Fourier Transformation

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1 Basics

1.1 Bases

The function bases are a collection of basic functions that can be spanned to form complicated functions in the function spaces. Some basic bases are

***Definition* 1.1.1** (Function Bases)
The standard function bases (orthogonal):
Polynomial Basis: A basis for polynomial functions,

$$\{1, x, x^2, x^3, \dots\}$$

Trigonometry Basis: A basis for sinusoidal functions,

$$\{1, \cos(2x), \sin(2x), \cos(3x), \sin(3x), \dots\}$$

The standard inner product for sinusoidal functions is

$$\langle f(x), g(x)\rangle = \int_0^{2\pi} f(x)g(x)\ dx$$

Two functions are orthogonal if their inner product is 0. Here come some handy orthogonality properties for Fourier Transformation,

***Proposition* 1.1.2** (Orthogonality)

$$\int_0^{2\pi} \cos(kx)\sin(jx)\ dx = 0, \forall k, j \in \mathbb{Z}^+$$
$$\int_0^{2\pi} \cos(kx)\cos(jx)\ dx = 0, \forall k \neq j \in \mathbb{Z}^+$$
$$\int_0^{2\pi} \sin(kx)\sin(jx)\ dx = 0, \forall k \neq j \in \mathbb{Z}^+$$
$$\int_0^{2\pi} \cos(kx)\ dx = 0, \forall k \in \mathbb{Z}^+$$
$$\int_0^{2\pi} \sin(kx)\ dx = 0, \forall k \in \mathbb{Z}^+$$

As a property of orthogonal basis, the inner product of any two distinct functions in the basis is 0. Note that they are not necessarily orthonormal.

1.2 Complex Numbers

A complex number $z \in \mathbb{C}$ could be written in the polar form

$$z = r(\cos(\theta) + i\sin(\theta))$$

where r is the norm of z and θ is the argument (angle) of z. The Euler's Formula states that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \qquad \text{(Euler's Formula)}$$

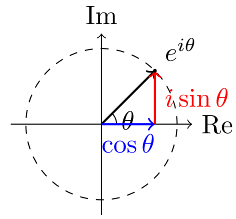

Figure 1.1: Complex Plane

The complex number could be written as $z = re^{i\theta}$. Note that its modulus is $|e^{i\theta}| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$.
By evenness of cos and oddness of sin we have

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

So

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{cos}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{sin}$$

The Nth root of unity W over 2π is defined as

$$W = e^{\frac{i2\pi}{N}} \tag{Nth root of unity}$$

where

$$W^N = e^{i2\pi} = 1$$

Specifically, the root n is

$$W^n = e^{\frac{i2\pi n}{N}}, n = 0, 1, \ldots, N-1$$

whose angle is θ is $\frac{2\pi n}{N}$. Note that the N roots equally share the 2π period, i.e. they have uniform spacing.

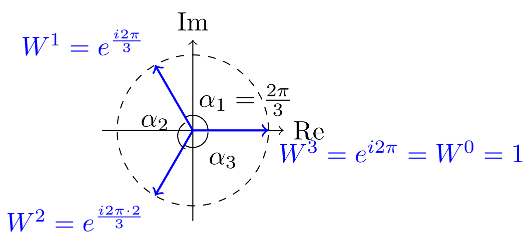

Figure 1.2: Cube Root of Unity

In Cube Root of Unity over 2π, since there are $N = 3$ roots of unity that uniformly divide the 2π plane, the inter-root angles are $\alpha_1 = \alpha_2 = \alpha_3 = \frac{2\pi}{3}$. The angle of each root is $\theta_n = n\frac{2\pi}{3}, n = 0, 1, 2$.
We can also extent to a larger plane such as 4π. Let the period be $2k\pi$ for integer k. Then

$$W^k = e^{\frac{i2k\pi}{N}} \quad \text{is still the } N\text{th root of unity} \tag{Power of Root of Unity}$$

Consider the example of 5 samples over 4π, 5th Root of Unity over 4π. The blue roots are in the first layer $[0, 2\pi]$, whereas the red roots are in the second layer $[2\pi, 4\pi]$.

