

# Kirchhoff's Current Law Can Be Exact

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## Abstract

Kirchhoff's current law is thought to describe the translational movement of charged particles through resistors. But Kirchhoff's law is widely used to describe movements of current through resistors in high speed devices. Current at high frequencies/short times involves much more than the translation of particles. Transients abound. Augmentation of the resistors with *ad hoc* 'stray' capacitances is often used to introduce transients like those in real resistors. But augmentation hides the underlying problem, rather than solves it: the location, value and dielectric properties of the stray capacitances are not well determined. Here, we suggest a more general approach, that is well determined. If current is redefined as in Maxwell's equations independent of the properties of dielectrics, Kirchhoff's law is exact and transients arise automatically without ambiguity.

Kirchhoff's current law says, in a crude representation, that the current that flows into a node, must flow out. In textbooks, Kirchhoff's current law describes the translational movement of charges through resistors that might be called the flux of electrons. The resistors are ideal, described by single real numbers. The current is carried by charges that have mass, e.g., electrons, and the movements are slow, without transients, nearly at DC [1-13].<sup>1</sup>

Kirchhoff's law is used today to describe currents through resistors on the nanosecond time scale. Indeed, it is the main design tool for the circuits of our high-speed technology. The currents in high speed circuits have transients not seen when movements are slow near DC. Current through resistors on the nanosecond time scale involves delays and overshoots: it is a complex phenomena [14-22], not just the movement of electrons in wires, in resistors or into capacitors. Kirchhoff's current law is viewed as approximate for these reasons, as is clear from its derivations [13, 14, 23, 24].

Engineers have dealt with these difficulties by *ad hoc* augmentation of DC circuits [25]. They construct wideband circuit models made of

- (1) idealized resistors with current strictly equal  $V_R/R$ , where  $R$  is the resistance and  $V_R$  is the voltage across the resistor

in parallel with

- (2) ideal capacitors, often characterized as stray [14, 26], sometimes as 'parasitic' [27].

These ideal capacitors carry currents strictly equal to  $C \partial V / \partial t$ , where  $C$  is the capacitance,  $V$  is the voltage across the capacitor and  $t$  is time.

The size and location of the stray capacitances are chosen empirically so the augmented circuits more or less fit measurements of high-speed transients.

Chosen values and locations are neither exact nor unique. They are often crude approximations, because actual currents deviate significantly from  $C \partial V / \partial t$  or  $V_R/R$  on the time scales of our digital technology, in ways important in practice [18, 19, 21, 22, 28-34].

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<sup>1</sup> Sommerfeld [14], p. 101 describes the origin of Kirchhoff's laws, as Kirchhoff's solution to a problem posed in a seminar led by Neumann.

**Exact treatments have advantages.** Empirical and imprecise modifications of a circuit seem a poor substitute for an exact treatment, derivable from electrodynamics, if that is possible.

An exact treatment is possible using a rederivation of the law of conservation of current (eq. 3 below and ref. [25, 35-39]). We show (eq. 4, below) that Kirchhoff's law can be as exact as the Maxwell equations themselves<sup>2</sup>, once current is defined as in the Maxwell equations, independent of the dielectric properties of matter [35-37, 40].

The role of 'current' was evidently a key issue in Maxwell's development of electrodynamics, according to the historical literature [41-44]. Maxwell defined current as we have, according to his successors at Trinity College Cambridge UK, Jeans and Whitaker, [45], p.511; [46], p. 280, respectively. Lorrain and Corson [1], p. 276 eq. 6-148, use that definition as well.

