Analysis Fall 2025 - Homework 1

Definition 1. A *norm* on a vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ such that

(i) $\|v\| \geq 0$ for all $v \in V$, with equality if and only if $v = 0$

(ii) $\|\lambda v\| = |\lambda| \cdot \|v\|$ for all $\lambda \in \mathbb{R}$ and $v \in V$

(iii) $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$

Definition 2. If $[a, b] \subset \mathbb{R}$, define

$$C^r([a, b]) = \left\{ f : [a, b] \to \mathbb{R} \text{ such that } f^(k) \text{ is continuous for all } 0 \leq k \leq r \right\}$$

and

$$\|f\|_{C^0} = \sup_{x \in [a,b]} |f(x)| \qquad \|f\|_{C^r} = \sum_{k=0}^{r} \left\| f^{(k)} \right\|_{C^0}$$

Problem 1

If V is a fixed vector space, show that the equivalence of norms is an equivalence relation on the space of norms on V.

Recall that for $\|\cdot\| \sim \|\cdot\|'$, then $\sim$ is an equivalence relation if

1. *Reflexivity*: for each norm $\|\cdot\|$, it is true that $\|\cdot\| \sim \|\cdot\|$.
2. *Symmetry*: $\|\cdot\| \sim \|\cdot\|'$ implies $\|\cdot\|' \sim \|\cdot\|$
3. *Transitivity*: $\|\cdot\| \sim \|\cdot\|'$ and $\|\cdot\|' \sim \|\cdot\|''$ implies $\|\cdot\| \sim \|\cdot\|''$.

Recall that a norm $\|\cdot\|$ on V is equivalent to $\|\cdot\|'$ if

$$\exists C \in \mathbb{R}\ \forall v \in V : \frac{1}{C}\|v\| \leq \|v\|' \leq C\|v\|$$

Denote norm equivalence with the operator $=$. We argue this is an equivalence relation.

We argue $=$ follows the reflexivity property.

Let $\|\cdot\|$ be an arbitrary norm on a vector space V, and let $v \in V$. Let $C = 1$. Then

$$\begin{aligned} \|v\| \leq \|v\| \leq \|v\| \Rightarrow \\ \frac{1}{1} \cdot \|v\| \leq \|v\| \leq 1 \cdot \|v\| \Rightarrow \\ \frac{1}{C} \cdot \|v\| \leq \|v\| \leq C \cdot \|v\| \end{aligned}$$

Thus $\|\cdot\|$ is equal to itself, and thus reflexive.

We argue $=$ follows the symmetric property.

Let $\|\cdot\|$ and $\|\cdot\|'$ be arbitrary norms on a vector space V, and let $v \in V$. Then $\exists C \in \mathbb{R}$ such that

$$\frac{1}{C} \cdot \|v\| \leq \|v\|' \leq C \cdot \|v\|$$

Then note that basic algebra shows us

$$\frac{1}{C} \cdot \|v\|' \leq \|v\| \leq C \cdot \|v\|$$

We argue $=$ follows the transitive property.

Let $\|\cdot\|$, $\|\cdot\|'$ and $\|\cdot\|''$ be arbitrary norms on a vector space V, and let $v \in V$. Then $\exists C_1, C_2 \in \mathbb{R}$ such that

$$\frac{1}{C_1}\|v\| \leq \|v\|' \leq C_1\|v\| \qquad \frac{1}{C_2}\|v\|' \leq \|v\|'' \leq C_2\|v\|'$$

Let $C_3 = C_1 \cdot C_2$. Then basic algebra shows us

$$\frac{1}{C_3} \cdot \|v\| \leq \|v\|'' \leq C_3 \cdot \|v\|$$

Problem 2

Show that $K := \left\{ v \in \mathbb{R}^n : \sum_{i=1}^n |v_i| = 1 \right\}$ is compact. Conclude that for all norms on $\mathbb{R}^n$, $\{v \in \mathbb{R}^n : \|v\| = 1\}$ is compact. [*Hint*: you may use the fact that all norms on $\mathbb{R}^n$ are equivalent]

We first prove the second half, then prove the first half.

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. We have previously proved several useful properties:
1. $\|\cdot\|$ is continuous.
2. If $f : X \to Y$ is continuous and $A \subset Y$ is closed, then $f^{-1}(A)$ is closed in X.
3. *Heine-Borel Theorem*: a subset $U \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

We argue that $S = \{v \in \mathbb{R}^n : \|v\| = 1\}$ is compact. Since it is a subset of $\mathbb{R}^n$, it suffices to show it is closed and bounded.