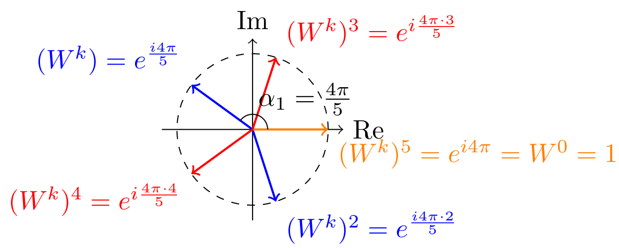

Figure 1.3: 5th Root of Unity over 4π

A useful lemma is that roots of unity sum to 0. For example, in Cube Root of Unity, the roots are

$$\begin{aligned}
&W^0 = W^3 = 1 \\
&W = e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\
&W^2 = e^{\frac{i4\pi}{3}} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}
\end{aligned}$$

and apparently $W^0 + W + W^2 = 0$. A more rigorous proof goes as

***Lemma* 1.2.1** (Summation of Roots of Unity)
Let $W^k = e^{\frac{i2k\pi}{N}}$ denote the Nth root of unity over $2k\pi$ for integer k. Then

$$\sum_{n=0}^{N-1} W^n = \begin{cases} 1 & \text{if } N = 1 \\ 0 & \text{if } N > 1 \end{cases}$$

Proof. By definition $W^N = 1$ thus the statement holds for $N = 1$. Suppose $N > 1$.

$$\begin{aligned}
\sum_{n=0}^{N-1} W^{kn} &= \sum_{n=0}^{N-1} \left(e^{\frac{i2k\pi}{N}}\right)^n \\
&= \frac{1 - \left(e^{\frac{i2k\pi}{N}}\right)^N}{1 - e^{\frac{i2k\pi}{N}}} \quad \text{by finite sum of geo series} \\
&= \frac{1 - e^{i2k\pi}}{1 - e^{\frac{i2k\pi}{N}}} \\
&= \frac{1 - 1}{1 - e^{\frac{i2k\pi}{N}}} \\
&= 0
\end{aligned}$$

□

1.3 Periodic Functions

A periodic function $f(t) : \mathbb{R} \to \mathbb{R}$ with period T seconds satisfies

$$f(t+T) = f(t)$$

such that $f(t)$ repeats itself after every period T. This is referred to be the *Time Domain* of $f(t)$, where t is denoted as *time* (in seconds). The *period* T is the length of a *cycle*, i.e. the time taken to complete a full *cycle*, which is a complete repetition of the function's repeated patterns. The *frequency* $f = \frac{1}{T}$ is the number (or fraction) of cycles completed in one unit time interval, i.e. time length of 1.

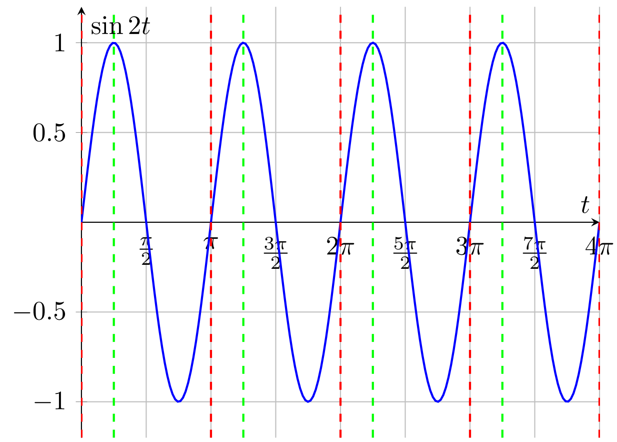

Figure 1.4: $\sin(2t)$

Referring to the graph of $\sin(2t)$, a subplot between two adjacent red (or green) vertical lines is a complete cycle, with period π and frequency $\frac{1}{\pi}$, meaning that it moves $\frac{1}{\pi}$ cycles in 1 second. Note that $\sin(kt)$ for any $k \in \mathbb{R}$ has a period of $\frac{2\pi}{k}$ and frequency $\frac{k}{2\pi}$, thus we can write any sine function with period T as

$$f(t) = \sin\left(\frac{2\pi t}{T}\right) = \sin(2\pi f t), f = \frac{1}{T}$$

Specifically, any periodic function with period T could be written as

$$f(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi k t}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi k t}{T}\right)$$

Note that k is an integer. This states that a periodic function can be decomposed into a summation of sin and cos functions with periods $\frac{T}{k}$ and amplitudes a_k, b_k respectively, for $k = 1, 2, \ldots$, i.e. periods $T, \frac{T}{2}, \frac{T}{3}, \ldots$, or equivalently, frequencies $\frac{1}{T}, \frac{2}{T}, \frac{3}{T}, \ldots$. For example, consider the following graph 1.5. Note that its function is

$$2\sin(3x) + 3\cos(2x)$$

However, if we do not know its explicit expression, we will not be able to obtain this decomposition by observing the graph. The Fourier Transformation helps analyze this information by transforming the *Time Domain* data into *Frequency Domain* data for better manipulations.

