Stochastikvertiefung: Finanzmathematik WS 19/20

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25. Oktober 2019

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Vorwort

Kapitel I

Introduction

1. The central issue of financial mathematics

Valuation:

Valuation of derivatives and hedge against the risks which emerges from the purchase / sale,

Definition (Derivative)
Financial product whose payouts are derived from price of one or more basic goods derived (underlying) derivative.

■ **Beispiel**

- Right to get 100,000 GBP in 3 months against 125,000 EUR (Call-Option, Underlying: Exchange rate GBP/EUR)
- Right within the next year to consume 100,000 Mw / h of electric energy at the price of 30EUR/Mwh with minimum order quantity of 50,000 Mwh (Swing-Option, Underlying: electricity price)
- buying and selling options on stock (Underlying: equity price)

Issue: What is the ”fair” price for such a derivative? (“Pricing”). How can the sellers protect themselves against the ... risks? (“Hedging”)

Optimal investment

Gathering Porftolios that are optimal for risk-return approach.

- How do I weigh risk against profit?
- What exactly is ”optimal”?
- Solution of the resulting optimization problems

Risk management + Risk measurement

- Legal rules (Basel + Solvency) should ensure stability of the banking system/insurance system even in the face of various risks

  $\Longrightarrow$ Mathematical theory of convex + coherent risk measures

Mathematical tools: Probability Theory + stochastic processes (dynamics in time, some linear algebra, optimization, measure theory).

2. Mathematical Financial Market Model

We consider:

1. Probability space $(\Omega, \mathscr{F}, \mathbb{P})$, later more probability measures $Q, \ldots$ on the same measure space $(\Omega, \mathscr{F}), \omega \in \Omega$ basic events or "scenarios".

2. Timeline I is either $I = \{t_0, t_1, \ldots, t_N = T\}$ N-period model (discrete model) or $I = [0, T]$ (continious-time model), where $T =$ time-horizon
   A stochastic process S is a measurable mapping $S : (\Omega, \mathscr{F}) \to \mathbb{R}^d$ mit $(\omega, t) \mapsto S_t(\omega)$
   Especially:
   - $t \mapsto S_t(\omega)$ function $I \to \mathbb{R}^d$ for every $\omega \in \Omega$ ("path")
   - $\omega \mapsto S_t(\omega)$ random variable $\Omega \to \mathbb{R}^d$ for every $t \in I$

3. Percolation a sequence of ω-algebras $(F_t)_{t \in I}$ with the property $\mathscr{F}_S \subseteq F_t \quad \forall s, t \in I, x \leq t$ and $F_t \subseteq F \quad \forall t \in I$
   Interpretation: $F_t =$ market participant at time t known/available information
   Events $A \in F_t$ are considered known 'at time t'
   A R^d-valued RV X is called F_t-measurable, if $E = X^{-1}(B) \in F_t \quad \forall$ Borel sets $B \subseteq R^d$ (E is actually the preimage of B).

   ■ **Beispiel**
   A stochastic process $(S_t)_{t \in I}$ on $(\Omega, \mathscr{F})$ is called adaptiert regarding a percolation $(\mathscr{F}_t)_{t \in I}$, wenn gilt:
   $$S_t \text{ is } F_t - \text{measurable} \quad \forall t \in I$$
   Interpretation: "the value S_t is known at time t"
   Why percolation in the financial mathematics (FiMa)?
   - Differentiation between future/past
   - Different information (Insider/Outsider) corresponds to different percolation $(F_t)_{t \in I}$ or $(G_t)_{t \in I}$

   S^i= price of the i-th asset at the time t

4. Assets R^{d+1}-valued stochastic process with components
   $$S^i : (\Omega \times I) \to \mathbb{R} \quad (\omega, t) \mapsto S^i_t(\omega) \text{ mit } i \in \{0, 1, \ldots, d\}$$
   where $S^i_t =$ price of the i-th asset at time t
   $S^i, i \in \{1, \ldots, d\}$ is typically
   - Stock, company share
   - Currency or exchange rate
   - Commodity such as oil, noble metal, electricity,..
   - Bond ...

   Principal assumption: S^i is liquid traded (eg on exchange), ie purchase/sale for the price S^i_t possible at any time.

