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Diploma Thesis

Evaluating Expansion Methods for Realized Moments in Heston Model Simulations

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0.1 The First 4 Moments

0.1.1 Moments and central moments

For a random variable X, the expectation, also referred to as the first moment, is given by

$$\mu = \mathbb{E}(X)$$

This expectation is estimated by the sample mean of the observed values x:

$$\hat{\mu} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Variance serves as a measure of the dispersion of the random variable X. In the special case where $\mu = 0$, the variance simplifies to

$$\sigma^2 = \mathbb{E}(X^2)$$

and is also referred to as the second moment. If $\mu \neq 0$, the variance is defined as

$$\begin{aligned} \sigma^2 &= \mathbb{E}((X-\mu)^2) \\ &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \end{aligned} \tag{0.1.1}$$

which is also known as the centered second moment. The variance is estimated using

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

where the denominator $n-1$ represents Bessel's correction, which improves the estimation of variance Radziwill, 2017. Analogously, the r-th moment is given by

$$\mathbb{E}(X^r)$$

and the corresponding centered r-th moment is

$$\mathbb{E}((X-\mu)^r)$$

Skewness measures the asymmetry of a distribution and is defined as the standardized third moment:

$$\begin{aligned} \gamma_1 &= \frac{\mathbb{E}((X-\mu)^3)}{\sigma^3} \\ &= \frac{\mathbb{E}(X^3) - 3\mathbb{E}(X)\mathbb{E}(X^2) + 2\mathbb{E}(X)^3}{\sigma^3} \end{aligned} \tag{0.1.2}$$

Different methods exist for estimating skewness, such as those proposed by Joanes and Gill, 1998:

$$
\begin{aligned}
b_1 &= \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^3}{\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{3/2}} \\
g_1 &= \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^3}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{3/2}} \\
G_1 &= \frac{n^2}{(n-1)(n-2)}b_1 = \frac{\sqrt{n(n-1)}}{n-2}g_1\hat{\gamma}_1 \qquad = \frac{n}{(n-1)(n-2)}\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\hat{\sigma}}\right)^3
\end{aligned}
$$

where b_1 and g_1 are estimators of the population skewness, while G_1 and $\hat{\gamma}_1$ estimate the skewness of a sample. The estimator G_1 is implemented in statistical software such as Excel, SAS, and SPSS Doane and Seward, 2011. Kurtosis measures the tailedness of a distribution and is defined as the standardized fourth moment:

$$
\begin{aligned}
\gamma_2 = \frac{\mathbb{E}((X-\mu)^4)}{\sigma^4} \\
= \frac{\mathbb{E}(X^4) - 4\mathbb{E}(X^3)\mathbb{E}(X) + 6\mathbb{E}(X^2)\mathbb{E}(X)^2 - 3\mathbb{E}(X)^4}{\sigma^4}
\end{aligned} \tag{0.1.3}
$$

Various estimation methods for kurtosis exist, such as those presented by Joanes and Gill, 1998:

$$
\begin{aligned}
g_2 &= \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^4}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{2}} \\
\hat{\gamma}_2 &= \frac{n(n+1)}{(n-1)(n-2)(n-3)}\sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\hat{\sigma}}\right)^4
\end{aligned}
$$

where g_2 estimates the kurtosis of a population and G_2 estimates the kurtosis of a sample. A commonly used alternative is excess kurtosis, which is obtained by subtracting 3:

$$
\begin{aligned}
\gamma_2^* &= \gamma_2 - 3 \\
G_2 &= \frac{n-1}{(n-2)(n-3)}[(n+1)g_2 + 6]
\end{aligned}
$$

This adjustment is motivated by the fact that for a standard normally distributed variable X, the kurtosis is $\gamma_2 = 3$ and the excess kurtosis is $\gamma_2^* = 0$. In general, due to the high powers involved in the definitions of skewness and kurtosis, these estimators are highly sensitive to outliers. In the following, the term moments will be used broadly to include related measures such as variance, skewness, and kurtosis.