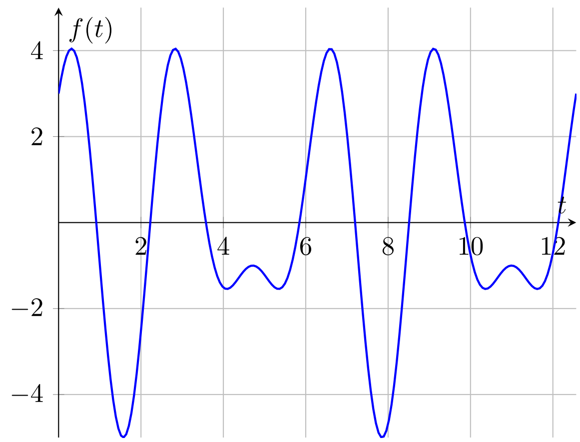

Figure 1.5: Complicated Periodic Function $f(t)$

2 Fourier Series

2.1 Finding Coefficients

Any periodic function $f(t)$ can be decomposed into an infinite sum of cos and sin functions from the Trigonometry Basis in Function Bases

$$f(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt)$$

Now we wish to find the coefficients c_0, a_k, b_k. With the properties from Orthogonality, we have

$$\begin{aligned}\int_0^{2\pi} f(t)\ dt &= \int_0^{2\pi} c_0\ dt + \int_0^{2\pi} \sum_{k=1}^{\infty} a_k \cos(kt)\ dt + \int_0^{2\pi} \sum_{k=1}^{\infty} b_k \sin(kt)\ dt \\ &= c_0(2\pi - 0) + \cancel{\sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos(kt)\ dt} + \cancel{\sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin(kt)\ dt} \\ &= 2\pi c_0\end{aligned}$$

Hence

$$c_0 = \frac{\int_0^{2\pi} f(t)\ dt}{2\pi}$$

which is the average of $f(t)$ over $[0, 2\pi]$.

For a_k, we multiply by a $\cos(jt), j \in \mathbb{Z}^+$ to them and integrate over the period,

$$\begin{aligned}\int_0^{2\pi} f(t)\cos(jt)\ dt &= \int_0^{2\pi} c_0 \cos(jt)\ dt + \int_0^{2\pi} \sum_{k=1}^{\infty} a_k \cos(kt)\cos(jt)\ dt + \int_0^{2\pi} \sum_{k=1}^{\infty} b_k \sin(kt)\cos(jt)\ dt \\ &= \cancel{c_0 \int_0^{2\pi} \cos(jt)\ dt} + \sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos(kt)\cos(jt)\ dt + \cancel{\sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin(kt)\cos(jt)\ dt} \\ &= a_j \int_0^{2\pi} \cos^2(jt)\ dt \quad \text{only } k = j \text{ term remains}\end{aligned}$$

Hence

$$a_j = \frac{\int_0^{2\pi} f(t)\cos(jt)\ dt}{\int_0^{2\pi} \cos^2(jt)\ dt}, \forall j \in \mathbb{Z}^+$$

For b_k, similarly, multiply by $\sin(jt), j \in \mathbb{Z}^+$ and integrate,

$$
\begin{aligned}
\int_0^{2\pi} f(t)\sin(jt)\ dt &= \int_0^{2\pi} c_0 \sin(jt)\ dt + \int_0^{2\pi} \sum_{k=1}^{\infty} a_k \cos(kt)\sin(jt)\ dt + \int_0^{2\pi} \sum_{k=1}^{\infty} b_k \sin(kt)\sin(jt)\ dt \\
&= \cancel{c_0 \int_0^{2\pi} \sin(jt)\ dt} + \cancel{\sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos(kt)\sin(jt)\ dt} + \sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin(kt)\sin(jt)\ dt \\
&= b_j \int_0^{2\pi} \sin^2(jt)\ dt
\end{aligned}
$$

