For Reference Only

NOT Standard Answer

Assignment II: Fictitious Play

[535523] Game Theory and Its Applications by Prof. Li-Hsing Yen

Q1. (10%) One pure-strategy Nash Equilibrium

See the following game matrix

	c_1	c_2
r_1	(-1,-1)	(1,0)
r_2	(0,1)	(3,3)

It has only one pure-strategy Nash equilibrium (r_2, c_2). Can it converge to the pure-strategy Nash equilibrium by fictitious play? Please justify your answer clearly.

Yes, it can converge to the pure-strategy Nash equilibrium (r_2, c_2) by fictitious play.

Show your explanation with implementation source code here

Explanation for reference:

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 times and c_2 1 time. Player 2 begins with the prior belief that Player 1 has played r_1 0 times and r_2 1 time. After initializing the prior belief setting, we can obtain the following result.

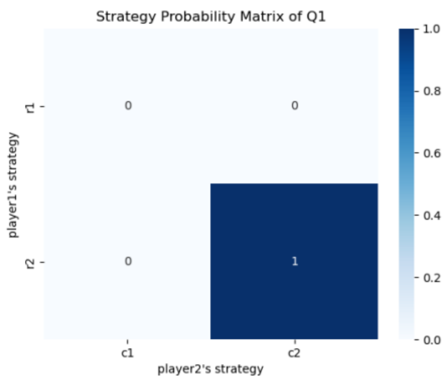

Fig Q1-1. two players converge to (r_2, c_2)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,1)	(0,1)	1 3	1 3
1	r_2	c_2	(0,2)	(0,2)	2 6	2 6
2	r_2	c_2	(0,3)	(0,3)	3 9	3 9
…	…	…	…	…	…	…
1000	r_2	c_2	(0,1001)	(0,1001)	1001 3003	1001 3003

Table Q1-1. two players converge to (r_2,c_2)

When analyzing this game, it is concluded that there exists one pure-strategy Nash equilibrium, namely (r_2,c_2). Two players can converge to the pure-strategy Nash equilibrium by fictitious play in this game. It can converge to the Nash equilibrium immediately, which means that player 1 will play r_2 and player 2 will play c_2 from the beginning. In Fig Q1-1, it is shown how many times it converges to which Nash equilibrium out of 1000 times. Actually, it does not matter what the players' initial beliefs of the other player is. This makes sense because if player 1 believes that player 2 will play c_1, the best choice for player 1 is r_2 since the payoff of 0 is higher than -1 and if player 1 believes that player 2 will play c_2, the best choice for player 1 is also r_2 since the payoff of 3 is higher than 1. The same reasoning goes for player 2. If player 2 believes that player 1 will play r_1, the best choice for player 2 is c_2 since the payoff of 0 is higher than -1 and if player 2 believes that player 1 will play r_2, the best choice for player 2 is also c_2 since the payoff of 3 is higher than 1. Thus, in all cases the best choice for the players is choosing r_2 and c_2. So, in this case it does not matter what the initial beliefs of the players is set to as the players will respond in the same way for every initial belief. In Table Q1-1 the results are shown of fictitious play for the game, with the initial beliefs in which Player 1 with the prior belief that Player 2 has played c_1 0 times and c_2 1 time and Player 2 with the prior belief that Player 1 has played r_1 0 times and r_2 1 time. From the results, it is seen that the beliefs of each player can converge rapidly. Therefore, both players will play r_2 and c_2 immediately in the first round, that is round 1, and never change their individual strategy permanently.

Q2. (10%) Two or more pure-strategy NE

See the following game matrix

	c_1	c_2
r_1	(2,2)	(1,0)

	c_1	c_2
r_2	(0,1)	(3,3)

It has two pure-strategy Nash equilibria (r_1, c_1), (r_2, c_2) respectively. Can it converge to both of the pure-strategy Nash equilibria by fictitious play? Or just one of them? Please justify your answer.

Yes, it can converge to both of the pure-strategy Nash equilibria (r_1, c_1), (r_2, c_2) by fictitious play.

Show your explanation with implementation source code here

Explanation for reference:

- Pure-strategy NE: (r_1, c_1)

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 1 time and c_2 0 times. Player 2 begins with the prior belief that Player 1 has played r_1 1 time and r_2 0 times. After initializing the prior belief setting, we can obtain the following result.

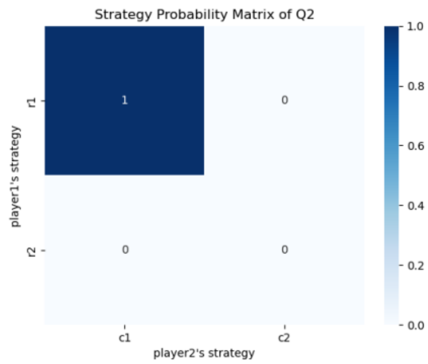

Fig Q2-1. two players converge to (r_1, c_1)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(1,0)	(1,0)	2 0	2 0
1	r_1	c_1	(2,0)	(2,0)	3 0	3 0
2	r_1	c_1	(3,0)	(3,0)	4 0	4 0
…	…	…	…	…	…	…
1000	r_1	c_1	(1001,0)	(1001,0)	2002 0	2002 0

Table Q2-1. two players converge to (r_1, c_1)

When player 1 believes that player 2 is more likely to play c_1, that is $b_1 = (1,0)$ and at the same time player 2 believes that player 1 is more likely to play r_1, that is $b_2 = (1,0)$, then it converges immediately to (r_1, c_1). It is shown in Fig Q2-1. This makes sense because when $b_1 = (1,0)$, the payoff for player 1 choosing strategy r_1 is $2 \times 1 + 1 \times 0 = 2$ and for strategy r_2 is $0 \times 1 + 3 \times 0 = 0$. Obviously, since the payoff for selecting strategy r_1 is better for player 1. Similarly, when $b_2 = (1,0)$, the payoff for player 2 choosing strategy c_1 is $2 \times 1 + 1 \times 0 = 2$ and for strategy c_2 is $0 \times 1 + 3 \times 0 = 0$. Thus, selecting strategy c_1 is better for player 2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_1 for player 1 and c_1 for player 2. We also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q2-1. It shows that the beliefs of each player can converge rapidly. Therefore, both players will play r_1 and c_1 immediately in the first round, that is round 1, and never change their individual strategy permanently.

- Pure-strategy NE: (r_2, c_2)

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 times and c_2 1 time. Player 2 begins with the prior belief that Player 1 has played r_1 0 times and r_2 1 time. After initializing the prior belief setting, we can obtain the following result.

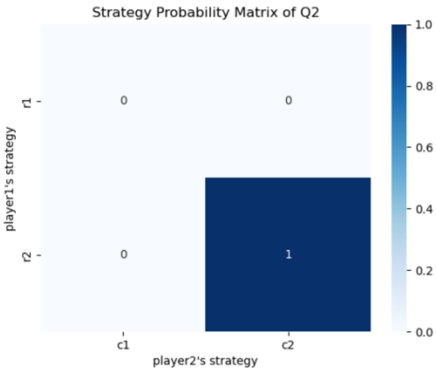

Fig Q2-2. two players converge to (r_2, c_2)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,1)	(0,1)	1 3	1 3
1	r_2	c_2	(0,2)	(0,2)	2 6	2 6
2	r_2	c_2	(0,3)	(0,3)	3 9	3 9
…	…	…	…	…	…	…
1000	r_2	c_2	(0,1001)	(0,1001)	1001 3003	1001 3003

Table Q2-2. two players converge to (r_2, c_2)

When player 1 believes that player 2 is more likely to play c_2, that is $b_1 = (0,1)$ and at the same time player 2 believes that player 1 is more likely to play r_2, that is $b_2 = (0,1)$, then it converges immediately to (r_2, c_2). It is shown in Fig Q2-2. This makes sense because when $b_1 = (0,1)$, the payoff for player 1 choosing strategy r_1 is $2 \times 0 + 1 \times 1 = 1$ and for strategy r_2

is $0 \times 0 + 3 \times 1 = 3$. Obviously, since the payoff for selecting strategy r_2 is better for player 1. Similarly, when $b_2 = (0,1)$, the payoff for player 2 choosing strategy c_1 is $2 \times 0 + 1 \times 1 = 1$ and for strategy c_2 is $0 \times 0 + 3 \times 1 = 3$. Thus, selecting strategy c_2 is better for player 2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_2 for player 1 and c_2 for player 2. We also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q2-2. It shows that the beliefs of each player can converge rapidly. Therefore, both players will play r_2 and c_2 immediately in the first round, that is round 1, and never change their individual strategy permanently. In addition to the two pure-strategy NE cases mentioned above, one more interesting thing is that there exists a mixed-strategy NE with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$ in this game matrix. We will then demonstrate the procedure and discuss the result as following.

- Mixed-strategy NE with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 time and c_2 0.5 times. Player 2 begins with the prior belief that Player 1 has played r_1 0.5 times and r_2 0 time. After initializing the prior belief setting, we can obtain the following result.

