Stochastikvertiefung: Finanzmathematik WS 19/20

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Vorwort

Kapitel I

Introduction

1. The central issue of financial mathematics

Valuation:

Valuation of derivatives and hedge against the risks which emerges from the purchase / sale,

> **Definition (Derivative)**
> Financial product whose payouts are derived from price of one or more basic goods derived (underlying) derivative.

■ **Beispiel**

- Right to get 100,000 GBP in 3 months against 125,000 EUR (Call-Option, Underlying: Exchange rate GBP/EUR)
- Right within the next year to consume 100,000 Mw / h of electric energy at the price of 30EUR/Mwh with minimum order quantity of 50,000 Mwh (Swing-Option, Underlying: electricity price)
- buying and selling options on stock (Underlying: equity price)

Issue: What is the "fair" price for such a derivative? ("Pricing"). How can the sellers protect themselves against the ... risks? ("Hedging")

Optimal investment

Gathering Porftolios that are optimal for risk-return approach.

- How do I weigh risk against profit?
- What exactly is "optimal"?
- Solution of the resulting optimization problems

Risk management + Risk measurement

- Legal rules (Basel + Solvency) should ensure stability of the banking system/insurance system even in the face of various risks

  $\Longrightarrow$ Mathematical theory of convex + coherent risk measures

Mathematical tools: Probability Theory + stochastic processes (dynamics in time, some linear algebra, optimization, measure theory).

2. Mathematical Financial Market Model

We consider:

1. Probability space $(\Omega, \mathscr{F}, \mathbb{P})$, later more probability measures $Q, \ldots$ on the same measure space $(\Omega, \mathscr{F}), \omega \in \Omega$ basic events or "scenarios".

2. Timeline I is either $I = \{t_0, t_1, \ldots, t_N = T\}$ N-period model (discrete model) or $I = [0, T]$ (continious-time model), where $T =$ time-horizon
   A stochastic process S is a measurable mapping $S : (\Omega, \mathscr{F}) \to \mathbb{R}^d$ mit $(\omega, t) \mapsto S_t(\omega)$
   Especially:
   - $t \mapsto S_t(\omega)$ function $I \to \mathbb{R}^d$ for every $\omega \in \Omega$ ("path")
   - $\omega \mapsto S_t(\omega)$ random variable $\Omega \to \mathbb{R}^d$ for every $t \in I$

3. Percolation a sequence of ω-algebras $(F_t)_{t \in I}$ with the property $\mathscr{F}_S \subseteq F_t \quad \forall s, t \in I, x \leq t$ and $F_t \subseteq F \quad \forall t \in I$
   Interpretation: $F_t =$ market participant at time t known/available information
   Events $A \in F_t$ are considered known 'at time t'
   A R^d-valued RV X is called F_t-measurable, if $E = X^{-1}(B) \in F_t \quad \forall$ Borel sets $B \subseteq R^d$ (E is actually the preimage of B).

   ■ **Beispiel**
   A stochastic process $(S_t)_{t \in I}$ on $(\Omega, \mathscr{F})$ is called adaptiert regarding a percolation $(\mathscr{F}_t)_{t \in I}$, wenn gilt:
   $$S_t \text{ is } F_t - \text{measurable} \quad \forall t \in I$$
   Interpretation: "the value S_t is known at time t"
   Why percolation in the financial mathematics (FiMa)?
   - Differentiation between future/past
   - Different information (Insider/Outsider) corresponds to different percolation $(F_t)_{t \in I}$ or $(G_t)_{t \in I}$

   S^i= price of the i-th asset at the time t

4. Assets R^{d+1}-valued stochastic process with components
   $$S^i : (\Omega \times I) \to \mathbb{R} \quad (\omega, t) \mapsto S^i_t(\omega) \text{ mit } i \in \{0, 1, \ldots, d\}$$
   where $S^i_t =$ price of the i-th asset at time t
   $S^i, i \in \{1, \ldots, d\}$ is typically
   - Stock, company share
   - Currency or exchange rate
   - Commodity such as oil, noble metal, electricity,..
   - Bond ...

   Principal assumption: S^i is liquid traded (eg on exchange), ie purchase/sale for the price S^i_t possible at any time.

$S^0 \ldots$ "numeraire" has a special role: describes interest rate of not in $(S^1, .., S^d)$ invested capital; is mostly considered to be risk-free.

Definition I.1 (Finance market model)
A finance market model (FMM) with a time axis I is given by

1. a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with percolation $(F_t)_{t\in I}$
2. an adapted to $(F_t)_{t\in I}$, R^{d+1}-valued stochastic process $S_t = (S_t^0, S_t^1, \ldots, S_t^d), t \in I$

■ **Beispiel (Cox-Rubinstein (CRR)-model (discrete-time))**

- $S_n^0 = (1+r)^n$, meaning interest at a constant rate r
- $S_n^1 = S_0^1 \prod_{k=1}^n (1 + Ru)$, wobei $(R_1, R_2, \ldots)$ independent random variables with two possible values $a < b$
  Image: " "recombined tree " with events ω corresponding paths in the tree

■ **Beispiel (Block-Scholes-modell (continious-time))**

- $S_t^0 = e^{rt}$, meaning interest at a constant rate r
- $S_t^1 = S_0^1 \cdot \exp((\mu - \frac{\sigma^2}{2}t + \sigma\beta_t)$ mit $\mu \in \mathbb{R}, \sigma > 0, S_0^1 > 0$ und β_t corresponds to Brownian motion(stochastic process in continuous time) and $\mu - \frac{\omega^2}{2}$ corresponds to trend component

Image: Exchange curve $= S_t(\omega)$, wherein time-continuous model for infinite probability space

3. Bonds and basic examples of derivatives

We consider here only one basic good $S_t = S_t^1$

1. Bond: (more precisely: zero-coupon bond) Der issuer of a bond with maturity T guaranties to pay the buyer the amount N (EUR/USD/...) at time T.
   Typical issuers:
   - States [government bond]
   - Companies (as an alternative to taking a credit)

   After emission bonds are resold on the secondary market, ie liquid traded securities/stocks
   Price of the emission: $B(0, T)$
   Price of the resale at time $t \leq T$: $B(t, T)$
   We always standardise $N = 1 \implies B(T, T) = 1$
   Bonds of West / North / Central European countries + USA / Cananda are regarded as risk-free (secure payment).
   Otherwise: credit risk
   Risk free bonds can be used as numeral $S_t^0 = B(t, T)$
   Image: can not describe it now:/

2. Forward contract
   From buyers view: agreement to buy a unit of basic good S for price at a certain future date T (purchase required)
   Popular with raw materials + electricity

Payout profile: $F_T = S_T - K$
Image: " A straight line with the intersection of the x axis at K and intersection of the y axis at S_T $ge0$, is only a 1st order polynomial"
Price at time t: F_t

3. European Put-/Call-Option: Right at a future time T to sell (put) or to buy (call) a unit of basic good S at the price K **no supply/purchase obligation**

   - Call:

$$C_T := \begin{cases} S_T - K & S_T \geq K \\ 0 & S_T < K \end{cases} = (S_T - K)_+$$

     ▶ **Bemerkung**

$$X_+ = \max(X, 0) \quad X_+ - X_- = X$$
$$X_- = \min(X, 0) \quad X_+ + X_- = |X|$$

     Image: (hockey stick function)

   - Put:

$$P_t = \begin{cases} 0 & S_T \geq K \\ K - S_T & S_t < K \end{cases} = (K - S_T)_+$$

     Bild: "inversed" hockey stick function

4. American Put/Call-Option: As put/call but with exercising at any time $t \in [0, T]$
   Price at time t: P_t^{AM}, C_t^{AM}
   Payout profile at time τ: $(S_\tau - K), (K - S_\tau)_+$
   Time τ must generally be determined as a solution of a stochastic optimization problem ("Optimal stop problem")

4. Elementary replications and arbitrage-arguments

What can we state (with elementary means) on the "fair" prices $B(t, T), F_T, C_T, P_t$
? We use:

- Replication principle: Two identical future cash flows have the same value even today. (A cash flow "replicates" the other)
- No-arbitage-principle: "Without capital investment, secure profit can be achieved without risk of loss"

- Arbitrage: risk-free profit

- Weaker form of replication principle:
  Super position principle: is a cash flow in any case greater than another, therefore also today it has the greater value

stark	Rep. principle	limited applicability
$\downarrow$	Super rep. principle	$\uparrow$
light	No-arbitrage-principle	always applicable

Lemma I.2
Für den preis C_t des europäischen Calls gilt:

$$(S_t - K \cdot B(t,T))_+ \leq C_t \leq S_t$$

Beweis.
- lower boundary: For contradiction $S_t - K \cdot (B(t,T)) - C_t = \varepsilon > 0$

Portofolio	value in t	value in T, $S_t \leq K$	value in T, $S_t > K$
Buy Call	C_t	0	$S_T - K$
Sell basic good	$-S_t$	$-S_T$	$-S_T$
Buy a bond	$\varepsilon + K \cdot B(t,T)$	$\frac{\varepsilon}{B(t,T)} + K$	$\frac{\varepsilon}{B(t,T)} + K$
Σ	0	$K - S_T + \frac{\varepsilon}{B(t,T)} > 0$	$\frac{\varepsilon}{B(t,T)} > 0$
	no initial capital	sure profit	sure profit

$\implies$ In contradiction to no-arbitrage
$\implies$ $S_t - K \cdot B(t,T) \leq C_t$ and moreover $C_t \geq 0$ $\implies$ $C_t \geq (S_t - K \cdot B(t,T))_+$

- upper boundary: UE □

Lemma I.3 (Put-Call-parity)
For put P_t, Call C_t with the same exercise price K and basic good S_t it holds

$$C_t - P_t = S_t - B(t,T)K$$

Image: need to add ..., but should be fast to do ...

Beweis. with replication:

Portofolio 1	value in t	value in T, $S_t \leq K$	valuet in T, $S_t > K$
Buy Call	C_t	0	$S_T - K$
Buy bond	$K \cdot B(t,T)$	K	K
Value portofolio 1	$C_t + K \cdot B(t,T)$	K	S_T

Portofolio 2	value in t	value in T, $S_t \leq K$	value in T, $S_t > K$
Buy put	P_t	$K - S_T$	0
Buy basic good	S_t	S_T	S_T
Value portofolio 2	$P_t + S_t$	K	S_T

Replication principle: $C_t + K \cdot B(t,T) = P_t + S_t$

$\implies C_t - P_t = S_t - K \cdot B(t,T)$ □

5. Conditional expectation values and Martingale

5.1. Conditional density and conditional expected value

Motivation: Given: Two random variables (X, Y) with values in $\mathbb{R}^m \times \mathbb{R}^n$ and joint density $f_{XY}(x,y)$. From f_{XY} we can derive:

- $f_Y(y) := \int_{\mathbb{R}^m} f_{XY}(x,y)\,\mathrm{d}x$ with marginal distribution of Y
- $S_Y := \{y \in \mathbb{R}^n : f_Y(y) > 0\}$ carrier of Y - Image?

