Analysis Fall 2025: Homework 3

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If V is a Banach space, an operator (on V) is a linear function $T : V \to V$. T is called bounded if $\|T\| := \sup_{\|v\|=1} \|T(v)\| < \infty$. T is called compact if $T(B(0,1))$ is precompact in V, where $B(0,1) = \{v \in V : \|v\| \leq 1\}$ is the unit ball in V.

Problem 1

Show that if an operator is compact, then it is bounded.

Let T be a compact operator. Then note that $\{v \in V : \|v\| = 1) \subseteq B(0,1)$, and further note that $T(B(0,1))$ is a precompact set.

By the Arzela-Ascoli Theorem, $T(B(0,1))$ is bounded and equicontinuous; i.e., there exists some $M \in \mathbb{R} : \forall T(v)$ where $\|v\| \leq 1$, then $T(v) \leq M$.

We know $\|T\| := \sup_{\|v\|=1} \|T(v)\| < \infty$. The above definition thus obviously implies that there exists some $M \in \mathbb{R} : \|v\| = 1 \Rightarrow T(v) \leq M$. Then $\|T\| = \sup_{\|v\|=1} \|T(v)\|$ is not infinity.

Problem 2

Show that the indefinite integration function I is a compact operator on $C^0([0,1])$ (with the C^0-norm), where

$$(I(f))(x) := \int_0^x f(t)\,\mathrm{d}t$$

(You must both show that it is an operator and compact) *Find the image I, and prove your answer is correct.

We argue that I is an operator.

We argue that $I(f+g) = I(f) + I(g)$ for $f, g \in C^0([0,1])$.

$$I(f+g) = \int_0^x f(t) + g(t)\,\mathrm{d}t = \int_0^x f(t)\,\mathrm{d}t + \int_0^x g(t)\,\mathrm{d}t = I(f) + I(g)$$

We argue that $I(\lambda f) = \lambda I(f)$ for $f \in C^0([0,1])$ and $\lambda \in \mathbb{R}$.

$$I(\lambda f) = \int_0^x \lambda f(t)\,\mathrm{d}t = \lambda \int_0^x f(t)\,\mathrm{d}t = \lambda I(f)$$

We argue that I is a compact operator; i.e., $I(B(0,1))$ is precompact. By the Arzela-Ascoli Theorem, it suffices to prove $I(B(0,1))$ is bounded and equicontinuous.

We argue $I(B(0,1))$ is bounded.

Let $M = 1$. Let $F \in I(B(0,1))$. Then note that $F' = f$ where $f(x) \leq 1$ for all x. Since integration preserves ordering, then $F(x) - F(0) \leq x$. Additionally, note that $F(0) = \int_0^0 f(t)\,\mathrm{d}t = 0$; and thus, $F(x) \leq x$. Finally, note that $x \leq 1$. Therefore, $\sup_{x \in [0,1]} |F(x)| \leq 1 \Rightarrow \|F\| \leq 1$. Thus $I(B(0,1))$ is bounded.

We argue $I(B(0,1))$ is equicontinuous.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. Let $x, y \in [0,1]$ such that $|x - y| < \delta$.

Let $F \in I(B(0,1))$. We argue that $|F(x) - F(y)| < \varepsilon$.

Note that $F' = f$ for some $f \in B(0,1)$. Therefore, $\forall x : F'(x) \leq 1$. By the Mean Value Theorem, this implies that ${}^{F(x)-F(y)}\!/_{x-y} \leq 1$.

Therefore, $|F(x) - F(y)| \leq |x - y| < \delta = \varepsilon$.

Problem 3

Show that the function $T : C^0([a, b]) \to C^0([a, b])$ defined by

$$(T(f))(x) = f\left(\frac{x}{2}\right)$$

is a bounded operator on $C^0([a, b])$ (with the C^0-norm), but not compact.

Let $f \in C^0([a, b])$ such that $\|f\| = 1$. Then $\sup_{x \in [a,b]} |f(x)| = 1$. Note that $\forall x \in [a, b]\ f(x) \leq 1$. Then $f(x/2) \leq 1$ on $[a, b]$ as well. Therefore T must be a bounded operator.

We argue that T is not compact. It suffices to find a sequence in $T(B(0, 1))$ that has no convergent subsequence.

> Let $f_n(x) = \cos(nx)$. This has no convergent subsequence; as n gets infinitely large, the sequence oscillates rapidly, and will never converge. Note that $T(f_n) = \cos(n/2 x)$, which suffers from the same problem.