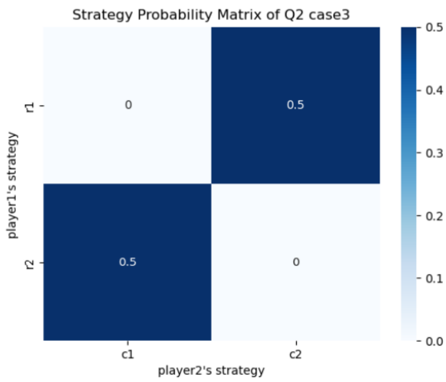

Fig Q2-3. two players converge to mixed-strategy NE

with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,0.5)	(0.5,0)	0.5 1.5	1 0
1	r_2	c_1	(1,0.5)	(0.5,1)	2.5 1.5	2 3
2	r_1	c_2	(1,1.5)	(1.5,1)	3.5 4.5	4 3
…	…	…	…	…	…	…
1000	r_1	c_2	(500,500.5)	(500.5,500)	1500.5 1501.5	1501 1500

Table Q2-3. two players converge to mixed-strategy NE

with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

When player 1 believes that player 2 is more likely to play c_2, that is $b_1 = (0,0.5)$ and at the same time player 2 believes that player 1 is more likely to play r_1, that is $b_2 = (0.5,0)$. The result of picture under such parameter setting is shown in Fig Q2-3. When initializing the prior belief $b_1 = (0,0.5)$ for player 1, the payoff for player 1 choosing strategy r_1 is $2 \times 0 + 1 \times 0.5 = 0.5$ and for strategy r_2 is $0 \times 0 + 3 \times 0.5 = 1.5$. Obviously, since the payoff for selecting strategy r_2 is better for player 1. Similarly, when setting the prior belief $b_2 = (0.5,0)$ for player 2, the payoff for player 2 choosing strategy c_1 is $2 \times 0.5 + 1 \times 0 = 1$ and for strategy c_2 is $0 \times 0.5 + 3 \times 0 = 0$. Thus, selecting strategy c_1 is better for player 2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_2 for player 1 and c_1 for player 2. And then, we also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q2-3. As the number of rounds tends to infinity the empirical distribution of each player will converge to (0.5,0.5), which is the same as the mixed-strategy Nash equilibrium. Actually, you can obtain the same result under different prior belief setting and we remain this kind of condition or critical value to readers. To our surprise, we can observe that two players always change their strategy simultaneously as Fig Q2-4. In the first round, that is round 1, player 1 select r_2 and player 2 select c_1 as their action respectively according to their payoff. Then, they shift their strategy from original one to the other one rapidly in the next round, that is player 1 choose r_1 and player 2 choose c_2 as their action respectively. It makes sense since both of the players take their action with fictitious play at the same time. If they take the action with fictitious play alternately, that is make decision individually in turns, then it is obvious to conclude that it would not get the same result as this situation. In other word, it can only converge to both of the pure-strategy NE instead of mixed-strategy NE in the long round.

	c_1	c_2
r_1	(2,2)	(1,0)
r_2	(0,1)	(3,3)

Fig Q2-4. two players change their strategy simultaneously

By the way, the original question 2 is as follows.

	c_1	c_2
r_1	(2,2)	(0,0)

	c_1	c_2
r_2	(0,0)	(3,3)

Although there exists a mixed-strategy in this game matrix due to typo error, it does not affect your answer for finding the pure-strategy NE. No matter you can find out the mixed one in this question, it does not affect your score in this part. The key point is to request you to take the fictitious play to find out both of the pure-strategy NE in such a game matrix.

Q3. (10%) Two or more pure-strategy NE (Conti.)

See the following game matrix

	c_1	c_2
r_1	(1,1)	(0,0)
r_2	(0,0)	(0,0)

It also has two pure-strategy Nash equilibria (r_1, c_1), (r_2, c_2) respectively. Can it converge to the both of the pure-strategy Nash equilibria by fictitious play? Or just one of them? Please justify your answer.

No, it can only converge to one Nash equilibrium (r_1, c_1) by fictitious play.

Show your explanation with implementation source code here

Explanation for reference:

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 1 time and c_2 0 times. Player 2 begins with the prior belief that Player 1 has played r_1 1 time and r_2 0 times. After initializing the prior belief setting, we can obtain the following result.

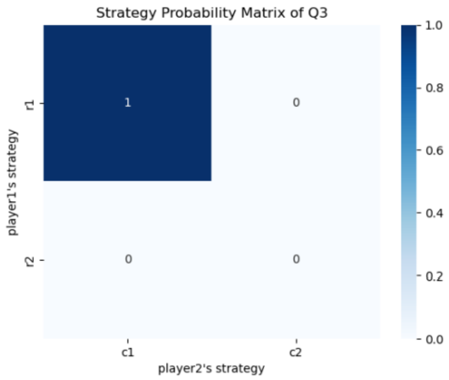

Fig Q3-1. two players converge to (r_1, c_1)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(1,0)	(1,0)	$\underline{1}$ 0	$\underline{1}$ 0
1	r_1	c_1	(2,0)	(2,0)	$\underline{2}$ 0	$\underline{2}$ 0
2	r_1	c_1	(3,0)	(3,0)	$\underline{3}$ 0	$\underline{3}$ 0
…	…	…	…	…	…	…
1000	r_1	c_1	(1001,0)	(1001,0)	$\underline{1001}$ 0	$\underline{1001}$ 0

Table Q3-2. two players converge to (r_1, c_1)

When player 1 believes that player 2 is more likely to play c_1, that is $b_1 = (1,0)$ and at the same time player 2 believes that player 1 is more likely to play r_1, that is $b_2 = (1,0)$. The result of picture under such parameter setting is shown in Fig Q3-1. When initializing the prior belief $b_1 = (1,0)$ for player 1, the payoff for player 1 choosing strategy r_1 is $1 \times 1 + 0 \times 0 = 1$ and for strategy r_2 is $0 \times 1 + 0 \times 0 = 0$. Obviously, since the payoff for selecting strategy r_1 is better for player 1. Similarly, when setting the prior belief $b_2 = (1,0)$ for player 2, the payoff for player 2 choosing strategy c_1 is $1 \times 1 + 0 \times 0 = 1$ and for strategy c_2 is $0 \times 1 + 0 \times 0 = 0$. Thus, selecting strategy c_1 is better for player 2. And then, we also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q3-1. As you can see, both of players can converge to (r_1, c_1) immediately in the first round, that is round 1, and never change their individual strategy permanently. When analyzing this game, it is concluded that there exist two pure-strategy Nash equilibrium, namely (r_1, c_1) and (r_2, c_2). However, no matter how to initialize the prior belief both of the player can only converge to one pure-strategy NE, that is (r_1, c_1) in this game matrix by fictitious play. It totally makes sense because the fictitious play adopts the best response strategy. Since the payoff of selecting r_2 for player 1 is definitely zero, it is theoretically impossible for player 1 to always select r_2. Once player 1 pick r_1 as its strategy at any round, it will continue to choose the same one and never change its decision on and on. In addition, Player 2 also has the same situation since it also takes the best response strategy in the fictitious play process. Even if player 1 has numerous prior belief c_2 for player 2 and player 2 has lots of prior belief r_1 for player 1 from the beginning, the convergence result (r_1, c_1) is determined in all of the cases.

Q4. (10%) Mixed-Strategy Nash Equilibrium

See the following game matrix

	c_1	c_2
r_1	(0,1)	(2,0)
r_2	(2,0)	(0,4)

Although it has no pure-strategy Nash equilibrium, there exists a mixed-strategy Nash equilibrium, i.e., $P(r_1) = \frac{4}{5}$, $P(r_2) = \frac{1}{5}$ for Player 1 and $P(c_1) = \frac{1}{2}$, $P(c_2) = \frac{1}{2}$ for Player 2. Can it converge to the mixed-strategy Nash equilibrium by fictitious play? Please justify your answer.

Yes, it can converge to the mixed-strategy Nash equilibrium by fictitious play.

Show your explanation with implementation source code here

Explanation for reference:

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 times and c_2 1 time. Player 2 begins with the prior belief that Player 1 has played r_1 1 time and r_2 0 times. After initializing the prior belief setting, we can obtain the following result.

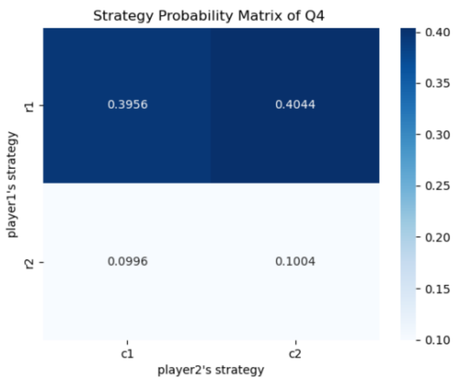

Fig Q4-1. two players converge to mixed-strategy NE

with $P(r_1) = 0.8, P(r_2) = 0.2$ and $P(c_1) = 0.5, P(c_2) = 0.5$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,1)	(1,0)	$\underline{2}$ 0	$\underline{1}$ 0
1	r_1	c_1	(1,1)	(2,0)	$\underline{2}$ 2	$\underline{2}$ 0
2	r_1	c_1	(2,1)	(3,0)	2 $\underline{4}$	$\underline{3}$ 0
…	…	…	…	…	…	…
1000	r_1	c_2	(496,505)	(782,219)	$\underline{1010}$ 992	782 $\underline{876}$