$S^0 \ldots$ "numeraire" has a special role: describes interest rate of not in $(S^1, .., S^d)$ invested capital; is mostly considered to be risk-free.

Definition I.1 (Finance market model)
A finance market model (FMM) with a time axis I is given by

1. a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with percolation $(F_t)_{t\in I}$
2. an adapted to $(F_t)_{t\in I}$, R^{d+1}-valued stochastic process $S_t = (S_t^0, S_t^1, \ldots, S_t^d), t \in I$

■ **Beispiel (Cox-Rubinstein (CRR)-model (discrete-time))**

- $S_n^0 = (1+r)^n$, meaning interest at a constant rate r
- $S_n^1 = S_0^1 \prod_{k=1}^n (1+Ru)$, wobei $(R_1, R_2, \ldots)$ independent random variables with two possible values $a < b$
  Image: " "recombined tree " with events ω corresponding paths in the tree

■ **Beispiel (Block-Scholes-modell (continious-time))**

- $S_t^0 = e^{rt}$, meaning interest at a constant rate r
- $S_t^1 = S_0^1 \cdot \exp((\mu - \frac{\sigma^2}{2}t + \sigma\beta_t)$ mit $\mu \in \mathbb{R}, \sigma > 0, S_0^1 > 0$ und β_t corresponds to Brownian motion(stochastic process in continuous time) and $\mu - \frac{\omega^2}{2}$ corresponds to trend component

Image: Exchange curve $= S_t(\omega)$, wherein time-continuous model for infinite probability space

3. Conditional expectation values and Martingale

3.1. Conditional density and conditional expected value

Motivation: Given: Two random variables (X, Y) with values in $\mathbb{R}^m \times \mathbb{R}^n$ and joint density $f_{XY}(x, y)$. From f_{XY} we can derive:

- $f_Y(y) := \int_{\mathbb{R}^m} f_{XY}(x, y)\,\mathrm{d}x$ with marginal distribution of Y
- $S_Y := \{y \in \mathbb{R}^n : f_Y(y) > 0\}$ carrier of Y - Image?

Definition (Conditional density of X with respect to Y)
Conditional density from X with respect to Y is defined as

$$f_{X|Y}(x, y) = \begin{cases} \frac{f_{XY}(x,y)}{f_Y(y)} & y \in S_Y \\ 0 & y \notin S_Y \end{cases}$$

Consider the following problem:
What is the best forecast from X if an observation $Y = y$ is given?
Criteria:
Minimiye the quadratic distance/second moment/ L_2-norm.
Vorhersage:
Measurable function $g : \mathbb{R}^n \to \mathbb{R}^m$ mit $y \mapsto g(y)$, meaning,.

$$\min\left\{\mathbb{E}[(X - g(Y))]^2 : g \text{ messbar } \mathbb{R}^n \to \mathbb{R}^m\right\} \tag{min-1}$$

Satz I.2
When (X, Y) have a joint density with $\mathbb{E}[|X|^2] < \infty$, then (min-1) is going to be minimized through the conditional expected value

$$g(y) = \mathbb{E}[X \mid Y = y] := \int_{\mathbb{R}^m} x f_{X|Y}(x, y) \,\mathrm{d}x$$

(where $\mathbb{E}[X \mid Y = y]$ "expected value of X conditioned on $Y = y$")

In general, it holds:

Theorem I.3
Let (X, Y) be random variables with joint density on$\mathbb{R}^m \times \mathbb{R}^n$, $h : \mathbb{R}^m \to \mathbb{R}^n$ measurable with $\mathbb{E}[h(X, y)^2]$. Then that is going to be minimisation problem

$$\min\left\{\mathbb{E}[(h(X, Y) - g(y))^2]\right\} \quad g\text{measurable from } \mathbb{R}^n \text{ towards } \mathbb{R}$$

solved through

$$g(y) = \mathbb{E}[h(X, Y) \mid Y = y] = \int_{\mathbb{R}^m} h(X, Y) f_{X|Y}(x, y) \,\mathrm{d}x$$

