Pricing of an American Call Option on a Dividend Paying Stock

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1. Background

1.1 Financial Background

Call Option is a financial product that provides the option owner rights to buy a specific stock at an exercise price *K* on or before an expiration date *T*. There are two types of call option, American and European call option. The difference between them is the owner can only exercise the option at the expiration date for European call option. For American call option, the owner can exercise the option on or before the expiration date.

For a dividend paying stock, it is assumed that the stock price will drop by the amount of dividend *D* after the ex-dividend date. To price American call options, an exercise policy is assumed which an option is to exercise just prior to an ex-dividend date if and only if the stock price exceeds a threshold price.

1.2 Mathematical Background

Stock price is one of the factors controlling the option price. In order to price an option, the payoff could be maximized using Monte Carlo simulation. To model stock prices without dividends, it is suggested to use Geometric Brownian Motion (GBM), which satisfies (i) memoryless property, and (ii) positive for the function domain.

$$\widetilde{S_t} = \widetilde{S_0}\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z\right) \equiv h\left(Z; \widetilde{S_0}, t, \mu, \sigma\right)$$

$$where\ Z\ is\ a\ standard\ normal\ r.v., \mu\ is\ mean\ rate\ of\ return\ and\ \sigma\ is\ volatility$$

By including the price movement effects from dividend pays, which will be discussed in section 2, it is possible to simulate the actual stock price movement.

According to the exercise policy, if the stock price at the ex-dividend date t_j, denoted by S_{t_j}, is higher than the threshold value s_j of the option owner, the owner would exercise the option and the payoff is $S_{t_j} - K$, where K is the exercise price. If the predicted stock price is not higher than the threshold value at any moment before the expiration date, the payoff is $(S_T - K)^+$, which is the difference between the stock price at expiration date S_T and the exercise price. To combine these two parts, considering the present value of the payoff, the payoff function becomes

$$J_T = e^{-rT}\left(\sum_{i=1}^{\eta(T)}\left[\prod_{j=1}^{i-1} 1_{\left\{S_{t_j^-} \le s_j\right\}}\right] 1_{\left\{S_{t_i^-} > s_i\right\}}\left(S_{t_i^-} - K\right) e^{r(T-t_i)} + \prod_{j=1}^{\eta(T)} 1_{\left\{S_{t_j^-} \le s_j\right\}}(S_T - K)^+\right.$$

$$where\ \eta(T)\ denotes\ the\ total\ number\ of\ ex-dividend\ dates\ and\ defines\ t_0 = 0, t_{\eta(T)+1} = T$$

2 Optimization

2.1 Problem Statement

Given an American call option with strike price $K = 50$ on a dividend paying stock with ex-dividend dates $t_1 = 0.3, t_2 = 0.6$ and expiration date $T = 0.9$. The initial stock price is $S_0 = 50$, the mean rate of return is $\mu = 0.1$. The initial threshold prices are $s_1 = s_2 = 60$ and the dividends are $D_1 = D_2 = 2$. The interest rate is taken as $r = 0.1$ and the call option price is inferred to be 4.85. Find the volatility such that it is the optimal fit to the actual price.

2.2 Optimization

The problem can be framed as an optimization problem as the following:

A loss function J is defined as,

$$J(\sigma) = \frac{1}{2}E[(J_T(\sigma) - A)^2]\ where\ A\ is\ the\ target\ option\ price$$

It is desired to find an optimal σ^* s.t.

$$\sigma^* = argmin_{\sigma} \mathrm{J}(\sigma) = argmin_{\sigma} \frac{E[(J_T(\sigma) - A)^2]}{2}$$

By gradient descent algorithm, following update rule could be used for finding the optimal σ^*,

$$\sigma_{n+1} = \sigma_n + \varepsilon_n \mathrm{G}(\sigma_n) = \sigma_n - \varepsilon_n \nabla J(\sigma_n) \approx \sigma_n - \varepsilon_n E\left[\frac{\partial}{\partial\sigma}J_T(\sigma_n)\right] E[(J_T(\sigma_n) - A)]$$

2.3 Algorithm

From the above update rule, $E\left[\frac{\partial}{\partial\sigma}J_T(\sigma_n)\right]$ is computationally difficult. To circumvent the difficulty of differentiating the payoff function, the $E[(J_T(\sigma_n) - A)]$, which is computationally easy, could provide a direction of update for finding the optimal σ.