Definition (Conditional density of X with respect to Y)
Conditional density from X with respect to Y is defined as

$$f_{X|Y}(x,y) = \begin{cases} \frac{f_{XY}(x,y)}{f_Y(y)} & y \in S_Y \\ 0 & y \notin S_Y \end{cases}$$

Consider the following problem:
What is the best forecast from X if an observation $Y = y$ is given?
Criteria:
Minimiye the quadratic distance/second moment/ L_2-norm.
Vorhersage:
Measurable function $g : \mathbb{R}^n \to \mathbb{R}^m$ mit $y \mapsto g(y)$, meaning,.

$$\min\left\{\mathbb{E}[(X - g(Y))]^2 : g \text{ messbar } \mathbb{R}^n \to \mathbb{R}^m\right\} \tag{min-1}$$

Satz I.4
When (X, Y) have a joint density with $\mathbb{E}[|X|^2] < \infty$, then (min-1) is going to be minimized through the conditional expected value

$$g(y) = \mathbb{E}[X \mid Y = y] := \int_{\mathbb{R}^m} x f_{X|Y}(x,y)\,\mathrm{d}x$$

(where $\mathbb{E}[X \mid Y = y]$ “expected value of X conditioned on $Y = y$”)

In general, it holds:

Theorem I.5
Let (X,Y) be random variables with joint density on$\mathbb{R}^m \times \mathbb{R}^n$, $h : \mathbb{R}^m \to \mathbb{R}^n$ measurable with $\mathbb{E}[h(X,y)^2]$. Then that is going to be minimisation problem

$$\min\left\{\mathbb{E}[(h(X,Y) - g(y))^2]\right\} \quad g\text{measurable from } \mathbb{R}^n \text{ towards } \mathbb{R}$$

solved through

$$g(y) = \mathbb{E}[h(X,Y) \mid Y = y] = \int_{\mathbb{R}^m} h(X,Y) f_{X|Y}(x,y)\,\mathrm{d}x$$

Beweis (only proposition, the Theorem is analogous, for $n = 1$). Set $g(y) = \int_{\mathbb{R}} f_{X|Y}(x,y)\,\mathrm{d}x$. Sei $p : \mathbb{R} \to \mathbb{R}$ arbitrary measurable function with $\mathbb{E}[p(y)^2] < \infty$. Set $g_\varepsilon(y) = g(y) + \varepsilon p(y)$. Minimize

$$\begin{aligned}
F(\varepsilon) &:= \mathbb{E}[(X - g_\varepsilon(y))^2] = \mathbb{E}[(X - g(y) - \varepsilon p(y))^2] \\
&= \mathbb{E}[(X - g(y))^2] - 2\varepsilon\mathbb{E}[(X - g(y))p(y)] + \varepsilon^2\mathbb{E}[p(y)^2] \\
\frac{\partial F}{\partial \varepsilon}(\varepsilon) &= 2\varepsilon\mathbb{E}[p(y)^2] - 2\mathbb{E}[(X - g(y))p(y)] \\
&\implies \varepsilon_* := \frac{\mathbb{E}[(X - g(y))p(y)]}{\mathbb{E}[p(y)^2]} = \frac{A}{B}
\end{aligned}$$

wobei

$$\begin{aligned}
A &= \mathbb{E}[Xp(y)] - \mathbb{E}[g(y)p(y)] \\
&= \int_{\mathbb{R}\times\mathbb{R}^n} xp(y) f_{XY}(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{S_y} g(y)p(y)f_Y(y) = [\text{Einsetzen von } g + \text{Fubini}] \\
&= \int_{\mathbb{R}\times\mathbb{R}^n} xp(y) f_{XY}(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}\times S_y} xp(y)\underbrace{f_{X|Y}(x,y)f_Y(y\,\mathrm{d}y)}_{=f_{XY}(x,y)} = 0
\end{aligned}$$

so $\varepsilon^* = 0$ independent from $p \implies g(y)$ minimizes (min-1). □

- **Beispiel**
  Let (X,Y) normaly distributed on $\mathbb{R} \times \mathbb{R}$ with

$$\mu = (\mu_x, \mu_y)^T \quad \Sigma = \begin{pmatrix} \sigma x^2 \rho\sigma_x\sigma_y & \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} \mathbb{V}\mathrm{ar}(X) & \mathbb{C}\mathrm{ov}(X,Y) \\ \mathbb{C}\mathrm{ov}(X,Y) & \mathbb{V}\mathrm{ar}(Y) \end{pmatrix} \text{ mit } \rho \in [-1,1]$$

  Then the arbitraty density is $f_{X|Y}(x,y)$. (Σ covariance matrix). Once more the density of a normaly distributed random variable with

$$\begin{aligned}
\mathbb{E}[X \mid Y = y] &= \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y) \\
\mathbb{V}\mathrm{ar}(X \mid Y = y) &= \sigma_x^2(1 - \rho^2)
\end{aligned}$$

  (is ÜA!). The mapping $y \mapsto \mu_x + g(y)\frac{\sigma_x}{\sigma_y}(y - \mu_y)$ is called regression line for X given $Y = y$.
  Image: μ_x, μ_y are values on x,y-axis and the σ's build the Triangle slope (slope is known substan-

tially by ρ)
For disrete random variables, i.e. when X, Y accept only finitely many $\{x_1, \dots, x_m\}$ or $\{y_1, \dots, y_m\}$ annehmen then with similar considerations we obtain as a solution of (min-1)

$$\mathbb{E}[X \mid Y = y_j] = \sum_{i=1}^{m} X_i \mathbb{P}(X = x_i \mid Y = y_j)$$

wherein directly the conditional probabilities

$$\mathbb{P}(X = x_i \mid Y = y_j) = \begin{cases} \frac{\mathbb{P}(X = x_i \wedge Y = y_j)}{\mathbb{P}(Y = y_j)} & \text{wenn } \mathbb{P}(Y = y_j) > 0 \\ 0 & \text{wenn } \mathbb{P}(Y = y_j) = 0 \end{cases}$$

5.2. Conditional expectation - measure theoretical access

We consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For random variables $X : \Omega \to \mathbb{R}$ und $p \in [1, \infty)$ we define the L_p-norm

$$\|X\|_p = \mathbb{E}[|X|^p]^{1/p} = \left(\int_{\Omega} |X(\omega)|^p \, \mathrm{d}\mathbb{P}(\omega) \right)^{1/p}$$

and L_p-space $L_p(\Omega, \mathscr{F}, \mathbb{P}) := \Big\{ X : \Omega \to \mathbb{R} \colon \mathscr{F} - \text{measurable}, \|X\|_p < \infty \Big\}$. We identificate random variables which differ only at zero amounts, ie $\mathbb{P}(X \neq X') = 0 \implies X = X'$ (in L_p).
From measure theory it is known: (?)
The spaces $L_p(\Omega, \mathscr{F}, \mathbb{P})$ with norm $\|\cdot\|_p, p \in [1, \infty)$ are always BANACH-spaces (linear, complete, normed vector spaces). For $p = 2$ also hilbert spaces with inner product

$$\langle X, Y \rangle = \mathbb{E}[XY] = \int_{\Omega} X(\omega) Y(\omega) \, \mathrm{d}\mathbb{P}(\omega)$$

Für $\mathscr{G} \subseteq \mathscr{F}$ Unter-σ-slgebra is $L_p(\Omega, \mathscr{F}, \mathbb{P}) \subseteq L_p(\Omega, \mathscr{F}, \mathbb{P})$ closed subspace.
We generalize "prediction problem" from the last section (1.3?)
Given are random variables X from $L_2(\Omega, \mathscr{F}, \mathbb{P})$ is $\mathscr{G} \subseteq \mathscr{F}$ Sub-σ-algebra.
What is the best $\mathscr{G}$-measurable forecast for Y?

$$\min\Big\{ \mathbb{E}[(X - G)^2] \colon G \in L_2(\Omega, \mathscr{F}, \mathbb{P}) \Big\} \tag{min-2}$$

wobei $\mathbb{E}[(X - G)^2] = \|X - G\|_2^2$.
From hilbert-space theory:
(min-2) possesses a unique solution $G_* \in L_2(\mathscr{F}, \mathscr{G}, \mathbb{P})$. G_* is optimization (with respect to $\langle \cdot, \cdot \rangle$) from $X \in L_2(\Omega, F, P)$ on closed subspace $L_2(\Omega, G, P)$
Image: maybe from Eric (Orthogonal projection on the subspace)
We denote the conditioned expected value $\mathbb{E}[X \mid \mathscr{G}]$ of X with recpect to $\mathscr{G}$ with G_*.

Theorem I.6
Let $X, Y \in L_2(\Omega, \mathscr{F}, \mathbb{P})$ and $\mathscr{G} \subseteq F$ sub-σ-algebra. Then it holds

1. (Linearity) $\mathbb{E}[aX + bY] = a\mathbb{E}[X \mid \mathscr{G}] + b\mathbb{E}[Y \mid \mathscr{G}]$
2. (Tower rule) For every further σ-algebra $\mathscr{H} \subseteq \mathscr{G}$ it holds
$$\mathbb{E}[E[X \mid \mathscr{G} \mid \mathscr{H}]] = \mathbb{E}[X \mid \mathscr{H}]$$
3. (Pullout-Property) $\mathbb{E}[XZ \mid \mathscr{G}] = Z\mathbb{E}[X \mid \mathscr{G}]$, if Z is bounded and $\mathscr{G}$-measurable.
4. (Monotonicity) $X \leq Y \implies \mathbb{E}[X \mid \mathscr{G}] \leq \mathbb{E}[Y \mid \mathscr{G}]$
5. (Δ-Inequality) $|\mathbb{E}[X \mid \mathscr{G}]| \leq \mathbb{E}[|X| \mid \mathscr{G}]$
6. (Independence) X independent from $G \implies \mathbb{E}[X \mid \mathscr{G}] = \mathbb{E}[X]$
7. (trivial σ-algebra) $\mathscr{G} = \{\varnothing, \Omega\} \implies \mathbb{E}[X \mid \mathscr{G}] = \mathbb{E}[X]$

Beweis. (without proof, see lecture probability theory with martingales or stochastics script SS19.) □

▶ **Bemerkung**

- The conditioned expectation value $\mathbb{E}[X \mid \mathscr{G}]$, which is defined for $X \in L_2(\Omega, \mathscr{F}, \mathbb{P})$, can be extended by approximation on all $X \in L_1(\Omega, \mathscr{F}, \mathbb{P})$. All properties from Theorem **???** remain the same!
- Let Y be a random variable and $\mathscr{G} = \sigma(Y)$ the σ-algebra which is generated by Y. We write:
$$\mathbb{E}[X \mid Y] = \mathbb{E}[X \mid \sigma(Y)] \quad \sigma\text{-measurable random variables}$$
- Measure theory: Doob-Dynkin-Lemma $\implies \exists$ measurable function $g : \mathbb{R}^n \to \mathbb{R}$ such that
$$\mathbb{E}[X \mid Y] = g(Y)$$
Where g is exactly the function from (min-1).