0.1.2 Cumulants

Throughout this work, the concept of cumulants will also be used, which provide an alternative representation of moments. The r-th cumulant is defined as the coefficient of t^r in the logarithm of the moment generating function of X. The moment generating function of X is given by

$$M_X(t) = \mathbb{E}(e^{tX})$$

The cumulant generating function of X is then defined as

$$K_X(t) = \log(M_X(t))$$

From this definition, the first four cumulants are given by

$$\kappa_1 = \mu \tag{0.1.4}$$
$$\kappa_2 = \sigma^2 \tag{0.1.5}$$
$$\kappa_3 = \gamma_1 \sigma^3 \tag{0.1.6}$$
$$\kappa_4 = \gamma_2^* \sigma^4 \tag{0.1.7}$$

0.1.3 Estimating the Moments of Low-Frequency Data using High-Frequency Data

For the pricing of financial derivatives, it is crucial to know the moments of returns, particularly those of monthly or quarterly returns Barro, 2006. However, estimating the moments of such low-frequency returns can be challenging due to the limited number of observations available Neuberger and Payne, 2021. Today, financial markets operate continuously, making it possible to obtain daily or even minute-level returns without difficulty. For example, the German stock index DAX is calculated every second Frankfurt, n.d. There are several approaches to estimating the moments of monthly or quarterly returns based on the moments of daily returns. One such method is proposed by Amaya et al., 2015. In this approach, the i-th intraday log return $r_{t,i}$ for day t is computed as

$$r_{t,i} = \log(P_{t,i/N}) - \log(P_{t,(i-1)/N})$$

where p represents the natural logarithm of the price, and N denotes the number of return observations within a trading day. The opening log-price on day t is given by $p_{t,0}$, while the closing log-price is $p_{t,1}$. Using five-minute returns, a standard 6.5-hour trading session results in $N = 78$. Based on this, the daily realized variance is computed as

$$\hat{\sigma}_t^2 = \sum_{i=1}^{N} r_{t,i}^2$$

This idea is not new and was first introduced by Andersen and Bollerslev, 1998. Building upon this approach, the daily realized skewness and kurtosis can be computed as

$$\hat{\gamma}_1 = \frac{\sqrt{N} \cdot \sum_{i=1}^{N} r_{t,i}^3}{\hat{\sigma}_t^3}$$
$$\hat{\gamma}_2 = \frac{N \cdot \sum_{i=1}^{N} r_{t,i}^4}{\hat{\sigma}_t^4}$$

To transition from daily realized moments to weekly or monthly moments, a moving average approach is applied. Choe and Lee, 2014 use variation processes to estimate low-frequency moments, specifically the quadratic variation of a semimartingale X:

$$[X]_t = X_t^2 - 2\int_0^t X_u \,\mathrm{d}X_u \tag{0.1.8}$$

and the quadratic covariation process of two semimartingales X and Y:

$$[X, Y]_t = X_t Y_t - \int_0^t X_u \,\mathrm{d}Y_u - \int_0^t Y_u \,\mathrm{d}X_u \tag{0.1.9}$$

For the log-return process R_t,

$$R_t = \log(P_t) - \log(P_0)$$

equations (0.1.8) and (0.1.9) can be approximated as follows:

$$[R]_T \approx \sum_{i=1}^{N} (R_i - R_{i-1})^2$$
$$[R, R^2]_T \approx \sum_{i=1}^{N} (R_i - R_{i-1})(R_i^2 - R_{i-1}^2)$$
$$[R^2]_T \approx \sum_{i=1}^{N} (R_i^2 - R_{i-1}^2)^2$$

From these, the moments follow:

$$\mathbb{E}(R_T^3) = \frac{3}{2}\mathbb{E}([R, R^2]_T)$$
$$\mathbb{E}(R_T^4) = \frac{3}{2}\mathbb{E}([R^2]_T)$$

The estimation of low-frequency variance follows the same approach as Andersen and Bollerslev, 1998 and Amaya et al., 2015. Neuberger and Payne, 2021 propose a novel method that only requires log prices to be martingales and stationary, meaning there is no drift. Under these conditions, they define new moment measures that approximate the standard definitions:

$$\begin{aligned}
var^L(r) &= \mathbb{E}(x^{(2L)}(r)), \text{ where } x^{(2L)}(r) = 2(e^r - 1 - r) \\
var^E(r) &= \mathbb{E}(x^{(2E)}(r)), \text{ where } x^{(2E)}(r) = 2(re^r - e^r + 1) \\
skew(r) &= \frac{\mathbb{E}(x^{(3)}(r))}{var^L(r)^{3/2}}, \text{ where } x^{(3)}(r) = 6((e^r + 1)r - 2(e^r - 1)) \\
kurt(r) &= \frac{\mathbb{E}(x^{(4)}(r))}{var^L(r)^2}, \text{ where } x^{(4)}(r) = 12(r^2 + 2(e^r + 2)r - 6(e^r - 1))
\end{aligned}$$

where r represents the log-return process:

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$$

The long-horizon returns process R is defined as

$$R_t(T) = \ln\left(\frac{P_t}{P_{t-T}}\right)$$

This establishes a connection between low-frequency moments and high-frequency log returns:

$$\begin{aligned}
var^L(R(T)) &= T \cdot var^L(r) \\
skew(R(T)) &= \left(skew(r) + 3\frac{cov(y^{(1)}, x^{(2E)}(r))}{var^L(r)^{3/2}}\right) T^{-1/2} \\
kurt(R(T)) &= \left(kurt(r) + 4\frac{cov(y^{(1)}, x^{(3)}(r))}{var^L(r)^2} + 6\frac{cov(y^{(2L)}, x^{(2L)}(r))}{var^L(r)^2}\right) T^{-1}
\end{aligned}$$

where

$$\begin{aligned}
y_t^{(j)} &= \sum_{u=1}^{T} \frac{x^{(j)}(R_{t-1}(u))}{T} \quad \text{for } j = 1, 2L \\
x^{(1)} &= e^r - 1
\end{aligned} \tag{0.1.10}$$

where equation (0.1.10) originates from Neuberger, 2012. In Neuberger, 2012, the aggregation property is also introduced: if g is a real function, X is a process, and for times $0 \le s \le t \le u \le T$,

$$\mathbb{E}_s(g(X_u - X_s)) = \mathbb{E}_s(g(X_u - X_t)) + \mathbb{E}_t(g(X_t - X_s))$$

then the pair (g, X) satisfies the aggregation property. This property is used, for instance, in estimating low-frequency variance by summing high-frequency variance. Fukasawa and Matsushita, 2021 build on this idea and derive formulas for realized cumulants that also satisfy the aggregation property.

0.1.4 The First 4 Moments of the Heston Model

Care must be taken with notation, as symbols such as μ and σ previously denoted the mean and variance but are now used to represent the drift and volatility of the Heston model (see Equations (**??**) and (**??**)). The noncentral moments are denoted by μ_1 through μ_4, while the central and standardized moments are denoted by ζ_1 through ζ_4.

Fortunately, moments for returns and other related measures have been derived. Okhrin et al., 2022 provide the unconditional noncentral moments for the log-return $r_t = \log(S_t) - \log(S_0)$:

$$\mu_1 = \left(\mu - \frac{\theta}{2}\right) t$$
$$\mu_2 = \frac{1}{4\kappa^3}\left(\exp(-\kappa t)\left[\exp(\kappa t)\left\{\kappa^3 t(t(\theta - 2\mu)^2 + 4\theta) - 4\kappa^2\rho\sigma t\theta + \kappa\sigma\theta(4\rho + \sigma t) - \sigma^2\theta\right\} + \sigma\theta(\sigma - 4\kappa\rho)\right]\right)$$

The expressions for μ_3 and μ_4 are too lengthy to be included here but can be found in Okhrin et al., 2022. The corresponding unconditional central and standardized moments are then given by:

$$\zeta_1 = \mu_1$$
$$\zeta_2 = \mathbb{E}\left[(r_t - \mu_1)^2\right]$$
$$= \frac{\theta[-4\kappa^2\rho\sigma t + 4\kappa^3 t + \sigma\exp(-\kappa t)(\sigma - 4\kappa\rho) + 4\kappa\sigma\rho + \kappa\sigma^2 t - \sigma^2]}{4\kappa^3}$$
$$\zeta_3 = \mathbb{E}\left[\left(\frac{r_t - \mu_1}{\zeta_1^{1/2}}\right)^3\right]$$
$$= \frac{3\kappa\sigma\theta\exp(\kappa t/2)(\sigma - 2\kappa\rho)}{\zeta_2^{3/2}}[4\kappa^2\{\exp(\kappa t)(\rho\sigma t + 1) + \rho\sigma t - 1\} - 4\kappa^3 t\exp(\kappa t) - \kappa\sigma\{\exp(\kappa t)(8\rho + \sigma t) - 8\rho$$
$$\zeta_4 = \mathbb{E}\left[\left(\frac{r_t - \mu_1}{\zeta_1^{1/2}}\right)^4\right]$$

The expression for ζ_4 is omitted here but can be found in Okhrin et al., 2022.