By considering a range of σ that encloses the σ^*, simulation results (fig.) suggested that approximately $E\left[\frac{\partial}{\partial\sigma}J_T(\sigma_n)\right] \ge 0$.

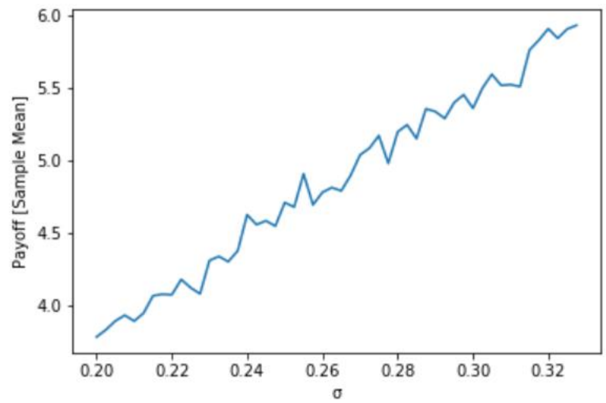

Fig. Plotting σ against payoff $\widehat{J_T}$

Therefore, a computationally efficient alternative update rule can be developed for optimizing σ,

$$\sigma_{n+1} = \sigma_n - \varepsilon_n \nabla J(\sigma_n) \approx \sigma_n - \varepsilon_n[(J_T(\sigma_n) - A)]$$

Intuitively speaking, the σ is updated toward the direction that helps $J_T(\sigma)$ approaches A by multiplying the learning rate ε_n.

2.4 Simulation

To conduct Monte Carlo simulation for pricing American call options, it is necessary to sample real stock price movements.

It is assumed that real stock prices $S(t)$ can be decomposed into two elements, namely present values of future dividends and intrinsic prices. The latter one can be simulated by GBM $\tilde{S}(t)$, and the former one is predetermined by the dividend scheme of a stock d_i.

Assumed a stock has a dividend scheme,

$$\{(D_1, t_1), (D_2, t_2), \ldots, (D_{\eta(T)}, t_{\eta(T)})\}$$

The real stock price can be modelled by the following piece-wise function,

$$S = \{S_i(t): 0 \le i \le \eta(T) + 1\}$$

$$where\ S_i(t) = \tilde{S}(t) + \sum_{j=1}^{\eta(T)} 1_{\{t \le t_j\}} D_j e^{-r(t_j - t)}\ for\ t_i^+ \le t \le t_{i+1}^-$$

Here are simulation results with different dividend schemes,

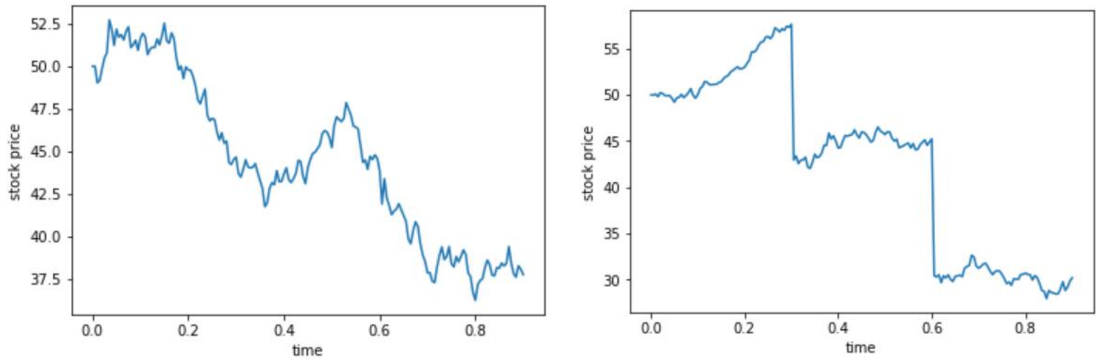

Fig1. Both graphs are generated from similar parameters except (Left) has a dividend scheme of $\{(2,0.3),(2,0.6)\}$ and, (Right) has a dividend scheme of $\{(15,0.3),(15,0.6)\}$

The growing trend of stock price on the right of Fig1. Is attributed to the relatively large dividend pay; the closer toward the ex-dividend date, higher the present value is for all future dividends.

3. Findings

3.1 Numerical Results

Using the gradient descent algorithm defined in section 2.3 and setting number of steps $n = 60000, \varepsilon = 0.00001, \sigma_0 = 0.15$, the last 10000 steps are sampled for estimating the payoff and the optimal volatility σ^*.