Summary:
Let X, Y from $L_1(\Omega, \mathscr{F}, \mathbb{R})$, $\mathscr{G} \subseteq \mathscr{F}$ sub-σ-algebra

1. $\mathbb{E}[X \mid Y = y]$ is a measurable function $g : \mathbb{R}^n \to \mathbb{R}^n$. If the conditioned density exists, then it holds:
$$\mathbb{E}[X \mid Y = y] = \int_{\mathbb{R}^m} f_{X|Y}(x, y) \,\mathrm{d}x$$
2. $\mathbb{E}[X \mid Y]$ is a $\sigma(y)$-measurable random variable, this can be represented as $g(Y)$. If the conditioned

density exists, then it holds

$$\mathbb{E}[X \mid Y](\omega) = \int_{\mathbb{R}^n} x f_{X|Y}(x, Y(\omega))\,\mathrm{d}x$$

3. $\mathbb{E}[X \mid \mathscr{G}]$ is a $\mathscr{G}$-measurable random variable. If $\mathscr{G} = \sigma(y)$ then 2) occurs.

In the general case $\mathbb{E}[\bar{X} \mid \cdot]$ can be interpreted as best forecast for X, given

1. Pointwise observation $Y = y$
2. Observation Y
3. Information $\mathscr{G}$

5.3. Martingale

Prototype of a "neutral" stochastic process, which has neither upward nor downward trend. Here only in discrete time $Z = \mathbb{N}_0$.

Definition (Martingale without a percolation)
Let $(X_n)_{n\in\mathbb{N}_0}$ be a stochastic process. If it holds

1. $\mathbb{E}[|X_n|] < \infty\ \forall n \in \mathbb{N}$
2. $\mathbb{E}[X_{n+1}, \ldots, X_n] = X_n\ \forall n \in \mathbb{N}$

then (X_n) is called a martingale. If we define $\mathscr{F}_n^* = \sigma(X_1, \ldots, X_n)$, then we can write 2) as

$$\mathbb{E}[X_{n+1} \mid \mathscr{F}_n^*] = X_n \quad \forall n \in \mathbb{N}$$

Interpretation:

- The best forecast for a future value X_{n+1}, based on the past $\sigma(X_1, \ldots, X_n)$ is the current value X_n.
- From the tower tule it follows

  $$\mathbb{E}[X_{n+k} \mid \mathscr{F}_n^*] = X_n \quad n, k \in \mathbb{N}_0$$

  since

  $$\mathbb{E}[X_{n+k} \mid \mathscr{F}_n^*] = \mathbb{E}[\mathbb{E}[X_{n+k} \mid \mathscr{F}_{n+k-1} \mid \mathscr{F}_n^*]] = \mathbb{E}[X_{n+k-1} \mid \mathscr{F}_n^*] = (k\text{-mal}) = X_n$$

It can be extended from $(\mathscr{F}_n)_{n\in\mathbb{N}}$ to arbitrary percolations $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$.

Definition (Martingale with percolation)
Let $(X_n)_{n\in\mathbb{N}_0}$ be a stochastic process, which is adapted to a percolation $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$. If it holds

1. $\mathbb{E}[|X_n|] < \infty\ \forall n \in \mathbb{N}_0$
2. $\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = X_n\ \forall n \in \mathbb{N}_0$

then $(X_n)_{n\in\mathbb{N}_0}$ is called a martingal with respect to percolation $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$

Interpretation:
The best forecast for future values X_{n+1}, based on the available information $\mathscr{F}_n$ is the current value X_n.

Definition (Supermatringale, Submartingale)
In in 2) instead of "=" the inequality $\leq$ oder $\geq$ holds, then $(X_n)_{n\in\mathbb{N}}$ is called a Supermartingale or a Submartingale.

First observation:

- X Martingale $\implies \mathbb{E}[X_n] = X_0$, i.e. $n \mapsto \mathbb{E}[X_n]$ is constant.
  Begründung:

  $$\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = X_n \implies \mathbb{E}[\mathbb{E}[X_{n+1} \mid \mathscr{F}_n]] = \mathbb{E}[X_n] = \mathbb{E}[X_{n+1}] \implies (n\text{-times aplied}\mathbb{E}[X_n] = X_0)$$

  Image: expected value is constant, but not a martingale.
- X Submartingale $\implies n \mapsto \mathbb{E}[X_n]$ is monotone increasing
- X Supermartingale $\implies n \mapsto \mathbb{E}[X_n]$ is monotone decreasing

In order to remember the difference between super and submartingale, here's a little help:
"Life is a supermartingale, expectations fall with time."

■ **Beispiel**

- Let $(Y_n)_{n\in\mathbb{N}}$ be independent random vriables in $L_1(\Omega, \mathscr{F}, \mathbb{P})$ mit $\mathbb{E}[Y_n] = 0$. Define $X_n := \sum_{k=1}^n Y_k$ mit $X_0 = 0$. Then $(X_n)_{n\in\mathbb{N}_0}$ is a martingale, since
  1. $\mathbb{E}[|X_n|] \leq \sum_{k=1}^n \mathbb{E}[|Y_k|] < \infty \quad \forall n \in \mathbb{N}$ ✓
  2. $$\begin{aligned}\mathbb{E}[X_{n+1} \mid \mathscr{F}_n^*] &= \mathbb{E}[Y_{n+1} + X_n \mid \mathscr{F}_n^*] \\ &= \mathbb{E}[Y_{n+1} \mid \mathscr{F}_n^*] = \mathbb{E}[X_n \mid \mathscr{F}_n^*] \quad (\text{ tower und } \mathscr{F}_n^*\text{-measurable}) \\ &= \underbrace{\mathbb{E}[Y_{n+1}]}_{=0} + X_n = X_n ✓\end{aligned}$$
- Further examples are to be found on the first exercise sheet!

Definition (predictable)
Let $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$ be a percolation. A stochastic process $(X_n)_{n\in\mathbb{N}}$ is called predictable with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$, if it holds:

$$H_n \text{ is } \mathscr{F}_{n-1}\text{-measurable} \quad \forall n \in \mathbb{N}$$

▶ **Bemerkung**
Stronger property than "adapted".

Definition (discrete stochastic integral)
Let X be adapted and H a predictable stochastical process with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$. Then

$$(H \bullet X)_n := \sum_{k=1}^{n} H_k(X_k - X_{k-1}) \qquad (*)$$

is called a discrete stochastic integral of H with respect to X.

▶ **Bemerkung**
Sums $(*)$ are in the analysis called Riemann-Stieltjes-summs. They are used for constructions of the RS-integral $\int h \, \mathrm{d}\rho$.

Definition (locally bounded)
A stochastic process $(H_n)_{n\in\mathbb{N}}$ is called locally bounded, if there exists a (defined) sequence $c_{\in}\mathbb{R}_{\geq 0}$ such that

$$|H_n| \leq c_n \text{ a.s.} \quad \forall n \in \mathbb{N}$$

Theorem I.7
Let X be adapted stochastic process (with respect to percolation $(\mathscr{F}_n)_{n\in\mathbb{N}}$). Then the following statements are equivalent:

1. X is a martingale
2. $(H \bullet X)$ is a martingale for all locally bounded, predictable $(H_n)_{n\in N}$

That means: the stochastic integral obrains the martingale-property.

Beweis. 8.11.2019! □

▶ **Bemerkung**
The random variable H is later going to be the investment strategy.

Beweis.
- $\Rightarrow$:
  - Adaptability: clear
  - Integrability: H is locally bounded, i.e. $|H_k| \leq c_k < \infty$ for all k.

    $$\mathbb{E}[|H_k(X_k - X_{k-1})|] \leq c_k * (\mathbb{E}[|x_k|] + \mathbb{E}[|X_{k+1}|]) < \infty$$

    With the triangle-inequality it follows $\mathbb{E}[|(H \bullet X)_n|] < \infty$.
  - Martignale property:

    $$\begin{aligned}\mathbb{E}[(H \bullet X)_n \mid \mathscr{F}_{n-1}] &= (H \bullet X)_{n-1} + \mathbb{E}[H_n(X_n - Xn - 1) \mid \mathscr{F}_{n-1}] \\ &= (H \bullet X)_{n-1} + H_n * \underbrace{(\mathbb{E}[X_n \mid \mathscr{F}_{n-1}] - Xn - 1)}_{=0} \\ &= (H \bullet X)_{n-1} \quad \forall n \in \mathbb{N}\end{aligned}$$

    Hence, also $(H \bullet X)$ is a martingale.

$(H \bullet X)_n = \sum_{k=1}^{n} H_k(X_k - X_{k-1})$.

$\Leftarrow$: Fix $N \in \mathbb{N}$. Set $H_n := \mathbb{1}_{n=N}$, this is locally bounded and deterministic (also predictable). One notices that $(H \cdot X)_n = 0$ for all $n \leq N-1$. Conversely, for all $n \geq N$ it holds $(H \cdot X)_n = X_N - X_{N-1}$. We check only the martingale property (the integrability follows from the triangle-inequality). We know that $(H \cdot X)$ is a martingale.

$$\begin{aligned}
0 = (H \cdot X)_{N-1} &= \mathbb{E}[(H \cdot X)_N \mid \mathscr{F}_{N-1}] \\
&= \mathbb{E}[x_N - X_{N-1} \mid \mathscr{F}_{N-1}] \\
&= \mathbb{E}[X_N \mid \mathscr{F}_{N-1}] - X_{N-1} \\
&\implies X_{N-1} = \mathbb{E}[X_N \mid \mathscr{F}_{N-1}] \text{ mit } N \in \mathbb{N} \text{ beliebig}
\end{aligned}$$

With that, X is a martingale. □

Folgerung I.8
Let $X = \{X_n\}n = 1, \dots, N$ be an adapted stochastic process with respect to a filtration $\{\mathscr{F}_n\}n = 1, \dots, N$. If $\mathbb{E}[(H \cdot X)_N] = 0$ for all locally bounded predictable processes H, thent X is a martingale with respect to $\{\mathscr{F}_n\}$.