Zhao et al., 2013 and Zhang et al., 2017 analyze moments for the continuously compounded return $R_t^T = \ln\left(\frac{S_T}{S_t}\right)$ and derive the conditional central variance:

$$\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^2 = \frac{1}{4}\mathrm{Var}_t(\int_t^T v_s \,\mathrm{d}s) + SW_{t,T} - \mathbb{E}_t(\int_t^T \sqrt{v_s}\,\mathrm{d}B_s(\int_t^T v_s \,\mathrm{d}s - \mathbb{E}_t(\int_t^T v_s \,\mathrm{d}s))) \tag{0.1.11}$$

where v_t represents the variance at time t, SW is the variance swap rate (the expectation of realized variance), and B_t is a Brownian motion. Zhang et al., 2017 further elaborate:

$$\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^2 = \int_t^T \mathbb{E}_t(v_s)\,\mathrm{d}s - \rho\sigma \int_t^T \frac{1-\exp(-\kappa(T-s))}{\kappa}\mathbb{E}_t(v_s)\,\mathrm{d}s + \frac{1}{4}\left(\sigma^2 \int_t^T \frac{(1-\exp(-\kappa(T-s}{\kappa^2}\right.$$

Using the expected instantaneous variance $\mathbb{E}_t(v_s) = \theta + (v_t - \theta)\exp(-\kappa(s-t))$, Mathematica yields the following results:

$$\int_t^T \mathbb{E}_t(V_s)\,\mathrm{d}s = \frac{v_t - \theta + \exp(\kappa(t-T))(-v_t+\theta) - t\theta\kappa + T\theta\kappa}{\kappa}$$

$$\rho\sigma \int_t^T \frac{1-\exp(-\kappa(T-s))}{\kappa}\mathbb{E}_t(v_s)\,\mathrm{d}s = \frac{\exp(-T\kappa)[\exp(t\kappa)(-v_t+2\theta+(t-T)(v_t-\theta)\kappa)+\exp(T\kappa)(v_t -}{\kappa^2}$$

$$\sigma^2 \int_t^T \frac{(1-\exp(-\kappa(T-s)))^2}{\kappa^2}\mathbb{E}_t(v_s)\,\mathrm{d}s = \frac{\exp(-2T\kappa)(\exp(2t\kappa)(-2v_t+\theta)+4\exp((t+T)\kappa)(\theta+(t-T)(}{2\kappa^3}$$

The third conditional central moment is given by

$$\mathbb{E}_t(R_t^T - \mathbb{E}_t(R_t^T))^3 = \mathbb{E}_t(X_T^3) - \frac{3}{2}\mathbb{E}_t(X_T^2 Y_T) + \frac{3}{4}\mathbb{E}_t(X_T Y_T^2) - \frac{1}{8}\mathbb{E}_t(Y_T^3)$$

where X_T and Y_T are defined as

$$X_T = \int_t^T \sqrt{v_s}\,\mathrm{d}B_s^S$$

$$Y_T = \int_t^T (v_s - \mathbb{E}_t(v_s))\,\mathrm{d}s = \sigma \int_t^T \frac{1-\exp(-\kappa(T-s))}{\kappa}\sqrt{v_s}\,\mathrm{d}B_s^v$$

where B^S and B^v are the Brownian motions associated with price and volatility, respectively.

