Theory of Computation

Example of NP

$SAT = \{ \Phi \mid \Phi$ is a satifiable boolean expression$\}$

$3SAT = \{ \Phi \mid \Phi$ is satisfiable 3 cnf boolean expression$\}$

$CLIQUE = \{ \langle G, K \rangle \mid G$ graph with clique of size $K \}$

$HAMPATH = \{ \langle G, s, t \rangle \mid G$ graph with a hamiltonian path from s to $t \}$

$3COLOR = \{ \langle G \rangle \mid$ Graph G, 3 colorable$\}$

$Subset\text{-}sum = \{ \langle S, t \rangle \mid S = \{ x_1 \ldots x_n \}, S' \subseteq S$; S' sum $= t \}$

Def^n L is NP-Complete

i) $L \in NP$

ii) $\forall L' \in NP, \; L' \leq_p L$

remark

If $\exists L$ NP-Complete, $L \in P$, then $P = NP$

Thm: If $A \in NP$ and $B \leq_p A$, B is NP complete, then A NP-Complete.

Thm: $3SAT \leq_p CLIQUE$ (since we proved $CLIQUE \in NP$, this will imply CLIQUE is NP-Complete)

Proof: Let ϕ be a k term boolean expression in 3CNF.

$$\phi = (a_1 \vee b_2 \vee c_2) \wedge \ldots \wedge (a_k \vee b_k \vee c_k)$$

We map this expression to some graph $\langle G, k \rangle$, $(\phi \longmapsto \langle G, k \rangle)$, such that

$$\phi \in 3SAT \iff \langle G, k \rangle \in CLIQUE$$

and f is computable in polytime

Let the vertex set of G, V be defined by

$$V = \{ V_{a_1}, V_{b_1}, V_{c_1}, \ldots, V_{a_k}, V_{b_k}, V_{c_k} \}$$

s.t. size of V, $|V| = 3k$.

and edge set of G, E be allmost fully connected except for....

1) Vertices in same Clause
2) Vertices x_i, $\overline{x_i}$ (negated versions of eachother)

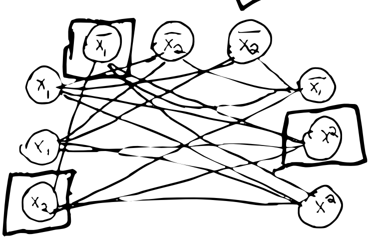
i.e.

$$E = \{(v, v') \mid v, v' \in V\} \setminus \bigcup_{i=1}^{k} \{(V_{a_i}, V_{b_i}), (V_{b_i}, V_{c_i}), (V_{c_i}, V_{a_i})\}$$
$$\setminus \{(v, v') \mid v \text{ and } v' \text{ correspond to negated literals i.e. } v = x_i, v' = \overline{x_i}\}$$

i.e.

$$\Phi = (x_1 \vee x_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2 \vee x_2)$$

transforms $\Downarrow$ to



$x_1 = F$, $x_2 = T$

Computation Time

- Input 3k literal

$$|V| = 3k = O(k)$$
$$|E| = \binom{3k}{2} = O(k^2)$$

So $f \in TIME(k^2) \subseteq P$

side remark

$A \iff B \equiv A \Rightarrow B \land B \Rightarrow A$

OR

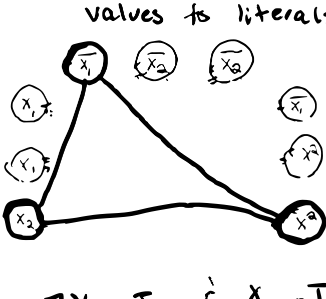
$(A \Rightarrow B) \land (\neg A \Rightarrow \neg B)$

1) $\phi \in 3SAT \Rightarrow \langle G, k \rangle \in CLIQUE$

Assume ϕ has a satisfying assignment. This means in each clause has a true literal. Choose one true literal in each clause. There are K clauses, so we picked K nodes. This forms a k clique because each pair of nodes come from different clauses, and does not fall under first exception. Secondly, both nodes correspond to a true literal, so cannot be $x_i, \overline{x_i}$ pair since both can't be true simultaneously.