Beweis. Fix $K \in [N] := \{1, 2, \dots, N\}$ and a set $A \in \mathscr{F}_{K-1}$. Define $H_n(\omega) = \mathbb{1}_A(\omega) * \mathbb{1}_{\{n=K\}}$, this is locally bounded and predictable. It holds $(H \cdot X)_n = 0$ for all $n \leq K-1$. For all $n \geq K$ it holds $(H \cdot X)_n = \mathbb{1}_A * (X_K - X_{K-1})$.

$$\begin{aligned}
0 = \mathbb{E}[(H \cdot X)_N] &= \mathbb{E}[\mathbb{1}_A(X_K - X_{K-1})] \\
&\overset{Tower}{=} \mathbb{E}[\mathbb{E}[\mathbb{1}_A(X_K - X_{K-1}) \mid \mathscr{F}_{K-1}]] \\
&= \mathbb{E}[\mathbb{1}_A * \left(\underbrace{\mathbb{E}[X_K \mid \mathscr{F}_{K-1]-X_{K-1}}]}_{Y_{K-1}} \right) \quad \forall A \in \mathscr{F}_{K-1} \\
&\implies \int_A Y_{K-1}(\omega)\, \mathrm{d}\mathbb{P}(\omega) \\
&= \int_A X_{K-1}(\omega)\, \mathrm{d}\mathbb{P}(\omega) \quad \forall A \in \mathscr{F}_{K-1} \\
&\implies Y_{K-1} = X_{K-1} \text{ almost surely} \\
&\implies \mathbb{E}[X_K \mid \mathscr{F}_{K-1]-X_{K-1}}] = X_{K-1}
\end{aligned}$$

for arbitrary K. Hence, X is a martingale. □

▶ **Bemerkung**
We write $[N] := \{1, 2, \dots, N\}$ and $[N]_0 := \{0, 1, 2, \dots, N\}$.

Kapitel II

Cox-Russ-Rubenstein-model

The Cox-Ross-Rubinstein-Model (short: CRR-Model) is also called a binomial model and is developed in 1979 from Cox, Ross and Rubinstein.
It deals with a model for development of the price of a security (paper) plus a offset account with constant interest (numeraire) in discrete time.
Parameter:

r	Rate of interes
b	Rate of return of the security up
a	Rate of return of the security down
$p \in (0,1)$	Probability for up
$S_0 > 0$	Price of the security at time zero
$N \in \mathbb{N}$	Number of time steps

Assumptions: $r > -1$, $b > a > -1$ We model security $\{S_k\}_{k \in N}$ and offset account $\{S_k\}_{k \in \mathbb{N}}$ as stochastic processes on a probability space $(\mathscr{F}, \mathbb{P})$.

- $S_0^0 = 1$ and $S_n^0 = (1+r)^n$
- We define the rate of return $R_n(\omega)$ in the n-th market period with

$$R_n = \begin{cases} b & \text{mit } p \\ a & \text{mit } 1-p \end{cases}$$

The rates of return $(R_1, \dots, R_N)$ are independent.

$$S_n = S_0 * \prod_{k=1}^{n} (1 + R_k)$$

The progress of S can be represented graphically as a binomial tree:

$$\begin{tikzcd}[column sep=small]
 & & S_0(1+b)^2 \\
 & S_0(1+b) \arrow[ur] \arrow[dr] & \\
S_0 \arrow[ur] \arrow[dr] & & S_0(1+a)(1+b) \\
 & S_0(1+a) \arrow[ur] \arrow[dr] & \\
 & & S_0(1+a)^2
\end{tikzcd}$$

One also names this as a 'recombined tree model'. It has the advantage that the number of nodes grows only linearly with n.

- Discountinious price process $\tilde{S}_n := \frac{S_n}{S_n^0} = S_0 * \prod_{k=1}^n \frac{1+R_k}{1+r}$.
- Filtration: natural filtration $\mathscr{F}_n = \sigma(S_1, \dots, S_n)$.

Satz II.1
Im CRR-Modell gilt:

1. The number of upward trends $U_n := \#\{k \in [n] \colon R_k = b\}$ is binomially distributed, i.e. $U_n \sim \mathrm{Bin}(n, p)$.
2. It holds
$$\log\left(\frac{\tilde{S}_n}{S_0}\right) = U_n \log\left(\frac{1+b}{1+a}\right) + n \log\left(\frac{1+a}{1+r}\right)$$
d.h. $\log\left(\frac{\tilde{S}_n}{S_0}\right)$ is per Skalen-Lagen-transformation binomially distributed.
3. The distribution of S_n is
$$\mathbb{P}\big(S_n = S_0(1+b)^k(1+a)^{n-k}\big) = \binom{n}{k} p^k (1-p)^{n-k}$$

Beweis. 1. clear
2. $\frac{\tilde{S}_n}{S_0} = \left(\frac{1+b}{1+a}\right)^{U_n} * \left(\frac{1+a}{1+r}\right)^n \implies \log\left(\frac{\tilde{S}_n}{S_0}\right) = U_n \log\left(\frac{1+b}{1+a}\right) + n \log\left(\frac{1+a}{1+r}\right)$
3. Es ist $S_n = S_0(1+b)^{U_1}(1+a)^{n-U_n}$. Also
$$\mathbb{P}\big(S_n = S_0(1+b)^k(1+a)^{n-k}\big) = \mathbb{P}(U_n = k) \overset{(a)}{=} \binom{n}{k} p^k (1-p)^{n-k}$$ □

▶ **Bemerkung**
Part (b) suggests convergence of $\log\left(\frac{\tilde{S}_n}{S_0}\right)$ towards the normal distribution for $n \to \infty$ (per scaling) $\rightsquigarrow$ Black-Scholes-Model (↗ Chapter 3).

Lemma II.2
A self-financed investment strategy $(\eta_n, \xi_n)_{n \in \mathbb{N}}$ with initial capital $w \in \mathbb{R}$ and value process Π_n are completelly defined through w and $(\xi_n)_{n \in \mathbb{N}}$.

- The discrete value process can be represented as
$$\tilde{\Pi}_n = w + \sum_{k=1}^n \xi_k (\tilde{S}_k - \tilde{S}_{k-1}) = w + (\xi \cdot \tilde{S})_n$$
- The amount η_n is uniquely given by
$$\eta_n = \tilde{\Pi}_n - \xi_n \tilde{S}_n$$

Beweis. klar! □

1. Replication/Hedging of derivated in CRR-Model

Derivative C with payout $h(S_1, S_2, \dots, S_N)$ at time N, i.e. $C = h(S_1, S_2, \dots, S_n)$ mit h measurable. We are looking for a strategy which replicated $(\xi_n)_{n\in[N]}$ and initial capital w, i.e.

- (ξ_n) predictable with discrete value process $\tilde{\Pi}_n = w + (\xi \cdot \tilde{S})_n$
- Replication condition

$$C = h(S_1, \dots, S_N) = \Pi_N \text{ f.s.} \tag{Rep}$$

Definition II.3

1. Derivative C is called reachable, if there exists a replication strategy.
2. A financial model is called complete, if every derivative is reachable.

Theorem II.4
Let $C = h(S_1, \dots, S_N)$ be a derivative im CRR-Model. Then C is reachable, i.e. $\exists w \in \mathbb{R}$ and $(\xi_n)_{n\in\mathbb{N}}$ with (Rep). It holds:

1. $\exists$ measurable function $f_n \colon \mathbb{R}^n \to \mathbb{R}, n \in [N]$ so that

$$\Pi_n = f_n(S_1, \dots, S_n)$$

and values of f_n along the paths in the binomial tree are recursively set through

$$Rek = \begin{cases} f_N(S_1, \dots, S_N) = h(S_1, \dots, S_N) = C \\ f_n(S_1, \dots, S_N) = \frac{1}{1+r}\left(\frac{r-a}{b-a} f_{n+1}^b + \frac{b-r}{b-a} f_{n+1}^a\right) \forall n \in [N]_0 \end{cases}$$

where $f_{n+1}^b = f_{n+1}(S_1, \dots, S_n(1+b))$ und $f_{n+1}^a = f_{n+1}(S_1, \dots, S_n(1+a))$

2. The strategy to be replicated is given by

$$\xi_n = \frac{f_n^b - f_n^a}{S_{n-1}(b-a)} \tag{Δ-Hedge}$$

Folgerung II.5
The CRR-Model is complete.

Folgerung II.6
If C is an european derivative, i.e. $C = h(S_N)$, with $h \colon \mathbb{R} \to \mathbb{R}$ measurable, then the following simplifications hold: It is sufficient to take $f_n \colon \mathbb{R} \to \mathbb{R}$ and it holds

$$\Pi_n = f_n(S_n) \quad f_{n+1}^b = f_{n+1}(S_n(1+b)) \quad f_{n+1}^a = f_{n+1}(S_n(1+a))$$

▶ **Bemerkung**

1. The recursions Rek corresponds to a backward iteration of the tree diagramm. Image: *tree diagram is missing :/* f_n is set as discontinious mean value of f_{n+1}^b and f_{n+1}^a. The weights $q_b = \frac{r-a}{b-a}, q_a = \frac{b-r}{b-a}$. It holds: $q_a + q_b = 1$

2. Originally the transition probabilities p do not play a role in the evaluation of C: it is replaced through the “risk-neutral” probabilities $q_b, q_a = 1 - q_b$
3. They can be efficiently implemented on the computer also for big trees
4. The formula for ξ_n is also denoted as “Delta-Hedge”

$$\xi_n = \frac{\text{“Price reduction derivative”}}{\text{“Price reduction basic goods”}} \quad \text{difference quotient}$$

5. Weitere Interpretation of ξ_n
   - $\xi_n > 0$ The price change derivative has same sign as the price reduction basic good, there is no need of short sale!
   - $\xi_n < 0$ Price alternation derivative has opposed sign as the price reduction basic good, there is noo need of short sale!
   - $\xi_n \approx 0$ Price alternation derivative barely depends from the price change basic good.