Dunn et al., 2014 derive the unconditional noncentral moments for the return $Q_{t+1} = \frac{S_{t+1}}{S_t}$:

$$\mathbb{E}(Q_{t+1}) = \mu_1 = 1 + \mu$$

$$\mathbb{E}(Q_{t+1}^2) = \mu_2 = (\mu+1)^2 + \theta$$

$$\mathbb{E}(Q_{t+1}^3) = \mu_3 = (\mu+1)^3 + 3\theta + 3\mu\theta$$

$$\mathbb{E}(Q_{t+1}^4) = \mu_4 = \frac{1}{\kappa(\kappa-2)}(\kappa^2\mu^4 + 4\kappa^2\mu^3 + 6\kappa^2\mu^2\theta - 2\kappa\mu^4 + 6\kappa^2\mu^2 + 12\kappa^2\mu\theta + 3\kappa^2\theta^2 - 8\kappa\mu^3 - 12\kappa\mu^2\theta +$$

Using Equations (0.1.1), (0.1.2), and (0.1.3), the central and standardized moments follow as

$$\begin{aligned}
\zeta_1 &= 1 + \mu \\
\zeta_2 &= \theta \\
\zeta_3 &= 0 \\
\zeta_4 &= 3\frac{\kappa^2\theta - 2\kappa\theta - \sigma^2}{\kappa\theta(\kappa - 2)}
\end{aligned}$$

Bibliography

Amaya, D., Christoffersen, P., Jacobs, K., & Vasquez, A. (2015). Does realized skewness predict the cross-section of equity returns? *Journal of Financial Economics*, *118*(1), 135–167. https://doi.org/10.1016/j.jfineco.2015.02.009

Andersen, T. G., & Bollerslev, T. (1998). Answering the Skeptics: Yes, Standard Volatility Models do Provide Accurate Forecasts. *International Economic Review*, *39*(4), 885–905. https://doi.org/10.2307/2527343

Barro, R. J. (2006). Rare Disasters and Asset Markets in the Twentieth Century*. *The Quarterly Journal of Economics*, *121*(3), 823–866. https://doi.org/10.1162/qjec.121.3.823

Choe, G. H., & Lee, K. (2014). High Moment Variations and Their Application. *Journal of Futures Markets*, *34*(11), 1040–1061. https://doi.org/10.1002/fut.21635

Doane, D. P., & Seward, L. E. (2011). Measuring Skewness: A Forgotten Statistic? *Journal of Statistics Education*, *19*(2). https://doi.org/10.1080/10691898.2011.11889611

Dunn, R., Hauser, P., Seibold, T., & Gong, H. (2014). Estimating Option Prices with Heston ' s Stochastic Volatility Model.

Frankfurt, B. (n.d.). So funktioniert die Börse.

Fukasawa, M., & Matsushita, K. (2021). Realized cumulants for martingales. *Electronic Communications in Probability*, *26*(none). https://doi.org/10.1214/21-ECP382

Joanes, D. N., & Gill, C. A. (1998). Comparing measures of sample skewness and kurtosis. *Journal of the Royal Statistical Society: Series D (The Statistician)*, *47*(1), 183–189. https://doi.org/10.1111/1467-9884.00122

Neuberger, A. (2012). Realized Skewness. *Review of Financial Studies*, *25*(11), 3423–3455. https://doi.org/10.1093/rfs/hhs101

Neuberger, A., & Payne, R. (2021). The Skewness of the Stock Market over Long Horizons (S. Van Nieuwerburgh, Ed.). *The Review of Financial Studies*, *34*(3), 1572–1616. https://doi.org/10.1093/rfs/hhaa048

Okhrin, O., Rockinger, M., & Schmid, M. (2022). Simulating the Cox–Ingersoll–Ross and Heston processes: Matching the first four moments. *Journal of Computational Finance*. https://doi.org/10.21314/JCF.2022.022

Radziwill, N. M. (2017). *Statistics (the Easier Way) with R*. Lapis Lucera.

Zhang, J. E., Zhen, F., Sun, X., & Zhao, H. (2017). The Skewness Implied in the Heston Model and Its Application. *Journal of Futures Markets*, *37*(3), 211–237. https://doi.org/10.1002/fut.21801

Zhao, H., Zhang, J. E., & Chang, E. C. (2013). The Relation between Physical and Risk-neutral Cumulants. *International Review of Finance*, *13*(3), 345–381. https://doi.org/10.1111/irfi.12013

Declaration of independence

I hereby declare that this thesis was written independently and without the use of any other resources than those stated. Any ideas taken literally or analogously from other sources are identified as such. I further declare that I have not submitted or will not submit this thesis as an examination paper to any other institution.

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