(\vec{x}, \delta_{\vec{x}}) \subset S$

$$U := \bigcup_{\vec{x} \in \mathrm{Int}(S)} B(\vec{x}, \delta_{\vec{x}}) \subset S$$

Conversely, $S \subset U \Rightarrow S = U$ Therefore, S is the union of open balls.

1.5 Theorem 1.2

$\mathrm{Int}(S)$ of any set $S \subset \mathbb{R}^n$ is an open set. Likewise, $\mathrm{Ext}(S)$ of S is also open.

Proof: Every $\vec{x} \in \mathrm{Int}(S)$ is an interior point of S

$\Rightarrow \exists \delta_{\vec{x}} > 0, s.t. B(\vec{x}, \delta_{\vec{x}}) \subset S$

$\Rightarrow U := \bigcup_{\vec{x} \in \mathrm{Int}(S)} B(\vec{x}, \delta_{\vec{x}})$ is open.

Thus, $\mathrm{Int}(S) \subset U$, because every $\vec{x} \in \mathrm{Int}(S)$ is contained in some open ball $B(\vec{x}, \delta_{\vec{x}}) \subset U$. For every $\vec{x} \in \mathrm{Int}(S), B(\vec{x}, \delta_{\vec{x}}) \subset \mathrm{Int}(S)$.

Since $B(\vec{x}, \delta_{\vec{x}})$ is open, every point in $B(\vec{x}, \delta_{\vec{x}})$ is an interior point of $B(\vec{x}, \delta_{\vec{x}}) \Rightarrow$ is an interior point of $S \supset B(\vec{x}, \delta_{\vec{x}})$.

1.6 Proposition 1.3

The union of any collection of (arbitrarily many) open sets is open.

1.7 Proposition 1.4

Finite intersections of open sets is open.

1.8 Theorem 1.5

Finite unions of closed sets is closed, and intersections of closed sets is closed.

2 Further properties

2.1 Proposition 2.1

Finite intersections of open sets is open.

2.2 Theorem 2.2

Finite unions of closed set is closed, and intersections of closed set is closed.

Proof:

$$\mathbb{R}^n \setminus (\bigcup_{\alpha \in \mathscr{A}} S_\alpha) = \bigcup_{\alpha \in \mathscr{A}} (\mathbb{R}^n \setminus S_\alpha)$$

$$\Rightarrow \vec{x} \in \mathbb{R}^n \setminus (\bigcup_{\alpha \in \mathscr{A}} S_\alpha)$$

$$\Leftrightarrow \vec{x} \notin \bigcup_{\alpha \in \mathscr{A}} S_\alpha$$

$$\Leftrightarrow \forall \alpha \in \mathscr{A}, \vec{x} \notin S_\alpha$$

$$\Leftrightarrow \forall \alpha \in \mathscr{A}, \vec{x} \in \bigcap_{\alpha \in \mathscr{A}} (\mathbb{R}^N \setminus S_\alpha)$$

2.3 Connectedness

2.3.1 Path-connectedness:

$\forall \vec{x}, \vec{y} \in S, \exists$ a continuous function $f : [0,1] \to S, s.t. f(0) = \vec{x}$ and $f(1) = \vec{y}$

2.3.2 Connectedness:

$S \subset \mathbb{R}, s.t. \exists$ open sets $U, V \subset \mathbb{R}^n, s.t. U \cap V = \varnothing, U, V \neq \varnothing$
$S \subset U \cup V$ and $S \cap V$ and $S \cup V$ are both non-empty.

2.4 Convergence

2.4.1 Definition

Let $\{\vec{x}_k\}_{k \in \mathbb{N}} = \vec{x}_1, \vec{x}_2, \ldots$ be a sequence in $\mathbb{R}^n$, we say that this sequence converges to $\vec{a} \in \mathbb{R}^n$ iff. $\forall \varepsilon > 0, \exists N \in \mathbb{N}, s.t. \forall k > N, ||\vec{x}_k - \vec{a}|| < \varepsilon$
$\Rightarrow \lim\limits_{k \to \infty} \vec{x}_k = \vec{a} \Leftrightarrow d(\vec{x}_k, \vec{a}) = 0$

2.4.2 Cauchy convergent sequences

A sequence $\{\vec{x}_k\}$ in $\mathbb{R}^n$ is called Cauchy convergent iff. $\forall \varepsilon > 0, \exists N, s.t. \forall j, k > N, d(\vec{x}, \vec{k}) < \varepsilon$