Table Q4-2. two players converge to mixed-strategy NE

with $P(r_1) = 0.8, P(r_2) = 0.2$ and $P(c_1) = 0.5, P(c_2) = 0.5$

When player 1 believes that player 2 is more likely to play c_2, that is $b_1 = (0,1)$ and at the same time player 2 believes that player 1 is more likely to play r_1, that is $b_2 = (1,0)$. The result of picture under such parameter setting is shown in Fig Q4-1. When initializing the prior belief $b_1 = (0,1)$ for player 1, the payoff for player 1 choosing strategy r_1 is $0 \times 0 + 2 \times 1 = 2$ and for strategy r_2 is $2 \times 0 + 0 \times 1 = 0$. Obviously, since the payoff for selecting strategy r_1 is better for player 1. Similarly, when setting the prior belief $b_2 = (1,0)$ for player 2, the payoff for player 2 choosing strategy c_1 is $1 \times 1 + 0 \times 0 = 1$ and for strategy c_2 is $0 \times 1 + 4 \times 0 = 0$. Thus, selecting strategy c_1 is better for player 2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_1 for player 1 and c_1 for player 2. And then, we also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q4-2. As the number of rounds tends to infinity the empirical distribution of each player will converge to some specific number, that is $P(r_1) = \frac{4}{5}, P(r_2) = \frac{1}{5}$ and $P(c_1) = \frac{1}{2}, P(c_2) = \frac{1}{2}$, which is the same as the mixed-strategy Nash equilibrium. Actually, it does not matter what the players' initial beliefs of the each other player is. It makes sense since when player 1 believes that player 2 is more likely to choose c_1, player 1 will choose r_2. Then in return if player 2 believes that player 1 will choose r_2 with a chance higher than 0.2, it will choose c_2. When player 1 believes that player 2 is more likely to choose c_2, it will choose r_1. As a response, when player 2 believes that player 1 will choose r_1 with a chance higher than 80%, it will choose c_1 which returns to the beginning. Thus, the players will keep changing their strategies alternatively, which lead to the mixed-strategy Nash equilibrium, namely $P(r_1) = \frac{4}{5}, P(r_2) = \frac{1}{5}$ for player 1 and $P(c_1) = \frac{1}{2}, P(c_2) = \frac{1}{2}$ for player 2.

Q5. (10%) Best-reply path

See the following game matrix

	c_1	c_2
r_1	(0,1)	(1,0)
r_2	(1,0)	(0,1)

Although it has no pure-strategy Nash equilibrium, there exists a finite best-reply path. Can it converge to the mixed-strategy Nash equilibrium by fictitious play? Please justify your answer.

Yes, it can converge to the mixed strategy Nash equilibrium by fictitious play.

Show your explanation with implementation source code here

Explanation for reference:

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 times and c_2 1 time. Player 2 begins with the prior belief that Player 1 has played r_1 1 time and r_2 0 times. After initializing the prior belief setting, we can obtain the following result.

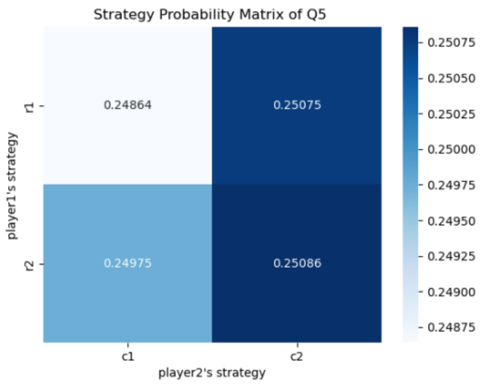

Fig Q5-1. two players converge to mixed-strategy NE

with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,1)	(1,0)	<u>1</u> 0	<u>1</u> 0
1	r_1	c_1	(1,1)	(2,0)	<u>1</u> 1	<u>2</u> 0
2	r_1	c_1	(2,1)	(3,0)	1 <u>2</u>	<u>3</u> 0
…	…	…	…	…	…	…
1000	r_2	c_2	(519, 482)	(503,498)	482 <u>519</u>	498 <u>503</u>

Table Q5-2. two players converge to mixed-strategy NE

with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

When player 1 believes that player 2 is more likely to play c_2, that is $b_1 = (0,1)$ and at the same time player 2 believes that player 1 is more likely to play r_1, that is $b_2 = (1,0)$. The result of picture under such parameter setting is shown in Fig Q5-1. When initializing the prior belief $b_1 = (0,1)$ for player 1, the payoff for player 1 choosing strategy r_1 is $0 \times 0 + 1 \times 1 = 1$ and for strategy r_2 is $1 \times 0 + 0 \times 1 = 0$. Obviously, since the payoff for selecting strategy r_1 is better for player 1. Similarly, when setting the prior belief $b_2 = (1,0)$ for player 2, the payoff for player 2 choosing strategy c_1 is $1 \times 1 + 0 \times 0 = 1$ and for strategy c_2 is $0 \times 1 + 1 \times 0 = 0$. Thus, selecting strategy c_1 is better for player 2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_1 for player 1 and c_1 for player 2. And then, we also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q5-2. As the number of rounds tends to infinity the empirical distribution of each player will converge to some specific number, that is $P(r_1) = \frac{1}{2}, P(r_2) = \frac{1}{2}$ and $P(c_1) = \frac{1}{2}, P(c_2) = \frac{1}{2}$, which is the same as the mixed-strategy Nash equilibrium. Actually, it does not matter what the players' initial beliefs of the other player is. It makes sense since when player 1 believes that player 2 is more likely to choose c_1, player 1 will choose r_2. Then, in return if player 2 believes that player 1 is more likely to choose r_2, player 2 will choose c_2. When player 1 believes that player 2 is more likely to choose c_2, it will choose r_1. As a response, when player 2 believes that player 1 is more likely to choose r_1, player 2 will choose c_1 which returns to the beginning. Thus, the players will keep changing their strategies alternatively, which lead to the best reply path as following Fig Q5-2. And two player's action can converge to the mixed-strategy Nash equilibrium, namely $P(r_1) = \frac{1}{2}, P(r_2) = \frac{1}{2}$ for player 1 and $P(c_1) = \frac{1}{2}, P(c_2) = \frac{1}{2}$ for player 2.

	c_1	c_2
r_1	(0,1)	(1,0)
r_2	(1,0)	(0,1)

Fig Q5-2. two players' behavior as best reply path

Q6. (10%) Pure-Coordination Game

There are two main characteristics for the pure-coordination game. One is that both players prefer the same Nash equilibrium outcome. The other is that both of them have identical interest [8]. See the following game matrix

	c_1	c_2
r_1	(10,10)	(0,0)
r_2	(0,0)	(10,10)

There exist two pure-strategy Nash equilibria, including (r_1, c_1) and (r_2, c_2). In addition, it also has a mixed-strategy Nash equilibrium, i.e., $P(r_1) = \frac{1}{2}$, $P(r_2) = \frac{1}{2}$ and $P(c_1) = \frac{1}{2}$, $P(c_2) = \frac{1}{2}$. Can it converge to the pure-strategy Nash equilibria by fictitious play? Or converge to mixed-strategy Nash equilibrium? Please justify your answer.

It can converge to all three of them.

Show your explanation with implementation source code here

Explanation for reference:

For such pure-coordination game, there are three kinds of result for taking fictitious play process, including converging both of the pure-strategy NE and mixed-strategy NE. See following explanation for more details.

- Pure-strategy NE: (r_1, c_1)

Assume that Player 1 begins this game with the prior belief that Player 2 has played c_1 1 time and c_2 0 times. Player 2 begins with the prior belief that Player 1 has played r_1 1 time and r_2 0 times. After initializing the prior belief setting, we can get the following result.

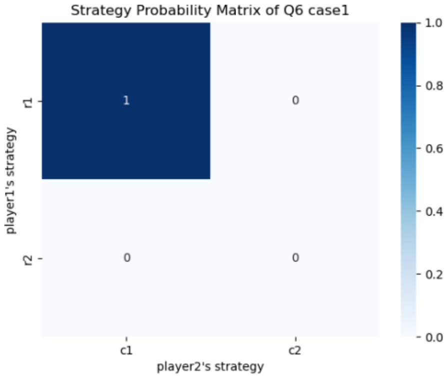

Fig Q6-1. two players converge to (r_1, c_1)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(1,0)	(1,0)	10 0	10 0
1	r_1	c_1	(2,0)	(2,0)	20 0	20 0
2	r_1	c_1	(3,0)	(3,0)	30 0	30 0
…	…	…	…	…	…	…
1000	r_1	c_1	(1001,0)	(1001,0)	10010 0	10010 0

Table Q6-1. two players converge to (r_1, c_1)

When player 1 believes that player 2 is more likely to play c_1, that is $b_1 = (1,0)$ and at the same time player 2 believes that player 1 is more likely to play r_1, that is $b_2 = (1,0)$, then it converges immediately to (r_1, c_1). It is shown in Fig Q6-1. This makes sense because when $b_1 = (1,0)$, the payoff for player 1 choosing strategy r_1 is $10 \times 1 + 0 \times 0 = 10$ and for strategy r_2 is $0 \times 1 + 10 \times 0 = 0$. It is clear for player 1 to select strategy r_1 since the payoff for choosing r_1 is better for player 1. Similarly, when $b_2 = (1,0)$, the payoff for player 2 choosing strategy c_1 is $10 \times 1 + 1 \times 0 = 10$ and for strategy c_2 is $0 \times 1 + 10 \times 0 = 0$. Therefore, selecting strategy c_1 is better for player 2 than selecting strategy c_2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect each player's payoff to select r_1 for player 1 and c_1 for player 2. Furthermore, we also show the result of fictitious play for 1000 times and the whole procedure can be seen in Table Q6-1. It shows that the beliefs of each player can converge to (r_1, c_1) rapidly. Thus, both of the players will play r_1 and c_1 immediately in the first round, that is round 1, and never change their individual strategy permanently.

- Pure-strategy NE: (r_2, c_2)

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 times and c_2 1 time. Player 2 begins with the prior belief that Player 1 has played r_1 0 times and r_2 1 time. After initializing the prior belief setting, we can obtain the following result.

