Let there be continuous trait axis with niches at 0 and 1. Assume that trait-niche affinity is a Gaussian function of their difference. Species 1 and 2 exist with trait values 0 and 1 respectively. Species 3 comes in with a trait value of .5. How big does the niche breadth have to be in order for species 3 to be able to invade?

Let the system be as follows:

$$\frac{dN_1}{dt} = c_{11}R_1N_1 + c_{12}R_2N_1 - d_1N_1$$

$$\frac{dN_2}{dt} = c_{21}R_1N_2 + c_{22}R_2N_2 - d_2N_2$$

$$\frac{dN_3}{dt} = c_{31}R_1N_3 + c_{32}R_2N_3 - d_2N_3$$

$$\frac{dR_1}{dt} = I_1 - E_1R_1 - c_{11}R_1N_1 - c_{21}R_1N_2 - c_{31}R_1N_3$$

$$\frac{dR_2}{dt} = I_2 - E_2R_2 - c_{12}R_2N_1 - c_{22}R_2N_2 - c_{32}R_2N_3$$

Where $c_{ij} = e^{\frac{(x_i - x_j)^2}{\sigma_j}}$

Assume the resource are identical, i.e. $I_1 = I_2$, $E_1 = E_2$, $\sigma_1 = \sigma_2$. Also assume $d_1 = d_2 = d_3$.

Let f_i the fitness of species i be the sum of all trait affinities:

$$f_i = \sum_j c_{ij} = \sum_j e^{\frac{(x_i - x_j)^2}{\sigma}}$$

The fitness of species 1 is as follows:

$$f_{1,2} = e^{-\frac{(0-0)^2}{\sigma}} + e^{-\frac{(0-1)^2}{\sigma}} = 1 + e^{-\frac{1}{\sigma}}$$

The fitness of species 3 is:

$$f_3 = e^{-\frac{(.5-0)^2}{\sigma}} + e^{-\frac{(.5-1)^2}{\sigma}} = 2e^{-\frac{.25}{\sigma}}$$

In order for $f_3 > f_{1,2}$:

$$2e^{-\frac{.25}{\sigma}} = 1 + e^{-\frac{1}{\sigma}}$$

The numerical solution for this is $\sigma \approx .41$. So we have:

- If $\sigma > .41$, then $f_3 > f_{1,2}$
- If $\sigma < .41$, then $f_3 < f_{1,2}$

From simulation results, it happens that:

- When $\sigma > .41$, species 3 exists and species 1 and 2 go extinct
- When $\sigma < .41$, species 1 and 2 coexist and species 3 goes extinct

In fact, we can be even more general than this. We want to find niche breadth σ, such that the fitness function peaks at 0 and 1. We have:

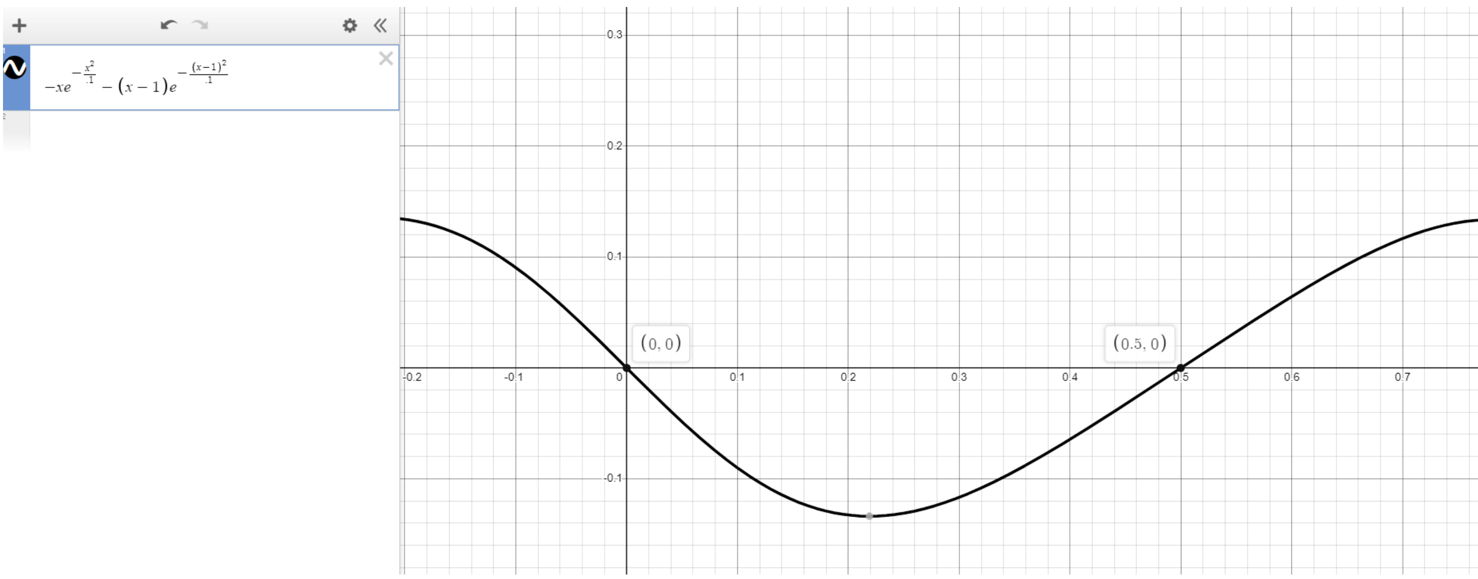
$$f(x) = e^{-\frac{x^2}{\sigma}} + e^{-\frac{(x-1)^2}{\sigma}}$$

