Introduction to Optimization: Linear Programming

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Contents

1 Basic Definitions of Linear Programming

A general LP can be defined as the following model:

$$\textbf{(LP)} \;\; \max\{c^T x : A_1 x \le b_1, A_2 x = b_2, x \ge 0\}$$

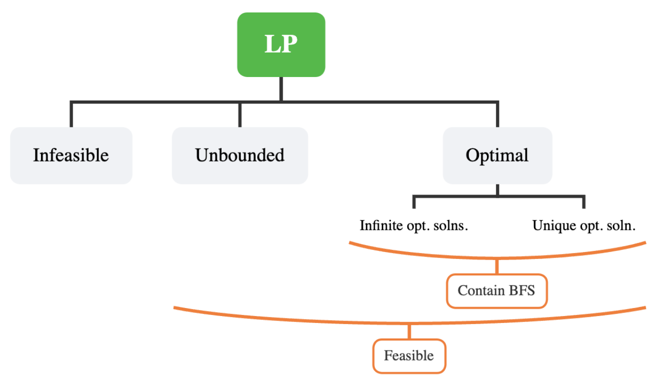
A LP (in SEF) can be described as the following model:

$$\textbf{(LP)} \;\; \max\{c^T x : Ax = b, x \ge 0\}$$

where $c^T x$ $s.t.$ $c, x \in \mathbb{R}^n$ is the objective function (linear), $Ax = b$ $s.t.$ $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ is the constraints (set of affine functions), along with the non-negativity requirements. We aim to maximize the objective value $c^T x$ by choosing the maximizer x from the feasible region such that x satisfies $Ax = b, x \ge 0$, where the feasible region of **LP** is defined as $\mathcal{F} = \{x : Ax = b, x \ge 0\}$.

Fundamental Theorem of Linear Programming (FTLP)

1. A LP is either infeasible, unbounded, or optimal.
2. If a LP is optimal, then there must exist an optimal solution that is basic, which is a vertex of the feasible region. If there exist at least two optimal solutions, then there will be infinite optimal solutions which include all points on the line segment between them.



1.1 Standard Equality Form (SEF)

Suppose we have a general NLP

$$\textbf{(NLP)}\ \min\{c^T x : A_1 x ? b_1, x ? 0\}$$

Recall that the standard equality form is

$$\textbf{(LP)}\ \max\{c^T x : Ax = b, x \geq 0\}$$

We aim to transform the NLP to LP. Now assume the condition is messy such that the constraints consist of equalities and inequalities, the objective is minimization, and some variables are non-negative, some are non-positive or free. We specify the approach to clear them up step by step.

Objective Function When dealing with minimization problems, directly reverse the direction of optimization by multiplying a -1 to the objective function,

$$\min c^T x \implies \max -c^T x$$

But remember that this objective value is the negative of the original objective value, so once obtain an objective z, the real value is $-z$.

Constraints If a constraint that is already in equality then no change is required. For those inequality constraints, we introduce the *slackness variable* for balance. For example,

$$x_1 + x_2 + x_3 \leq 5$$

Add a new non-negative slack variable s_1 such that

$$x_1 + x_2 + x_3 + s_1 = 5$$

Similar for $\geq$ constraints,

$$x_1 + x_2 + 2x_3 \geq 3 \implies x_1 + x_2 + 2x_3 - s_2 = 3$$

Note that if there are multiple inequality constraints, we need a slack variable for each of them. Thus if $Ax?b$ consists of inequalities only, we have

$$(A| \pm I)z = b, A \in \mathbb{R}^{m\times n}, (A| \pm I) \in \mathbb{R}^{m\times(n+m)}, z = \begin{bmatrix} x \\ s \end{bmatrix} \in \mathbb{R}^{n+m}, s, b \in \mathbb{R}^m$$

where the $\pm I$ is a $m \times m$ diagonal matrix whose diagonal entry $i, i = 1, ..., m$ is

$$\pm I_{ii} = \begin{cases} 1 & \text{if } i\text{th constraint is } \leq. \\ -1 & \text{if } i\text{th constraint is } \geq. \end{cases}$$

This ensures the equality constraints.