Beweis (only proposition, the Theorem is analogous, for $n = 1$). Set $g(y) = \int_{\mathbb{R}} f_{X|Y}(x, y) \,\mathrm{d}x$. Sei $p : \mathbb{R} \to \mathbb{R}$ arbitrary measurable function with $\mathbb{E}[p(y)^2] < \infty$. Set $g_\varepsilon(y) = g(y) + \varepsilon p(y)$. Minimize

$$\begin{aligned}
F(\varepsilon) &:= \mathbb{E}[(X - g_\varepsilon(y))^2] = \mathbb{E}[(X - g(y) - \varepsilon p(y))^2] \\
&= \mathbb{E}[(X - g(y))^2] - 2\varepsilon \mathbb{E}[(X - g(y))p(y)] + \varepsilon^2 \mathbb{E}[p(y)^2] \\
\frac{\partial F}{\partial \varepsilon}(\varepsilon) &= 2\varepsilon \mathbb{E}[p(y)^2] - 2\mathbb{E}[(X - g(y))p(y)] \\
&\implies \varepsilon_* := \frac{\mathbb{E}[(X - g(y))p(y)]}{\mathbb{E}[p(y)^2]} = \frac{A}{B}
\end{aligned}$$

wobei

$$\begin{aligned}
A &= \mathbb{E}[Xp(y)] - \mathbb{E}[g(y)p(y)] \\
&= \int_{\mathbb{R} \times \mathbb{R}^n} xp(y) f_{XY}(x, y) \,\mathrm{d}x \,\mathrm{d}y - \int_{S_y} g(y)p(y) f_Y(y) = [\text{Einsetzen von } g + \text{Fubini}] \\
&= \int_{\mathbb{R} \times \mathbb{R}^n} xp(y) f_{XY}(x, y) \,\mathrm{d}x \,\mathrm{d}y - \int_{\mathbb{R} \times S_y} xp(y) \underbrace{f_{X|Y}(x, y) f_Y(y \,\mathrm{d}y)}_{= f_{XY}(x,y)} = 0
\end{aligned}$$

so $\varepsilon^* = 0$ independent from $p \implies g(y)$ minimizes (min-1). □

■ **Beispiel**

Let (X, Y) normaly distributed on $\mathbb{R} \times \mathbb{R}$ with

$$\mu = (\mu_x, \mu_y)^T \quad \Sigma = \begin{pmatrix} \sigma x^2 \rho\sigma_x\sigma_y & \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} \mathbb{V}\mathrm{ar}(X) & \mathbb{C}\mathrm{ov}(X,Y) \\ \mathbb{C}\mathrm{ov}(X,Y) & \mathbb{V}\mathrm{ar}(Y) \end{pmatrix} \text{ mit } \rho \in [-1, 1]$$

Then the arbitraty density is $f_{X|Y}(x, y)$. (Σ covariance matrix). Once more the density of a normaly distributed random variable with

$$\mathbb{E}[X \mid Y = y] = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y - \mu_y)$$
$$\mathbb{V}\mathrm{ar}(X \mid Y = y) = \sigma_x^2(1 - \rho^2)$$

(is ÜA!). The mapping $y \mapsto \mu_x + g(y)\frac{\sigma_x}{\sigma_y}(y - \mu_y)$ is called regression line for X given $Y = y$.
Image: μ_x, μ_y are values on x, y-axis and the σ's build the Triangle slope (slope is known substantially by ρ)
For disrete random variables, i.e. when X, Y accept only finitely many $\{x_1, \dots, x_m\}$ or $\{y_1, \dots, y_m\}$ annehmen then with similar considerations we obtain as a solution of (min-1)

$$\mathbb{E}[X \mid Y = y_j] = \sum_{i=1}^{m} X_i \mathbb{P}(X = x_i \mid Y = y_j)$$

wherein directly the conditional probabilities

$$\mathbb{P}(X = x_i \mid Y = y_j) = \begin{cases} \frac{\mathbb{P}(X = x_i \wedge Y = y_j)}{\mathbb{P}(Y = y_j)} & \text{wenn } \mathbb{P}(Y = y_j) > 0 \\ 0 & \text{wenn } \mathbb{P}(Y = y_j) = 0 \end{cases}$$