Problem 4

Show that if $\mathcal{F} \subseteq C^1([a,b])$ is C^1-closed, C^1-bounded and the collection $\{f' : f \in \mathcal{F}\}$ is equicontinuous, then $\mathcal{F}$ is compact (with respect to the C^1-norm). [*Note*: You can use the C^0-version of Arzela-Ascoli in your proof]

We argue $\mathcal{F}$ is compact with respect to the C^1 norm. Let (f_n) be a sequence in $\mathcal{F}$. We argue (f_n) has a convergent subsequence with respect to the C^1 norm.

We argue that the sequence $(f_n)'$ has a convergent subsequence with respect to the C^0 norm.

> Because $\mathcal{F}$ is C^1-bounded, then $\exists M \in \mathbb{R}$ such that $\|f_n\|_{C^1} \leq M$; thus, $\|f_n'\|_{C^0} \leq M$ for all M. Further note that this sequence is equicontinuous, as given in the problem statement. Thus, by the C^0 Arzela-Ascoli Theorem, it has a convergent subsequence with respect to the C^0 norm.

We further argue the sequence (f_n) has a convergent subsequence with respect to the C^0 norm.

> Note that because $\mathcal{F}$ is C^1-bounded, then $\exists M \in \mathbb{R}$ such that $\forall n\ f_n' \leq M$. Then each (f_n) is Lipschitz with a coefficient of M; i.e., (f_n) is equicontinuous. Furthermore, because $\mathcal{F}$ is C^1-bounded, it must be C^0-bounded, so by the Arzela-Ascoli Theorem, it has a convergent subsequence.

Finally, we argue (f_n) has a convergent subsequence with respect to the C^1 norm.

> Suppose (f_n) has a convergent subsequence $f_{n_k} \to f$ and $(f_n)'$ has a convergent subsequence $f_{n_k}' \to v$ with respect to the C^0-norm. Then note that $v = f'$, and further, the distance metric
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> $$\left\|f_n - f_{n_k}\right\|_{C^1} = \left\|f_n - f_{n_k}\right\|_{C^0} + \left\|f_n' - f_{n_k}'\right\|_{C^0}$$
>
> Because each term becomes arbitrarily small, the overall distance becomes arbitrarily small. Thus $\left(f_{n_k}\right)$ converges in the C^1 norm.

Problem 5

Show that if a subset $\mathcal{F} \subseteq C^1([a, b])$ is compact (with respect to the C^1-norm), then it is C^1-closed, C^1-bounded and the collection $\{f' : f \in \mathcal{F}\}$ is equicontinuous. [*Note*: You can use the C^0-version of Arzela-Ascoli in your proof]

Suppose that $\mathcal{F} \subseteq C^1([a, b])$ is compact with respect to the C^1-norm.

We argue $\mathcal{F}$ is C^1-closed. This has been proved in class: every compact set is closed.

We argue $\mathcal{F}$ is C^1-bounded.

> Fix $f \in \mathcal{F}$. Note that because the distance between any two points is finite, then the collection
>
> $$\bigcup_{n=1}^{\infty} B(0, n)$$
>
> forms an open cover of $\mathcal{F}$. Because $\mathcal{F}$ is compact, given any open cover of $\mathcal{F}$, there exists a finite subcover of $\mathcal{F}$. Then there must exist some $k \in \mathbb{N}$ such that
>
> $$\bigcup_{n=1}^{k} B(0, n)$$
>
> still forms an open cover of $\mathcal{F}$. Then k is an upper bound on the distance between 0 and $\mathcal{F}$, and thus $\mathcal{F}$ is bounded.

We argue that $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is equicontinuous.

> Note that our definition of equicontinuity is not dependent on the norm of the function; i.e., the set is equicontinuous in C^1 if and only if it is equicontinuous in C^0. By the Arzela-Ascoli Theorem, it suffices to prove that $\mathcal{F}'$ is compact in C^0.
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> Note that the image of a compact set must be compact. We know that $\mathcal{F}$ is C^1-compact and C^0-compact. Further note that $\mathcal{F}'$ is the image of $\mathcal{F}$ under the function $(D(f))(x) = f'(x)$. It thus suffices to show that D is continuous.
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> We have previously proven that every linear operator is continuous. Note that the derivative is a linear operator:
> 1. $(f + g)' = f' + g'$
> 2. $(\lambda f)' = \lambda f'$
>
> ...and thus the derivative is continuous.