Variables If a variable x_i is already non-negative, no change is required. If a variable $x_i \leq 0$, reverse the corresponding objective entry c_i to $-c_i$ as well as the ith column of A, then change the variable to $x_i \geq 0$. It is more complicated if x_i is free such that, it can be negative, positive, or zero. To handle this, split and turn the variable into the difference of two non-negative variables such that $x_i = x^+ - x^-, x^+, x^- \geq 0$, and then augment the the objective function and constraint as well. For example,

$$
\begin{gathered}
c = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\
A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 1 & 3 \end{bmatrix} \\
x_1 \leq 0, x_2 \geq 0, x_3 \text{ free}
\end{gathered}
$$

then

$$
\begin{gathered}
c = \begin{bmatrix} -1 & 2 & 3 & -3 \end{bmatrix} \\
A = \begin{bmatrix} -1 & 1 & 4 & -4 \\ -2 & 1 & 3 & -3 \end{bmatrix} \\
x_1, x_2, x^+, x^- \geq 0
\end{gathered}
$$

Now we have completed the transformation. The SEF is significant in various aspects, which we will see later.

Exercise 1.1.1. *Transform the following system into SEF.*

$$
\begin{gathered}
\min \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} x \\
\begin{bmatrix} 3 & 2 & 8 & 4 & 2 \\ 1 & 0 & 2 & 7 & 3 \\ 5 & 3 & 1 & 9 & 4 \end{bmatrix} x \begin{bmatrix} \geq \\ \leq \\ = \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \\
x_1 \geq 0, x_2, x_3 \leq 0, x_4, x_5 \textit{ free.}
\end{gathered}
$$

(Don't worry if the system is inconsistent, just obtain the SEF).

1.2 Feasibility

A LP is said to be feasible if there exist feasible solutions, that said the feasible region $\mathcal{F} \neq \emptyset$. Thus a certificate of feasibility is any feasible solution $\bar{x} \in \mathcal{F}$ such that $A\bar{x} = b, \bar{x} \geq 0$. The feasible region $\mathcal{F}$ is a convex polytope which is the intersection of finitely many half-spaces, each defined by an inequality constraint if assuming that the LP is not in SEF. We now look at the boundedness of $\mathcal{F}$, The feasible region $\mathcal{F}$ is bounded if, and only if, $\|x\|_2 \leq M \; \forall x \in \mathcal{F}$, for some $M \geq 0 \in \mathbb{R}$.

If $\mathcal{F}$ is bounded then the objective value will be limited in some fixed range, which will never exceed $M\|c\|$ or below $-M\|c\|$ by the Cauchy-Schwartz Inequality $|c^T x| \leq \|c\| \|x\|$, but the bound might not be tight.

We call a LP infeasible if no feasible solution exists such that $\mathcal{F} = \emptyset$.

A certificate of infeasibility is

$$y^T A \geq 0^T, y^T b < 0$$

for some $y \in \mathbb{R}^m$.

Note that y is free. Assume the LP is feasible where there exists feasible solution x satisfying $Ax = b, x \geq 0$. Then $y^T Ax = y^T b$. If $y^T A \geq 0^T$ then $y^T Ax \geq 0$ thus it is impossible to have $y^T b < 0$. In fact, the result comes from Farkas' Lemma,

Farkas' Lemma
Exactly one of the two following conditions is true:

1. (feasible) $\exists x \in \mathbb{R}^n$ such that $Ax = b, x \geq 0$.
2. (infeasible) $\exists y \in \mathbb{R}^m$ such that $y^T A \geq 0^T, y^T b < 0$.

Example 1.2.1. $\mathcal{F} = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 5, x_2 \geq 7, x \geq 0\}$ *is infeasible.*
Set up the matrix in SEF will be

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

A possible certificate is $y = (1, -1)^T$, *for which* $y^T A = (1, 0, 1, 1) \geq 0^T, y^T b = -2 < 0$.

1.3 Basis

As stated in FTLP, we are interested in finding basic optimal solutions. In fact, when performing the simplex algorithm, we are actually looking for an *optimal basis*, whose corresponding basic feasible solution carries an optimal value. We first look at the definition of a basis.