Hence

$$b_j = \frac{\int_0^{2\pi} f(t)\sin(jt)\ dt}{\int_0^{2\pi} \sin^2(jt)\ dt}, \forall j \in \mathbb{Z}^+$$

2.2 Fourier Transform

We want to convert

$$f(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt)$$

into a more condensed format

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

with complex coefficients c_k. Use the property of cos and sin,

$$
\begin{aligned}
c_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt) &= c_0 + \sum_{k=1}^{\infty} a_k \frac{e^{ikt} + e^{-ikt}}{2} + \sum_{k=1}^{\infty} b_k \frac{e^{ikt} - e^{-ikt}}{2i} \\
&= c_0 + \sum_{k=1}^{\infty} (\frac{a_k}{2} + \frac{b_k}{2i}) e^{ikt} + \sum_{k=1}^{\infty} (\frac{a_k}{2} - \frac{b_k}{2i}) e^{-ikt} \\
&= c_0 + \sum_{k=1}^{\infty} (\frac{a_k}{2} + \frac{b_k}{2i}) e^{ikt} + \sum_{k=-\infty}^{-1} (\frac{a_k}{2} - \frac{b_k}{2i}) e^{ikt} \quad \text{assuming } b_k = b_{-k} \\
&= c_0 + \sum_{k=1}^{\infty} \frac{a_k - ib_k}{2} e^{ikt} + \sum_{k=-\infty}^{-1} \frac{a_k + ib_k}{2} e^{ikt} \\
&= \sum_{k=-\infty}^{\infty} c_k e^{ikt}
\end{aligned}
$$

Therefore, for $k = 1, 2, \ldots, \infty$,

$$c_k = \frac{a_k - ib_k}{2}, c_{-k} = \frac{a_k + ib_k}{2}, |c_k| = |c_{-k}| = \frac{1}{2}\sqrt{a_k^2 + b_k^2}$$

Using the orthogonality property

$$\int_0^{2\pi} e^{ikt} e^{-ijt}\ dt = \begin{cases} 0 & \text{if } k \neq j \\ 2\pi & \text{if } k = j \end{cases} \quad \text{for integer } j$$

Proof. For $k = j$ it gives $\int_0^{2\pi} 1 \; dt = 2\pi$. For $k \neq j$,

$$\begin{aligned}
\int_0^{2\pi} e^{ikt}e^{-ijt} \; dt &= \int_0^{2\pi} e^{i(k-j)t} \; dt \\
&= \int_0^{2\pi} \cos((k-j)t) + i\sin((k-j)t) \; dt \quad \text{by Euler's Formula} \\
&= \frac{1}{k-j}\sin((k-j)t)\Big|_0^{2\pi} + \frac{-i}{k-j}\cos((k-j)t)\Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

□

This also implies $\int_0^{2\pi} e^{ikt} \; dt = 0$ for $k \neq 0$ and 2π for $k = 0$. Multiply by e^{-ijt} and integrate

$$\begin{aligned}
\int_0^{2\pi} f(t)e^{-ijt} \; dt &= \sum_{k=-\infty}^{\infty} c_k \int_0^{2\pi} e^{ikt}e^{-ijt} \; dt \\
&= 2\pi c_j
\end{aligned}$$

Therefore

$$c_k = \frac{\int_0^{2\pi} f(t)e^{-ikt} \; dt}{2\pi}, k \in \mathbb{Z}$$

The c_k's are known as *Fourier Coefficients*, where each k is referred to be a bin of the Fourier coefficients, such that bin k stores the amplitude of functions with frequency $\frac{T}{k}$.

***Proposition* 2.2.1** (Summary of Fourier Coefficients)
For the two types of decomposition:

1. $$f(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt)$$

   we have
   $$c_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t) \; dt, a_j = \frac{\int_0^{2\pi} f(t)\cos(jt) \; dt}{\int_0^{2\pi} \cos^2(jt) \; dt}, b_j = \frac{\int_0^{2\pi} f(t)\sin(jt) \; dt}{\int_0^{2\pi} \sin^2(jt) \; dt}, \forall j \in \mathbb{Z}^+$$

2. $$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

   we have
   $$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt} \; dt, \forall k \in \mathbb{Z}$$

As well,
$$c_j = \frac{a_j - ib_j}{2}, c_{-j} = \frac{a_j + ib_j}{2}, |c_j| = |c_{-j}| = \frac{1}{2}\sqrt{a_j^2 + b_j^2}, \forall j \in \mathbb{Z}^+$$