2.5 Theorem 2.4

Let $\{\vec{x}_k\}$ be a sequence in $\mathbb{R}^n$, we adopt the following notion:

$$\vec{x}_k = (x_{k,1}, x_{k,2}, \ldots, x_{k_n})$$

Then $\lim\limits_{k \to \infty} \vec{x}_k = \vec{a} \Leftrightarrow \forall j = 1, \ldots, n, \lim\limits_{k \to \infty} x_{k,j} = a_j$ **Similar to Cauchy convergence**
N.B. In $\mathbb{R}^n$, Cauchy convergent **doesn't mean** convergent in S (∂S).

2.6 Lemma 2.5

$S \subset \mathbb{R}^n, \vec{a} \in \mathbb{R}^n, \vec{a}$ is adherent point of $S \Leftrightarrow \{\vec{x}_k\}$ in S converge to $\vec{a}$

2.7 Lemma 2.6

$\vec{a}$ is a limit point $\Leftrightarrow \{\vec{x_k}\}$ in $S \setminus \{\vec{a}\}$ converges to $\vec{a}$

2.8 Theorem 2.7

$S \subset \mathbb{R}^n$ is closed $\Leftrightarrow S$ contains all of its adherent points.

2.9 Diameter of a set

2.9.1 Definition

$diam(S) := \sup\limits_{\vec{x},\vec{y} \in S} d(\vec{x},\vec{y}) \to R \in (0,+\infty) \cup \{+\infty\}$

2.9.2 Bounded sets and sequences

$diam(S) < \infty$

2.10 Cantor's intersection theorem

$\{F_k\}_{k\in\mathbb{N}}$ of non-empty, closed *(without this is not true)* subsets of $\mathbb{R}^n$ nested in the following way:

$$\mathbb{R}^n \supset F_1 \supset F_2 \supset ... \supset F_k \supset ...$$

If $\lim\limits_{k\to\infty} diam(F_k) = 0$, then the $\bigcup\limits_{k=1}^{\infty} F_k$ is a set consisting a single element.

3 Vector function

3.1 Function

f is a subset of $A \times B$:

$$\forall a \in A, \exists! b \in B, s.t. (a,b) \in f \Leftrightarrow f(a) = b$$

3.2 Scalar and vector valued functions

3.2.1 Definition

$f : \Omega_1 \to \Omega_2$ between a set $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ is called an **n-variable, m-dimensioned vector valued** function. Especially when $m = 1$, it is a scalar function.

3.2.2 Algebraic constructions

$$\begin{aligned}
&\text{For } \Omega \subset \mathbb{R}^n, \mu : \Omega \to \mathbb{R}, \boldsymbol{f}, \boldsymbol{g} : \Omega \to \mathbb{R}^n : \\
&\mu \boldsymbol{f} : \Omega \to \mathbb{R}^n, \vec{x} \mapsto \mu(\vec{x})\boldsymbol{f}(\vec{x}) \\
&\boldsymbol{f} + \boldsymbol{g} : \Omega \to \mathbb{R}^n, \vec{x} \mapsto \boldsymbol{f}(\vec{x}) + \boldsymbol{g}(\vec{x})
\end{aligned}$$

3.2.3 Composition

$$\begin{aligned}
&\boldsymbol{f} : \Omega_1 \subset \mathbb{R}^n \to \Omega_2 \subset \mathbb{R}^m, \boldsymbol{g} : \Omega_2 \to \mathbb{R}^l := \\
&\boldsymbol{g} \circ \boldsymbol{f} : \Omega_1 \to \mathbb{R}^l, \vec{x} \mapsto \boldsymbol{g}(\boldsymbol{f}(\vec{x}))
\end{aligned}$$

Practically requires $\boldsymbol{f}(\Omega_1) = \boldsymbol{f}(\vec{x})|\vec{x} \in \Omega_1$ is a subset of the domain of $\boldsymbol{g}$