Suppose the LP is in SEF with full row rank (delete redundant rows). Normally the constraint matrix in SEF is underdetermined, i.e. has more columns than rows. Although the matrix itself cannot be non-singular, we can pick some linearly independent columns to form a basis for $\mathbb{R}^m$.

A basis of A is a $m \times m$ non-singular matrix formed with m linearly independent columns of A, where m is the number of rows of A as well as its rank.

Recall the basis of $\mathbb{R}^m$ can map all vectors in $\mathbb{R}^m$, so does basis of A. If A has m rows, then there should be exactly m columns in the basis. Thus the number of bases is bounded which is at most $\binom{n}{m}$ as well as the maximum number of simplex iterations.

After determining the basis, we can use it to find the corresponding *basic solution*.

The basic solution with respect to basis B is defined as:

$$\begin{aligned} x_{\mathcal{B}} &= A_{\mathcal{B}}^{-1} b \\ x_N &= 0 \end{aligned}$$

where $A_{\mathcal{B}}$ is the submatrix formed with columns in the basis B, x_B is the subvector of the basic entries (entries corresponding to the basic columns), x_N is a subvector of the rest entries.

As $A_{\mathcal{B}}$ is non-singular, the corresponding basic solution is unique. According to the definition, the basic solution is actually using the basic columns to map b while remaining other entries 0. This will be emphasized in the following example.

Example 1.3.1.

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$
$$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}$$

We see that it can have infinitely many solutions, but we are only interested in basic solutions. We start with picking a valid basis. Recall that a basis only consists of linearly independent columns, so for example, $\mathcal{B} = \{1, 2, 3\}$ *is a valid basis, with corresponding basic solution*

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$
$$x = \begin{bmatrix} \underline{2} & \underline{4} & \underline{8} & 0 & 0 & 0 \end{bmatrix}$$

The underlined entries are the basic entries of x*, which is* $x_{\mathcal{B}}$*, and the following entries are* x_N*. Now it is clear that we are finding a solution using only the basic columns. There are* *many choices of bases, as shown below,*

$$
\begin{aligned}
\mathcal{B} = &\{4,5,6\} \\
&\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \\
x = &\begin{bmatrix} 0 & 0 & 0 & \underline{1} & \underline{2} & \underline{4} \end{bmatrix} \\
\\
\mathcal{B} = &\{1,5,6\} \\
&\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \\
x = &\begin{bmatrix} \underline{2} & 0 & 0 & 0 & \underline{2} & \underline{4} \end{bmatrix} \\
&\dots
\end{aligned}
$$

However, $\mathcal{B} = \{1,4,5\}$ *is not a valid basis since columns* $1,4$ *are linearly dependent.*

Exercise 1.3.1. *Find some other valid/invalid bases in 1.3.1.*

For a LP, a basic solution is feasible if and only if it is non-negative, however, in a general program it is feasible as long as it satisfies all constraints. We only focus on LP,

Suppose the system $Ax = b$ is consistent and $\bar{x}$ is a basic solution with respect to basis $\mathcal{B}$. Then it is a basic feasible solution (BFS) if and only if $\bar{x} \geq 0$.

One can easily understand the basic infeasible, non-basic feasible, non-basic infeasible solutions as well. As $x_N = 0$ is enforced, we can fit the above definition to

A basic solution $\bar{x}$ is feasible if and only if $\bar{x}_{\mathcal{B}} \geq 0$.

The feasibility and other properties of a basis can be very apparent if we convert the LP to canonical form, which we will discuss below.

1.4 Canonical Form

Suppose $\mathcal{B}$ is a valid basis of LP. Recall the LP

$$\textbf{(LP)} \ \max\{c^T x : Ax = b, x \geq 0\}$$

We define the canonical form:

The canonical form of LP with respect to basis $\mathcal{B}$ is in the shape of

$$\bar{c}^T\bar{x} + \bar{z}$$
$$\bar{A}\bar{x} = \bar{b}$$

where $\bar{c}_{\mathcal{B}} = 0, \bar{A}_{\mathcal{B}} = I_m, \bar{x}_{\mathcal{B}} = \bar{b}, \bar{x}_N = 0$.