Let $A = \{x \in \mathbb{R} : x = 1\}$. Note that since this is a singleton set, it is obviously closed. Furthermore, it is obviously bounded in $\mathbb{R}^n$. Therefore, since S is the inverse image of A, then S is closed.

Furthermore, S is obviously bounded. The value norm is constant within the set, which means that the distance metric defined by the norm is also constant within the set.

Therefore, S is compact.

Note, additionally, that the set K as defined in the problem half is a special-case of S on the L_0 norm. Therefore, all findings apply, and K is compact.

Problem 3

Show that $\|\cdot\|_{C^r}$ is a norm on $C^r([a,b])$. [*Hint*: First show it for $r = 0$, and use this to prove it for other values of r].

We prove via induction on r.

We first argue that $\|\cdot\|_{C^0}$ is a norm on $C^0([a,b])$.

1. We argue that $\|\cdot\|_{C^0} \geq 0$ for all $v \in C^0([a,b])$, with equality if and only if $v = 0$.

   Let $f \in C^0([a,b])$. Then $\|f\|_{C^0} = \sup_{x\in[a,b]}|f(x)|$. Note that $|f(x)| \geq 0$.

   Suppose $g(x) = 0 \in C^0([a,b])$. Then $\|g\|_{C^0} = \sup_{x\in[a,b]}|g(x)| = \sup_{x\in[a,b]}|0| = 0$.

   Suppose $h(x) \in C^0([a,b])$ and $\exists w \in [a,b]\ h(x) \neq 0$. Then $\|h\|_{C^0} = \sup_{x\in[a,b]}|h(x)|$. Since $h(w) \neq 0$, then $|h(w)| > 0$, and $\|h\|_{C^0} \geq |h(w)| > 0$.

2. We argue that $\|\lambda v\|_{C^0} = |\lambda| \cdot \|v\|_{C^0}$ for all $\lambda \in \mathbb{R}$ and $v \in C^0([a,b])$.

   Let $f \in C^0([a,b])$ and $\lambda \in \mathbb{R}$.

   Then $\|\lambda f\|_{C^0} = \sup_{x\in[a,b]}|\lambda \cdot f(x)| = |\lambda| \cdot \sup_{x\in[a,b]}|f(x)| = |\lambda| \cdot \|f\|_{C^0}$.

3. We argue $\|v + w\|_{C^0} \leq \|v\|_{C^0} + \|w\|_{C^0}$ for all $v, w \in C^0([a,b])$.

   Let $f, g \in C^0([a,b])$. Then

   $$\|f + g\|_{C^0} = \sup_{x\in[a,b]}|f(x) + g(x)| \leq \sup_{x\in[a,b]}(|f(x)| + |g(x)|) \leq$$
   $$\sup_{x\in[a,b]}|f(x)| + \sup_{x\in[a,b]}|g(x)| \leq \|f\|_{C^0} + \|g\|_{C^0}$$

We additionally argue that if $\|\cdot\|_{C^r}$ is a norm on $C^r([a,b])$, then $\|\cdot\|_{C^{r+1}}$ is a norm on $C^{r+1}([a,b])$.

Note that $\|f\|_{C^{r+1}} = \left\|f^{(r+1)}\right\|_{C^0} + \|f\|_{C^r}$. Additionally, note that our base case has proved the left operand is a norm, and by assumption, the right operand is a norm. It therefore suffices to show that the sum of two norms is a norm.

Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on $C^r([a,b])$. We argue that their sum, $\|\cdot\| + \|\cdot\|'$ is still a norm.

1. Since $\forall h \in C^r([a,b])$, then $\|h\| \geq 0$ and $\|h\|' \geq 0$ with equality if and only if $h(x) = 0$, then $\|h\| + \|h\|' \geq 0$ with equality if and only if $h(x) = 0$.
2. Let $\lambda \in \mathbb{R}$ and $f \in C^r([a,b])$. Then $\|\lambda v\| + \|\lambda v\|' = |\lambda|\|v\| + |\lambda|\|v\|' = |\lambda|(\|v\| + \|v\|')$.
3. Let $f, g \in C^r([a,b])$. We know $\|f + g\| \leq \|f\| + \|g\|$ and $\|f + g\|' \leq \|f\|' + \|g\|'$. Then obviously $\|f + g\| + \|f + g\|' \leq (\|f\| + \|f\|') + (\|g\| + \|g\|')$.