3.2. Conditional expectation - measure theoretical access

We consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For random variables $X : \Omega \to \mathbb{R}$ und $p \in [1, \infty)$ we define the L_p-norm

$$\|X\|_p = \mathbb{E}[|X|^p]^{1/p} = \left(\int_\Omega |X(\omega)|^p \,\mathrm{d}\mathbb{P}(\omega)\right)^{1/p}$$

and L_p-space $L_p(\Omega, \mathscr{F}, \mathbb{P}) := \left\{X : \Omega \to \mathbb{R} : \mathscr{F} - \text{measurable}, \|X\|_p < \infty\right\}$. We identificate random variables which differ only at zero amounts, ie $\mathbb{P}(X \neq X') = 0 \implies X = X'$ (in L_p).
From measure theory it is known: (?)
The spaces $L_p(\Omega, \mathscr{F}, \mathbb{P})$ with norm $\|\cdot\|_p, p \in [1, \infty)$ are always BANACH-spaces (linear, complete, normed vector spaces). For $p = 2$ also hilbert spaces with inner product

$$\langle X, Y\rangle = \mathbb{E}[XY] = \int_\Omega X(\omega)Y(\omega) \,\mathrm{d}\mathbb{P}(\omega)$$

Für $\mathscr{G} \subseteq \mathscr{F}$ Unter-σ-slgebra is $L_p(\Omega, \mathscr{F}, \mathbb{P}) \subseteq L_p(\Omega, \mathscr{F}, \mathbb{P})$ closed subspace.
We generalize "prediction problem" from the last section (1.3?)
Given are random variables X from $L_2(\Omega, \mathscr{F}, \mathbb{P})$ is $\mathscr{G} \subseteq \mathscr{F}$ Sub-σ-algebra.

What is the best $\mathscr{G}$-measurable forecast for Y?

$$\min\{\mathbb{E}[(X-G)^2] \colon G \in L_2(\Omega, \mathscr{F}, \mathbb{P})\} \tag{min-2}$$

wobei $\mathbb{E}[(X-G)^2] = \|X-G\|_2^2$.
From hilbert-space theory:
(min-2) possesses a unique solution $G_* \in L_2(\mathscr{F}, \mathscr{G}, \mathbb{P})$. G_* is optimization (with respect to $\langle\cdot,\cdot\rangle$) from $X \in L_2(\Omega, F, P)$ on closed subspace $L_2(\Omega, G, P)$
Image: maybe from Eric (Orthogonal projection on the subspace)
We denote the conditioned expected value $\mathbb{E}[X \mid \mathscr{G}]$ of X with recpect to $\mathscr{G}$ with G_*.

Theorem I.4
Let $X, Y \in L_2(\Omega, \mathscr{F}, \mathbb{P})$ and $\mathscr{G} \subseteq F$ sub-σ-algebra. Then it holds

1. (Linearity) $\mathbb{E}[aX + bY] = a\mathbb{E}[X \mid \mathscr{G}] + b\mathbb{E}[Y \mid \mathscr{G}]$
2. (Tower rule) For every further σ-algebra $\mathscr{H} \subseteq \mathscr{G}$ it holds
$$\mathbb{E}[E[X \mid \mathscr{G} \mid \mathscr{H}]] = \mathbb{E}[X \mid \mathscr{H}]$$
3. (Pullout-Property) $\mathbb{E}[XZ \mid \mathscr{G}] = Z\mathbb{E}[X \mid \mathscr{G}]$, if Z is bounded and $\mathscr{G}$-measurable.
4. (Monotonicity) $X \leq Y \implies \mathbb{E}[X \mid \mathscr{G}] \leq \mathbb{E}[Y \mid \mathscr{G}]$
5. (Δ-Inequality) $|\mathbb{E}[X \mid \mathscr{G}]| \leq \mathbb{E}[|X| \mid \mathscr{G}]$
6. (Independence) X independent from $G \implies \mathbb{E}[X \mid \mathscr{G}] = \mathbb{E}[X]$
7. (trivial σ-algebra) $\mathscr{G} = \{\varnothing, \Omega\} \implies \mathbb{E}[X \mid \mathscr{G}] = \mathbb{E}[X]$

Beweis. (without proof, see lecture probability theory with martingales or stochastics script SS19.) □