**Current in the Maxwell equations.** Current appears in the equations of Maxwell in his generalization of Ampere's law.

#### **Maxwell's Version of Ampere's Law**

$$\frac{1}{\mu_0} \mathbf{curl} \mathbf{B} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (1)$$

$$\mathbf{J} = (\epsilon_r - 1)\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_{everything\ else} \quad (2)$$

See texts [1, 2, 4, 5, 47] for the standard formulation of Maxwell's equations. See ref. [35] for an update to Maxwell's equation that includes a more realistic (and general) description of polarization and permanent charge (not present in Maxwell's original formulation).  $\epsilon_0 \partial \mathbf{E} / \partial t$  is written separately in eq. (1) because it is a property of space, not matter, as discussed below. The variable  $\epsilon_r - 1$  and  $\mathbf{J}_{everything\ else}$  are properties of matter, not space, see below on p. 6.

$\mathbf{B}$  describes the magnetic field with magnetic constant (permeability of vacuum)  $\mu_0$ .  $\mathbf{E}$  describes the electric field, with electric constant  $\epsilon_0$  (permittivity of vacuum).  $\epsilon_r$  is the relative dielectric coefficient of perfect dielectrics, a single real positive constant  $\geq 1$ .  $\mathbf{J}$  is the current produced by all translation of mass—

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<sup>2</sup> Ref [19-23, 37, 41-43] apply the Maxwell equations within atoms using Bohm's version of quantum mechanics.

including all movements of mass with charge, however small or transient the movement.  $\mathbf{J}$  includes the polarization currents of dielectrics, ideal and real. The polarization of idealized dielectrics  $(\epsilon_r - 1)\epsilon_0 \partial\mathbf{E}/\partial t$  is isolated from other currents in eq. (2) only for convenience in relating our results to the literature [2, 5, 47-55]. Most of the literature of electrodynamics is written as if all dielectrics are ideal; Robinson [56] is a welcome exception that most resembles the treatment here.

Eq. (1)-(2) require a complete description of  $\mathbf{J}$ . A set of experimental results can serve this purpose. Theories or simulations of  $\mathbf{J}$  can also serve this purpose if they fit the experimental data.<sup>3</sup>

$\mathbf{J}$  includes charge carried by the flux of particles, as in most textbooks.  $\mathbf{J}_{everything\ else}$  includes the flux of particles and the nonideal polarization of real materials. It also includes other movements of charge, described below, and quantum effects [18-22, 36, 57-59] although those are not our focus.

**Current.** The current described as ‘Everything else’ in eq. (2) includes current produced by

- (1) transport (flux) of electrons or charged particles, as in classical Kirchhoff’s law of DC (or low frequency) circuit analysis.

‘Everything else’ also includes

- (2) the nonideal properties of dielectrics;
- (3) polarization of matter in general, time dependent, nonlinear or whatever;
- (4) currents driven by other fields, like diffusion, heat, and convection;
- (5) any other movement (of any type) of charge with mass, including quantum effects [18-22, 36, 57-59].

‘Everything else’ does *not* include

- (6) the properties of dielectrics idealized by  $(\epsilon_r - 1)\epsilon_0 \partial\mathbf{E}/\partial t$ . That is treated separately as is the custom in the classical literature and textbooks.
- (7) the polarization of the vacuum  $\epsilon_0 \partial\mathbf{E}/\partial t$ . That ethereal current is written separately from  $\mathbf{J}$  and  $\mathbf{J}_{everything\ else}$  in eq. (2).

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<sup>3</sup> Those theories and simulations are most useful if they are transferable from one set of experimental conditions to another, using just one set of parameters. Not all theories and simulations have that property (see the note [61] and its documentation in [62]). In chemical kinetics, parameters are customarily adjusted as conditions change so a favored equation—the law of mass action—always fits data.