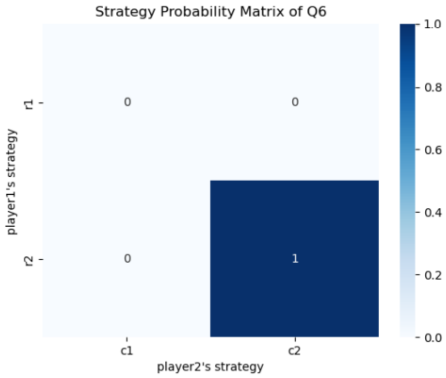

Fig Q6-2. two players converge to $(\mathrm{r}_2, \mathrm{c}_2)$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,1)	(0,1)	0 10	0 10
1	r_2	c_2	(0,2)	(0,2)	0 20	0 20
2	r_2	c_2	(0,3)	(0,3)	0 30	0 30
…	…	…	…	…	…	…
1000	r_2	c_2	(0,1001)	(0,1001)	0 10010	0 10010

Table Q6-2. two players converge to $(\mathrm{r}_2, \mathrm{c}_2)$

When player 1 believes that player 2 is more likely to play c_2, that is $\mathrm{b}_1 = (0,1)$ and at the same time player 2 believes that player 1 is more likely to play r_2, that is $\mathrm{b}_2 = (0,1)$, then it can converge to $(\mathrm{r}_2, \mathrm{c}_2)$ immediately. The final result can be seen in Fig Q6-2. It makes sense since when $\mathrm{b}_1 = (0,1)$, the payoff for player 1 choosing strategy r_1 is $10 \times 0 + 0 \times 1 = 0$ and for strategy r_2 is $0 \times 0 + 10 \times 1 = 10$. Clearly, since the payoff for selecting strategy r_2 is better, player 1 definitely choose r_2 as his action in the next round. Similarly, when the prior belief for player 2, that is $\mathrm{b}_2 = (0,1)$, the payoff for player 2 choosing strategy c_1 is $10 \times 0 + 0 \times 1 = 0$ and the other strategy c_2 is $0 \times 0 + 10 \times 1 = 10$. Thus, selecting strategy c_2 is better than selecting strategy c_1 for player 2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_2 for player 1 and c_2 for player 2. In Table Q6-2, we demonstrate the procedure of fictitious play for 1000 times in detail and the overall result as well. It shows that fictitious with such the prior belief setting for each player can converge to (r_2, c_2) rapidly. Hence, both of the players will play r_2 and c_2 respectively in the first round, that is round 1, and do not change their individual strategy permanently. In addition to both of two pure-strategy NE cases mentioned above, one more interesting thing is that there also exists a mixed-strategy NE with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$ in this game matrix. We will then show the procedure and discuss the detail as following.

- Mixed-strategy NE with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 time and c_2 0.5 times. Player 2 begins with the prior belief that Player 1 has played r_1 0.5 times and r_2 0 time. After initializing the prior belief setting, we can then conclude the following result.

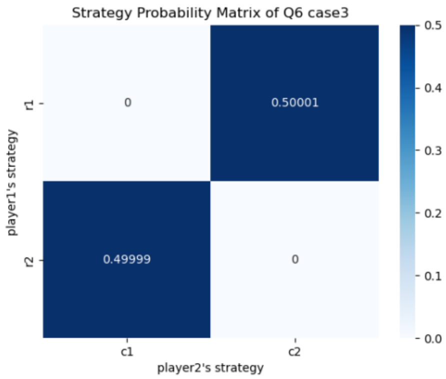

Fig Q6-3. two players converge to mixed-strategy NE

with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,0.5)	(0.5,0)	0 5	5 0
1	r_2	c_1	(1,0.5)	(0.5,1)	10 5	5 10
2	r_1	c_2	(1,1.5)	(1.5,1)	10 15	15 10
...	...	...	...	...	...	...
1000	r_1	c_2	(500,500.5)	(500.5,500)	5000 5005	5005 5000

Table Q6-3. two players converge to mixed-strategy NE

with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

When player 1 believes that player 2 is more likely to play c_2, that is $b_1 = (0,0.5)$. Similarly, at the same time player 2 believes that player 1 is more likely to play r_1, that is $b_2 = (0.5,0)$. The result of picture under such parameter setting is shown in Fig Q6-3. After initializing the prior belief $b_1 = (0,0.5)$ for player 1, the payoff for player 1 choosing strategy r_1 is $10 \times 0 + 0 \times 0.5 = 0$ and for strategy r_2 is $0 \times 0 + 10 \times 0.5 = 5$. Obviously, since the payoff for selecting strategy r_2 is better for player 1. Similarly, when setting the prior belief $b_2 = (0.5,0)$ for player 2, the payoff for player 2 choosing strategy c_1 is $10 \times 0.5 + 0 \times 0 = 5$ and for strategy c_2 is $0 \times 0.5 + 10 \times 0 = 0$. After calculating the payoff for player 2, it is easy to see that selecting strategy c_1 is better for player 2. This jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_2 for player 1 and c_1 for player 2. Then, we present the result of fictitious play for 1000 times and the overall procedure can be seen in Table Q6-3. As the number of rounds tends to infinity the empirical distribution of each player will converge to (0.5,0.5), which is the same as the mixed-strategy Nash equilibrium. Actually, you may obtain almost the same conclusion under different prior belief setting and we remain this kind of condition or critical value to readers. There is no doubt that we can observe that two players always change their strategy simultaneously as Fig Q6-4. In the first round, that is round 1, player 1 select r_2 and player 2 select c_1 as their action respectively according to their payoff. Then, they change their individual strategy from the original one to the other one rapidly in the next round, that is player 1 select r_1 and player 2 select c_2 as their action respectively. It makes sense since both of the players take their action by fictitious play process at the same time. If taking the action with fictitious play alternately instead of simultaneously, that is make decision individually in turns, then it is obvious to conclude that it cannot obtain the same result as such situation. In other word, it can only converge to both of the pure-strategy NE instead of mixed-strategy NE in the long round.

	c_1	c_2
r_1	(10,10)	(0,0)
r_2	(0,0)	(10,10)

Fig Q6-4. two players' change strategy simultaneously

Q7. (10%) Anti-Coordination game

Compared to pure coordination game, it can be thought of as the opposite of a coordination game. In pure-coordination game, sharing the resource creates a benefit for all. In contrast, the resource is rivalrous but non-excludable in anti-coordination games and sharing comes at a cost. See the following game matrix

	c_1	c_2
r_1	(0,0)	(1,1)
r_2	(1,1)	(0,0)

There exist three Nash equilibria as well, including two pure-strategy Nash equilibria and one mixed-strategy Nash equilibrium. The pure-strategy equilibria are in the upper right corner (r_1, c_2) and the lower left corner (r_2, c_1) respectively. And the mix-strategy Nash equilibrium can be simply obtained by partial derivative, which is $P(r_1) = \frac{1}{2}$, $P(r_2) = \frac{1}{2}$ for Player 1 and $P(c_1) = \frac{1}{2}$, $P(c_2) = \frac{1}{2}$ for Player 2. Can it converge to the pure-strategy Nash equilibrium by fictitious play? Or converge to mixed-strategy Nash equilibrium? Please justify your answer.

It can converge to all three of them.

Show your explanation with implementation source code here

Explanation for reference:

For such anti-coordination game, there are three kinds of result to adopt fictitious play for each player, including both of the pure-strategy NE and mixed-strategy NE. Look at the following explanation for more details.

- Pure-strategy NE: (r_2, c_1)

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 1 time and c_2 0 times. Player 2 begins with the prior belief that Player 1 has played r_1 0 times and r_2 1 time. After initializing the prior belief setting, we can get the following result.

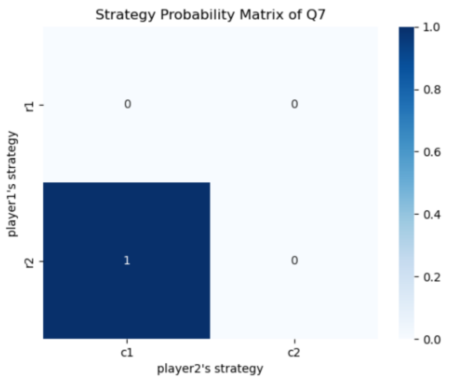

Fig Q7-1. two players converge to (r_2, c_1)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(1,0)	(0,1)	0 1	1 0
1	r_2	c_1	(2,0)	(0,2)	0 2	2 0
2	r_2	c_1	(3,0)	(0,3)	0 3	3 0

…	…	…	…	…	…	…
1000	r_2	c_1	(1001,0)	(0,1001)	0 1001	1001 0

Table Q7-1. two players converge to (r_2, c_1)

When player 1 believes that player 2 has played c_1 1 time and played c_2 0 times, that is $b_1 = (1{,}0)$. At the same time, player 2 believes that player 1 has played r_1 0 times and r_2 1 time, that is $b_2 = (0{,}1)$. In Fig Q7-1, it shows that both of the players can converges to (r_2, c_1) by the fictitious play. It makes sense because when $b_1 = (1{,}0)$, the payoff for player 1 choosing strategy r_1 is $0 \times 1 + 1 \times 0 = 0$ and for strategy r_2 is $1 \times 1 + 0 \times 0 = 1$. Without doubt, since the payoff for selecting strategy r_2 is better than strategy r_1, player 1 must choose strategy r_2 as his action in the next round. Similarly, when $b_2 = (0{,}1)$, the payoff for player 2 choosing strategy c_1 is $0 \times 0 + 1 \times 1 = 1$ and $0 \times 1 + 1 \times 0 = 0$ for choosing strategy c_2. Thus, selecting strategy c_1 is much better for player 2. Without loss of generality, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_2 for player 1 and c_1 for player 2. We also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q7-1. It shows that the beliefs of each player can converge rapidly. Therefore, both players will play r_2 and c_1 immediately in the first round, that is round 1, and never change their individual strategy permanently.