It is clear that if $A_{\mathcal{B}} = I$ then obviously $x_{\mathcal{B}} = b$ such that the augmented b is already a basic solution. For this convenience, we need to convert $A_{\mathcal{B}}$ to I. A straightforward approach will be multiply a $A_{\mathcal{B}}^{-1}$ (which always exists as $A_{\mathcal{B}}$ is non-singular) to the constraint

$$A_{\mathcal{B}}^{-1}Ax = A_{\mathcal{B}}^{-1}b$$

The new matrix $\bar{A}_{\mathcal{B}}$ will then be I, thus the basic solution is $x_{\mathcal{B}} = A_{\mathcal{B}}^{-1}b$. Recall from the previous lemma, we can verify the feasibility of the basis:

A basis $\mathcal{B}$ is (primal) feasible if and only if $A_{\mathcal{B}}^{-1}b$ is non-negative.

That said, the basis is infeasible if $A_{\mathcal{B}}^{-1}b$ contains negative entries. It is also essential to enforce $\bar{c}_{\mathcal{B}} = 0$ since

$\bar{c}^T\bar{x} = \bar{c}_{\mathcal{B}}^T\bar{x}_{\mathcal{B}} + \bar{c}_N^T\bar{x}_N = 0$ where $\bar{c}_{\mathcal{B}} = 0, \bar{x}_N = 0$.

That said the inner product of canonical $\bar{c}, \bar{x}$ is always 0, since whatever what $\bar{x}_{\mathcal{B}}, \bar{c}_N$ is, they will be mapped to 0 by $\bar{c}_{\mathcal{B}}, \bar{x}_N$. The reason why we are making this assertion is because that based on this, we can come up with a necessary condition for optimality,

In the canonical form with respect to basis $\mathcal{B}$, if $\bar{c} \leq 0$, then $\mathcal{B}$ is optimal as well as the basic solution $\bar{x}$, and $\bar{z}$ which is the constant following $\bar{c}^T\bar{x}$ is the optimal value.

Assume $\bar{c} \leq 0$ holds, and as $\bar{x} \geq 0$, we have $\bar{c}^Tx \leq 0$ thus 0 is the maximum value it can reach, that said $\bar{c}^Tx + \bar{z} = 0 + \bar{z} = \bar{z}$ is the optimal value. This is also referred to be the *dual feasibility* such that, if a basis is both primal and dual feasible, then it is an optimal basis. It is also the sign of the termination of simplex algorithm, and can lead to further optimality conditions such as complementary slackness theorem.

After understanding the importance of canonical form, we now continue focusing on how to transform c to this way. Keep in mind that, when converting c, we cannot change its attribute but transform it to our desired shape, that said the objective functions before and after transformation must remain equivalent such that $c^Tx = \bar{c}^Tx + \bar{z}$. Recall that the basic entries $\bar{c}_{\mathcal{B}} = 0$, so to achieve this, we still use the constraints. Define a vector $y \in \mathbb{R}^m$, which is referred to be the dual variable, and

$$y^TAx = y^Tb \implies y^TAx - y^Tb = 0$$

We first notice that, as $y^TAx - y^Tb$ has a constant value 0, there will be no penalty if we add or subtract this from the objective function. Thus

$$c^Tx - (y^TAx - y^Tb) = (c^T - y^TA)x + y^Tb$$

is exactly the same function. Make use of the basic columns of $y^T A$, as we want to make $\bar{c}_{\mathcal{B}} = 0$, we have

$$y^T A_{\mathcal{B}} = c_{\mathcal{B}}^T$$

such that if this equality holds, then $\bar{c} = (c^T - y^T A)$ is exactly the shape we want, as we do not care about the non-basic entries $\bar{c}_N$. Since $A_{\mathcal{B}}$ is non-singular, we know that there exists a unique y such that $y = A_{\mathcal{B}}^{-T} c_{\mathcal{B}}$. Substitute this y back to the equation, we have

$$\begin{aligned} \bar{c}_N^T &= c_N^T - y^T A_N \\ &= c_N^T - c_{\mathcal{B}}^T A_{\mathcal{B}}^{-1} A \\ \bar{c}_{\mathcal{B}}^T &= 0^T \end{aligned}$$

Look back to the optimality conditions, since $\bar{c}_{\mathcal{B}}^T = 0$ is enforced, it can be fit to

> In the canonical form with respect to primal feasible basis $\mathcal{B}$, if $\bar{c}_N \leq 0$ then $\mathcal{B}$ is an optimal basis.

