Analysis Fall 2025 - Homework 2

If V is a normed vector space, a function $\mathcal{L} : V \to \mathbb{R}$ is called a *bounded functional* if $\mathcal{L}$ is linear (i.e., $\mathcal{L}(v + w) = \mathcal{L}(v) + \mathcal{L}(w)$ and $\mathcal{L}(\lambda v) = \lambda \mathcal{L}(v)$ and $\sup_{\|v\|=1} |\mathcal{L}(v)| < \infty$.

Problem 1

If X is a compact metric space, a function $f : X \to \mathbb{R}$ is called L-Lipschitz if for every $x, y \in X$, $|f(x) - f(y)| \leq L|x - y|$. Show that $K_{L,B} = \{f \in C^0(X) : f \text{ is } L\text{-Lipschitz and } \|f\|_{C^0} \leq B\}$ is compact for every L and B.

Fix $L, B \geq 0$. By the Arzela-Ascoli Theorem, the set $K_{L,B}$ is compact if and only if

1. $K_{L,B}$ is closed.
2. $K_{L,B}$ is bounded.
3. $K_{L,B}$ is equicontinuous.

We first argue that $K_{L,B}$ is closed.

Recall that the intersection of two closed sets is closed, and observe that $K_{L,B}$ is the intersection of $\{g \in C^0(X) : \|g\|_{C^0} \leq B\}$ and $\{h \in C^0(X) : h \text{ is } L\text{-Lipschitz}\}$.

We argue that $\{g \in C^0(X) : \|g\|_{C^0} \leq B\}$ is closed. Recall that for a continuous function $\alpha : X \to Y$, if $A \subset Y$ is closed, then $\alpha^{-1}(A) \subset X$ is closed. Further recall that norms are continuous. Observe that the set $[0, B] \subset \mathbb{R}$ is closed, and thus its inverse image is closed.

We argue that $D = \{g \in C^0(X) : g \text{ is } L\text{-Lipschitz}\}$ is closed. Recall that if every sequence in D converges to a point in D, then D is closed. Let (h_n) be a sequence of Lipschitz functions in D converging to h. We argue $h \in D$.

Let $x, y \in X$. Let $\varepsilon > 0$, and note that $\exists n$ such that $\|h_n - h\|_{C^0} < {}^{\varepsilon}/_{2}$. Since the C^0-norm defines a supremum, this implies that $\forall z \in X : |h_n(x) - h(x)| < {}^{\varepsilon}/_{2}$. Additionally, note that h_n is L-Lipschitz; thus $|h_n(x) - h_n(y)| \leq L|x - y|$.

Finally, $|h(x) - h(y)| \leq |h(x) - h_n(x)| + |h_n(x) - h_n(y)| + \left|h_{n(y)} - h(y)\right|$. Then $|h(x) - h(y)| < L|x - y| + \varepsilon$. Since this holds $\forall \varepsilon > 0$, then $|h(x) - h(y)| \leq L|x - y|$. Thus h is L-Lipschitz.

We secondly argue that $K_{L,B}$ is bounded.

Note that $\forall f \in K$, then $\|f\|_{C^0} \leq B$. Since the norm defines the distance metric, the distance of each value from the origin is also bounded above by B. Thus, $K_{L,B}$ is bounded.

We finally argue that $K_{L,B}$ is equicontinuous.

Recall that a set S is equicontinuous if and only if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \forall x, y \text{ with } d(x, y) < \delta \text{ and } \forall f \in S, \text{then } |f(x) - f(y)| < \varepsilon$$

Let $\varepsilon > 0$. We choose a δ that meets the above properties. Namely, let $\delta = {}^{\varepsilon}/_{L}$.

Let $x, y \in X$ such that $d(x, y) < \delta$, and let $f \in K_{L,B}$.

Note that because f is L-Lipschitz, then $|f(x) - f(y)| < L \cdot d(x, y) < L \cdot \delta = \varepsilon$.

Problem 2

The C^1-unit ball is the set $\{f \in C^1([a,b]) : \|f\|_{C^1} \leq 1\}$. Show that the C^1-unit ball is pre-compact in $C^0([a,b])$ with respect to $\|f\|_{C^0}$, but not compact.