Problem 4

Show that if X is a compact metric space, $f_n \in C^0(X)$ and $f_n \to f$ with respect to $\|\cdot\|_{C^0}$ for some function f on X, then $f \in C^0(x)$. [*Hint*: Fix an $\varepsilon > 0$ then fix a large n such that $\|f_n - f\|_{C^0}$ is small. Choose a δ based on the continuity of f_n. You should use the triangle inequality three times.]

Let $\varepsilon > 0$. We argue that $\exists \delta \in \mathbb{R} : \forall x, y \in X \; d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

We know that $f_n \to f$; thus, $\exists n$ such that $\|f_n - f\|_{C^0} < \frac{\varepsilon}{3}$. Since the C^0 norm is the supremum of the absolute value of the function, then $\forall x \; |f_n(x) - f(x)| \leq \|f_n - f\| < \frac{\varepsilon}{3}$ as well.

We know that $f_n(x)$ is continuous; thus, $\exists \delta$ such that $d(x, y) < \delta) \Rightarrow f(x) - f(y) < \frac{\varepsilon}{3}$. We choose this delta as well.

Finally, we conclude that

$$\begin{aligned} |f(x) - f(y)| \leq\ & |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \leq \\ & |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f(y) - f_n(y)| < \\ & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon \end{aligned}$$

Problem 5

Show that the C^1-norm and C^0-norm on $C^\infty([0,1])$ are not equivalent. [*Hint*: Find a sequence of nonzero C^∞ functions f_n such that the ratio of the two norms either goes to ∞ or 0]

Note that two norms $\|\cdot\|, \|\cdot\|'$ are equivalent if

$$\exists C \in \mathbb{R}\ \forall v \in V : \frac{1}{C} \cdot \|v\| \leq \|v\|' \leq C \cdot \|v\|$$

We have already constructed the requested sequence in class. Let

$$f_n(x) = \frac{1}{n} \sin(n^2(x)) \text{ on } [0,1]$$

Since $\frac{1}{n} \to 0$ and $\sin(n^2(x))$ is bounded, then $f_n \xrightarrow{C^0} 0$. Since every convergent sequence is Cauchy, then (f_n) is Cauchy on the C^0 norm.

However, note that

$$(f_n)' = \frac{1}{n} \cdot n^2 \cos(n^2 x) = n \cos(n^2 x)$$

This is obviously unbounded. Therefore, for $\|f\|_{C^1} = |f'|_{C^0} + |f|_{C^0}$, the sequence f_n cannot be Cauchy. (This is what we demonstrated in class.)

We have previously demonstrated via a theorem that if $\|\cdot\|$ and $\|\cdot\|'$ are equivalent, then they have the same set of Cauchy sequences. Since there exists a sequence that is Cauchy on the C^0 norm but not the C^1 norm, then the norms cannot be equivalent.

Problem 6

Show that $\{f \in C^0([0,1]) : \|f\|_{C^0} \leq 1\}$ is closed and bounded in $C^0(X)$ with respect to $\|\cdot\|_{C^0}$, but not compact.

We argue that $K = \{f \in C^0([0,1]) : \|f\|_{C^0} \leq 1\}$ is closed and bounded in $C^0(x)$ with respect to $\|\cdot\|_{C^0}$.

Since the norm of all $f \in K$ is less than or equal to 1, then the set is obviously bounded.

Note that the C^0 norm is continuous, and therefore, if $A \subset \mathbb{R}$ is closed, then $f^{-1}(A)$ is closed in $C^0([0,1])$. Further note that $K = f^{-1}([0,1])$. The set $[0,1] \subset \mathbb{R}$ is obviously closed by the Heine-Borel Theorem, and thus K is closed.

We argue that K is not compact. Assume for the sake of contradiction that K were compact. Then if $f : K \to \mathbb{R}$ is continuous, then $f(K)$ must also be compact.

Recall again the pathological sequence from Problem 5

$$f_n(x) = \frac{1}{n} \sin\left(n^2(x)\right) \text{ on } [0,1]$$

Note that since $\frac{1}{n} \leq 1$ and $\sin\left(n^2(x)\right) \leq 1$, then all (f_n) are in K.

Furthermore, recall that the C^1 norm is continuous (as it is a norm), and furthermore, that under the C^1 norm, the sequence $\|f_n\|$ is unbounded. Since the image of K under the C^1 norm is unbounded, it cannot be compact. Therefore, K itself cannot be compact.

Problem 7

For $d \in \mathbb{N}$, let $\mathcal{P}_d([0,1])$ be the polynomials of degree d, considered as functions defined on $[0,1]$.