The converse also holds. The proof will be presented in later chapters on optimality. Now we update and conclude the transformation of canonical form,

> For a LP
>
> $$\textbf{(P)}\ \max\{c^T x : Ax = b, x \geq 0\},$$
>
> The canonical form of LP with respect to basis $\mathcal{B}$ is in the shape of
>
> $$\begin{aligned} \bar{c}^T \bar{x} + \bar{z} &= (c^T - \bar{y}^T A)\bar{x} + \bar{y}^T b \\ \bar{A}\bar{x} &= \bar{b} \end{aligned}$$
>
> where $\bar{x}_{\mathcal{B}} = \bar{b} = A_{\mathcal{B}}^{-1} b, \bar{x}_N = 0, \bar{A} = A_{\mathcal{B}}^{-1} A, \bar{y} = A_{\mathcal{B}}^{-T} c_{\mathcal{B}}$.

1.5 Duality

For a primal LP, P,

$$\textbf{(P)}\ \max\{c^T x : Ax = b, x \geq 0\},$$

its dual LP is defined as

$$\textbf{(D)}\ \min\{b^T y : A^T y \geq c, y \text{ free}\}$$

Observe that the system (those things involving y) when transforming into canonical form is in fact the dual. We can now conclude the properties of a basis.

> A basis $\mathcal{B}$ of A
>
> 1. is non-singular;
> 2. carries unique primal&dual solutions $\bar{x}$ and $\bar{y}$;
> 3. has unique primal&dual objective values $c^T \bar{x}$ and $b^T \bar{y}$

The further properties of primal-dual pair on optimality will be discussed later.

2 Optimality

2.1 Duality and Optimality

Prior to optimality analysis, we first introduce some basic duality theory. Recall that for a primal LP,

$$\textbf{(P)}\ \max\{c^T x : Ax = b, x \geq 0\},$$

its dual LP is defined as

$$\textbf{(D)}\ \min\{b^T y : A^T y \geq c, y \text{ free}\}$$

A significant assertion is that $c^T x \leq b^T y$, that the primal objective value is always less than or equal to the dual's. To show this,

$$y^T b = y^T Ax \geq c^T x$$

The reason why we define a dual is to construct a game, such that $\max c^T x \leq \min b^T y$ which can help us find an optimal solution. That said, when $z = c^T x = b^T y$ such that they are equal, then z is the maximum objective value of P and the minimum objective value of D. Thus we can define that,

A certificate of optimality is

$$c^T x = b^T y$$

such that x, y are a feasible solution to P, D respectively.

This states that x (resp. y) is also a maximizer(resp. minimizer) to P (resp. D), which is the optimal solution. This result comes from weak duality,

Weak Duality Theorem

Let x, y be a feasible solution to P, D respectively, then

1. $c^T x \leq b^T y$
2. If $c^T x = b^T y$ then x, y are an optimal solution to P, D respectively.

Note that the necessary condition is that x, y must be feasible, otherwise the equality does not guarantee an optimal value. What follows is strong duality,

Strong Duality Theorem

1. If both primal and dual are feasible, then they are optimal.
2. If one of the primal-dual pair is optimal, then the other one is also optimal, and their optimal values are equal.

Let us go back to the optimality condition in section 1.2, In the canonical form with respect to primal feasible basis $\mathcal{B}$, if $\bar{c} \leq 0$ then $\mathcal{B}$ is an optimal basis.

We now prove it using weak duality. Assume $\bar{c}_N \leq 0$. First we claim that,

$\bar{c}^T = (c^T - y^T A) \leq 0$ is the dual constraint of the primal LP, P, such that $c^T \leq y^T A$. If there exists any y that satisfies this constraint, then y is feasible.