Call the C^1-unit ball S. By the Arzela-Ascoli Theorem, S is precompact with respect to C^0 if and only if S is bounded and equicontinuous. It thus suffices to prove that S is bounded, S is equicontinuous, and S is not compact.

We claim that S is bounded.

> Note that $\forall s \in S$, then $\|s\|_{C^1} = \|s\|_{C^0} + \|s'\|_{C^0}$. Since the C^1 norm is an upper bound for the C^0 norm, then $\|s\|_{C^0} \leq 1$. Therefore, S is bounded on the C^0 norm.

We claim that S is equicontinuous.

> Recall that S is equicontinuous if and only if
>
> $$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \forall x, y \text{ with } d(x,y) < \delta \text{ and } \forall f \in S, \text{then } |f(x) - f(y)| < \varepsilon$$
>
> Let $\varepsilon > 0$, and choose $\delta = \varepsilon$. Let $x, y \in [a,b]$ such that $d(x,y) < \delta$. Let $f \in S$.
>
> Note that $\|f\|_{C^1} = \|f\|_{C^0} + \|f'\|_{C^0}$. Since the C^0 norm defines the supremum of the absolute value of the function, and the C^1 norm is bounded above by 1, then the absolute value of the first derivative of f is bounded above by 1; that is, $\forall z \in [a,b] : |f'(z)| \leq 1$.
>
> Further note that by the Mean Value Theorem, $\exists c \in [a,b] : {}^{|f(x)-f(y)|}\!/_{|x-y|} = f'(c)$. Because $f'(c)$ is bounded, ${}^{|f(x)-f(y)|}\!/_{|x-y|} \leq 1$, thus, $|f(x) - f(y)| \leq |x - y|$.

We claim that S is not closed.

> It suffices to prove there exists a sequence in S which has no convergent subsequence. Consider the sequence $f_n(x) = {}^1\!/_3\sqrt{x^2 + {}^1\!/_n}$. Without loss of generality, assume the interval is $[-1,1]$ (i.e., we may generalize to $[a,b]$ by scaling the coefficients of terms and the overall function, which would require tedious bookkeeping around whether $a, b \leq 1$ or $a, b \geq 0$, etc.).
>
> Since $x^2 \leq 1$ and ${}^1\!/_n \leq 1$, then $f_n(x) \leq {}^1\!/_2$. Additionally, $f_n'(x) = {}^x\!/_{3 \cdot \sqrt{x^2 + {}^1\!/_n}}$. Note that $x \leq \sqrt{x^2 + {}^1\!/_n}$, so $f_n'(x) \leq {}^1\!/_3$. Therefore, $f_n(x)$ is in S.
>
> However, $f_n(x) \to f(x)$ where $f(x) = {}^1\!/_3|x|$. This is *not* differentiable at $x = 0$. Thus, it is not in S.

Problem 3

Show that if X is a compact metric space and $V \subset C^0(X)$ is a finite-dimensional subspace, then $\{f \in V : \|f\|_{C^0} \leq 1\}$ is compact. [*Hint*: Establish the assumptions of Arzela-Ascoli by fixing some basis of V]

Let $V \subset C^0(X)$ be a finite-dimensional subspace. Let $B = \{b_1, b_2, ..., b_n\}$ be a basis of V, and let $S = \{f \in V : \|f\|_{C^0} \leq 1\}$. Note that by the Arzela-Ascoli Theorem, S is compact if

1. S is closed.
2. S is bounded.
3. S is equicontinuous.

We argue S is closed.

> Note the intersection of two closed sets is closed, and S is the intersection of $f \in V$ and the C^0 unit ball. It thus suffices to prove these individually are closed.
>
> We argue V is closed. Let $x_n = \overrightarrow{a_n} \cdot B$ be a sequence converging to x (where $a_n \in \mathbb{R}^n$). Note that because convergence is preserved across addition and scaling of vector sequences, this is true if and only if $a_n \to a \in \mathbb{R}^n$ and $x = a \cdot B$. However, this implies $x \in V$, as it is expressed as a sum of basis vectors. Thus V is closed.
>
> We argue the C^0 unit ball is closed. Note that for a continuous function, the inverse image of a closed set must be closed. Further note that C^0 norm is continuous with respect to itself, and that $[0, 1] \in \mathbb{R}$ is closed. Finally, note that the C^0 unit ball is the inverse image of $[0, 1]$ with respect to the C^0 norm. Thus, the C^0 unit ball is closed.