Beweis. With backwards induction over $n \in [N]_0$

1. For $n \in \mathbb{N}$ it holds: $\Pi_N = C = h(S_1, \dots, S_N)$ (Rep) so $\Pi_N = f_N(S_1, \dots, S_N)$ with $f_N = h$
2. Induction step from SF-condition follows

$$\begin{aligned} \tilde{\Pi}_{n+1} = \tilde{\Pi}_n &= \xi_{n+1}(\tilde{S}_{n+1} - \tilde{S}_n) \quad |(1+r)^{n-1} \\ \implies \Pi_{n+1} - (1+r)\Pi_n &= \xi_{n+1}(S_{n+1} - (1+r)S_n) \end{aligned} \tag{$*$}$$

Per induction condition it holds $\Pi_{n+1} = f_{n+1}(S_1, \dots, S_{n+1}) = f_{n+1}(S_1, \dots, S_n, S_n(1 + R_{n+1}))$ (because of the definition of CRR and $S_{n+1} = S_n(1 + R_n)$). The second cases $R_k = b$ and $R_k = a$ can occur respectively with strictly positive probability.

Case 1: $\Pi_{n+1} = F_{n+1}(S_1, \dots, S_n, S_n(1 + b)) = f^b_{n+1}$, substitution in ($*$), yields

$$f^b_{n+1} - (1+r)\Pi_n = \xi_{n+1}S_n(b - r) \tag{I}$$

Case 2: $S_{n+1} = S_n(1 - a)$ und $\Pi_{n+1} = f_{n+1}(S_1, \dots, S_n, S_n(1 + a)) = f^a_{n+1}$, substitution in ($*$), yields

$$f^a_{n+1} - (1+r)\Pi_n = \xi_{n+1}S_n(a - r) \tag{II}$$

□

Π_n is ξ_{n+1} $\mathscr{F}_n$-measurable, meaning independent from R_{n-1} $\implies$ (II) (I)

- (II) - (I): $f^b_{n+1} - f^a_{n+1} = \xi_{n+1}S_n(b-a)$, then $\xi_{n+1} = \frac{f^b_{n+1} - f^a_{n+1}}{S_n(b-a)}$ so (Δ-Hedge) ✓
- in (I) $f^b_{n+1} - (1+r)\Pi_n = \frac{b-r}{b-a}(f^b_{n+1} - f^a_{n+1})$, then $\Pi_n = \frac{1}{1+r}\left(\frac{r-a}{b-r}f^a_{n+1} + \frac{b-r}{b-a}f^a_{n+1}\right) \implies$ (Rep) ✓

▶ Bemerkung

Systems of linear equations (I) + (II) can be written as

$$\begin{pmatrix} 1+r & b-r \\ 1+r & a-r \end{pmatrix} \begin{pmatrix} \Pi_n \\ \xi_{n+1}S_n \end{pmatrix} = \begin{pmatrix} f^b_{n+1} \\ f^a_{n+1} \end{pmatrix} \tag{SLE-1}$$

■ **Beispiel**

"Asiatic call-option", payout:

$$C = (\overline{S}_N - K)_+ \text{ mit } \overline{S}_N = \frac{1}{1+N}\sum_{k=0}^{N} S_k$$

Path dependent derivative. Evaluation in CRR-model with $N = 2$ with parameter:

$$b = 0,3 \quad a = 0,3 \quad r = 0,2 \quad S_0 = 100 \quad K = 100$$

Binomial tree:

$$\begin{tikzcd}
 & & 169 \\
 & 130 \arrow[ur] \arrow[r] & 91 \\
100 \arrow[ur] \arrow[r] & 70 \arrow[ur] \arrow[dr] & \\
 & & 49
\end{tikzcd}$$

$$\begin{aligned}
C &= h(S_1, S_2) \text{ mit } h = f_2 \\
h(130, 169) &= (\frac{399}{3} - 100)_+ = 33 \\
h(130, 91) &= (\frac{321}{3} - 100)_+ = 7 \\
h(70, 91) &= (\frac{261}{3} - 100)_+ = 0 \\
h(70, 49) &= (\frac{219}{3} - 100)_+ = 0
\end{aligned}$$

Recursion:

$$\text{Auxiliary calculation: } q = \frac{r-a}{b-a} = \frac{0,4}{0,6} = 2/3 \text{ und } 1-a = 1/3$$

$$\begin{aligned}
f_1(130) &= \frac{1}{1+r}(q \cdot f^b + (1-q)f^a) \\
&= \frac{1}{1,1}(2/333 + 1/37) = \frac{1}{1,1} \cdot 73/3 \\
&\approx 22,12 \\
f_1(70) &= \frac{1}{1,1}(2/30 + 1/30) = 0 \\
f_0 &= \frac{1}{1,1}(2/3\frac{1}{1,1}73/3 + 1/\cdot 0) \approx 13,41
\end{aligned}$$

Strategy:

$$\begin{aligned}
\xi_2(130) &= \frac{f_2^b - f_2^a}{S_1(b-a)} = \frac{33-7}{130 \cdot 0,6} = \frac{26}{13 \cdot 6} = 1/3 \\
\xi_2(70) &= \frac{0-0}{70 \cdot 0,6} = 0 \\
\xi_1 &= \frac{f_1^b - f_1^a}{S_0(b-a)} = \frac{\frac{1}{1,1}73/3 - 0}{100 \cdot 0,6} = \frac{73}{3 \cdot 11 \cdot 6} = \frac{73}{196} \approx 0,37
\end{aligned}$$

2. Martingale and arbitrage in CRR-model

Consider a CRR-model on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Set another probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$. I.e. we leave the structure of the tree unaltered, but we change the transition probability:

$$\begin{aligned}
\text{von } p &= \mathbb{P}(R_n = b) \\
\text{zu } q &= \mathbb{Q}(R_n = b)
\end{aligned}$$

Notation: $\mathbb{E}^{\mathbb{Q}}[\cdot]$ expected value under $\mathbb{Q}$.

Definition II.7

A probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ is called equivalent martingale measure (EMM) for the CRR-model, if it holds

1. $\mathbb{Q} \sim \mathbb{P}$ ($\mathbb{Q}$ equivalent to $\mathbb{P}$)
2. discrete price process $(\tilde{S}_n)_{n \in [N]}$ is $\mathbb{Q}$-martingale, i.e.

$$\mathbb{E}^{\mathbb{Q}}[\tilde{S}_{n+1} \mid \mathscr{F}_n] = \tilde{S}_n \quad \forall n \in [N-1]_0$$

▶ **Erinnerung II.8**

$\mathbb{P}, \mathbb{Q}$ probability measures on $(\Omega, \mathscr{F})$

- $\mathbb{P} \sim \mathbb{Q} :\Leftrightarrow (\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0 \forall A \in \mathscr{F}$ (equivalent)
- $\mathbb{Q} << \mathbb{P} \Leftrightarrow (\mathbb{P}(A) = 0 \implies \mathbb{Q}(A) = 0 \forall A \in \mathscr{F}$ ($\mathbb{Q}$ absolute continious with respect to $\mathbb{P}$)
- It holds: $\mathbb{Q} \sim \mathbb{P} \Leftrightarrow (\mathbb{Q} << \mathbb{P} \wedge \mathbb{P} << \mathbb{Q})$

Theorem II.9

1. In CRR-model there exists a EMM iff it holds $a < r < b$
2. The EMM $\mathbb{Q}$ is unique and it holds

$$q := \mathbb{Q}(R_n = b) = \frac{r-a}{b-a}$$
$$1 - q = \mathbb{Q}(R_n = a) = \frac{b-r}{b-a} \quad \forall n \in [N]$$

▶ **Bemerkung**

q and $1-q$ are exactly the risk-neutral weights, which appear in (Rep)

Beweis. Let $\mathbb{Q}$ be an arbitrary probability measure on $(\Omega, \mathscr{F})$. Set

$$\begin{aligned} q_n &:= \mathbb{Q}(R_n = b \mid \mathscr{F}_{n-1}) \in [0,1] \\ \mathbb{E}^{\mathbb{Q}}[\tilde{S}_n \mid \mathscr{F}_{n-1}] &= \mathbb{E}^{\mathbb{Q}}[\tilde{S}_n \cdot (\frac{1+R_n}{1+r}) \mid \mathscr{F}_{n-1}] \\ &= \tilde{S}_{n-1} \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[1 + R_n \mid \mathscr{F}_{n-1}] \\ &= \tilde{S}_{n-1} \cdot \frac{1}{1+r} (q_n(1+b) + (1-q_n)(1+a)) \end{aligned}$$

Then

$$\begin{aligned} (\tilde{S}_{n\in[N]}) \text{ ist } \mathbb{Q}\text{-martingale} \Leftrightarrow \\ \frac{1}{1+r}(q_n(1+b) + (1-q_n)(1-a) = 1 \quad \forall n \\ q_n b = (1-q_n)a = r \\ q_n(b-a) = r - a \implies q_n = \frac{r-a}{b-a} \end{aligned}$$

$$\begin{aligned} q_n \in [0,1] \Leftrightarrow a \leq r \leq b \\ \mathbb{Q} \sim \mathbb{P} \colon q_n \in (0,1) \Leftrightarrow a < r < b \end{aligned}$$

i.e. $\mathbb{Q}$ is EMM $\Leftrightarrow a < r < b$. □

Theorem II.10 (Risk-neutral evaluation formula)
Let $C = h(S_1, \dots, S_n)$ be a derivative in CRR-model with EMM $\mathbb{Q}$. For the price process $(\Pi_n)_{n\in[N]}$ of C it holds:

$$\Pi_n = (1+r)^{-(N-n)} \cdot \mathbb{E}^{\mathbb{Q}}[C \mid \mathscr{F}_n]$$

It especially holds

$$w = \Pi_0 = (1+r)^{-N} \cdot \mathbb{E}^{\mathbb{Q}}[C]$$

In words: The fair price of C is unique and given by the discontinious expected value of C under the martingale measure $\mathbb{Q}$.