- Pure-strategy NE: (r_1, c_2)

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 times and c_2 1 time. Player 2 begins with the prior belief that Player 1 has played r_1 1 time and r_2 0 times. After initializing the prior belief setting, we can obtain the following result.

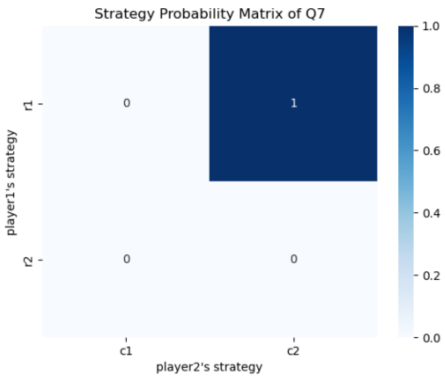

Fig Q7-2. two players converge to (r_1, c_2)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,1)	(1,0)	1 0	0 1
1	r_1	c_2	(0,2)	(2,0)	2 0	0 2
2	r_1	c_2	(0,3)	(3,0)	3 0	0 3

…	…	…	…	…	…	…
1000	r_1	c_2	(0,1001)	(1001,0)	1001 0	0 1001

Table Q7-2. two players converge to (r_1, c_2)

When player 1 believes that player 2 has played c_1 0 time and played c_2 1 time, that is $b_1 = (0,1)$. At the same time, player 2 believes that player 1 has played r_1 1 time and r_2 0 times, that is $b_2 = (1,0)$. In Fig Q7-2, it shows that both of the players can converges to (r_1, c_2) by the fictitious play. This makes sense because when $b_1 = (0,1)$, the payoff for player 1 choosing strategy r_1 is $0 \times 0 + 1 \times 1 = 1$ and for strategy r_2 is $1 \times 0 + 0 \times 1 = 0$. Obviously, since the payoff for selecting strategy r_1 is better than strategy r_2, it is for sure for player 1 to select strategy r_1. Similarly, when the prior belief for player 2 $b_2 = (0,1)$, the payoff for player 2 choosing strategy c_1 is $1 \times 0 + 0 \times 1 = 0$ and for strategy c_2 is $1 \times 1 + 0 \times 0 = 1$. Clearly, selecting strategy c_2 is better for player 2 instead of strategy c_1. Thus, it jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_1 for player 1 and c_2 for player 2. Then, we show the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q7-2. It can be seen that such the prior beliefs for each player can converge immediately. Therefore, both players will play r_2 and c_2 respectively in the first round, that is round 1, and never change their individual strategy permanently. In addition to the two pure-strategy NE cases mentioned above, one more interesting thing is that there exists a mixed-strategy NE with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$ in this game matrix. We then present the procedure and discuss the result as following.

- Mixed-strategy NE with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0.5 times and c_2 0 times. Player 2 begins with the prior belief that Player 1 has played r_1 0.5 times and r_2 0 times. After initializing the prior belief setting, we can obtain the following result.

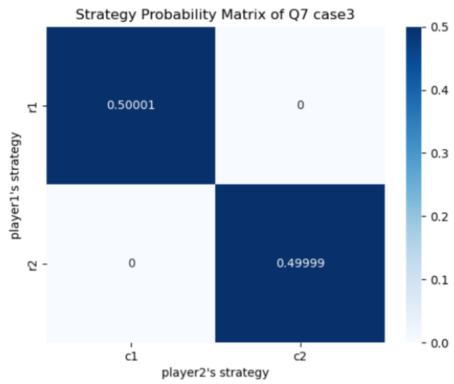

Fig Q7-3. two players converge to mixed-strategy NE

with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0.5,0)	(0.5,0)	0 0.5	0 0.5
1	r_2	c_2	(0.5,1)	(0.5,1)	1 0.5	1 0.5
2	r_1	c_1	(1.5,1)	(1.5,1)	1 1.5	1 1.5
…	…	…	…	…	…	…
1000	r_1	c_1	(500.5,500)	(500.5,500)	500 500.5	500 500.5

Table Q7-3. two players converge to mixed-strategy NE

with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

When player 1 believes that player 2 is more likely to play c_1, that is $b_1 = (0.5,0)$ and at the same time player 2 believes that player 1 is more likely to play r_1, that is $b_2 = (0.5,0)$. The result of picture under such parameter setting is shown in Fig Q7-3. When initializing the prior belief $b_1 = (0.5,0)$ for player 1, the payoff for player 1 choosing strategy r_1 is $0 \times 0.5 + 1 \times 0 = 0$ and for strategy r_2 is $1 \times 0.5 + 0 \times 0 = 0.5$. Obviously, since the payoff for selecting strategy r_2 is better for player 1. Similarly, when setting the prior belief $b_2 = (0.5,0)$ for player 2, the payoff for player 2 choosing strategy c_1 is $0 \times 0.5 + 1 \times 0 = 0$ and for strategy c_2 is $1 \times 0.5 + 0 \times 0 = 0.5$. Thus, selecting strategy c_2 is better than strategy c_1 for player 2. without loss of generality, it jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_2 for player 1 and c_1 for player 2. And then, we show the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q7-3. As the number of rounds tends to infinity the empirical distribution of each player will converge to (0.5,0.5), which is the same as the mixed-strategy Nash equilibrium. Actually, you can obtain the same result under different prior belief setting and we remain this kind of condition or critical value to readers. Not surprisingly, we can also observe that two players can always shift their strategy simultaneously as Fig Q7-4. In the first round, player 1 select r_2 and player 2 select c_2 as their action respectively according to their payoff. Then, they change their strategy from original one to the other one immediately in the next round, that is player 1 select r_1 and player 2 select c_2 as their action respectively. It makes sense since both of the players take their action by fictitious play at the same time. If they can take the action alternately with fictitious play, that is change their decision respectively in turns, then it is no doubt to conclude that it would not get the same result as this situation. In other word, it can only converge to both of the pure-strategy NE instead of mixed-strategy NE in the long round.

	c_1	c_2
r_1	(0,0)	(1,1)
r_2	(1,1)	(0,0)

Fig Q7-4. two players' change strategy simultaneously

Q8. (10%) Battle of the Sexes

It is a two-player coordination game with different utility functions. Different from pure-coordination game, one player is happier than the other in any Nash equilibrium. See the following game matrix

	c_1	c_2
r_1	(3,2)	(0,0)
r_2	(0,0)	(2,3)

There co-exists two pure-strategy and one mixed-Strategy Nash equilibria. Can it converge to the pure-strategy Nash equilibria by fictitious play? Or converge to mixed-strategy Nash equilibrium? Please justify your answer.

It can converge to all three of them.

Show your explanation with implementation source code here

Explanation for reference:

For such battle of the sexes game matrix, there are three kinds of convergence result for fictitious play, including both of the pure-strategy NE and mixed-strategy NE. See the following cases for more details.

- Pure-strategy NE: (r_1, c_1)

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 1 time and c_2 0 times. Player 2 begins with the prior belief that Player 1 has played r_1 1 time and r_2 0 times. After initializing the prior belief setting, we can obtain the following result.

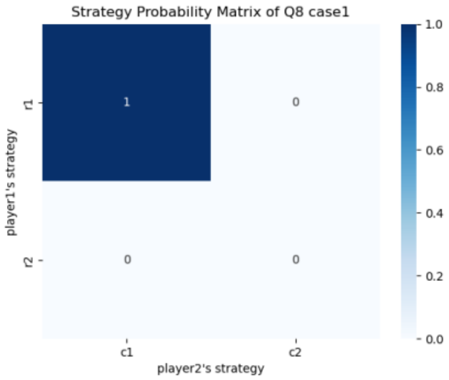

Fig Q8-1. two players converge to (r_1, c_1)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(1,0)	(1,0)	3 0	2 0
1	r_1	c_1	(2,0)	(2,0)	6 0	4 0
2	r_1	c_1	(3,0)	(3,0)	9 0	6 0
…	…	…	…	…	…	…
1000	r_1	c_1	(1001,0)	(1001,0)	3003 0	2002 0

Table Q8-1. two players converge to (r_1, c_1)

When player 1 believes that player 2 has played c_1 1 time and played c_2 0 times, that is $b_1 = (0,1)$. On the other hand, player 2 believes that player 1 has played r_1 1 time and r_2 0 times, that is $b_2 = (1,0)$. In Fig Q8-1, it shows that both of the players can converges to (r_1, c_1) by the fictitious play. It makes sense because when $b_1 = (1,0)$, the payoff for player 1 choosing strategy r_1 is $3 \times 1 + 0 \times 0 = 3$ and for strategy r_2 is $0 \times 1 + 2 \times 0 = 0$. Obviously, since the payoff for selecting strategy r_1 is better for player 1. At the same time, when $b_2 = (1,0)$, the payoff for player 2 choosing strategy c_1 is $2 \times 1 + 0 \times 0 = 2$ and for strategy c_2 is $0 \times 1 + 3 \times 0 = 0$. Thus, selecting strategy c_1 is better for player 2. Without doubt, it means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_1 for player 1 and c_1 for player 2. And then, we also show the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q8-1. It shows that the beliefs of each player can converge rapidly. Therefore, both players under such prior belief setting will play r_1 and c_1 immediately in the first round, and never change their individual strategy permanently.