As y is free, it is feasible once the dual constraint (the inequality) holds. Thus the $\bar{y}$ obtained in the canonical form is feasible. We also show that

$$c^T \bar{x} = b^T \bar{y}$$

since

$$c^T \bar{x} = (c^T - \bar{y}^T A)\bar{x} + \bar{y}^T b = 0 + \bar{y}^T b = \bar{y}^T b$$

Hence by weak duality theorem, they are optimal. Notice that, this equality states that during the transformation of canonical form with respect to any basis, $c^T \bar{x}$ always equals $b^T \bar{y}$. Thus once find that $\bar{x}, \bar{y}$ are both feasible, we can conclude the optimality. Rearrange these we obtain a stronger theorem,

If a basis $\mathcal{B}$ is both primal and dual feasible, then it is an optimal basis.

In fact this is similar to the proof of weak duality theorem. The converse is also true. The strong duality theorem states that, when optimal, both the primal-dual pair must be optimal. This enforces the feasibility as well. Therefore, we conclude that

Optimality Condition

A basis $\mathcal{B}$ is optimal if and only if it is both primal and dual feasible.

2.2 Complementary Slackness Conditions

The C.S. conditions are strongly related to optimality.

Complementary Slackness Conditions

1. $x_j = 0$ or the corresponding jth dual constraint is tight.
2. $y_i = 0$ or the corresponding ith primal constraint is tight.

3 Unboundedness

3.1 Unboundedness of LP

A certificate of unboundedness of **LP** is

$$c^T d > 0 \text{ (\textbf{a.k.a} } d \text{ is an objective direction)}$$
$$Ad = 0 \text{ (\textbf{a.k.a} } d \in \text{Null } A)$$
$$d \geq\neq 0 \text{ (\textbf{a.k.a} } d \text{ is a feasible direction)}$$

If a LP is unbounded, then we can find two non-negative feasible solutions $x_1 \leq\neq x_2$ such that $Ax_1 = b, Ax_2 = b$ with $c^T x_1 < c^T x_2$. That said, x_2 has at least one entry that is greater than that of x_1 and the objective value $c^T x_2$ is also greater than $c^T x_1$. If we keep increasing x in this direction, the objective value is likely increasing as well which is desired in the max problem. Rearrange above, we call $\vec{d} := x_2 - x_1 (d \geq\neq 0)$ the *direction of increment* such that

$$Ad = A(x_2 - x_1) = Ax_2 - Ax_1 = b - b = 0$$

by linearity and given that x_1, x_2 are feasible, and

$$c^T d = c^T (x_2 - x_1) = c^T x_2 - c^T x_1 > 0.$$

This demonstrates the derivation of the unboundedness certificate. The following example shows how d looks.

Example 3.1.1.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

Note the existence of positive and negative columns such that there might exist some non-negative null vector. That said, as x_3 grows infinitely large, x_1, x_2 can always be increased to cancel out x_3. For example, $d = t(1,1,1)^T$ is a feasible direction for any $t \geq 0$ including $\lim_{t\to\infty} t(1,1,1)^T$. Hence the LP is unbounded once $c^T d > 0$ is satisfied.

Example 3.1.2.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The nullspace is $\{(1,1,-1)^T\}$ which does not contain a feasible d, thus the feasible region is bounded (Note that $d := (1,1,-1)^T$ is also an unbounded direction if the $x \geq 0$ constraint is not present). An optimal solution exists if the LP is feasible. The proof will be presented below.

Condition 1 The first necessary condition of unboundedness is that the feasible region is unbounded. To illustrate this, we focus on d itself. Normalize $\hat{d}$ to demonstrate the direction only, and define a special line $\vec{x} = \bar{x} + td, t \geq 0 \in \mathbb{R}, \bar{x} \in \mathcal{F}$, where t is the length of the line and $\bar{x}$ is any point in the feasible region. That said, $\vec{x}$ is a line in direction d starting at any feasible point $\bar{x}$ which is obviously unbounded and feasible (i.e. $\vec{x} \in \mathcal{F}$ where any point on $\vec{x}$ is feasible),

$$A\vec{x} = A(\bar{x} + td) = A\bar{x} + tAd = b + t \cdot 0 = b, \vec{x} \geq 0.$$

This proves that $\mathcal{F}$ must be unbounded. In fact, if there exists any feasible direction d, then $\mathcal{F}$ is unbounded. The contrapositive states that any bounded $\mathcal{F}$ must contain an optimal solution, assuming that $\mathcal{F} \neq \emptyset$. Furthermore, a LP could have various feasible directions, where in this case are the non-negative null vectors, that said there might exist multiple lines $\vec{x}$.