Beweis. The probability space for the CRR-model is finite, i.e. $|\Omega| = 2^N < \infty$ (finitely many paths in CRR-model). Hence, every random variable is bounded and especially C and $(\xi_n)_{n\in[N]}$. Let (ξ_n) be a replication strategy for C with discontinious value process $(\tilde{\Pi}_n)$, i.e.

$$\tilde{\Pi}_n = w + \sum_{k=1}^{n} \xi_k(\tilde{S}_k - \tilde{S}_{k-1}) = w + (\xi \cdot \tilde{S})_n$$

und

$$\tilde{\Pi}_n = (1+r)^{-N} C$$

$\mathbb{Q}$ is EMM $\implies$ $(\tilde{S}_n)$ is $\mathbb{Q}$-martingale. With Theorem 1.6 $(\xi \cdot \tilde{S})_n$ is $\mathbb{Q}$-martingale. Hence it follows $\tilde{\Pi}_n$ is $\mathbb{Q}$-martingale.

$$\begin{aligned}\Pi_n &= (1+r)^n \cdot \tilde{\Pi}_n = (1+r)^n \mathbb{E}^{\mathbb{Q}}[\Pi_N \mid \mathscr{F}_n] \quad \text{martingale} \\ &= (1+r)^{-(N-n)} \cdot \mathbb{E}^{\mathbb{Q}}[C \mid \mathscr{F}_n].\end{aligned}$$ □

▶ **Bemerkung (to martingale condition for $\mathbb{Q}$)**
We write (somewhat incoveniently)

- $q_b = \mathbb{Q}(R_n = b)$ und $q_a = \mathbb{Q}(R_n = a)$
- $\mathbb{Q}$-measure: $q_a + q_b = 1 \Leftrightarrow q_b(1+r) + q_a(1+r) = 1 + r$
- Martingale condition:

$$(1+b)q_b + (1+a)q_a = 1 - r \Leftrightarrow q_b(b-r) + q_a(a-r) = 0$$

As a system of linear equations:

$$\begin{pmatrix} 1+r & 1+r \\ b-r & a-r \end{pmatrix} \begin{pmatrix} q_b \\ q_a \end{pmatrix} = \begin{pmatrix} 1+r \\ 0 \end{pmatrix} \tag{SLE-2}$$

is a condition for martingale measure. Compare with

$$\begin{pmatrix} 1+r & 1+r \\ b-r & a-r \end{pmatrix} \begin{pmatrix} \Pi_{n+1} \\ \xi_n \cdot S_{n-1} \end{pmatrix} = \begin{pmatrix} f_n^b \\ f_n^a \end{pmatrix}$$

the latter is again (SLE-1), the same matrix but transposed $\implies$ duality!

Arbitrage in CRR-model

Definition II.11

An investment strategy $(\xi_n)_{n\in[N]}$ with time horizon N and discontinious value process $(\tilde{\Pi}_n)_{n\in[N]}$ is called an arbitrage, if it holds:

1. $\tilde{\Pi}_0 = 0$ (no initial capital)
2. $\mathbb{P}(\tilde{\Pi}_N \geq 0) = 1$ (no risk of loss)
3. $\mathbb{P}(\tilde{\Pi}_N > 0) > 0$ (positive profit with positive probability)

We negotiate the 3 conditions (arb.)

Theorem II.12

In CRR-model are equivalent

1. There does not exist an arbitrage (NA = "No-arbitrage")
2. There exists an EMM $\mathbb{Q}$

▶ **Bemerkung**

This Theorem basically holds in all financial models (discrete, continious, ...). It is also called 1. Main theorem of the price theory.

Beweis.

- b) $\implies$ a) with contradiction. Let $\mathbb{Q}$ be an EMM and (ξ) arbitrage. Because of $\mathbb{Q} \sim \mathbb{P}$ it follows from (arb):

$$\begin{aligned} \mathbb{Q}(\tilde{\Pi}_N \geq 0) &= 1 \\ \mathbb{Q}(\tilde{\Pi}_N > 0) &> 0 \\ \implies \mathbb{E}^{\mathbb{Q}}[\tilde{\Pi}_N] &> 0 \end{aligned} \tag{$*$}$$

Otherwise: $\tilde{\Pi}_N = 0 + (\xi \cdot \tilde{S})_N$. $\tilde{S}$ is $\mathbb{Q}$-martingale $\implies$ $(\xi \cdot \tilde{S})$ is $\mathbb{Q}$-martingale, then

$$\mathbb{E}^{\mathbb{Q}}[\tilde{\Pi}_N] = \mathbb{E}^{\mathbb{Q}}((\xi \cdot \tilde{S})_N) = 0$$

and that is a contradiction to $(*)$. □

Kapitel III

Block-Scholes-model

THe goal is to transition from CRR-model (in discrete time) to Block-Scholes (BS-)model (in continious time) through fomation of limit.

- Derivation of Block-Scholes-formula for price of european put- and call-options.

Consider the time interval $[0, T]$, for every $N \in \mathbb{N}$ divided in steps of length $\Delta_n = \frac{T}{N}$. Choose a parameter $r \in \mathbb{R}, \mu \in \mathbb{R}$ (trend parameter), $\sigma > 0$ (violatility). Define a sequence of CRR-models $(S^N)_{N\in\mathbb{N}}$ embedded in $[0, T]$ with parameters

$$r_N = r \cdot \Delta_n \quad b_N = \mu\Delta_n + \sigma\sqrt{\Delta_n} \quad a_N = \mu\Delta_n - \sigma\sqrt{\Delta_n}\, p \in (0, 1),\ s > 0$$

i.e. $S_0^N = s$, $S_{t_k}^N = s \cdot \prod_{i=1}^{k}(1 + R_i^N)$ with $t_k = k \cdot \Delta_n$, or $\tilde{S}_0^N = s$ and hence $\tilde{S}_{t_k}^N = s \cdot \prod_{i=1}^{k} \frac{1+R_i^N}{1+r_N}$, where $\mathbb{P}(R_i^N = n_N) = p, \mathbb{P}(R_i^N = a_N) = 1 - p$. Denote the sequence with CRR_N. If its necessasry, we interpolate between the grid points with

$$S_t^N = S_{t_k}^N \quad t \in [t_k, t_{k+1}]$$

Calculate the risk-neutral probabilities

$$q_N = \mathbb{Q}_N(R_i^N = b_N) = \frac{r_N - a_N}{b_N - a_N} = \frac{(r-\mu)\Delta_n + \sigma\sqrt{\Delta_n}}{2\sigma\sqrt{\Delta_n}} = \frac{1}{2} - \frac{\lambda}{2}\sqrt{\Delta_n}$$

with $\lambda := \frac{\mu - r}{\sigma}$

▶ **Bemerkung**

- If $\mu = r$, then $q_N = \frac{1}{2}$ and in generall $\lim_{k\to\infty} a_N = \frac{1}{2}$
- $\lambda := \frac{\mu-r}{\sigma}$ is called "Sharp-ratio" or market risk price

Question: convergence of the distribution of S_T^N under $\mathbb{Q}_N$ for $N \to \infty$?
Transition to logarithm:

$$\mathcal{Z}_N := \log(\frac{S_T^N}{S_0}) = \sum_{k=1}^{N} \underbrace{\log(1 + R_k^N)}_{L_k^N}$$

Sum the independent, identically distributed random variables, then use the central convergence theorem?

There exists a so-called Triangle-scheme

$$\begin{aligned} \mathcal{Z}_1 &= L_1^1 \\ \mathcal{Z}_2 &= L_2^1 \quad + L_2^2 \\ \mathcal{Z}_3 &= L_3^1 \quad + L_2^3 \quad + L_3^3 \end{aligned}$$

Random variables in a row are stochastically independent.

Theorem III.1 (Central convergence theorem for triangle-scheme)
Let $L^N := (L_1^N, L_2^N, \ldots, L_N^N)$ be a vector of random variables for every $N \in \mathbb{N}$ (“triangle-scheme”) with the following properties:

1. $\forall N \in \mathbb{N}$, $(L_1^n, \ldots, L_N^N)$ are independent with identicall distribution
2. $\exists$ Sequence of (deterministic) constants $C_N \to 0$, such that
$$\left|L_k^N\right| \leq C_N \quad \forall k \in [N]$$
3. With $\mathcal{Z}_N = L_1^N + \cdots + L_N^N$ it holds
$$\begin{aligned} \mathbb{E}[\mathcal{Z}_N] &\to m \in \mathbb{R} \\ \mathbb{V}\mathrm{ar}(\mathcal{Z}_N) &\to s^2 > 0 \text{ für } N \to \infty \end{aligned}$$

Then $(\mathcal{Z}_N)_{N\in\mathbb{N}}$ converges in distribution towards the normally distributed random variable $\mathcal{Z}$ with $\mathbb{E}[\mathcal{Z}] = m$ und $\mathbb{V}\mathrm{ar}(\mathcal{Z}) = s^2$

Beweis. Without a proof, see eg Probability theory with martingale. □

► **Bemerkung**
Compare 2nd exercise sheet/ 1st exercise.

► **Erinnerung III.2**
The density of the standard normal distribution is:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and the distribution function:

$$\Phi(x) = \int_{-\infty}^{x} \varphi(y)\,\mathrm{d}y = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,\mathrm{d}y$$

The normal distribution with expecte value m and variance s^2 has distribution function $\Phi(\frac{x-m}{s})$

Definition III.3
A strict positive random variable X is called log-normally distributed with parameter m, s^2, if it holds

$$\log(X) \sim \mathcal{N}(m, s^2)$$

Theorem III.4
Consider the sequence $(S^N)_{N\in\mathbb{N}}$ of CRR-models, as described in CRR_N. Then S_T^N converges in distribution under $\mathbb{Q}_N$ towards a random variable S_T and S_T/S_0 is log-normally distributed with paraeters $n = T(r - \sigma^2/2)$ and $s^2 = T\sigma^2$. Equivalently it holds $\mathcal{Z}_N = \log(S_T^N/S_0)$

$$\mathbb{Q}_N(\mathcal{Z}_N \leq x) \xrightarrow{N\to\infty} \Phi\left(\frac{x - T(r-\sigma^2/2)}{\sigma\sqrt{T}}\right)$$

Beweis. The triangle-scheme $L^N = (L_1^N, \dots, L_N^N)$ with $L_k^N = \log(1 + R_k^N)$ obviously satisfies the condition 1. and 2. from theorem 3.1, (under $\mathbb{Q}_N$). Choose as eg.:

$$C_N = \max(\left|\log(1 + \mu\Delta_n + \sigma\sqrt{\Delta_n})\right|, \left|\log(1 + \mu\Delta_n - \sigma\sqrt{\Delta_n})\right|)$$

We calculate the expected value and variance of L_k^N or $\mathcal{Z}_N$. Use the Taylor expansion:

$$\log(1+x) = x - x^2/2 + x^3/3 + \mathcal{O}(x^4) \quad (x \to 0)$$

I.e.

$$\log(1 + \underbrace{\mu\Delta_n \pm \sigma\sqrt{\Delta_n}}_{b_N \text{ or } a_N}) = \pm\sigma\sqrt{\Delta_n} + \mu\Delta_n - \sigma^2/2\Delta_n + \mathcal{O}(\Delta_n^{3/2})$$