- Pure-strategy NE: (r_2, c_2)

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 times and c_2 1 time. Player 2 begins with the prior belief that Player 1 has played r_1 0 times and r_2 1 time. After initializing the prior belief setting, we can obtain the following result.

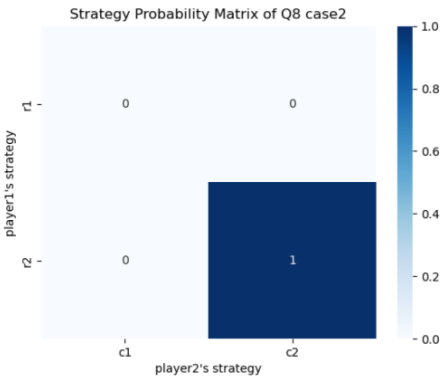

Fig Q8-2. two players converge to (r_2, c_2)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,1)	(0,1)	0 2	0 3
1	r_2	c_2	(0,2)	(0,2)	0 4	0 6
2	r_2	c_2	(0,3)	(0,3)	0 6	0 9
…	…	…	…	…	…	…
1000	r_2	c_2	(0,1001)	(0,1001)	0 2002	0 3003

Table Q8-2. two players converge to (r_2, c_2)

When player 1 believes that player 2 has played c_1 0 times and played c_2 1 time, that is prior belief $b_1 = (0,1)$. On the other hand, player 2 believes that player 1 has played r_1 0 times and r_2 1 time, that is prior belief $b_2 = (1,0)$. In Fig Q8-2, it shows that both of the players can converges to (r_2, c_2) by the fictitious play. This makes sense because when $b_1 = (0,1)$, the payoff for player 1 choosing strategy r_1 is $3 \times 0 + 0 \times 1 = 0$ and for strategy r_2 is $0 \times 0 + 2 \times 1 = 2$. Obviously, since the payoff for selecting strategy r_2 is much better than strategy r_1, player 1 is for sure to select strategy r_2 as his action in the next round. At the same time, when prior belief $b_2 = (0,1)$, the payoff for player 2 choosing strategy c_1 is $2 \times 0 + 0 \times 1 = 0$ and for strategy c_2 is $0 \times 0 + 3 \times 1 = 3$. Thus, selecting strategy c_2 is better for player 2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_2 for player 1 and c_2 for player 2. We also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q8-2. It shows that the beliefs of each player can converge rapidly. Therefore, both players will play r_2 and c_2 immediately in the first round, that is round 1, and never change their individual strategy permanently. In addition to the two pure-strategy NE cases mentioned above, one more interesting thing is that there exists a mixed-strategy NE with $P(r_1) = 0.6, P(r_2) = 0.4$ and $P(c_1) = 0.4, P(c_2) = 0.6$ in this game matrix. We will then present the procedure and discuss the result as following.

- Mixed-strategy NE with $\mathrm{P}(\mathrm{r}_1) = 0.6, \mathrm{P}(\mathrm{r}_2) = 0.4$ and $\mathrm{P}(\mathrm{c}_1) = 0.4, \mathrm{P}(\mathrm{c}_2) = 0.6$

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0.5 times and c_2 0 times. Player 2 begins with the prior belief that Player 1 has played r_1 0 times and r_2 0.5 time. After initializing the prior belief setting, we can obtain the following result.

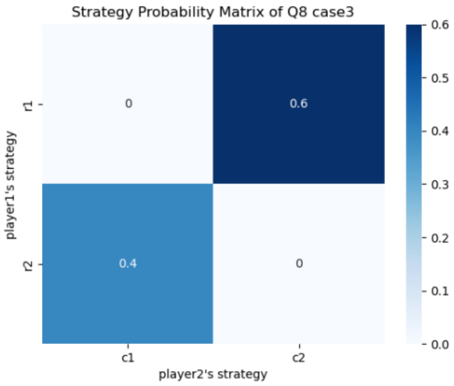

Fig Q8-3. two players converge to mixed-strategy NE

with $\mathrm{P}(\mathrm{r}_1) = 0.6, \mathrm{P}(\mathrm{r}_2) = 0.4$ and $\mathrm{P}(\mathrm{c}_1) = 0.4, \mathrm{P}(\mathrm{c}_2) = 0.6$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0.5,0)	(0,0.5)	1.5 0	0 1.5
1	r_1	c_2	(0.5,1)	(1,0.5)	1.5 2	2 1.5
2	r_2	c_1	(1.5,1)	(1,1.5)	4.5 2	2 4.5
...	...	...	...	...	...	...
1000	r_2	c_1	(400.5,600)	(600,400.5)	1201.5 1200	1200 1201.5

Table Q8-3. two players converge to mixed-strategy NE

with $\mathrm{P}(\mathrm{r}_1) = 0.6, \mathrm{P}(\mathrm{r}_2) = 0.4$ and $\mathrm{P}(\mathrm{c}_1) = 0.4, \mathrm{P}(\mathrm{c}_2) = 0.6$

When player 1 believes that player 2 is more likely to play c_1, that is $\mathrm{b}_1 = (0.5,0)$ and at the same time player 2 believes that player 1 is more likely to play r_2, that is $\mathrm{b}_2 = (0,0.5)$. The result of picture under such parameter setting can be seen in Fig Q8-3. When initializing the prior belief $\mathrm{b}_1 = (0.5,0)$ for player 1, the payoff for player 1 choosing strategy r_1 is $3 \times 0.5 + 0 \times 0 = 1.5$ and for strategy r_2 is $0 \times 0.5 + 2 \times 0 = 0$. Obviously, since the payoff for selecting strategy r_1 is better for player 1. Similarly, when setting the prior belief $\mathrm{b}_2 = (0.5,0)$ for player 2, the payoff for player 2 choosing strategy c_1 is $2 \times 0 + 0 \times 0.5 = 0$ and for strategy c_2 is $0 \times 0 + 3 \times 0.5 = 1.5$. Therefore, selecting strategy c_2 is better for player 2. And then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_1 for player 1 and c_2 for player 2. By the way, we also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q8-3. As the number of rounds tends to infinity the empirical distribution of each player will converge, which is the same as the mixed-strategy Nash equilibrium, that is $P(r_1) = 0.6, P(r_2) = 0.4$ and $P(c_1) = 0.4, P(c_2) = 0.6$. Actually, you can obtain the same result with different prior belief setting and we still remain such kind of condition or critical value for readers. To our expectation, we can observe that two players always change their strategy simultaneously as Fig Q8-4. In the first round, that is round 1, player 1 select r_1 and player 2 select c_2 as their action respectively according to their payoff. Then, they shift their strategy from original one to the other one rapidly in the next round, that is player 1 choose r_2 and player 2 choose c_1 as their action respectively. It makes sense since both of the players take their action with fictitious play at the same time. If they take the action with fictitious play alternately, that is make decision individually in turns, then it is obvious to conclude that it would not get the same result as this situation. In other word, it can only converge to both of the pure-strategy NE instead of mixed-strategy NE in the long round.

	c_1	c_2
r_1	(3,2)	(0,0)
r_2	(0,0)	(2,3)

Fig Q8-4. two players' change strategy simultaneously

Q9. (10%) Stag Hunt Game

One main characteristic is that one pure-strategy Nash equilibrium is the best for all but no other Nash equilibria. It differs from the prisoner's dilemma in that there are two pure-strategy Nash equilibria: one where both players cooperate, and one where both players defect. One famous application is the power control game. See the following game matrix

	c_1	c_2
r_1	(3,3)	(0,2)
r_2	(2,0)	(1,1)

There exist three Nash equilibria as well, including two pure-strategy Nash equilibria, i.e. (r_1, c_1) and (r_2, c_2), and one mixed-strategy Nash equilibrium, i.e. $P(r_1) = \frac{1}{2}$, $P(r_2) = \frac{1}{2}$ and $P(c_1) = \frac{1}{2}$, $P(c_2) = \frac{1}{2}$. Can it converge to the pure-strategy Nash equilibria by fictitious play? Or converge to mixed-strategy Nash equilibrium? Please justify your answer.

It can converge to all three of them.

Show your explanation with implementation source code here

Explanation for reference:

For such stag hunt game matrix, which is almost the same as Q2, there are three cases of convergence result for fictitious play, including both of the pure-strategy NE and mixed-strategy NE. Look at the following content for more explanations.

- Pure-strategy NE: (r_1, c_1)

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 times and c_2 1 time. Player 2 begins with the prior belief that Player 1 has played r_1 0 times and r_2 1 time. After initializing the prior belief setting, we can obtain the following result.