We argue S is bounded.

> Note that $\forall s \in S$, then $\|s\|_{C^0} \leq 1$. Since the norm defines distance, the distance between all points in the set is also obviously bounded. Therefore, S is bounded on the C^0 norm.

We argue S is equicontinuous.

> Recall that S is equicontinuous if and only if
>
> $$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \forall x, y \text{ with } d(x, y) < \delta \text{ and } \forall f \in S, \text{then } |f(x) - f(y)| < \varepsilon$$
>
> Let $\varepsilon > 0$. Choose δ (as described below). Let $x, y \in X$ such that $d(x, y) < \delta$. Let $f \in S$ and note that
>
> $$|f(x) - f(y)| \leq |(a \cdot B)(x) - (a \cdot B)(y)| \leq |a \cdot B(x - y)|$$
>
> Since the sup-norm of f is less than 1, then $(a \cdot B)(x) \leq (1 \cdot B)(x)$ for every x. Thus,
>
> $$|f(x) - f(y)| \leq |(1 \cdot B)(x - y)|$$
>
> Note that because $(1 \cdot B)$ is a linear function, it has a bound L. Choose $\delta = {}^{\varepsilon}/_{L}$. Then
>
> $$|f(x) - f(y)| \leq |(1 \cdot B)(x - y)| \leq L|x - y| < L \cdot \delta = \varepsilon$$

Problem 4

Let V be a Banach space with respect to norm $\|\cdot\|$ and $W \subset V$ is a vector subspace, then $\overline{W}$ is a Banach space with respect to $\|\cdot\|$. [*Note*: You need to show that $\overline{W}$ is a vector subspace in addition to being complete]

First, we show that $\overline{W}$ is a vector subspace.

> We argue that $x, y \in \overline{W} \Rightarrow x + y \in \overline{W}$. Note since $x \in \overline{W}$, then $\exists (x_n) \in W : x_n \to x$ and symmetrically $\exists (y_n) \in W : y_n \to y$. Note that $\forall n$, then $x_n + y_n \in W$, as W is a vector subspace (i.e., it is a sequence in W). Additionally note that $x_n + y_n \to x + y$. Therefore, $x + y \in \overline{W}$.
>
> We argue that $x \in \overline{W}$ and $\lambda \in \mathbb{R} \Rightarrow \lambda x \in \overline{W}$. Note since $x \in \overline{W}$, then $\exists (x_n) \in W : x_n \to x$. Note that $\forall n$, then $\lambda x_n \in W$, as W is a vector subspace (i.e., it is a sequence in W). Additionally note that $\lambda x_n \to \lambda x$. Therefore, $\lambda x \in \overline{W}$.

Secondly, we argue that $\overline{W}$ is a Banach space.

> Let x_n be a Cauchy sequence in $\overline{W}$. We argue that x_n converges. Note that $\overline{W} \subseteq V$ and V is a Banach space. Therefore, x_n converges in V. However, since $\overline{W}$ is closed, every convergent sequence converges to a point in $\overline{W}$. Therefore, x_n must converge to a point in $\overline{W}$. Thus, $\overline{W}$ is a Banach space.

Problem 5

Show that if $C^0([0,1])$ is given the C^0-norm and $x \in [0,1]$, the function $\delta_x(f) = f(x)$ is a bounded functional.

Fix $x \in [0,1]$ and let $g, h \in C^0([0,1])$. Let $\delta_x(f) = f(x)$, and let $\lambda \in \mathbb{R}$.

Note that $\delta_x(g+h) = (g+h)(x) = g(x) + h(x) = \delta_x(g) + \delta_x(h)$, because g and h are continuous.

Note that $\delta_x(\lambda g) = (\lambda g)(x) = \lambda \cdot g(x) = \lambda \cdot \delta_x(g)$, because g is continuous.