**Polarization of the vacuum.** The term  $\epsilon_0 \partial \mathbf{E} / \partial t$  is treated separately because it has such a different origin.  $\epsilon_0 \partial \mathbf{E} / \partial t$  describes the ‘polarization of the vacuum’ [35] that allows (or should one say ‘supports’?) the ethereal propagation of electromagnetic waves through a vacuum devoid of mass. The polarization of the vacuum is a consequence of general physical laws [2, 47, 49, 50, 60-62], not a property of matter, and should really be called ‘the polarization of space’. The polarization  $\epsilon_0 \partial \mathbf{E} / \partial t$  makes charge relativistically invariant, independent of velocity, even at velocities approaching the speed of light. References include p. 553 of [2] ; p. 228, eq. 5-110, of [1]. See also ref. [47, 50, 60, 63, 64].

Charge is different from mass, length, and time, and most fundamental quantities. They vary with velocity according to the Lorentz transformation. Charge does not [2, 47, 49, 50, 60-62].

**Polarization of real materials.** Polarization of real materials is too complex to approximate usefully with a constant  $\epsilon_r$  [40]. Polarization is nonlinear in significant applications, particularly optical [56, 65-70]. Even when the polarization is linear, it involves complicated delays and varies too much with frequency/time and conditions to allow description with a constant  $\epsilon_r$ .

An abundant and classical literature—prominent since at least 1928 [71, 72]—reports the actual properties of polarization of real dielectrics, mostly linear [36, 71-100]. Polarization has been studied in great detail because it is a major determinant of the forces between molecules [101, 102].

Parsegian [101] discusses at length (and with admirable clarity) the connection between polarization and the spectra observed when light interacts with molecules. Spectra are used to identify molecules—more or less as successfully as fingerprints identify people—because spectra (and polarization) are sensitive to details of chemical structure and thus remarkably diverse, almost as diverse as the molecules themselves [101, 103-107]. The diverse polarization and spectra of real materials obviously cannot be described by a single number or dielectric constant.

Despite this literature,  $\epsilon_r$  has been treated as a single real positive constant  $\geq 1$  in textbooks of electrodynamics [1-5, 47, 49, 50, 61, 62] for many years [108, 109]; Robinson [56] is a welcome exception.

The properties of charge movement in matter are so complex that Feynman concluded that nothing much could be said in general (on p. 10-7 of [5]). It is necessary “... to exhibit in every case all the charges, whatever their origin, [so] the equations [of electrodynamics] are always correct.” In fact, something important can be said about electrodynamics in general, independent of the properties of matter. Current, as defined by Maxwell, is universally conserved [35-39].

**Conservation of current can be derived without reference to matter:**

Conservation of current  $\mathbf{J}_{total}$  is in fact a general and exact property of the Maxwell equations as general as the Maxwell equations themselves, independent of any properties of matter [35-39]. The dielectric constant  $\epsilon_r$  is not involved in the derivation of conservation of current at all. The derivation involves no statement or approximation to dielectric properties or polarization, nonlinear or linear, or any other properties of matter whatsoever.

**Conservation of current.** A general statement of conservation of current—eq. (3) below—can be derived [35-39] because the divergence of the curl in eq. (1) is identically zero, independent of any properties of matter, whenever Maxwell's equations can be used. The crucial term that produces universal conservation of current is the polarization of space  $\epsilon_0 \partial \mathbf{E} / \partial t$ . The polarization of space has nothing to do with matter because  $\epsilon_0$  has nothing to do with matter.  $\epsilon_0$  is a property of space, not matter.

Physically, the polarization of the vacuum creates the ethereal current  $\epsilon_0 \partial \mathbf{E} / \partial t$ . The ethereal current is an output of the Maxwell equations that varies so the total current  $\mathbf{J}_{total} = \mathbf{J} + \epsilon_0 \partial \mathbf{E} / \partial t$  is conserved [35]. The ethereal current allows current to be conserved no matter what physics is involved—see eq. 4 of [39] and Fig. 2 of [37]—even quantum physics (see eq. 45 of [36]).

$$\text{Conservation of Current} \quad \mathbf{div} \left( \overbrace{\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}}^{\text{Current}} \right) = 0 \quad (3)$$

**Novel derivation, apparently.** This general statement of conservation of current is not easily found in textbooks or the literature of electrodynamics.