Condition 2 Assuming that the first necessary condition holds, the second necessary condition of unboundedness is that the objective value $c^T x$ keeps increasing in the desired direction (e.g. growing in a max problem or declining in a min problem) as x continuously moves forward along the line $\vec{x}$, which we call an *objective direction.* For convenience we focus on max problems. Say $x := \bar{x} + td$ is a point on the line $\vec{x}$ where x moves along $\vec{x}$ as t increases, and we can see the change in objective value as t grows $\bar{t}$ is $c^T(\bar{t}d) = \bar{t}(c^T d)$. Hence, once $c^T d > 0$, any increment in t will result in an increase in the objective value. Equivalently, as t grows infinitely, the objective value will be

$$\lim_{t\to\infty} c^T(\bar{x} + td) = c^T\bar{x} + \lim_{t\to\infty} t(c^T d) = \lim_{t\to\infty} t(c^T d)$$

ignoring the constant $c^T\bar{x}$ and, if $c^T d > 0$, the objective value goes to ∞ which is unbounded.

Exercise 3.1.1. *Analyze the optimality and boundedness of the cases $c^T d \leq 0$. Where will the optimal solution be at? What about min problems?*

Exercise 3.1.2. *Assume that the LP model in 3.1.1 is feasible. Determine some objective vectors c such that the LP is optimal/unbounded. Do the same for*

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

3.2 Unboundedness of free LP

To better understand unboundedness, we will look at a more general model. Obtain the new model by removing the non-negativity constraint,

$$\textbf{(LPfree)} \;\; \max\{c^T x : Ax = b\}.$$

Without the $x \geq 0$ bounds, x can go any direction. Assume that x_1, x_2 are two feasible solutions such that $Ax_1 = b, Ax_2 = b$. Then both $x_2 \geq x_1, x_2 \leq x_1$ are fine, just make sure $x_1 \neq x_2$, and thus the new feasible direction will be $d := x_2 - x_1 \neq 0$, where the angle θ of d is not limited to be in the range $\theta \in [0, \frac{\pi}{2}]$ only. We claim that,

Any non-zero vector in the nullspace of A is a feasible direction.

That said if the nullity of A is n, then there will be n linearly independent feasible directions d. The following also come from the fundamanetal linear algebra,

If d is a feasible direction, so does $-d$. As well, $c^T d > 0$ iff $c^T(-d) < 0$.

We still use the definition of the special line from above, $\vec{x} := \bar{x} + td$. Recall that $\bar{x} \in \mathcal{F}$, $t \geq 0$, and d is normalized. The only difference is that now $\vec{x}$ could be negative but is still feasible in this model such that $\vec{x} \in \mathcal{F}$.

Assume that the above first necessary condition holds. We analyze the change in the objective value $c^T x$ as x moves forward along the line $\vec{x}$. Recall, the ending objective value depends on

$$\lim_{t \to \infty} c^T(\bar{x} + td) = \lim_{t \to \infty} t(c^T d)$$

ignoring the constant $c^T \bar{x}$. If $c^T d > 0$ then the LP is obviously unbounded. However, the above lemma states that, when $c^T d < 0$, pick $d^- = -d$ is still an objective direction such that $c^T d^- > 0$, where we know d^- is also an unbounded direction. Rearrange these we have that any $c^T d \neq 0$ will guarantee the unboundedness. To conclude,

A certificate of unboundedness of ***LPfree*** is

$$\begin{aligned} c^T d &\neq 0 \\ Ad &= 0 \\ d &\neq 0 \end{aligned}$$

Exercise 3.2.1. *Repeat 3.1.2 using the model **LPfree**, such that the $x \geq 0$ constraint is removed.*

Exercise 3.2.2. *Suppose a LPfree only has one feasible direction d. If $c^T d = 0$, where will the optimal solution be at?*

4 Union of Polyhedra

5 Sensitivity Analysis