The risk-neutral probabilities are

$$q_N = \frac{1}{2} - \frac{\lambda}{2}\sqrt{\Delta_n} \quad 1 - q_N = \frac{1}{2} + \frac{\lambda}{2}\sqrt{\Delta_n}$$

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_N}[L_k^N] &= \mathbb{E}^{\mathbb{Q}_N}[\log(1 + R_k^N)] = q_N \log(1 + b_N) + (1 + p_N)\log(1 + a_N) \\
&= (\mu - \sigma^2/2)\Delta_n - \lambda\sigma\Delta_n + \mathcal{O}(\Delta_n^{3/2}) \quad \text{mit } \lambda = \frac{\mu - r}{\sigma} \\
&= (\mu - (\mu - r) - \sigma^2/2)\Delta_n + \mathcal{O}(\Delta_n^{3/2}) \\
&= (r - \sigma^2/2)\Delta_n + \mathcal{O}(\Delta_n^{3/2}) \\
\mathbb{E}^{\mathbb{Q}_N}[(L_k^N)^2] &= q_N \log^2(1 + b_N) + (1 - q_N)\log^2(1 + a_N) \\
&= \sigma^2\Delta_n + \mathcal{O}(\Delta_n^{3/2}) \\
\mathbb{V}\text{ar}^{\mathbb{Q}_N}(L_k^N) &= \mathbb{E}^{\mathbb{Q}_N}[(L_k^N)^2] - \mathbb{E}^{\mathbb{Q}_N}[L_k^N]^2 = \sigma^2\Delta_n + \mathcal{O}(\Delta_n^{3/2})
\end{aligned}$$

So, it holds

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_N}[\mathcal{Z}_N] &= N \cdot \mathbb{E}^{\mathbb{Q}_N}[L_k^N] = (r - \sigma^2/2)T + \mathcal{O}(N^{-1/2}) \xrightarrow{N\to\infty} (r - \sigma^2/2)T =: m \\
\mathbb{V}\text{ar}^{\mathbb{Q}_N}[\mathcal{Z}_N] &= N \cdot \mathbb{V}\text{ar}^{\mathbb{Q}_N}[L_k^N] = \sigma^2 T + \mathcal{O}(N^{-1/2}) \xrightarrow{N\to\infty} \sigma^2 T =: s^2
\end{aligned}$$

The result follows with the central limit theorem (Theorem 3.1). □

Asymptotics of put- and call-option

Fix the duration T and the strike price K and write:

- $C_N(t, S_t^N)$ ... price of a european call-option in CRR_N model, dependant of time t and basic good S_t^N
- $P_N(t, S_t^N)$ ... analogously for put

Theorem III.5 (Block-Scholes-formula)
The prices C_N, P_N converge for $N \to \infty$ towards a BS-price

$$C_{BS}(t, S_t) = \lim_{N\to\infty} C_N(t, S_t^N)$$
$$P_{BS}(t, S_t) = \lim_{N\to\infty} P_N(t, S_t^N)$$

and the following Block-Scholes-formula holds:

$$C_{BS}(t, S_t) = S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2)$$
$$P_{BS}(t, S_t) = S_t\Phi(-d_1) - e^{-r(T-t)}K\Phi(-d_2)$$

where

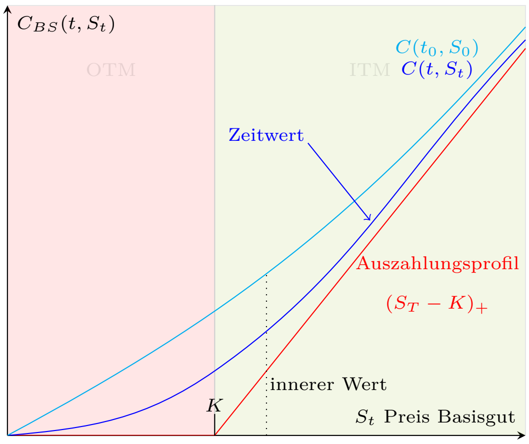
$$d_1 = d_1(t, S_t) = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_2(t, S_t) = \frac{\log(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

▶ **Bemerkung**

- Geschlossener Ausdruck für Bewertung von europäischen Put- und Call-Optionen
- Herleitung als Grenzwert aus dem CRR-Modell entspricht nicht der ursrünglichen Herleitung von Black & Scholes mittels stochastische Analysis ($\implies$ VL stoch. Calculus)
- Für Entwicklung von BS-Formel und BS-Modell erhielten Scholes & Merton dem Wirtschaftsnobel(gedenk)preis 1997
- Der Parameter σ heißt Voliatitität und entspricht der Schwankungsbreite der Preisänderung

Skizze vom BS-Call-Preis

BS-Call-Preis



- innere Wert: $(S_t - K)_+$ konvergiert gegen Auszahlungsprofil: $(S_T - K)_+$, für $t \to T$
- Zeitwert: $C_{BS}(t, S_t) - (S_t - K)_+ \geq 0$ konvergiert gegen Null für $t \to T$
-
  - out of the money (OTM): Innere Wert $= 0$ bei $S_t < K$
  - in the money (ITM): Innere Wert > 0 bzw. $S_t > K$
  - at the money (ATM): Grenzfall $S_t = K$
- Zeitwert ist am größten für ATM-Optionen
- $t \mapsto C_{BS}(t, S_t)$ ist streng monoton fallend bzw.

$$\frac{\partial C_{BS}(t, S_t)}{\partial t} < 0$$

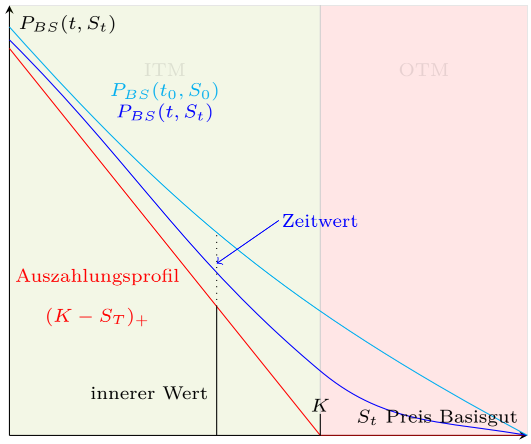
- $S_t \mapsto C_{BS}(t, S_t)$ ist streng monoton steigend und konvex bzw

$$\frac{\partial C_{BS}(t, S_t)}{\partial S}(t, S_t) > 0 \text{ und } \frac{\partial^2 C_{BS}}{\partial S^2}(t, S_t) > 0$$

- Für den Put ist das ganze Symmetrisch

Skizze vom BS-Put-Preis

BS-Put-Preis



- innere Wert: $(K - S_t)_+$ konvergiert gegen Auszahlungsprofil: $(K - S_T)_+$, für $t \to T$
- Zeitwert: $P_{BS}(t, S_t) - (K - S_t)_+ \geq 0$ konvergiert gegen Null für $t \to T$
-
  - out of the money (OTM): Innere Wert $= 0$ bei $S_t > K$
  - in the money (ITM): Innere Wert > 0 bzw. $S_t < K$
  - at the money (ATM): Grenzfall $S_t = K$
- Zeitwert ist am größten für ATM-Optionen
- $t \mapsto C_{BS}(t, S_t)$ ist streng monoton fallend bzw.

$$\frac{\partial C_{BS}(t, S_t)}{\partial t} < 0$$

- $S_t \mapsto C_{BS}(t, S_t)$ ist streng monoton fallend und konvex bzw

$$\frac{\partial C_{BS}(t, S_t)}{\partial S}(t, S_t) < 0 \text{ und } \frac{\partial^2 C_{BS}}{\partial S^2}(t, S_t) > 0$$

Beweis (**???**). Wir beweisen das Resultat für $t = 0$: andere Zeitpunkte $t \in [0, T]$ können analog behandelt werden.

- Nach **???**, gilt für Preis der Put-Option im CRR_N-Modell

$$\begin{aligned} P^N(0, S_0^N) &= (1 + r\Delta_n)^{-N} \cdot \mathbb{E}^{\mathbb{Q}}[(K - S_T^N)_+] \\ &= (1 + r\Delta_n)^{-N} \cdot \mathbb{E}^{\mathbb{Q}}[(K - S_0 e^{\mathcal{Z}_N = \log(\frac{S_T^N}{S_0})})] \\ &= (1 + r\Delta_n)^{-N} \cdot \mathbb{E}^{\mathbb{Q}}[f(\mathcal{Z}_N)] \end{aligned}$$

  mit $f(z) = (K - S_0 e^z)_+$ stetig und beschränkt. Aus Stochastik ist bekannt $\mathcal{Z}_N \to \mathcal{Z}$ in Verteilung, dann folgt $\mathbb{E}[f(\mathcal{Z}_N)] \to \mathbb{E}[f(\mathcal{Z})] \quad \forall f \in C_b(\mathbb{R})$.
  - $\lim_{N\to\infty}(1 + r\Delta_n)^{-N} = \lim_{N\to\infty}(1 + rT/N)^{-N} = e^{-rT}$

– $\lim_{N\to\infty} \mathbb{E}^{\mathbb{Q}}[f(\mathcal{Z}_N)] = \mathbb{E}[f(Z)]$ mit $\mathcal{Z} \sim \mathcal{N}((r-\sigma^2/2)T, \sigma^2 T) = \mathcal{N}(mT, \sigma^2 T)$

$$\begin{aligned}
\mathbb{E}[f(\mathcal{Z})] &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{T}} \int_{-\infty}^{\infty} (K - S_0 e^{\mathcal{Z}})_+ \exp(-\frac{(\mathcal{Z}-mt)^2}{2\sigma^2 T}) \,\mathrm{d}z \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\log(K/S_0)} (K - S_0 e^{\mathcal{Z}}) \exp(-\frac{1}{2}(\frac{\mathcal{Z}-mT}{\sigma\sqrt{T}})^2) \,\mathrm{d}z \\
&= \begin{pmatrix} y = \frac{\mathcal{Z}-mT}{\sigma\sqrt{T}} \\ \mathrm{d}y = \frac{\mathrm{d}\mathcal{Z}}{\sigma\sqrt{T}} \end{pmatrix} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} (K - S_0 \exp(y\sigma\sqrt{T} + mT) e^{y^2/2} \,\mathrm{d}y \\
&= K\Phi(-d_2) - S_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} \exp(-y^2/2 + y\sigma\sqrt{T} + mT) \,\mathrm{d}y
\end{aligned}$$