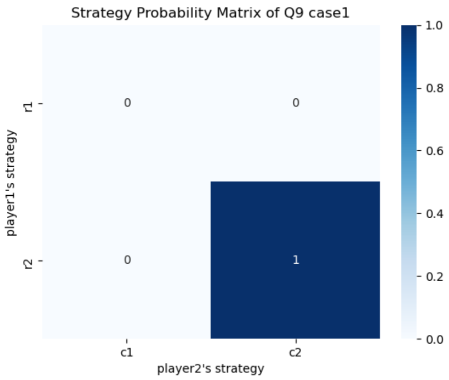

Fig Q9-1. two players converge to (r_2, c_2)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,1)	(0,1)	0 1	0 1
1	r_2	c_2	(0,2)	(0,2)	0 2	0 2
2	r_2	c_2	(0,3)	(0,3)	0 3	0 3
…	…	…	…	…	…	…
1000	r_2	c_2	(0,1001)	(0,1001)	0 1001	0 1001

Table Q9-1. two players converge to (r_2, c_2)

When player 1 believes that player 2 is more likely to play c_1, that is $b_1 = (0,1)$ and at the same time player 2 believes that player 1 is more likely to play r_1, that is $b_2 = (0,1)$, then it converges immediately to (r_2, c_2). It is shown in Fig Q9-1. This makes sense because when $b_1 = (0,1)$, the payoff for player 1 choosing strategy r_1 is $3 \times 0 + 0 \times 1 = 0$ and for strategy r_2 is $2 \times 0 + 1 \times 1 = 1$. Obviously, since the payoff for selecting strategy r_2 is better for player 1. Similarly, when $b_2 = (0,1)$, the payoff for player 2 choosing strategy c_1 is $3 \times 0 + 0 \times 1 = 0$ and for strategy c_2 is $2 \times 0 + 1 \times 1 = 1$. Thus, selecting strategy c_2 is better for player 2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_2 for player 1 and c_2 for player 2. We also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q9-1. It shows that the beliefs of each player can converge rapidly. Therefore, both players will play r_2 and c_2 immediately in the first round, that is round 1, and never change their individual strategy permanently.

- Pure-strategy NE: (r_2, c_2)

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 1 time and c_2 0 times. Player 2 begins with the prior belief that Player 1 has played r_1 1 time and r_2 0 times. After initializing the prior belief setting, we can obtain the following result.

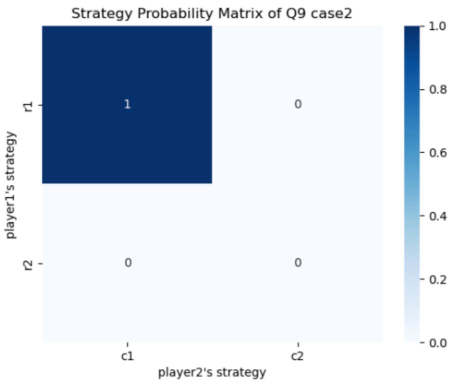

Fig Q9-2. two players converge to (r_1, c_1)

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(1,0)	(1,0)	$\underline{3}$ 2	$\underline{3}$ 2
1	r_1	c_1	(2,0)	(2,0)	$\underline{6}$ 4	$\underline{6}$ 4
2	r_1	c_1	(3,0)	(3,0)	$\underline{9}$ 6	$\underline{9}$ 6
…	…	…	…	…	…	…
1000	r_1	c_1	(1001,0)	(1001,0)	$\underline{3003}$ 2002	$\underline{3003}$ 2002

Table Q9-2. two players converge to (r_1, c_1)

When player 1 believes that player 2 is more likely to play c_2, that is $b_1 = (1,0)$ and at the same time player 2 believes that player 1 is more likely to play r_2, that is $b_2 = (1,0)$, then it converges immediately to (r_1, c_1). It is shown in Fig Q9-2. This makes sense because when $b_1 = (1,0)$, the payoff for player 1 choosing strategy r_1 is $3 \times 1 + 0 \times 0 = 3$ and for strategy r_2 is $2 \times 1 + 1 \times 0 = 2$. Obviously, since the payoff for selecting strategy r_1 is better for player 1. Similarly, when $b_2 = (1,0)$, the payoff for player 2 choosing strategy c_1 is $3 \times 1 + 0 \times 0 = 3$ and for strategy c_2 is $2 \times 1 + 1 \times 0 = 2$. Thus, selecting strategy c_1 is better for player 2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_1 for player 1 and c_1 for player 2. We also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q9-2. It shows that the beliefs of each player can converge rapidly. Therefore, both players will play r_1 and c_1 immediately in the first round, that is round 1, and never change their individual strategy permanently. In addition to the two pure-strategy NE cases mentioned above, one more interesting thing is that there exists a mixed-strategy NE with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$ in this game matrix. We will then demonstrate the procedure and discuss the result as following.

- Mixed-strategy NE with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 time and c_2 0.5 times. Player 2 begins with the prior belief that Player 1 has played r_1 0.5 times and r_2 0 time. After initializing the prior belief setting, we can obtain the following result.

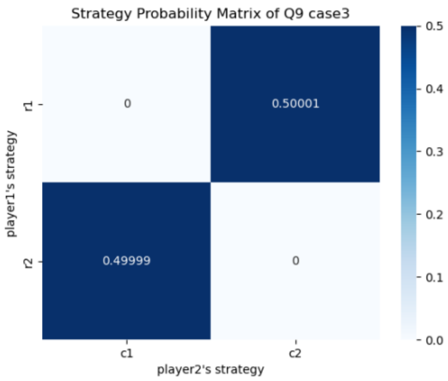

Fig Q9-3. two players converge to mixed-strategy NE with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,0.5)	(0.5,0)	0 0.5	1.5 1
1	r_2	c_1	(1,0.5)	(0.5,1)	3 2.5	1.5 2
2	r_1	c_2	(1,1.5)	(1.5,1)	3 3.5	4.5 4
…	…	…	…	…	…	…
1000	r_1	c_2	(500,500.5)	(500.5,500)	1500 1500.5	1501.5 1501

Table Q9-3. two players converge to mixed-strategy NE with $P(r_1) = 0.5, P(r_2) = 0.5$ and $P(c_1) = 0.5, P(c_2) = 0.5$

When player 1 believes that player 2 is more likely to play c_2, that is $b_1 = (0,0.5)$ and at the same time player 2 believes that player 1 is more likely to play r_1, that is $b_2 = (0.5,0)$. The result of picture under such parameter setting is shown in Fig Q9-3. When initializing the prior belief $b_1 = (0,0.5)$ for player 1, the payoff for player 1 choosing strategy r_1 is $3 \times 0 + 0 \times 0.5 = 0$ and for strategy r_2 is $2 \times 0 + 1 \times 0.5 = 0.5$. Obviously, since the payoff for selecting strategy r_2 is better for player 1. Similarly, when setting the prior belief $b_2 = (0.5,0)$ for player 2, the payoff for player 2 choosing strategy c_1 is $3 \times 0.5 + 0 \times 0 = 1.5$ and for strategy c_2 is $2 \times 0.5 + 1 \times 0 = 1$. Thus, selecting strategy c_1 is better for player 2. Then, this jointly means that the initial prior belief b_1 and b_2 can affect player's payoff to select r_2 for player 1 and c_1 for player 2. And then, we also demonstrate the procedure of fictitious play for 1000 times and the overall result can be seen in Table Q9-3. As the number of rounds tends to infinity the empirical distribution of each player will converge to (0.5,0.5), which is the same as the mixed-strategy Nash equilibrium. Actually, you can obtain the same result under different prior belief setting and we remain this kind of condition or critical value to readers. Same as Q6-8, we can observe that two players always change their strategy simultaneously as Fig Q9-4. In the first round, that is round 1, player 1 select r_2 and player 2 select c_1 as their action respectively according to their payoff. Then, they shift their strategy from original one to the other one rapidly in the next round, that is player 1 choose r_1 and player 2 choose c_2 as their action respectively. It makes sense since both of the players take their action with fictitious play at the same time. If they take the action with fictitious play alternately, that is make decision individually in turns, then it is obvious to conclude that it would not get the same result as this situation. In other word, it can only converge to both of the pure-strategy NE instead of mixed-strategy NE in the long round.

	c_1	c_2
r_1	(3,3)	(0,2)
r_2	(2,0)	(1,1)

Fig Q9-4. two players' change strategy simultaneously

Q10. (10%) Observation and Conclusion

According to the observation based on above results, fictious play is still a useful iterative algorithm for us to find out the Nash equilibrium no matter pure-strategy or mixed-strategy. However, is it reliable to find Nash equilibrium by fictious play for every game matrix? If yes, please explain your reason in detail to justify it. If no, please provide a concrete counter-example.

No, it is not reliable to find out Nash equilibrium by fictitious play for every game matrix.

Show your counter-example and explanation
with implementation source code here

Explanation for reference:
see the counter-example below.

	c_1	c_2
r_1	(1,0)	(0,0)
r_2	(0,0)	(0,1)

- Case 1 with $P(r_1) = 1, P(r_2) = 0$ and $P(c_1) = \frac{1}{2}, P(c_2) = \frac{1}{2}$

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 1 time and c_2 0 times. Player 2 begins with the prior belief that Player 1 has played r_1 1 time and r_2 0 times. After initializing the prior belief setting, we can obtain the following result.

