Theory of Complex Systems: Assignment

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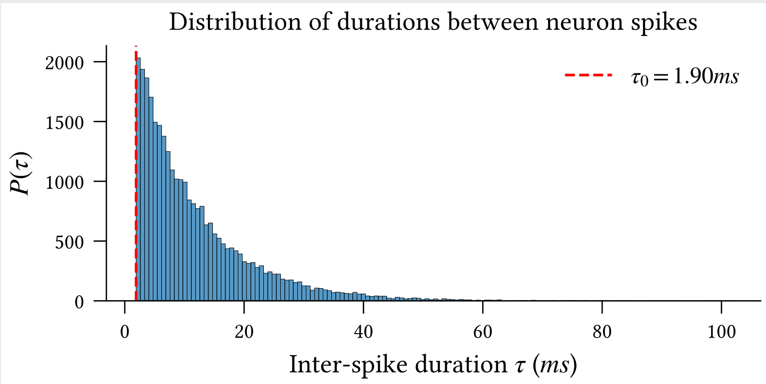
1 Modelling the activity of a single neuron

(a) Can you plot the distribution $P(\tau)$ of the time intervals τ between successive spikes? Check that there is indeed a rafactory period, i.e., a time interval τ_0 after each spike, during which the neuron doesn't spike again. What is the duration τ_0 for this time interval?



Refractory period duration $\tau_0 = 1.9ms$ calculated as the minimum observed duration between neuron spikes.

(b) Can you check that the decay of the distribution $P(\tau)$ of inter-spike intervals is indeed exponential? Measure the corresponding decay rate λ

(c) Can you deduce an analytical expression for the distribution of inter-spike time interval $P(\tau)$ of the delayed Poisson process as a function of λ and τ_0? Compare your model distribution to the one obtained from the data.

(d) Using your model, can you generate another 1000 (spike times) datapoints?

(e) What is the average spiking rate f of the neuron in the data? How is f analytically related to τ_0 and λ that you have previously measured?

2 Modelling binary data with the Ising model

2.A Pairwise spin model

(a) How many terms are in the sum over the pair(i, j)? Can you deduce what is the number of parameters in the vector $g = (h_1, ..., h_n, J_{1,2}, ..., J_n$? Can you re-write the sum over the pair(i, j) as a double sum over i and j (without counting twice each pair)?

(b) Can you write down explicitly the terms in the exponential of Eq. (1) for a system with $n = 3$ spins?

(c) In Eq. (1), we can recognize the Boltzmann distribution, in which the parameter $\beta = 1/(k_b T)$ was taken equal to 1 (more precisely, the constant k_B was taken equal to 1, and the temperature parameter T was absorbed in the parameters h_i and J_{ij}). What is the energy function associated with the Boltzmann distribution in that case? What is the partition function and what is its general expression?

(d) Take a spin s_i: if h_i is positive, which direction will s_i tend to turn to, i.e., which direction of s_i will minimize the associated energy $-h_i s_i$? Take a pair of spins s_i and s_j: if J_{ij} is positive, which configurations of (s_i, s_j) minimize the coupling energy $-J_{ij} s_i s_j$?

Assume that we have inferred the best parameters h_i and J_{ij} for the US supreme court dataset discussed in section 2. How would you interpret the sign of the inferred parameters h_i and J_{ij} in this context?

2.B Observables

(a) Given a stationary probability distribution of the state $p_{g(\boldsymbol{s})}$, what are the definitions of $\langle s_i \rangle$ and of $\langle s_i s_j \rangle$?

(b) Consider a dataset $\hat{\boldsymbol{s}}$ composed of N independent observations of the spins: $\hat{\boldsymbol{s}} = \left(\boldsymbol{s}^{(1)}, ..., \boldsymbol{s}^{(N)}\right)$. Let us denote by $\langle s_i \rangle_D$ and $\langle s_i s_j \rangle_D$ the empirical averages of s_i and of $s_i s_j$ respectively (i.e., their average values in the dataset). How would you compute $\langle s_i \rangle_D$ and $\langle s_i s_j \rangle_D$ from the data?

