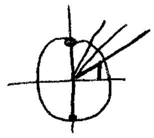
AM I. Informática

Teste 1 - 27.nov.2013

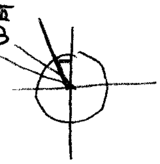
(1)

a) $f\left(\frac{4\pi}{3}\right) = \operatorname{sen}\left(\frac{\pi}{6}\right) + \cos\left(\frac{8\pi}{3}\right) = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0$



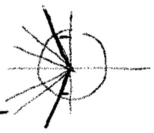
$\frac{6\pi}{3} \leq \frac{8\pi}{3} \leq \frac{9\pi}{3}$

$\hookrightarrow \frac{8\pi}{3} = 3\pi - \frac{\pi}{3}$



$\frac{8\pi}{3}$

b) $f(x) = 0 \iff \operatorname{sen}\left(\frac{\pi}{6}\right) + \cos(2x) = 0$



$\iff \cos(2x) = -\frac{1}{2}$

$\iff 2x = \frac{2\pi}{3} + k\cdot 2\pi \ \vee\ 2x = -\frac{2\pi}{3} + k\cdot 2\pi$

$\iff x = \frac{\pi}{3} + k\cdot\pi \ \vee\ x = -\frac{\pi}{3} + k\pi,\ k \in \mathbb{Z}$

c) $f\left(\frac{\pi}{3}\right) = f\left(-\frac{\pi}{3}\right) = 0$ pelo que f não é injectiva.

A restrição de injectividade de f é definida pela restrição principal do cosseno.

$0 \leq 2x \leq \pi \longrightarrow 0 \leq x \leq \frac{\pi}{2}$

$$[0, \pi/2] \xrightarrow{\ f\ }$$

$$x \longmapsto y = \frac{1}{2} + \cos(2x)$$

d)

$-1 \leq \cos(2x) \leq 1$

$-\frac{1}{2} \leq \frac{1}{2} + \cos(2x) \leq 3/2$

$CD_f = \left[-\frac{1}{2}, \frac{3}{2}\right]$

$f^{-1}(y) = ?$ $\quad y = \frac{1}{2} + \cos(2x) \Rightarrow y - \frac{1}{2} = \cos(2x)$

$\Rightarrow \arccos\left(y - \frac{1}{2}\right) = 2x$

$\Rightarrow x = \frac{1}{2}\arccos\left(y - \frac{1}{2}\right)$

$$\left[0, \frac{\pi}{2}\right] \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \left[-\frac{1}{2}, \frac{3}{2}\right]$$

$$\frac{1}{2}\arccos\left(y - \frac{1}{2}\right) = x \longleftrightarrow y = \frac{1}{2} + \cos(2x)$$

2

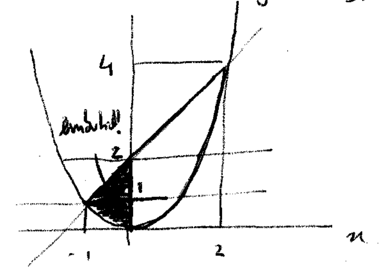
a) $\begin{cases} y = x^2 \\ y = x+2 \end{cases} \Rightarrow x^2 = x+2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x = \dfrac{1 \pm \sqrt{1+8}}{2} \Rightarrow x = \dfrac{1 \pm 3}{2}$

$\Rightarrow x = 2 \vee x = -1$

$A = \{ (x,y) \in \mathbb{R}^2 : -1 \leq x \leq 2 \wedge x^2 \leq y \leq x+2 \}$

b) i) Área $|A| = \int_{-1}^{2} (x+2) - (x^2) \, dx$

ii) $y = x^2 \Rightarrow x = +\sqrt{y}$

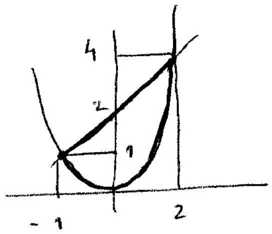


$y = x+2$

$\Rightarrow x = y-2$

$$V(A_{oy}) = \pi \int_0^2 (\sqrt{y})^2 - (0)^2 \, dy + \pi \int_2^4 (\sqrt{y})^2 - (y-2)^2 \, dy$$

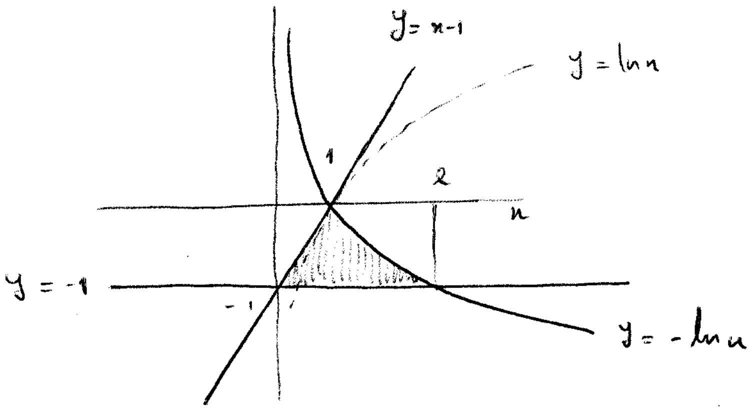
iii)



$$\text{Perímetro} = \sqrt{3^2 + 3^2} + \int_{-1}^{2} \sqrt{1 + [(x^2)']^2} \, dx$$
$$= \sqrt{18} + \int_{-1}^{2} \sqrt{1 + (2x)^2} \, dx$$
$$= \sqrt{18} + \int_{-1}^{2} \sqrt{1 + 4x^2} \, dx$$

