Theory of Complex Systems: Assignment

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1 Modelling the activity of a single neuron

(a) Can you plot the distribution $P(\tau)$ of the time intervals τ between successive spikes? Check that there is indeed a rafactory period, i.e., a time interval τ_0 after each spike, during which the neuron doesn't spike again. What is the duration τ_0 for this time interval?

(b) Can you check that the decay of the distribution $P(\tau)$ of inter-spike intervals is indeed exponential? Measure the corresponding decay rate λ

(c) Can you deduce an analytical expression for the distribution of inter-spike time interval $P(\tau)$ of the delayed Poisson process as a function of λ and τ_0? Compare your model distribution to the one obtained from the data.

(d) Using your model, can you generate another 1000 (spike times) datapoints?

(e) What is the average spiking rate f of the neuron in the data? How is f analytically related to τ_0 and λ that you have previously measured?

2 Modelling binary data with the Ising model

2.A Pairwise spin model

(a) How many terms are in the sum over the $\text{pair}(i, j)$? Can you deduce what is the number of parameters in the vector $g = (h_1, ..., h_n, J_{1,2}, ..., J_n$? Can you re-write the sum over the $\text{pair}(i, j)$ as a double sum over i and j (without counting twice each pair)?

(b) Can you write down explicitly the terms in the exponential of Eq. (1) for a system with $n = 3$ spins?

(c) In Eq. (1), we can recognize the Boltzmann distribution, in which the parameter $\beta = 1/(k_b T)$ was taken equal to 1 (more precisely, the constant k_B was taken equal to 1, and the temperature parameter T was absorbed in the parameters h_i and J_{ij}). What is the energy function associated with the Boltzmann distribution in that case? What is the partition function and what is its general expression?

(d) Take a spin s_i: if h_i is positive, which direction will s_i tend to turn to, i.e., which direction of s_i will minimize the associated energy $-h_i s_i$? Take a pair of spins s_i and s_j: if J_{ij} is positive, which configurations of (s_i, s_j) minimize the coupling energy $-J_{ij} s_i s_j$?

Assume that we have inferred the best parameters h_i and J_{ij} for the US supreme court dataset discussed in section 2. How would you interpret the sign of the inferred parameters h_i and J_{ij} in this context?

2.B Observables

(a) Given a stationary probability distribution of the state $p_{g(\boldsymbol{s})}$, what are the definitions of $\langle s_i \rangle$ and of $\langle s_i s_j \rangle$?

(b) Consider a dataset $\hat{\boldsymbol{s}}$ composed of N independent observations of the spins: $\hat{\boldsymbol{s}} = \left(\boldsymbol{s}^{(1)}, ..., \boldsymbol{s}^{(N)}\right)$. Let us denote by $\langle s_i \rangle_D$ and $\langle s_i s_j \rangle_D$ the empirical averages of s_i and of $s_i s_j$ respectively (i.e., their average values in the dataset). How would you compute $\langle s_i \rangle_D$ and $\langle s_i s_j \rangle_D$ from the data?

(c) Assume that the data is stationary and that eah datapoint has been randomly sampled from $p(\boldsymbol{s})$. Can you show that the empirical averages, $\langle s_i \rangle_D$ and $\langle s_i s_j \rangle_D$, converge to the model averages, respectively $\langle s_i \rangle$ and $\langle s_i s_j \rangle$, as the number N of datapoints goes to infinity? (very large dataset)

2.C Maximum Entropy models

(a) Consider a spin system with stationary probability distribution $p(\boldsymbol{s})$. Can you recall the definition of the Shannon entropy $S[p(\boldsymbol{s})]$? As mentioned above for the Boltzmann distribution, we will take $k_b = 1$.

The Ising model in Eq. (1) can be seen as a *Maximum Entropy Model*, constrained to reproduce the data local magnetisation and local correlation, i.e., constrained to reproduce all the data averages $\langle s_i \rangle_D$ and $\langle s_i s_j \rangle_D$ (for all spins s_i and s_j). We also want $p(\boldsymbol{s})$ to be normalised, which introduces the additional constraint $\sum_{\boldsymbol{s}} p(\boldsymbol{s}) = 1$. To summarise, we are looking for the set of 2^n probabilities $p(\boldsymbol{s})$ such that $S[p(\boldsymbol{s})]$ is maximal, and such that

$$\sum_{\boldsymbol{s}} p(\boldsymbol{s}) = 1 \quad \text{and} \quad \sum_{\boldsymbol{s}} p(\boldsymbol{s}) s_i(\boldsymbol{s}) = \langle s_i \rangle_D \quad \text{and} \quad \sum_{\boldsymbol{s}} p(\boldsymbol{s}) s_i(\boldsymbol{s}) s_j(\boldsymbol{s}) = \langle s_i s_j \rangle_D \tag{1}$$

where $\langle s_i \rangle_D$ and $\langle s_i s_j \rangle_D$ are constants that are computed from the data for all distinct s_i and s_j. Note that to be more precise, we wrote $s_i(\boldsymbol{s})$ (instead of just s_i) to specify that this is the value of s_i in the state $\boldsymbol{s}$ (this will help with the next questions).

(b) How many constraints are there in total?

To find the shape of the distributions $p(\boldsymbol{s})$ that maximises the entropy while satisfying these constraints, we introduce an auxiliary function:

$$\begin{aligned} U[p(\boldsymbol{s})] = S[p(\boldsymbol{s})] + \lambda_0 \left(\sum_s p(\boldsymbol{s}) - 1 \right) + \sum_{i=1}^{n} \alpha_i \left(\sum_{\boldsymbol{s}} p(\boldsymbol{s}) s_i(\boldsymbol{s}) - \langle s_i \rangle_D \right) \\ + \sum_{\text{pair}(i,j)}^{n} \eta_{ij} \left(\sum_{\boldsymbol{s}} p(\boldsymbol{s}) s_i(\boldsymbol{s}) s_j(\boldsymbol{s}) - \langle s_i s_j \rangle_D \right) \end{aligned} \tag{2}$$

where we have introduced a parameter in front of each constraint we want to impose. These parameters (λ_0, α_i, and η_{ij}) are called Lagrange multipliers. To find $p(\boldsymbol{s})$ one must maximise this auxiliary function with respect to the 2^n probabilities $p(\boldsymbol{s})$.

(c) Let us fix a choice of a state $\boldsymbol{s}$. The probability $p_{\boldsymbol{s}} = p(\boldsymbol{s})$ is a parameter of $U[\boldsymbol{p}]$ where $\boldsymbol{p}$ is the vector of the 2^n probabilities. Can you show that:

$$\frac{\partial U[\boldsymbol{p}]}{\partial p_s} = -\log(p_s) - 1 + \lambda_0 + \sum_{i=1}^{n} \alpha_i s_i(\boldsymbol{s}) + \sum_{\text{pair}(i,j)} \eta_{ij} s_i(\boldsymbol{s}) s_j(\boldsymbol{s}) \tag{3}$$

(d) Can you show that the most general expression of $p_{\boldsymbol{s}}$ with maximal entropy that satisfying the constraints in Eq. (2) is Eq. (1)? Give the relation between λ_0 and the partition function Z. How are the parameters α_i or η_{ij} related to the parameters h_i and J_{ij}?

2.D Statistical inference: model with no couplings

(a)

(b)

(c)

2.E Statistical inference: maximising the log-likelihood function

(a)

(b)

3 Application to the analysis of the US supreme Court

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

(i)