3.3 Taking limits

Let $\Omega \subset \mathbb{R}^n$ be non-empty. Given a function $\boldsymbol{f} : \Omega \to \mathbb{R}^n$, a limit point $\vec{x_0}$ of Ω and $\vec{a} \in \mathbb{R}^m$. If

$$\begin{aligned}
&\forall \varepsilon > 0, \exists \delta > 0, s.t. \forall \vec{x} \in \Omega \\
&0 < ||\vec{x} - \vec{x_0}||_n < \delta \Rightarrow ||\boldsymbol{f}(\vec{x}) - \vec{a}||_m < \varepsilon \\
&\Leftrightarrow \vec{x} \in B(\vec{x_0}, \delta) \setminus \{\vec{x_0}\} \Rightarrow \boldsymbol{f}(\vec{x}) \in B(\vec{a}, \varepsilon) \Rightarrow \vec{x} \text{ limits to } \vec{x_0} \text{ in } \Omega . \boldsymbol{f}(\vec{x}) \text{ limits to } \vec{a} : \\
&\lim_{\Omega \ni \vec{x} \to \vec{x_0}} \boldsymbol{f}(\vec{x}) = \vec{a}
\end{aligned}$$

When limit point $\vec{x_0}$ is an interior point of $\Omega \cup \{\vec{x_0}\}$, then we can denote

$$\lim_{\vec{x} \to \vec{x_0}} \boldsymbol{f}(\vec{x})$$

Coordinate-wise denotation:

$$\lim_{\Omega\ni\vec{x}\to\vec{x_0}} \boldsymbol{f}(\vec{x}) = \vec{a} \Leftrightarrow \text{For } j = 1, ..., m, \lim_{\Omega\ni\vec{x}\to\vec{x_0}} f_j(\vec{x}) = a_j, \boldsymbol{f}(\vec{x}) = (f_1\vec{x}, ..., f_n(\vec{x}))$$

3.4 Scalar function properties

3.4.1 Inequality

If $\forall \vec{x} \in \Omega \setminus \{\vec{x_0}\}, f(\vec{x}) \le g(\vec{x})$, then

$$\lim_{\Omega\ni\vec{x}\to\vec{x_0}} f(\vec{x}) \le \lim_{\Omega\ni\vec{x}\to\vec{x_0}} g(\vec{x})$$

3.4.2 Non-negativity and Sandwich Theorem

3.4.3 Calculation

If $\lim_{\Omega\ni\vec{x}\to\vec{x_0}} \mu(\vec{x}), \boldsymbol{f}(\vec{x}), \boldsymbol{g}(\vec{x})$ all exist, then

$$\begin{aligned}
\lim_{\Omega\ni\vec{x}\to\vec{x_0}} (\mu\boldsymbol{f})(\vec{x}) &= \lim_{\Omega\ni\vec{x}\to\vec{x_0}} \mu(\vec{x}) \cdot \boldsymbol{f}(\vec{x}) \\
\lim_{\Omega\ni\vec{x}\to\vec{x_0}} (\boldsymbol{f}+\boldsymbol{g})(\vec{x}) &= \lim_{\Omega\ni\vec{x}\to\vec{x_0}} \boldsymbol{f}(\vec{x}) + \lim_{\Omega\ni\vec{x}\to\vec{x_0}} \boldsymbol{g}\vec{x} \\
\lim_{\Omega\ni\vec{x}\to\vec{x_0}} \frac{1}{\mu(\vec{x})} &= \frac{1}{\lim_{\Omega\ni\vec{x}\to\vec{x_0}} \mu(\vec{x})} (\mu \neq 0)
\end{aligned}$$

3.5 Uniqueness

If a limit exists, then it must be unique.

Proof: Given $\lim_{\Omega\ni\vec{x}\to\vec{x_0}} \boldsymbol{f}(\vec{x}) = \vec{a}, \vec{b}$,

$\Rightarrow$ A sequence $\{\vec{x_k}\}_{k\in\mathbb{N}}$ has its tail $\to \vec{a}, \vec{b}$

$\Rightarrow \vec{a}, \vec{b}$ are arbitrary close *(via triangle inequality)*

$\Rightarrow \vec{a} = \vec{b}$

3.6 Iterated limits / Repeated limits

Taking limits of a function one variable at a time.

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)$$

1. Compute $\lim_{y\to y_0} f(x,y)$ for all x in a small punctured ball $B(x_0,\delta)\setminus\{x_0\}$ around x_0
2. This produces a function $\varphi(x) := \lim_{x\to x_0} f(x,y)$ on $B(x_0,\delta)\setminus\{x_0\} \Rightarrow \lim_{x\to x_0}\varphi(x)$

$$\Rightarrow \lim_{x\to x_0}\lim_{y\to y_0} f(x,y) = \lim_{x\to x_0}\varphi(x) = \lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

If

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y) \neq \lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$$

then $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ doesn't exist.