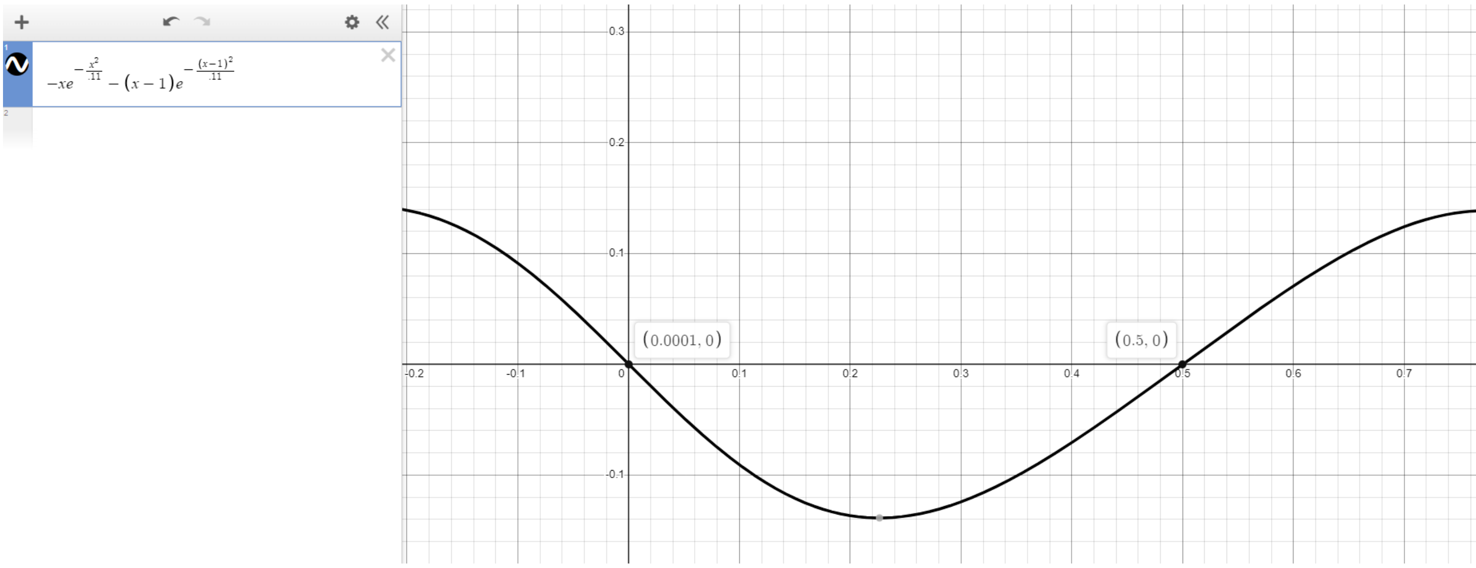
$$\frac{df}{dx} = -\frac{-2x}{\sigma}e^{-\frac{x^2}{\sigma}} - \frac{2(x-1)}{\sigma}e^{-\frac{(x-1)^2}{\sigma}} = 0$$

$$\rightarrow xe^{-\frac{x^2}{\sigma}} + (x-1)e^{-\frac{(x-1)^2}{\sigma}} = 0$$

This can't be analytically solved for x in terms of σ, but from graphing, I find that $\sigma \approx .1$ is where behavior changes. That is to say:

- $\sigma < .1$ means 0 and 1 the traits with maximal fitness
- $\sigma > .1$ means there is some intermediate trait value with higher fitness than 0 and 1





It also happens that when I place a species with trait value .0001 in when niche breadth is .11, it will out compete the species with trait value 0. So in general, having the highest total fitness seems to have some predictive power with respect to the outcome of competition. Is this something that can be/has been proven mathematically?

I originally did this because in order to do any work about persistence times, I need to condition the niche breadth such that no intermediate trait value species will persist indefinitely (coexist). Obviously, all results above are corollaries of the utilization function being the Gaussian, but it is still a little interesting to me that, for even a relatively small niche breadth, intermediate traits have maximal fitness in this system.

Now let's say that we restrict $\sigma < .1$ such that any species with intermediate trait value will go extinct. Is there any way we can analytically show that extreme trait values will have longer persistence times?