The usual derivations of conservation of current (and the continuity equation derived in [35]) involve the dielectric constant  $\epsilon_r$  and treat it as a single real constant number in contradiction to experimental measurements of dielectric properties [36, 71-100], spectra [103-107, 110], and nonlinear polarization [56, 65-70].

Usual derivations of conservation of current deal with  $\mathbf{J}$  and use a poor approximation— $\epsilon_r$  as a single constant—that does not fit experimental data [40], so it is natural to conclude (incorrectly) that conservation of current is a poor approximation that does not fit experimental data either. Indeed,  $\mathbf{J}$  is not universally conserved (because it neglects the ethereal current)!

However, current  $\mathbf{J}_{total}$  is conserved universally. Conservation of total current is not an approximation because  $\mathbf{J}_{total}$  includes the ethereal current  $\epsilon_0 \partial \mathbf{E} / \partial t$ , as shown in [35-39].

**Kirchhoff's current law** is the general conservation law eq. (3) rewritten for branched one dimensional networks.<sup>4</sup> One dimensional networks have the special property that  $\text{curl } \mathbf{B} = \mathbf{0}$ , as is apparent if one writes out the curl operator explicitly in one dimension [111, 112].

Kirchhoff's law is widely viewed as an approximation, needing derivation; see p. 8-10 of [14] and [13, 23, 24] for some derivations, along with less precise discussions in most textbooks describing circuits. But Kirchhoff's law need not be approximate if current is defined<sup>5</sup> to include the ethereal term  $\epsilon_0 \partial \mathbf{E} / \partial t$  [35-39].

$$\text{Proposed Definition:} \quad \text{Current} \triangleq \mathbf{J}_{total} = \mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (4)$$

With this definition of  $\mathbf{J}_{total}$ , all the current that flows into a node, flows out, exactly, at any time, no matter how brief, under all conditions in which the Maxwell equations apply.

'Current' is defined in many textbooks as the charge carried by the translation of charged particles [2, 5-8, 11, 12, 14, 15, 17, 47-55, 113, 114]. This ordinary translational current is just part of the current defined here. In eq. (2), another part of the current is classical, namely the current through an ideal dielectric  $(\epsilon_r - 1) \epsilon_0 \partial \mathbf{E} / \partial t$ .

The ordinary definition of current in Kirchhoff's law as only the flux of charged particles causes practical difficulty when applied to high speed circuits made only of idealized resistors (described by constant real numbers) [13, 14, 23-25]. Such idealized circuits do not have the transients, delays, or overshoots found in real circuits made of resistors [18, 23, 24, 28-34]. Idealized circuits cannot deal with the actual behavior of circuits observed on the nanosecond time scale if they use the ordinary statement of Kirchhoff's law and the ordinary definition of current as  $\mathbf{J}$ . Of course, circuits can be made more realistic by adding fictitious circuit elements.

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<sup>4</sup> The precise definition of a network involves many issues beyond the scope of this paper [(6-12, 15-18, 114, 115].

<sup>5</sup> Lorrain and Corson [1], p. 276 eq. 6-148, use this definition of current. As far as I can tell, they do not discuss the approximate nature of  $\epsilon_r$ . If the approximate nature of  $\epsilon_r$  is not discussed, the reader will then naturally think (incorrectly) that conservation of current is as unrealistic as the idealization of a single dielectric constant. The development of Robinson [57] does not depend on the approximate nature of  $\epsilon_r$ , but it does not use this definition of current.



**Stray Capacitances.** Engineers routinely add capacitances—that are not actual distinct circuit components—to circuits of idealized resistors [25], so the circuits approximate the transients observed in real circuits [26, 27]. The location and values of the stray capacitances are at the choice of the engineer.