Note that because g is in $C^0([0,1])$, the function is bounded, and thus $\delta_x(g)$ is also bounded, and thus $\sup_{\|v\|=1} |\delta_x(v)| < \infty$.

Problem 6

Show that if $C^0([0,1])$ is given the C^0-norm, the function $\mathcal{I}_g(f) = \int_0^1 g(x)f(x)\,\mathrm{d}x$ is a bounded functional for every $g \in C^0([0,1])$ (the integral is the usual Riemann integral).

Fix $g \in C^0([0,1])$.

Let $h, i \in C^0([0,1])$. Note that

$$\mathcal{I}_g(h+i) = \int_0^1 g(x)(h+i)(x)\,\mathrm{d}x = \int_0^1 g(x)(h(x)+i(x))\,\mathrm{d}x = \int_0^1 g(x)h(x) + g(x)i(x)\,\mathrm{d}x =$$

$$\int_0^1 g(x)h(x)\,\mathrm{d}x + \int_0^1 g(x)i(x)\,\mathrm{d}x = \mathcal{I}_g(h) + \mathcal{I}_g(i)$$

Let $f \in C^0([0,1])$ and let $\lambda \in \mathbb{R}$. Note that

$$\mathcal{I}_g(\lambda f) = \int_0^1 g(x) \cdot \lambda f(x)\,\mathrm{d}x = \lambda \int_0^1 g(x)f(x)\,\mathrm{d}x = \lambda \mathcal{I}_g(f)$$

We seek to prove $\exists M \in \mathbb{R}$ such that $\forall f \in C^0([0,1])$ with $\|f\| = 1$, then $\int_0^1 g(x)f(x)\,\mathrm{d}x \leq M$. Denote $G(x)$ to be the antiderivative of $g(x)$, and set $M = G(1) - G(0)$. Note that $\|f\| = 1$, thus, $f(x) \leq 1$ for every x in $[0,1]$. Therefore,

$$\int_0^1 g(x)f(x)\,\mathrm{d}x \leq \int_0^1 g(x) \cdot 1\,\mathrm{d}x = G(1) - G(0)$$

Problem 7

Show that $\mathcal{L}$ is a bounded functional if and only if it is continuous from V to $\mathbb{R}$, where V uses its norm to define a distance. [As a reminder, continuity in this context means that for every $\varepsilon > 0$, find a $\delta > 0$ such that if $\|v - w\| < \delta$, then $|\mathcal{L}(v) - \mathcal{L}(w)| < \varepsilon$.]

Suppose that $\mathcal{L}$ is a bounded functional. We argue that $\mathcal{L}$ is continuous.

Let $\varepsilon > 0$.

Since $\mathcal{L}$ is a bounded functional, then $\sup_{\|v\|=1} |\mathcal{L}(v)| < \infty$. Let $B = \sup_{\|v\|=1} |\mathcal{L}(v)|$, and choose $\delta = {}^{\varepsilon}/_{B}$.

Let $v, w \in V$ such that $\|v - w\| < \delta$. Then

$$|\mathcal{L}(v) - \mathcal{L}(w)| = |\mathcal{L}(v) + \mathcal{L}(-w)| = |\mathcal{L}(v - w)| = \|v - w\| \cdot \left|\mathcal{L}\left(\frac{v - w}{\|v - w\|}\right)\right|$$

Note that $\left\| {}^{v-w}/_{\|v-w\|} \right\| = 1$, and thus, the second term is bounded above by B. Further, by definition, the first term is bounded above by δ. Thus

$$|\mathcal{L}(v) - \mathcal{L}(w)| < \delta \cdot B = \frac{\varepsilon}{B} \cdot B = \varepsilon$$

Suppose that $\mathcal{L}$ is continuous. We argue that $\mathcal{L}$ is a bounded functional.

I could not resolve how to solve this. In fact, I'm not sure I'm convinced it is true. Take, for example, $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^2$. Then $f(x + y) = x^2 + 2xy + y^2 \neq x^2 + y^2$. Furthermore, $f(\lambda x) = \lambda^2 x^2 \neq \lambda x^2$. Therefore, f cannot be a bounded functional.