2) $\langle G, k \rangle \in CLIQUE \Rightarrow \phi \in 3SAT$

Assume we have identified a clique. So we have k-nodes all connect. Since we know no nodes in same clause are connected, can't have 2 nodes in the same clause, thus each vertex corresponds to different clause (pigeonhole, k clauses, k nodes). Use these K nodes to assign true values to literals they represent. Since we can never select negated versions, and we have 1 true literal per clause, this is a valid and satisfying assignment.



$\blacksquare$

$\neg x_1 = T$; $x_2 = T \rightarrow x_1 = F$, $x_2 = T$

Thm

$3SAT \leq_p HAMPATH$ (which will prove HAMPATH is NP complete)

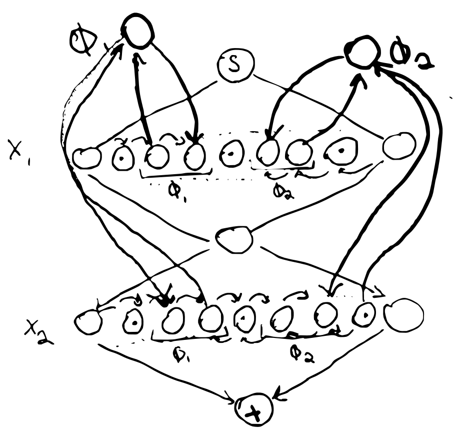
Proof:

Want a mapping

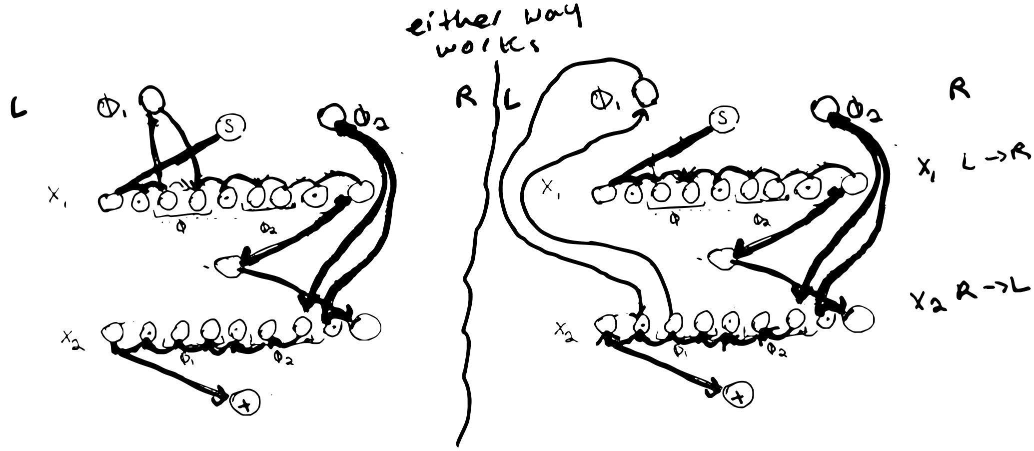
$$\underset{cnf}{\phi} \xmapsto{f} \langle G, s, t \rangle$$

s.t. $\phi \in 3SAT \iff \langle G, s, t \rangle \in HAMPATH$

ex $\phi = \underbrace{(x_1 \vee x_1 \vee \overline{x_2})}_{\phi_1} \wedge \underbrace{(\overline{x_1} \vee \overline{x_2} \vee \overline{x_2})}_{\phi_2}$



⊙ vertex is buffer vertex



for example

Since $X_1 = T$ traverse diamond L→R

and $X_2 = F$; traverse diamond R→L

not possible cuz we need to visit Φ_1

For each detour, for each clause Φ choose one of the true literals X_r or $\overline{X}_r$, then when traversing the diamond correspondingly to X_r, thake the detour to Φ_j. Since X_r makes the clause Φ_j true if

- $X_i = T$, then X_i in Φ_r we c/a traversing L→R and detour is hooked L→R so we can take detour
- $X_i = F$, $\overline{X}_i$ in Φ_j, then we are traversing R→L, this detour is R→L so we can take detour

Given a satisfying assiment $\phi \Rightarrow$ hampath G, s to t.

- Assuming a hampath $s \to t$ exists a satisfying assignment

Can extract a satisfying assignment

i.e. $L \to R$ through $x_1 = T$ if $L \leftarrow R$ then $x_i = F$.

↳ If we do detour at A_1 and go to diff diamond, then we can never incorporate the next node A_2 into a hamiltonian path.