The redefinition of current by equation (4) produces transients that are determined precisely. Reference [25], eq. 14, shows how to choose capacitances arising from  $\epsilon_0 \partial \mathbf{E} / \partial t$  once current is defined by eq. 4.

Transients in real circuits are more complicated than those predicted in this way. Polarization of real circuits involves more than  $\epsilon_r \epsilon_0 \partial \mathbf{E} / \partial t$  on the time scales of practical importance in our digital technology. Inductive effects of wires [20-24]; capacitances arising from materials like circuit boards [28-33], and the complex geometry of the real circuit contribute additional terms beyond those arising from  $\epsilon_0 \partial \mathbf{E} / \partial t$  or  $(\epsilon_r - 1) \epsilon_0 \partial \mathbf{E} / \partial t$ . Measurements need to be made of the polarization of real circuit boards because they are likely to depend on details of composition and construction, and perhaps on the class of signals in the circuits.

Reference [41] shows how to include *any* types of current flow—polarization, inductive, or anything else—in the analysis, so the theory can cope with the properties of real circuit boards.

**Role of the  $\mathbf{B}$  field.** The lack of magnetic field  $\mathbf{B}$  in branched one dimensional circuits is discussed at length in [35] where the speculation is made that the absence of  $\mathbf{B}$  fields makes circuit design (at high speeds) much easier.  $\mathbf{B}$  fields produce ‘leaks of current’ and cross talk that are difficult to deal with in circuit design, particularly because they are so variable and dependent on properties of the signals themselves, as well as details of layout, etc.

It seems to me that ground planes in high speed circuits might function better—i.e., more ideally, obeying the DC version of Kirchhoff’s law more accurately—if they too were built as branched one dimensional circuits, with minimal  $\mathbf{B}$  fields, cross talk, and current leakage in the grounds.

**Conclusion.** Stray capacitances, of unknown value, location, and unrealistic properties seem a poor substitute for currents defined exactly by the Maxwell equations. Transients arise naturally if current in Kirchhoff’s law is defined as in Maxwell’s equation, see eq. (1)-(4). Realistic transients, arising from nonideal properties of circuits, can be easily incorporated into our treatment [35].

## Appendix

### Networks of Chemical Reactions

Our discussion has focused on Kirchhoff's law in electrical networks. Networks and analogs of Kirchhoff's law are also used widely to describe interacting chemical reactions in chemistry, biochemistry and biology [115-122]. The hundreds of enzyme reactions in the intermediary metabolism that form 'the chemical factory of life' are a notable example. These networks of chemical reactions are used by thousands of scientists every day to explain medical and biological phenomena.

**Chemical networks.** Equations of chemical [115-120] and enzyme kinetics [121, 122] use analogs of Kirchhoff's law to connect chemical reactions. The analogs describe the flux of particles, not the flow of electric current. The analog equations use conservation of mass but rarely mention or use conservation of current. They do not include an  $\epsilon_r \epsilon_0 \partial \mathbf{E} / \partial t$  term and so cannot conserve  $\mathbf{J}_{total}$ . They also do not conserve  $\mathbf{J}_{total}$  in the steady-state, as is clear by using the law of mass action of chemical reactions to compute and compare steady state currents for two reactions in series [35, 37, 123, 124].

**Auxiliary conditions may be possible** that make equal the current  $\mathbf{J}_{total}$  in a series of chemical reactions but auxiliary conditions are not found in the standard references [115-122].

Rate 'constants' of a series of reactions are coupled even for chemical reactions that are distinct and disjoint in space, far apart on an atomic length scale, occurring in different structures. The atomic scale electric field in one reaction is influenced by distant reactions—and by even more distant macroscopic boundary conditions—to create 'flux coupling' that would not be present without this long range influence.