Conclusion: Iterated limits $\not\Leftrightarrow$ double limits

4 Continuity

4.1 Continuity

4.1.1 Definition

$\boldsymbol{f}:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m, \vec{x_0}\in\Omega, \boldsymbol{f}$ is continuous at $\vec{x_0} \Leftrightarrow$

$$\forall\varepsilon>0, \exists\delta>0, s.t. \forall\vec{x}\in\Omega, ||\vec{x}-\vec{x_0}||<\delta \Rightarrow ||\boldsymbol{f}(\vec{x})-\boldsymbol{f}(\vec{x_0})||<\varepsilon$$

$$(\vec{x}\in B(\vec{x_0},\delta)\cap\Omega \Rightarrow \boldsymbol{f}(\vec{x})\in B(\boldsymbol{f}(\vec{x_0}),\varepsilon))$$

If $\vec{x_0}\in\Omega$ is isolated, then $\boldsymbol{f}$ is always continuous at $\vec{x_0}$.

if $\vec{x_0}\in\Omega$ is a limit point ,then Continuity at $\vec{x_0}$ is equal to the following limit being true: $\lim_{\Omega\ni\vec{x}\to\vec{x_0}} \boldsymbol{f}(\vec{x}) = \boldsymbol{f}(\vec{x_0})$

4.1.2 Continuous function

$\boldsymbol{f} : \Omega \to \mathbb{R}^m$ is continuous at every $\vec{x} \in \Omega := C^0$

Scalar function: $C^0(\Omega) := C^0(\Omega; \mathbb{R})$

$\boldsymbol{f}(\vec{x})$ is continuous $\Leftrightarrow f_1(\vec{x}), f_2(\vec{x}), ..., f_m(\vec{x})$ are continuous.

4.2 Theorem 4.1

Given continuous functions $\boldsymbol{f} : \Omega_1 \subset \mathbb{R}^n \to \mathbb{R}^m, \boldsymbol{g} : \Omega_2 \subset \mathbb{R}^m \to \mathbb{R}^l$ such that $\boldsymbol{f}(\Omega_1) \subset \Omega_2$, then the function $\boldsymbol{g} \circ \boldsymbol{f} : \Omega_1 \to \mathbb{R}^l$ is continuous.

4.3 Theorem 4.2

Given a function $\boldsymbol{f} : \Omega \to \mathbb{R}^m$, where Ω is an open set. Then $\boldsymbol{f}$ is continuous iff. for every open set $U \subset \mathbb{R}^m$, the preimage set: $\boldsymbol{f}^{-1}(U) := \{\vec{x} \in \Omega | \boldsymbol{f}(\vec{x} \in U)\}$ is open.

If Ω isn't open, then $\boldsymbol{f}$ is open iff. for all open sets $U \subset \mathbb{R}^m$, the preimage set $\boldsymbol{f}^{-1}(U)$ is the intersection of Ω with an open set in $\mathbb{R}^n \supset \Omega$

4.4 Extreme value theorem

Given closed, bounded set $\Omega \subset \mathbb{R}^n$, continuous scalar function $f \in C^0(\Omega)$, then f attains its maximum/minimum on Ω:

$\Rightarrow$ There exists $m, M > 0$ s.t. $\forall \vec{x} \in \mathbb{R}^n$,

$$m \sum_{j=1}^{n} |x_j| \leq ||\vec{x}|| \leq M \sum_{j=1}^{n} |x_j|$$

4.5 Differentiability

$\boldsymbol{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at an interior point $\vec{x} \in \Omega$ iff. there exists $\mathbb{L} : \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$\lim_{\vec{h} \to 0} \frac{||\boldsymbol{f}(\vec{x} + \vec{h}) - \boldsymbol{f}(\vec{x}) - \mathbb{L}(\vec{h})||_m}{||\vec{h}||_n} = 0$$

We denote $\mathbb{L}$ by $\mathrm{d}\boldsymbol{f}|_{\vec{x}}$ or $\mathrm{d}\boldsymbol{f}(\vec{x})$, it is the derivative of $\boldsymbol{f}$ at $\vec{x}$. When Ω is open, $\boldsymbol{f}$ is differentiable iff. $\boldsymbol{f}$ is differentiable of every $\vec{x} \in \Omega$.

