Let there be continuous trait axis with niches at 0 and 1. Assume that trait-niche affinity is a Gaussian function of their difference. Species 1 and 2 exist with trait values 0 and 1 respectively. Species 3 comes in with a trait value of .5. How big does the niche breadth have to be in order for species 3 to be able to invade?

Let the system be as follows:

$$\frac{dN_1}{dt} = c_{11}R_1N_1 + c_{12}R_2N_1 - d_1N_1$$

$$\frac{dN_2}{dt} = c_{21}R_1N_2 + c_{22}R_2N_2 - d_2N_2$$

$$\frac{dN_3}{dt} = c_{31}R_1N_3 + c_{32}R_2N_3 - d_2N_3$$

$$\frac{dR_1}{dt} = I_1 - E_1R_1 - c_{11}R_1N_1 - c_{21}R_1N_2 - c_{31}R_1N_3$$

$$\frac{dR_2}{dt} = I_2 - E_2R_2 - c_{12}R_2N_1 - c_{22}R_2N_2 - c_{32}R_2N_3$$

Where $c_{ij} = e^{\frac{(x_i - x_j)^2}{\sigma_j}}$

Assume the resource are identical, i.e. $I_1 = I_2$, $E_1 = E_2$, $\sigma_1 = \sigma_2$. Also assume $d_1 = d_2 = d_3$.

Let f_i the fitness of species i be the sum of all trait affinities:

$$f_i = \sum_j c_{ij} = \sum_j e^{\frac{(x_i - x_j)^2}{\sigma}}$$

The fitness of species 1 is as follows:

$$f_{1,2} = e^{-\frac{(0-0)^2}{\sigma}} + e^{-\frac{(0-1)^2}{\sigma}} = 1 + e^{-\frac{1}{\sigma}}$$

The fitness of species 3 is:

$$f_3 = e^{-\frac{(.5-0)^2}{\sigma}} + e^{-\frac{(.5-1)^2}{\sigma}} = 2e^{-\frac{.25}{\sigma}}$$

In order for $f_3 > f_{1,2}$:

$$2e^{-\frac{.25}{\sigma}} = 1 + e^{-\frac{1}{\sigma}}$$

The numerical solution for this is $\sigma \approx .41$. So we have:

- If $\sigma > .41$, then $f_3 > f_{1,2}$
- If $\sigma < .41$, then $f_3 < f_{1,2}$

From simulation results, it happens that:

- When $\sigma > .41$, species 3 exists and species 1 and 2 go extinct
- When $\sigma < .41$, species 1 and 2 coexist and species 3 goes extinct

This brings up a more interesting question. For what σ does the fitness function peak at 0 and 1? Well as it turns out, it never does. The fitness function is as follows:

$$f(x) = e^{-\frac{x^2}{\sigma}} + e^{-\frac{(x-1)^2}{\sigma}}$$

And we have:

$$f'(x) = -\frac{-2x}{\sigma}e^{-\frac{x^2}{\sigma}} - \frac{2(x-1)}{\sigma}e^{-\frac{(x-1)^2}{\sigma}}$$

$$f'(0) = \frac{2}{\sigma}e^{-\frac{1}{\sigma}}$$

Since $f'(0)$ is positive, there must be some value ϵ such that $f(\epsilon) > f(0)$. A symmetrical argument can be made for $f(1)$ ($f'(1)$ is negative, which implies there exists some $f(1-\epsilon) > f(1)$). Also note that the same argument holds for an exponent of 4.