Nebenrechung:

$$\begin{aligned}
y^2/2 + y\sigma\sqrt{T}(+mT &= rT - \frac{1}{2}(y^2 - 2y\sigma\sqrt{T} + \sigma^2 T) = rT - \frac{1}{2}(y - \sigma\sqrt{T}) \\
&= K\Phi(-d_2) = S_0 e^{rT} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{(y-\sigma\sqrt{T})/2} \,\mathrm{d}y}_{\Phi(-d_2-\sigma\sqrt{T})} \\
&= K\Phi(-d_2) - S_0 e^{rT} \Phi(-d_1)
\end{aligned}$$

Dann folgt $\lim_{N\to\infty} P_N(0, S_0^N) = e^{-rT} K\Phi(-d_2) - S_0\Phi(-d_1)$ und das ist die Formel für den Put ✓

- Für Call: Nutze Put-Call-Parität

$$C^N(0,S_0) - \underbrace{P^N(0,S_0)}_{P_{BS}(0,S_0)} = \underbrace{S_0}_{S_0} - \underbrace{(1+r\Delta_n)^{-N}K}_{\to e^{-rT}K}$$

$$\begin{aligned}
C_{BS}(0,S_0) &= \lim_{N\to\infty} C^N(0,S_0) \\
&= P_{BS}(0,S_0) + S_0 - e^{rT}K \\
&= e^{-rT}K(\underbrace{\Phi(-d_2)-1}_{-\Phi(d_2)}) - S_0(\underbrace{\Phi(-d_1)-1}_{-\Phi(d_1)}) \\
&= S_0\Phi(d_1) - e^{-rT}K\Phi(d_2)
\end{aligned}$$

wobei wir die Symmetrie der Normalverteilung: $\Phi(-x) = 1 - \Phi(x)$ genutzt haben. Damit ist auch die BS-Formel für den Call gezeigt ✓. □

Wir haben gezeigt: CRR_N-Preise konvergieren gegen BS-Preise.
Frage: Was gilt für die Replikationsstrategie? Konvergiert diese auch?

Theorem III.6
Für die Replikationsstrategie $\xi_{t_N}^N$ der Put- bzw. Call-Optionen in CRR$_N$-Modell gilt:

- Put: $\lim_{N\to\infty} \xi_{t_N}^N = \frac{\partial P_{BS}}{\partial S}(t, S_t) = -\Phi(-d_1)$
- Call: $\lim_{N\to\infty} \xi_{t_N}^N = \frac{\partial C_{BS}}{\partial S}(t, S_t) = \Phi(d_1)$

Diese partielle Ableitungen heißen auch "Delta" des Put- bzw. Call-Preisen.

Beweis. BEtrachte nur $t = 0$, $t \in [0, T]$ kann analog behandelt werden. Nach **???** ist ξ_0^N für Put gegeben durch

$$\begin{aligned}\xi_0^N &= \frac{P_N(\Delta_N, S_0(1+b_N)) - P_N(\Delta_N, S_0(1+a_N))}{S_0(b_N - A_N)} \\ &= \frac{P_N(\Delta_N, S_0(1+\mu\Delta_n + \sigma\sqrt{\Delta_n})) - P_N(\Delta_N, S_0(1+\mu\Delta_N + \sigma\sqrt{\Delta_N}))}{2S_0\sigma\sqrt{\Delta_N}}\end{aligned}$$

Es gilt $\lim_{N\to\infty} P_N(\Delta_N, S_0(1+\mu\Delta_N)) = P_{BS}(0, S_0)$. Unter geeigneten Annahmen an gleichmäßige Konvergenz folgt

$$\lim_{N\to\infty} \xi_0^N = \frac{\partial P_{BS}}{\partial S}(t, S_t)$$

und analog für Call. Wir berechnen explizit:

$$\begin{aligned}\frac{\partial C_{BS}}{\partial S}(t, S) &= \Phi(d_1) + S\varphi(d_1) \cdot \frac{\partial d_1}{\partial S} - e^{-r(T-t)} K\varphi(d_2) \frac{\partial d_1}{\partial d_2} \\ &= \Phi(d_1) + \frac{\partial d_1}{\partial S}(S\varphi(d_1) - e^{-r(T-t)} K\varphi(d_2))\end{aligned}$$

Nebenrechung:

$$\begin{aligned}e^{-rt} K/S\varphi(d_2) &= e^{-r\tau} \frac{1}{\sqrt{2\pi}} K/S \exp(-\frac{1}{2} \frac{\log(S/K) + r\tau - \sigma^2 r/\tau}{\sigma\sqrt{\tau}}) \\ &= \frac{1}{\sqrt{2\pi}} e^{-r\tau} K/S \exp(-\frac{1}{2} \frac{(\log(S/K) + r\tau)^2}{\sigma^2\tau} - 2(\log(S/K) + r\tau) + \sigma^2\tau/4) \\ &= \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} \frac{(\log(S/K) + r\tau)^2}{\sigma^2\tau} + (\log(S/K) + r\tau + \sigma^2\tau/4) \\ &= \varphi(d_1)\end{aligned}$$

also

$$e^{-r(T-t)} K\varphi(d_2) = S\varphi(d_1)$$

Das heißt: $\frac{\partial C_{BS}}{\partial S}(t, S) = \Phi(d_1)$. Put folgt analog oder mit Put-Call-Parität. $\square$

► **Bemerkung**

- $\frac{\partial C_{BS}}{\partial S}$ bzw. $\frac{\partial P_{BS}}{\partial S}$ lassen sich auch interpretieren als Sensitivität des Call- bzw. Put-Preises gegenüber Preisänderungen des Basisguts.

Analog lassen sich die Sensitivitäten ("Greeks") nach den weiteren Parametern berechnen.

Definition III.7
Die "Greeks" des BS-Preises sind folgende partielle Ableitungen

Bezeichg.	Def. der part. Abl.	Call	Put	
Delta	$\frac{\partial}{\partial S}$	$\Phi(d_1)$	$-\Phi(-d_1)$	Bestimm
Gamma	$\frac{\partial^2}{\partial S^2}$	$\frac{\varphi(d_1)}{S\sigma\sqrt{T-t}}$	$\frac{\varphi(d_1)}{S\sigma\sqrt{T-t}}$	Sensitivität von Delta gg
Vega	$\frac{\partial}{\partial \sigma}$	$S_t\sqrt{T-t}\varphi(d_1)$	$S_t\sqrt{T-t}\varphi(d_1)$	Sensitivit
Theta	$\frac{\partial}{\partial t}$	siehe ÜA	siehe ÜA	
Rho	$\frac{\partial}{\partial r}$	$K(T-t)(e^{-r(T-t)})\Phi(d_2)$	$-K(T-t)(e^{-r(T-t)})\Phi(-d_2)$	Se

▶ **Bemerkung**
"Vega" ist kein Buchstabe des griechischen Alphabets :/

Folgerung III.8
Der BS-Preis $C_{BS}(t,S)$ erfüllt folgende partielle DGL

$$\frac{\partial C_{BS}}{\partial t} + rS\frac{\partial C_{BS}}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 C_{BS}}{\partial S^2} + rC_{BS} = 0 \qquad \text{(BS-PDE)}$$

wobei $(t,s) \in [0,T] \times \mathbb{R}_{\geq 0}$. Mit Endwertbedingung

$$\lim_{t\to T} C_{BS}(t,S) = (S-K)_+$$

Für P_{BS} gilt die gleiche PDE mit Endwertbedingung

$$\lim_{t\to T} P_{BS}(t,S) = (K-S)_+$$

Beweis. Siehe Übung 3.0. □

▶ **Bemerkung**
In Erweiterungen des BS-Modells gibt es keine geschlossene Ausdrücke für Put/Call-Preise, aber eine PDE ähnlich zu (BS-PDE) gilt weiterhin.

1. Implizite Volatilität/ Grenzen des BS-Modells

Wir schreiben etwas ausführlicher

$$C_{BS}(t,S_t,T,K,\sigma) := C_{BS}(t,S_t)$$

eine Abhängigkeit von (T,K,σ) zu verdeutlichen.

Theorem III.9 (Implizite Volatlität)
Sei $C_*(0, S_0, T, K)$ ein vorgegebener (beobachtbarer) Preis einer Call-Option mit Fälligkeit T, Ausübungspreis K welcher innerhalb der Arbitragegrenzen liegt

$$(S_0 - e^{-rT}K)_+ < C_*(0, S_0, T, K) < S_0$$

Dann existiert ein eindeutiges $\sigma_*(T, K) \in (0, \infty)$, die implizite Volativilität von C_* sodass

$$C_*(0, S_0, T, K) = C_{BS}(0, S_0, T, K, \sigma_*(T, K))$$

gilt.

▶ **Bemerkung**
$\sigma_*(T, K)$ ist Lösung eines inversen Problems.

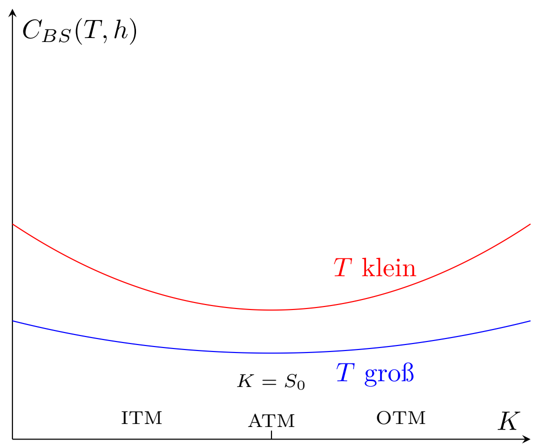
Vorwärtsproblem:	Parameter $\rightarrow$ Call-Preis	
inverses Problem:	Call-Preis $\rightarrow$ Parameter	

Kann zur impirischen Überpürfung des BS-Modells verwendet werden:

- BS-Modell passt gut zu Daten: $(T, K) \mapsto \sigma_*(T, K)$ ist annähernd konstant
- BS-Modell passt nicht gut zu Daten: $(T, K) \mapsto \sigma_*(T, K)$ variiert stark mit (T, K)

Typische tatsächliche Beobachtung:

Volativitäts-Smile



Eigenschaften:

- konvex
- assymetrisch (höher für große K)
- Minimum bei ATM

- flacher für lange Laufzeiten, steiler für kurze Laufzeiten

Form weist daraufhin, dass BS-Modell große Preissprünge des Basisguts unterschätzt. Form des Vola-Smiles in Modellen jeweils von BS $\implies$ aktuelles Forschungsthema.

Kapitel IV

Optimale Investition

Anhang

Literaturverzeichnis

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