▶ Bemerkung

- The conditioned expectation value $\mathbb{E}[X \mid \mathscr{G}]$, which is defined for $X \in L_2(\Omega, \mathscr{F}, \mathbb{P})$, can be extended by approximation on all $X \in L_1(\Omega, \mathscr{F}, \mathbb{P})$. All properties from Theorem **???** remain the same!
- Let Y be a random variable and $\mathscr{G} = \sigma(Y)$ the σ-algebra which is generated by Y. We write:
$$\mathbb{E}[X \mid Y] = \mathbb{E}[X \mid \sigma(Y)] \quad \sigma\text{-measurable random variables}$$
- Measure theory: Doob-Dynkin-Lemma $\implies$ $\exists$ measurable function $g : \mathbb{R}^n \to \mathbb{R}$ such that
$$\mathbb{E}[X \mid Y] = g(Y)$$
Where g is exactly the function from (min-1).

Summary:
Let X, Y from $L_1(\Omega, \mathscr{F}, \mathbb{R})$, $\mathscr{G} \subseteq \mathscr{F}$ sub-σ-algebra

1. $\mathbb{E}[X \mid Y = y]$ is a measurable function $g : \mathbb{R}^n \to \mathbb{R}^n$. If the conditioned density exists, then it holds:

$$\mathbb{E}[X \mid Y = y] = \int_{\mathbb{R}^m} f_{X|Y}(x, y)\,\mathrm{d}x$$

2. $\mathbb{E}[X \mid Y]$ is a $\sigma(y)$-measurable random variable, this can be represented as $g(Y)$. If the conditioned density exists, then it holds

$$\mathbb{E}[X \mid Y](\omega) = \int_{\mathbb{R}^n} x f_{X|Y}(x, Y(\omega))\,\mathrm{d}x$$

3. $\mathbb{E}[X \mid \mathscr{G}]$ is a $\mathscr{G}$-measurable random variable. If $\mathscr{G} = \sigma(y)$ then 2) occurs.

In the general case $\mathbb{E}[\bar{X} \mid \cdot]$ can be interpreted as best forecast for X, given

1. Pointwise observation $Y = y$
2. Observation Y
3. Information $\mathscr{G}$

3.3. Martingale

Prototype of a "neutral" stochastic process, which has neither upward nor downward trend. Here only in discrete time $Z = \mathbb{N}_0$.

Definition (Martingale without a percolation)
Let $(X_n)_{n\in\mathbb{N}_0}$ be a stochastic process. If it holds

1. $\mathbb{E}[|X_n|] < \infty\ \forall n \in \mathbb{N}$
2. $\mathbb{E}[X_{n+1}, \ldots, X_n] = X_n\ \forall n \in \mathbb{N}$

then (X_n) is called a martingale. If we define $\mathscr{F}_n^* = \sigma(X_1, \ldots, X_n)$, then we can write 2) as

$$\mathbb{E}[X_{n+1} \mid \mathscr{F}_n^*] = X_n \quad \forall n \in \mathbb{N}$$

Interpretation:

- The best forecast for a future value X_{n+1}, based on the past $\sigma(X_1, \ldots, X_n)$ is the current value X_n.
- From the tower tule it follows

  $$\mathbb{E}[X_{n+k} \mid \mathscr{F}_n^*] = X_n \quad n, k \in \mathbb{N}_0$$

  since

  $$\mathbb{E}[X_{n+k} \mid \mathscr{F}_n^*] = \mathbb{E}[\mathbb{E}[X_{n+k} \mid \mathscr{F}_{n+k-1} \mid \mathscr{F}_n^*]] = \mathbb{E}[X_{n+k-1} \mid \mathscr{F}_n^*] = (k\text{-mal}) = X_n$$

It can be extended from $(\mathscr{F}_n)_{n\in\mathbb{N}}$ to arbitrary percolations $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$.

Definition (Martingale with percolation)
Let $(X_n)_{n\in\mathbb{N}_0}$ be a stochastic process, which is adapted to a percolation $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$. If it holds

1. $\mathbb{E}[|X_n|] < \infty \ \forall n \in \mathbb{N}_0$
2. $\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = X_n \ \forall n \in \mathbb{N}_0$

then $(X_n)_{n\in\mathbb{N}_0}$ is called a martingal with respect to percolation $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$

Interpretation:
The best forecast for future values X_{n+1}, based on the available information $\mathscr{F}_n$ is the current value X_n.