Differentiability implies continuity

4.6 Theorem 4.3

If $\boldsymbol{f} : \Omega \to \mathbb{R}^m$ is differentiable at $\vec{x}$, then the derivative $\mathrm{d}\boldsymbol{f}|_{\vec{x}}$ at $\vec{x}$ is unique.

4.7 Partially differentiate functions

$$\frac{\partial f_i}{\partial x_j}(\vec{t}) := \frac{\partial f_i}{\partial x_j}|_{\vec{t}} := \lim_{h\to 0} \frac{f_i(t_1, ..., t_{j-1}, t_j + h, t_{j+1}, ..., t_n) - f_i(\vec{t})}{h}$$

4.7.1 Matrix expression for the total derivative

$$\left[\mathrm{d}\boldsymbol{f}|_{\vec{t}}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}|_{\vec{t}} & \cdots & \frac{\partial f_1}{\partial x_n}|_{\vec{t}} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}|_{\vec{t}} & \cdots & \frac{\partial f_m}{\partial x_n}|_{\vec{t}} \end{bmatrix}$$

Jacobian of $\boldsymbol{f}$ of $\vec{t}$

4.7.2 Little o notation

Given scalar functions $f, g : \Omega \to \mathbb{R}$, if

$$\lim_{\Omega \ni \vec{x} \to \vec{x}_0} \frac{|\boldsymbol{f}(\vec{x})|}{|\boldsymbol{g}(\vec{x})|} = 0$$

We can write:

$$||\boldsymbol{g}(\vec{h})|| = o(||\vec{h}||)(\vec{h} \to \vec{0})$$

4.7.3 Big O notation

Given scalar functions $f, g : \Omega \to \mathbb{R}$, $\exists C, \delta > 0$,

$$\forall \vec{x} \in B(\vec{x}_0, \delta) \setminus \{\vec{x}_0\} \cap \Omega, |\boldsymbol{f}(\vec{x})| \le C|\boldsymbol{g}(\vec{x})|$$

We can write:

$$\boldsymbol{f}(\vec{x}) = O(\boldsymbol{g}(\vec{x}))$$

5 Differentiability

5.1 Derivatives as linear functions

$\mathrm{d}\boldsymbol{f}|_{\vec{x}} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function:

For scalar functions:

$$\mathrm{d}f|_{\vec{t}} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}_{x_i}, \mathbb{R}^n \to \mathbb{R}^m$$

For vector functions:

$\boldsymbol{f} = (f_1, ..., f_m), \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$

$$\mathrm{d}\boldsymbol{f}|_{\vec{t}} = (\mathrm{d}f_1|_{\vec{t}}, ..., \mathrm{d}f_m|_{\vec{t}})$$

5.2 Chain Rule

Given $\boldsymbol{f} : \Omega_1 \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{g} : \Omega_2 \subset \mathbb{R}^m \to \mathbb{R}^l$ such that:

$\boldsymbol{f}(\Omega_1) \subset \Omega_2$, and $\boldsymbol{f}$ is differentiable at $\vec{t} \in \Omega_1$ and $\boldsymbol{g}$ is differentiable at $\boldsymbol{f}(\vec{t}) \in \Omega_2$, then

$$\mathrm{d}(\boldsymbol{g} \circ \boldsymbol{f})|_{\vec{t}} = (\mathrm{d}_{\boldsymbol{g}}|_{\boldsymbol{f}(\vec{t})}) \circ (\mathrm{d}\boldsymbol{f}|_{\vec{t}}) : \mathbb{R}^n \to \mathbb{R}^l$$

$$[\mathrm{d}(\boldsymbol{f} \circ \boldsymbol{g})|_{\vec{t}}] = [\mathrm{d}_{\boldsymbol{g}}|_{\boldsymbol{f}(\vec{t})}] \cdot [\mathrm{d}\boldsymbol{f}|_{\vec{t}}]$$

5.2.1 Notes

Continuous and partially differentiable $\not\Rightarrow$ differentiable

5.3 continuously differentiable functions

A function $\boldsymbol{f} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, where Ω is open, is called *continuously differentiable* at $\vec{x} \in \Omega$ iff.

$\boldsymbol{f}$ is partially differentiable at $\vec{x}$ and all of its partial Derivatives are continuous at $\vec{x}$.

If $\boldsymbol{f}$ is continuously differentiable at every point in Ω, we say that $\boldsymbol{f}$ is a *continuously differentiable function*, and adopt the notation $\boldsymbol{f} \in C^1(\Omega; \mathbb{R}^n)$