2.3 Discrete Fourier Transform (DFT)

Look back to the equation

$$f(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi kt}{T}\right)$$

which has generalized form

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{i2\pi kt}{T}}$$

Recall that $e^{i\frac{2\pi kt}{T}}$ is a periodic function $\cos(\frac{2\pi kt}{T}) + i\sin(\frac{2\pi kt}{T})$, whose frequency is $\frac{k}{T}$. Suppose a set of (even) N discrete and evenly spaced data points $f_0, f_1, \ldots, f_{N-1}$ are sampled from $f(t)$ at time steps $t = 0, t_1, \ldots, t_{N-1}$ respectively, where $t_n = n\frac{T}{N}, n = 0, 1, \ldots, N-1$ such that they equally share one whole period T. The approximate truncated Fourier Series is

$$f_n = \sum_{k=-N/2+1}^{N/2} c_k e^{\frac{i2\pi kt_n}{T}} = \sum_{k=-N/2+1}^{N/2} c_k e^{\frac{i2\pi kn}{N}}, n = 0, 1, \ldots, N-1 \tag{DFT}$$

Note that we replace the time step fraction $\frac{t_n}{T}$ with discrete fraction $\frac{n}{N}$, which have the same result. If $\theta = \frac{2\pi kn}{N}$ refers to the angle of $e^{i\theta}$, it states N samples over $2k\pi$ period, such that $e^{\frac{i2\pi kn}{N}}$ is the Nth root of unity over $2k\pi$.
The DFT system consists of N equations involving N unknowns, namely the c_k's, which can be computed by manipulating the formula of *Inverse Fourier Transform.*

2.4 Inverse Fourier Transform (IDFT)

For convenience we rearrange the DFT to a neat form

$$f_n = \sum_{k=0}^{N-1} F_k e^{\frac{i2\pi kn}{N}} \tag{IDFT}$$

where F_k depends on c_j's. The DFT coefficients are periodic such that $c_{j\pm N} = c_j$. We re-index the summation using the change of variable $j = k + N$.

$$\begin{aligned}
f_n &= \sum_{k=-N/2+1}^{N/2} c_k e^{\frac{i2\pi kn}{N}} \\
&= \sum_{k=0}^{N/2} c_k e^{\frac{i2\pi kn}{N}} + \sum_{k=-N/2+1}^{-1} c_k e^{\frac{i2\pi kn}{N}} \\
&= \sum_{k=0}^{N/2} c_k e^{\frac{i2\pi kn}{N}} + \sum_{j=N/2+1}^{N-1} c_{j-N} e^{\frac{i2\pi (j-N)n}{N}} \\
&= \sum_{k=0}^{N/2} c_k e^{\frac{i2\pi kn}{N}} + \sum_{j=N/2+1}^{N-1} c_j e^{\frac{i2\pi jn}{N}} \cancel{e^{\frac{-i2\pi Nn}{N}}} \\
&= \sum_{k=0}^{N-1} F_k e^{\frac{i2\pi kn}{N}} \quad \text{rename } c_k \text{ to } F_k \\
&= \sum_{k=0}^{N-1} F_k W^{nk} \quad \text{recall } N\text{th root of unity}
\end{aligned}$$

If given the Fourier Coefficients F_k, this formula transforms F_k's back to f_n, a.k.a. *Inverse Fourier Transform.* In order to derive the Forward Fourier Transform, we make use of the orthogonality property

$$\sum_{n=0}^{N-1} W^{nk} W^{-nj} = N\delta_{k,j}$$

Proof. Recall by Power of Root of Unity and Summation of Roots of Unity,

$$\sum_{n=0}^{N-1} W^{mn} = \begin{cases} N & \text{if integer } m = 0 \\ 0 & \text{if integer } m \neq 0 \text{, where } W^m \text{ is still } N\text{th root of unity} \end{cases} \tag{Sum of PRU}$$

With the orthogonality,

$$\begin{aligned} \sum_{n=0}^{N-1} W^{nk} W^{-nj} &= \sum_{n=0}^{N-1} W^{n(k-j)} \\ &= \sum_{n=0}^{N-1} W^{nm} \quad \text{where } m = k - j \\ &= N\delta_{k,j} \end{aligned}$$