Definition (Supermatringale, Submartingale)
In in 2) instead of "=" the inequality $\leq$ oder $\geq$ holds, then $(X_n)_{n\in\mathbb{N}}$ is called a Supermartingale or a Submartingale.

First observation:

- X Martingale $\implies \mathbb{E}[X_n] = X_0$, i.e. $n \mapsto \mathbb{E}[X_n]$ is constant.
  Begründung:

  $$\mathbb{E}[X_{n+1} \mid \mathscr{F}_n] = X_n \implies \mathbb{E}[\mathbb{E}[X_{n+1} \mid \mathscr{F}_n]] = \mathbb{E}[X_n] = \mathbb{E}[X_{n+1}] \implies (n\text{-times aplied}\mathbb{E}[X_n] = X_0)$$

  Image: expected value is constant, but not a martingale.
- X Submartingale $\implies n \mapsto \mathbb{E}[X_n]$ is monotone increasing
- X Supermartingale $\implies n \mapsto \mathbb{E}[X_n]$ is monotone decreasing

In order to remember the difference between super and submartingale, here's a little help:
"Life is a supermartingale, expectations fall with time."

■ **Beispiel**

- Let $(Y_n)_{n\in\mathbb{N}}$ be independent random vriables in $L_1(\Omega, \mathscr{F}, \mathbb{P})$ mit $\mathbb{E}[Y_n] = 0$. Define $X_n := \sum_{k=1}^n Y_k$ mit $X_0 = 0$. Then $(X_n)_{n\in\mathbb{N}_0}$ is a martingale, since
  1. $\mathbb{E}[|X_n|] \leq \sum_{k=1}^n \mathbb{E}[|Y_k|] < \infty \quad \forall n \in \mathbb{N}$ ✓
  2. $$\begin{aligned}\mathbb{E}[X_{n+1} \mid \mathscr{F}_n^*] &= \mathbb{E}[Y_{n+1} + X_n \mid \mathscr{F}_n^*] \\ &= \mathbb{E}[Y_{n+1} \mid \mathscr{F}_n^*] = \mathbb{E}[X_n \mid \mathscr{F}_n^*] \quad (\text{ tower und } \mathscr{F}_n^*\text{-measurable}) \\ &= \underbrace{\mathbb{E}[Y_{n+1}]}_{=0} + X_n = X_n ✓\end{aligned}$$
- Further examples are to be found on the first exercise sheet!

Definition (predictable)
Let $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$ be a percolation. A stochastic process $(X_n)_{n\in\mathbb{N}}$ is called predictable with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}_0}$, if it holds:

$$H_n \text{ is } \mathscr{F}_{n-1}\text{-measurable} \quad \forall n \in \mathbb{N}$$

▶ **Bemerkung**
Stronger property than "adapted".

Definition (discrete stochastic integral)
Let X be adapted and H a predictable stochastical process with respect to $(\mathscr{F}_n)_{n\in\mathbb{N}}$. Then

$$(H \cdot X)_n := \sum_{k=1}^{n} H_k(X_k - X_{k-1}) \qquad (*)$$

is called a discrete stochastic integral of H with respect to X.

▶ **Bemerkung**
Sums $(*)$ are in the analysis called Riemann-Stieltjes-summs. They are used for constructions of the RS-integral $\int h \, d\rho$.

Definition (locally bounded)
A stochastic process $(H_n)_{n\in\mathbb{N}}$ is called locally bounded, if there exists a (defined) sequence $c_{\in}\mathbb{R}_{\geq 0}$ such that

$$|H_n| \leq c_n \text{ a.s.} \quad \forall n \in \mathbb{N}$$

Theorem I.5
Let X be adapted stochastic process (with respect to percolation $(\mathscr{F}_n)_{n\in\mathbb{N}}$). Then the following statements are equivalent:

1. X is a martingale
2. $(H \cdot X)$ is a martingale for all locally bounded, predictable $(H_n)_{n\in N}$

That means: the stochastic integral obrains the martingale-property.

Beweis. 8.11.2019! □

▶ **Bemerkung**
The random variable H is later going to be the investment strategy.

Anhang

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