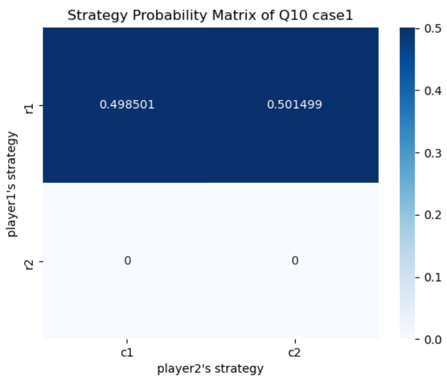

Fig Q10-1. two players convergence result with $b_1 = (1,0)$ and $b_2 = (1,0)$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(1,0)	(1,0)	$\underline{1}$ 0	$\underline{0}$ 0
1	r_1	c_1	(2,0)	(2,0)	$\underline{2}$ 0	0 $\underline{0}$

2	r_1	c_2	(2,1)	(3,0)	2 0	0 0
…	…	…	…	…	…	…
1000	r_1	c_2	(499,502)	(1001,0)	499 0	0 0

Table Q10-1. two players convergence procedure with $b_1 = (1,0)$ and $b_2 = (1,0)$

When player 1 believes that player 2 has played c_1 1 time and played c_2 0 times, that is prior belief $b_1 = (1,0)$. On the other hand, player 2 believes that player 1 has played r_1 1 time and r_2 0 times, that is prior belief $b_2 = (1,0)$. In Fig Q10-1, it shows that both of the players will converge with $P(r_1) = 1, P(r_2) = 0$ and $P(c_1) = \frac{1}{2}, P(c_2) = \frac{1}{2}$ by the fictitious play in the long run. When initializing the prior belief $b_1 = (1,0)$ for player 1, the payoff for player 1 choosing strategy r_1 is $1 \times 1 + 0 \times 0 = 1$ and for strategy r_2 is $0 \times 0 + 0 \times 0 = 0$. Obviously, since the payoff for selecting strategy r_1 is better than selecting strategy r_2, it is clear for player 1 to choose strategy r_1. At the same time, when setting the prior belief $b_2 = (1,0)$ for player 2, the payoff for player 2 choosing strategy c_1 is $0 \times 1 + 0 \times 0 = 0$ and for strategy c_2 is $0 \times 1 + 1 \times 0 = 0$. Since the payoff for selecting strategy c_1 is the same as selecting strategy c_2, player 2 will randomly pick one action from his best response set. In Table Q10-1, we also demonstrate the procedure of fictitious play in detail for 1000 times. Observe that no matter how many times the round number increase the payoffs for player 2 to take c_1 or c_2 is zero. That is, player 2 will pick either c_1 or c_2 randomly in each round. On the other hand, the player 1 continues to select strategy r_1 as hist best response in the long run. As the number of rounds tends to infinity, the empirical distribution for player 2 will converge to (0.5,0.5) while player 1 remains to select strategy r_1 permanently. Actually, you can obtain the same result under different prior belief setting and we remain this kind of condition or critical value for readers. To our surprise, the convergence result is neither pure-strategy NE nor mixed-strategy NE. Therefore, we can conclude that the fictitious play is not reliable for finding NE in such game matrix. Then, we also present another converge result to show the weird situation by fictitious play.

- Case 2 with $P(r_1) = \frac{1}{2}, P(r_2) = \frac{1}{2}$ and $P(c_1) = 0, P(c_2) = 1$

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 0 times and c_2 1 time. Player 2 begins with the prior belief that Player 1 has played r_1 0 times and r_2 1 time. After initializing the prior belief setting, we can obtain the following result.

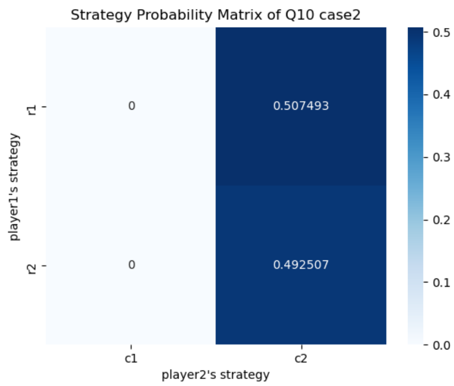

Fig Q10-2. two players convergence result with $b_1 = (0,1)$ and $b_2 = (0,1)$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(0,1)	(0,1)	0 0	0 1
1	r_2	c_2	(0,2)	(0,2)	0 0	0 2
2	r_1	c_2	(0,3)	(1,2)	0 0	0 2
…	…	…	…	…	…	…
1000	r_2	c_2	(0,1001)	(508,493)	0 0	0 493

Table Q10-2. two players convergence procedure with $b_1 = (0,1)$ and $b_2 = (0,1)$

When player 1 believes that player 2 has played c_1 0 times and played c_2 1 time, that is prior belief $b_1 = (0,1)$. On the other hand, player 2 believes that player 1 has played r_1 0 times and r_2 1 time, that is prior belief $b_2 = (0,1)$. In Fig Q10-2, it shows that both of the players will converge with $P(r_1) = \frac{1}{2}, P(r_2) = \frac{1}{2}$ and $P(c_1) = 0, P(c_2) = 1$ by the fictitious play in the long run. When initializing the prior belief $b_1 = (0,1)$ for player 1, the payoff for player 1 choosing strategy r_1 is $1 \times 0 + 0 \times 1 = 0$ and for strategy r_2 is $0 \times 0 + 0 \times 1 = 0$. Since the payoff for selecting strategy r_1 is the same as selecting strategy r_2, player 1 will randomly pick one action from his best response set. At the same time, when setting the prior belief $b_2 = (0,1)$ for player 2, the payoff for player 2 choosing strategy c_1 is $0 \times 0 + 0 \times 1 = 0$ and for strategy c_2 is $0 \times 0 + 1 \times 1 = 1$. Obviously, since the payoff for selecting strategy c_2 is better than selecting strategy c_1, it is clear for player 2 to choose strategy c_2 as his action in the next round. In Table Q10-2, we also demonstrate the procedure of fictitious play in detail for 1000 times. Observe that no matter how many times the round number increase the payoffs for player 1 to take r_1 or r_2 is zero. That is, player 1 will pick either r_1 or r_2 randomly in each round. On the other hand, the player 2 continues to select strategy c_2 as hist best response in the long run. As the number of rounds tends to infinity, the empirical distribution for player 1 will converge to (0.5,0.5) while player 2 remains to select strategy

c_1 permanently. Actually, you can obtain the same result under different prior belief setting and we remain this kind of condition or critical value for readers. The same as above, the convergence result is neither pure-strategy NE nor mixed-strategy NE. Therefore, we can conclude that the fictitious play is not reliable for finding NE in such game matrix as well. Although it is not suitable for fictitious play to find NE under such prior belief, we can also find one pure-strategy NE (r_1, c_2) through this model-free model method as follows.

- Case 3 with pure-strategy NE (r_1, c_2)

Assume that Player 1 begins the game with the prior belief that Player 2 has played c_1 1 time and c_2 1 time. Player 2 begins with the prior belief that Player 1 has played r_1 1 time and r_2 1 time. After initializing the prior belief setting, we can obtain the following result.

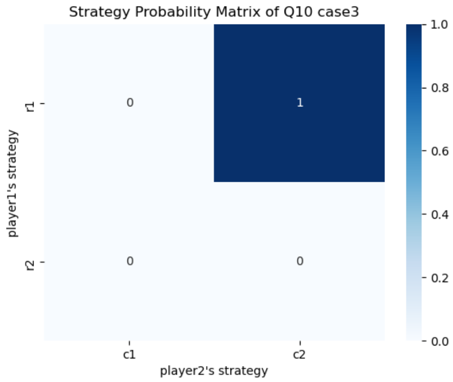

Fig Q10-3. two players convergence result with $b_1 = (1,1)$ and $b_2 = (1,1)$

Round	1's action	2's action	1's belief	2's belief	1's payoff	2's payoff
0	-	-	(1,1)	(1,1)	$\underline{1}$ 0	0 $\underline{1}$
1	r_1	c_2	(1,2)	(2,1)	$\underline{1}$ 0	0 $\underline{1}$
2	r_1	c_2	(1,3)	(3,1)	$\underline{1}$ 0	0 $\underline{1}$
…	…	…	…	…	…	…
1000	r_1	c_2	(1,1001)	(1001,1)	$\underline{1}$ 0	0 $\underline{1}$

Table Q10-3. two players convergence procedure with $b_1 = (1,1)$ and $b_2 = (1,1)$

When player 1 believes that player 2 has played c_1 1 time and played c_2 1 time, that is prior belief $b_1 = (1,1)$. On the other hand, player 2 believes that player 1 has played r_1 1 time and r_2 1 time, that is prior belief $b_2 = (1,1)$. In Fig Q10-3, it shows that both of the players can converge to pure-strategy NE, that is (r_1, c_2), by the fictitious play. When initializing the prior belief $b_1 = (1,1)$ for player 1, the payoff for player 1 choosing strategy r_1 is $1 \times 1 + 0 \times 1 = 1$ and for strategy r_2 is $0 \times 1 + 0 \times 1 = 0$. Obviously, since the payoff for selecting strategy r_1 is better than selecting strategy r_2, it is clear for player 1 to choose strategy r_1 as his action in the next round. At the same time, when setting the prior belief $b_2 = (1,1)$ for player 2, the payoff for player 2 choosing strategy c_1 is $0 \times 1 + 0 \times 1 = 0$ and for strategy c_2 is $0 \times 1 + 1 \times 1 = 1$. Since the payoff for selecting strategy c_2 is much better than selecting strategy c_1, it is no doubt for player 2 to choose strategy c_2 as his action in the next round. In Table Q10-3, we also demonstrate the procedure of fictitious play in detail for 1000 times. Observe that no matter how many times the round number increase the result would always converge to (r_1, c_2). In addition, it also shows that the beliefs of each player can converge rapidly. Therefore, both players will play r_1 and c_1 immediately in the first round, and never change their respective strategy permanently.

Besides, you can provide your own counter-example and explain the reason why it is not reliable for fictitious play according to the convergence result. In fact, the slide in this course also provide some useful examples as statement in the hint. One interesting example is coexistence of finite and infinite reply paths as the following game matrix and we remain the convergence result for readers.

	c_1	c_2	c_3
r_1	(0,1)	(1,0)	(0,0)
r_2	(1,0)	(0,1)	(0,0)
r_3	(0,0)	(0,0)	(0.5,0.5)