□

Therefore,

$$\begin{aligned} \sum_{n=0}^{N-1} f_n W^{-nj} &= \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} F_k W^{nk} W^{-nj} \\ &= \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} F_k W^{n(k-j)} \\ &= \sum_{k=0}^{N-1} F_k \sum_{j=0}^{N-1} W^{n(k-j)} \\ &= \sum_{k=0}^{N-1} F_k N \delta_{k,j} \\ &= N F_j \end{aligned}$$

Hence

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n W^{-nk}$$

***Proposition* 2.4.1** (Summary of Fourier Transform)
For $W = e^{\frac{i2\pi}{N}}$:

1. Forward Discrete Fourier Transform (DFT):

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n W^{-nk}, k = 0, 1, \ldots, N-1$$

2. Inverse Discrete Fourier Transform (IDFT):

$$f_n = \sum_{k=0}^{N-1} F_k W^{nk}, n = 0, 1, \ldots, N-1$$

Note that

$$F_0 = \frac{1}{N}\sum_{n=0}^{N-1} f_n W^0 = \frac{1}{N}\sum_{n=0}^{N-1} f_n$$

such that F_0 is always the average of the data points, just like the continuous Fourier Transform. Ignoring F_0, the rest Fourier Coefficients $F_1, \ldots, F_{N-1}$ are symmetric about $\frac{N}{2}$.

***Lemma* 2.4.2** (Conjugate Symmetry of Fourier Coefficients)
The Fourier Coefficients $F_1, \ldots, F_{N-1}$ are symmetric about $N/2$, such that $F_k = \bar{F}_{N-k}, k = 1, \ldots, N-1$.

Proof.

$$\begin{aligned}
F_{N-k} &= \frac{1}{N}\sum_{n=0}^{N-1} f_n W^{-n(N-k)} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{\frac{-i2\pi n(N-k)}{N}} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} f_n \cancel{e^{-i2\pi n}} e^{\frac{i2\pi nk}{N}} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} f_n W^{nk} && (\star) \\
&= \bar{F}_k \quad \text{where } W^{-j} = \overline{W^j}
\end{aligned}$$

□

They are also periodic and repeat after every N data points. It is because the coefficients are computed from a complete cycle of the periodic target function $f(t)$, thus the next cycle of $f(t)$ would also give the same Fourier Coefficients.

***Lemma* 2.4.3** (Periodicity of Fourier Coefficients)
Fourier Coefficients F_K for $k = 0, \ldots, N-1$ are periodic over N points.

Proof.

$$\begin{aligned}
F_{k+N} &= \frac{1}{N}\sum_{n=0}^{N-1} f_n W^{-n(k+N)} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{\frac{-i2\pi n(k+N)}{N}} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{\frac{-i2\pi nk}{N}} \cancel{e^{\frac{-i2\pi nN}{N}}} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{\frac{-i2\pi nk}{N}} \\
&= F_k
\end{aligned}$$

□

***Theorem* 2.4.4** (Parseval's Theorem)
Let $f_0, f_1, \ldots, f_{N-1}$ be complex discrete input data with DFT $F_0, F_1, \ldots, F_{N-1}$. Then

$$\sum_{k=0}^{N-1} F_k \bar{F}_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \bar{f}_n$$

Proof. Note that by Conjugate Symmetry of Fourier Coefficients,

$$F_k = \frac{1}{N} \sum_{n=0}^{n-1} f_n W^{-nk}$$
$$\bar{F}_k = \frac{1}{N} \sum_{n=0}^{n-1} \bar{f}_n W^{nk}$$

Therefore,

$$\begin{aligned}
\sum_{k=0}^{N-1} F_k \bar{F}_k &= \sum_{k=0}^{N-1} \frac{1}{N^2} \left(\sum_{n=0}^{N-1} f_n W^{-nk} \sum_{j=0}^{N-1} \bar{f}_j W^{jk} \right) \\
&= \sum_{k=0}^{N-1} \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} f_n \bar{f}_j W^{(j-n)k} \\
&\text{only } n = j = 0, \ldots, n = j = N-1 \text{ terms present} \\
&= \sum_{k=0}^{N-1} \frac{1}{N^2} \sum_{n=0}^{N-1} f_n \bar{f}_n \cancel{W^0} \\
&= \frac{N}{N^2} \sum_{n=0}^{N-1} f_n \bar{f}_n \\
&= \frac{1}{N} \sum_{n=0}^{N-1} f_n \bar{f}_n
\end{aligned}$$

□

2.5 Fast Fourier Transform (FFT)