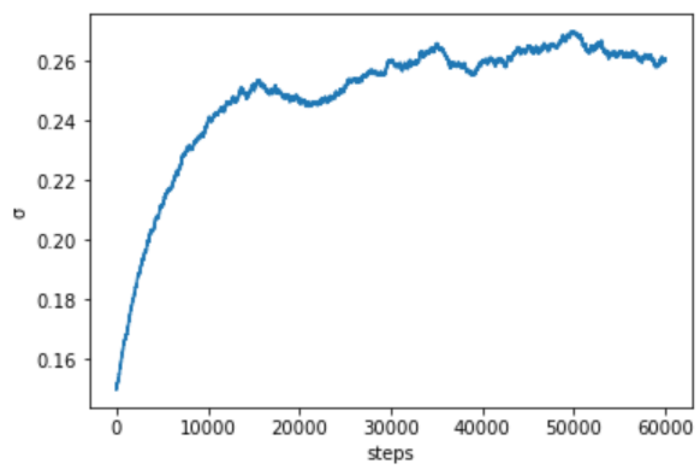

Fig. The optimal volatility search curve for 60000 steps.

By taking the sample mean and standard deviation of the last 10000 steps, an estimate for the payoff – ***Mean: 4.94*** and ***Standard Deviation: 6.64*** and volatility – ***Mean: 0.263*** and ***Standard Deviation: 0.0025*** are computed.

To verify the validity of the optimal volatility σ^* estimate, $\widehat{\sigma^*} = 0.263$ is plugged in to simulate the expected payoff of the American call option.

With 10000 repetitions of simulations of the payoff with $\widehat{\sigma^*} = 0.263$, the estimate of payoff is – ***Mean: 4.82*** and ***Standard Deviation: 6.60***, which is close to the desired payoff ***4.85***. It is reasonable to say that the optimization algorithm can recover the optimal volatility.

3.2 Discussion

In section 2.3, the rationale of why the gradient is always positive in a restricted range of volatility is justified by inspecting the payoff simulation results. To provide a more generalized approach for estimating the sign of gradient, finite difference (FD) method is considered and tested.

$$sign\big(\nabla J_T(\sigma)\big) \approx \begin{cases} +1 & if\ J_T(\sigma + \Delta) - J_T(\sigma - \Delta) > 0 \\ -1 & if\ J_T(\sigma + \Delta) - J_T(\sigma - \Delta) < 0 \end{cases}$$

An interesting phenomenon is observed – the optimal volatility converges to a different value.

By using the same setting as section 3.1 except for the update direction is determined by FD, a different optimal volatility – ***Mean: 0.0998*** and ***Standard Deviation: 0.000598*** – is found.

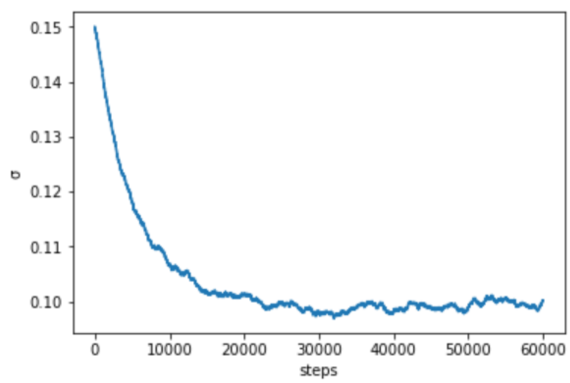

Fig. The optimal volatility search curve for 60000 steps using finite difference (FD) for gradient sign estimation.

The estimate for the payoff is biased, which is equals to ***Mean:1.99*** and ***Standard Deviation: 2.84***. It is realized that the mean of error $\overline{J_T(\sigma_n) - A}$ is close to $A - \widehat{J_T}$. This suggests that although the algorithm acknowledge the error for updating σ, it is probable that the sign of gradient estimates is not robust under stochasticity and exhibited limiting behavior.

Alternate schemes for gradient sign estimation, like sampling multiple more points with $\pm n\Delta$ to defend against stochasticity, could be tested in the future.

4. Conclusion

This report has covered the background for conducting Monte Carlo simulation to price American option. Simulation and optimization policies are devised and used according to find the implied volatility. Further research can be conducted to analyse the robustness and efficiency of gradient sign estimation.

Appendix