(1) Show that $\mathcal{P}_d([0,1])$ is a vector space. What is its dimension?
(2) Show that $\|\cdot\|_{C^0}$ is a norm on $\mathcal{P}_d([0,1])$.
(3) Show that $p(x) = a_d x^d + a_{d-1} x^{d-1} + ... + a_0$ is a polynomial, $\|p\| := \max\{|a_d|, ..., |a_0|\}$ defines a norm on $\mathcal{P}_d([0,1])$
(4) Show that for all $d \in \mathbb{N}$, there exists C_d with the following property: for all polynomials $p(x) = a_d x^d + a_{d-1} x^{d-1} + ... + a_0$ there exists $x \in [0,1]$ such that for every $i \in \{0, ..., d\}$,

$$|a_i| \leq C_d |p(x)|$$

Recall that a vector space must support *addition* and *scaling*. We argue $\mathcal{P}_d([0,1])$ satisfies these.

1. We argue if $f, g \in \mathcal{P}_d([0,1])$, then $f + g \in \mathcal{P}_d([0,1])$. This is trivial; the sum of two polynomials will sum their corresponding coefficients, and produce another polynomial of degree d. Addition, as previously proved, is associative, commutative, and there exists an additive inverse.
2. We argue if $f \in \mathcal{P}_d([0,1])$ and $\lambda \in \mathbb{R}$, then $\lambda f \in \mathcal{P}_d([0,1])$. This is similarly trivial; all coefficients are scaled, and thus normal properties apply.

We argue $\|\cdot\|_{C^0}$ is a norm on $\mathcal{P}_d([0,1])$. Note that all polynomials are continuous. We have previously proven that $\|\cdot\|_{C^0}$ is a norm on C^0. Since $\mathcal{P}_d([0,1]) \subseteq C^0$, then it is obviously also a norm on $\mathcal{P}_d([0,1])$.

We argue $p(x) = a_d x^d + a_{d-1} x^{d-1} + ... + a_0$ is a polynomial, $\|p\| := \max\{|a_d|, ..., |a_0|\}$ defines a norm on $\mathcal{P}_d([0,1])$.

1. Since $\|p\|$ takes the max of absolute values, it is true that $\|p\| \geq 0$.

   Suppose $p(x) = 0$. Then $\|p\| = \max\{0, ..., 0\} = 0$.

   Suppose $p(x) \neq 0$. Then $\|p\|$ will take the max of a set containing at least one non-zero value. Then $\|p\| \neq 0$.

2. Let $\lambda \in \mathbb{R}$. Then

$$\|\lambda p\| = \max\{|\lambda \cdot a_d|, ..., |\lambda \cdot a_0|\} = \max\{|\lambda| \cdot |a_d|, ..., |\lambda| \cdot |a_0|\} = \\ |\lambda| \cdot \max\{|a_d|, ..., |a_0|\} = |\lambda| \cdot \|p\|$$

3. Let $f = f_d x^d + ... + f_0$ and $g = g_d x^d + ... + g_0 \in \mathcal{P}_d([0,1])$.

   Then $\|f + g\| = \max\{|f_d + g_d|, ..., |f_0 + g_0|\}$. Recall that $|a + b| \leq |a| + a(b)$. Therefore,

$$\|f + g\| \leq \max\{|f_d| + |g_d|, ..., |f_0| + |g_0|\} \leq \max\{|f_d|, ..., |f_0|\} + \max\{|g_d|, ..., |g_0|\} \leq \|f\| + \|g\|$$

We argue for all $d \in \mathbb{N}$, there exists C_d such that for all polynomials $p(x) = a_d x^d + a_{d-1} x^{d-1} + ... + a_0$ there exists $x \in [0,1]$ such that for every $i \in \{0, ..., d\}$, then $|a_i| \leq C_d |p(x)|$.

Let $d \in \mathbb{N}$. Furthermore, let $C_d = d$.

Let $p(x) = a_d x^d + a_{d-1} x^{d-1} + ... + a_0$ be an arbitrary polynomial.

Let $x \in [0,1]$ such that $|p(x)| = \sup_{w \in [0,1]} |p(x)|$ (i.e., select the x-coordinate at which p achieves the highest magnitude.

Let $i \in \{0, ..., d\}$. Note that because we sum d digits, then the coefficient of any individual one cannot overpower the sum of the maximum value of $|p(x)|$. Therefore, $|a_i| \leq C_d |p(x)|$.