The auxiliary equations needed to conserve current interact with the equations of chemical kinetics that conserve mass. One set of conservation laws is not enough. Both must be solved together. One way to do this is to write separate networks for electric current and for mass flow. The networks together can be solved with the methods of the theory of complex fluids which are designed to deal with multiple simultaneous force and flow fields. In particular, the variational method of complex fluids [125-136] may allow one to create a new synthetic composite functional with units of energy. In this way, flux coupling can be dealt with consistently on all scales, i.e., with all variables satisfying all boundary conditions with one set of unchanging parameters.

**Flux coupling** plays a central role in many transport systems and chemical reactions in biology. Flux coupling allows the ‘unnatural’ uphill transport of ions (and solutes) to be driven by the natural downhill movement of other ions (and solutes). Active transport of this sort occurs throughout biological cells and organelles and is one of the fundamental mechanisms of life.

Flux coupling is a central mechanism in oxidative phosphorylation and photosynthesis, which provide the energy for life. In oxidative phosphorylation and photosynthesis, electron flow is coupled to the movement of ions and ‘protons’ across membranes of mitochondria.<sup>6</sup> It is clear that flux coupling is a combination of macroscopic and atomic scale phenomena involving their mutually generated electric field, as well as (perhaps) chemical interactions.

**Flux coupling depends on the experimental conditions.** Flux coupling depends on anything that can change the macroscopic electric field. In particular, flux coupling depends on the boundary conditions.

Boundary conditions are different in different setups used to measure flux and flux coupling. Some setups leave the system in its natural state, with transporters/channels in native structures, like mitochondria. Other setups insert the transporter/channel into a lipid bilayer.

Flux coupling will be different in the two setups because they impose different boundary conditions. In mitochondria, the sum of currents across a membrane is zero (as it is in cells or organelles shorter than a length constant or so [137]). The biological situation imposes the condition that the sum of all currents across the membrane is zero, because structures, like mitochondria, are so small.

Transporters are often studied, however, in non-native bilayer setups where currents are only constrained by the applied electrical and chemical potential (i.e., concentrations). Quite different results for flux coupling can occur in that case, quite different from the flux coupling that is found in the mitochondria, because the constraints on the fluxes (the boundary conditions produced by conservation of current) are so different.

**Flux Coupling and Channel Opening.** Manuel Landstorfer and I recently realized [36] that a different kind of coupling can occur between the voltage sensor of a voltage sensitive channel [138-143] and its conduction pore. Part of the current injected by the voltage sensor might flow through the conduction pore and trigger its opening.

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<sup>6</sup> ‘Proton’ is a nickname for the positively charged form of water, sometimes written as  $\text{H}_3\text{O}^+$ .

Similarly, flux coupling can occur when a transmitter binds to a receptor on a channel protein, like acetylcholine binding to the acetylcholine channel. Binding of a charged agonist to a charged receptor produces current and some of that current can be injected into the conduction pore, triggering its opening, even though the pore is far away from the receptor.

**Historical Note.** The reason  $J_{total}$  is not conserved in chemical kinetics [115-122] is historical, I suspect. Chemists were understandably focused on mass and its transformations not electric current and its flow. Conservation of mass is used in the derivation of the equations of enzyme kinetics, but conservation of current is not mentioned, to the best of my knowledge.

Conservation of mass allows accumulation of mass but is silent about conservation of current, whether steady-state or transient. In contrast, conservation of current, and the Maxwell equations, do not allow accumulation of the current  $J_{total}$ , whether steady-state or transient, not at all, not under any conditions or at any time. All the  $J_{total}$  that flows into a node flows out.

**Conclusion.** It seems wise to use network models in both chemistry and electronics that conserve  $J_{total}$  with as few *ad hoc* extensions to Maxwell's equations as possible: the artifacts in electrical potential and fluxes can be very large if current is not conserved. The artifact in the electric potential is large, because of the strength of the electric field (see the first page of Feynman [5] and Appendix of [37]). Fluxes often flow over large barriers, where flux is an exponential function of potential. Flux artifacts can then be exponentially large and are best avoided.

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