(c) Assume that the data is stationary and that eah datapoint has been randomly sampled from $p(\boldsymbol{s})$. Can you show that the empirical averages, $\langle s_i \rangle_D$ and $\langle s_i s_j \rangle_D$, converge to the model averages, respectively $\langle s_i \rangle$ and $\langle s_i s_j \rangle$, as the number N of datapoints goes to infinity? (very large dataset)

2.C Maximum Entropy models

(a) Consider a spin system with stationary probability distribution $p(\boldsymbol{s})$. Can you recall the definition of the Shannon entropy $S[p(\boldsymbol{s})]$? As mentioned above for the Boltzmann distribution, we will take $k_b = 1$.

The Ising model in Eq. (1) can be seen as a *Maximum Entropy Model*, constrained to reproduce the data local magnetisation and local correlation, i.e., constrained to reproduce all the data averages $\langle s_i \rangle_D$ and $\langle s_i s_j \rangle_D$ (for all spins s_i and s_j). We also want $p(\boldsymbol{s})$ to be normalised, which introduces the additional constraint $\sum_{\boldsymbol{s}} p(\boldsymbol{s}) = 1$. To summarise, we are looking for the set of 2^n probabilities $p(\boldsymbol{s})$ such that $S[p(\boldsymbol{s})]$ is maximal, and such that

$$\sum_{\boldsymbol{s}} p(\boldsymbol{s}) = 1 \quad \text{and} \quad \sum_{\boldsymbol{s}} p(\boldsymbol{s}) s_i(\boldsymbol{s}) = \langle s_i \rangle_D \quad \text{and} \quad \sum_{\boldsymbol{s}} p(\boldsymbol{s}) s_i(\boldsymbol{s}) s_j(\boldsymbol{s}) = \langle s_i s_j \rangle_D \tag{1}$$

where $\langle s_i \rangle_D$ and $\langle s_i s_j \rangle_D$ are constants that are computed from the data for all distinct s_i and s_j. Note that to be more precise, we wrote $s_i(\boldsymbol{s})$ (instead of just s_i) to specify that this is the value of s_i in the state $\boldsymbol{s}$ (this will help with the next questions).

(b) How many constraints are there in total?

To find the shape of the distributions $p(\boldsymbol{s})$ that maximises the entropy while satisfying these constraints, we introduce an auxiliary function:

$$\begin{aligned} U[p(\boldsymbol{s})] = S[p(\boldsymbol{s})] + \lambda_0 \left(\sum_{\boldsymbol{s}} p(\boldsymbol{s}) - 1 \right) + \sum_{i=1}^{n} \alpha_i \left(\sum_{\boldsymbol{s}} p(\boldsymbol{s}) s_i(\boldsymbol{s}) - \langle s_i \rangle_D \right) \\ + \sum_{\text{pair}(i,j)}^{n} \eta_{ij} \left(\sum_{\boldsymbol{s}} p(\boldsymbol{s}) s_i(\boldsymbol{s}) s_j(\boldsymbol{s}) - \langle s_i s_j \rangle_D \right) \end{aligned} \tag{2}$$

where we have introduced a parameter in front of each constraint we want to impose. These parameters (λ_0, α_i, and η_{ij}) are called Lagrange multipliers. To find $p(\boldsymbol{s})$ one must maximise this auxiliary function with respect to the 2^n probabilities $p(\boldsymbol{s})$.

(c) Let us fix a choice of a state $\boldsymbol{s}$. The probability $p_{\boldsymbol{s}} = p(\boldsymbol{s})$ is a parameter of $U[\boldsymbol{p}]$ where $\boldsymbol{p}$ is the vector of the 2^n probabilities. Can you show that:

$$\frac{\partial U[\boldsymbol{p}]}{\partial p_s} = -\log(p_s) - 1 + \lambda_0 + \sum_{i=1}^{n} \alpha_i s_i(\boldsymbol{s}) + \sum_{\text{pair}(i,j)} \eta_{ij} s_i(\boldsymbol{s}) s_j(\boldsymbol{s}) \tag{3}$$

(d) Can you show that the most general expression of $p_{\boldsymbol{s}}$ with maximal entropy that satisfying the constraints in Eq. (2) is Eq. (1)? Give the relation between λ_0 and the partition function Z. How are the parameters α_i or η_{ij} related to the parameters h_i and J_{ij}?

2.D Statistical inference: model with no couplings

(a)

(b)

(c)

2.E Statistical inference: maximising the log-likelihood function

(a)

(b)

3 Application to the analysis of the US supreme Court

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

(i)