3

a)

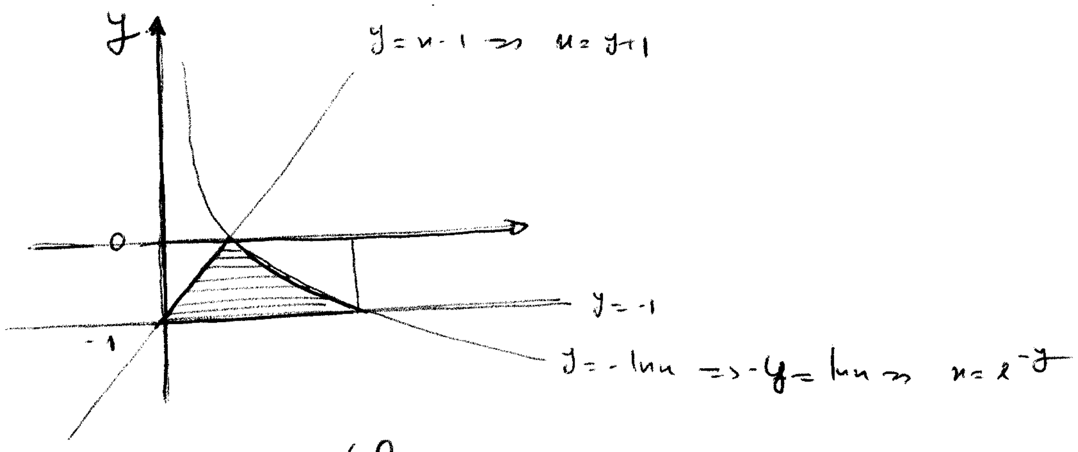


$\begin{cases} y = -1 \\ y = -\ln x \end{cases} \Rightarrow -\ln x = -1 \Rightarrow x = e^1$

b) $$V(B_{ox}) = \pi \int_0^1 (-1)^2 - (x-1)^2 \, dx + \pi \int_1^e (-1)^2 - (-\ln x)^2 \, dx$$

$$= \pi \int_0^1 1 - (x-1)^2 \, dx + \pi \int_1^e 1 + \ln^2 x \, dx$$

c)



$$\text{Área}(B) = \int_{-1}^{0} e^{-y} - (y+1)\, dy$$

4)

a) $\int f(x)\,dx = F(x) \iff F'(x) = f(x)$

$\Rightarrow \int \ln x \, dx = x(\ln x - 1) + c \iff \left(x(\ln x - 1) + c\right)' = \ln x$

$$\begin{aligned}\left(x(\ln x - 1) + c\right)' &= x' \cdot (\ln x - 1) + x \cdot (\ln x - 1)' \\ &= 1 \cdot (\ln x - 1) + x\left(\tfrac{1}{x} - 0\right) \\ &= \ln x - 1 + 1 \\ &= \ln x\end{aligned}$$

b) $Df = \{x \in \mathbb{R} : x > 0\} =]0, +\infty[$

$f(x) = \ln x$ é contínua em $\mathbb{R}^+$

i) $\int_0^1 \ln x \, dx$: $]0,1[\subset Df$ mas $\lim_{x \to 0^+} \ln x = -\infty$

pelo que o integral é impróprio de 2ª espécie

$\int_1^3 \ln x \, dx$: $[1,3] \subset Df$ e f é contínua em Df

pelo que o integral é definido

$\int_1^{+\infty} \ln x \, dx$: $[1, +\infty[\subset Df$ mas o intervalo de integração é <u>ilimitado</u>

pelo que o integral é impróprio de 1ª espécie

ii) $\int_1^{+\infty} \ln x \, dx = \lim_{B \to +\infty} \int_1^B \ln x \, dx = \lim_{B \to +\infty} \left[x(\ln x - 1)\right]_1^B$

$= \lim_{B \to +\infty} \left(B(\ln B - 1) - (-1)\right)$

$= +\infty \Rightarrow$ o integral é divergente.

⑤

a) $D_f = \{x \in \mathbb{R} : x \geq 0 \wedge \sqrt{x}\,(x+1) \neq 0\}$

$= \{x \in \mathbb{R} : x \geq 0 \wedge x \neq 0 \wedge x \neq -1\}$

$=]0, +\infty[$

$f(x)$ é contínua em D_f pois é o quociente de funções contínuas (radical e polinomial)

b) i) $\int_1^3 f(x)\,dx$ p.f. $[1,3] \subset D_f$

ii) $\int_0^1 f(x)\,dx$ p.s. $\lim_{x \to 0^+} \frac{1}{\sqrt{x}\,(x+1)} = +\infty$

iii) $\int_1^{+\infty} f(x)\,dx$ p.s. o intervalo de integração é ilimitado.

⑥

a) $\sqrt{1-y^2}\,dx - x\,dy = 0 \Leftrightarrow \sqrt{1-y^2} - x\frac{dy}{dx} = 0$

$\Leftrightarrow -x\frac{dy}{dx} = -\sqrt{1-y^2}$

$\Leftrightarrow \frac{dy}{dx} = \sqrt{1-y^2} \cdot \frac{1}{x}$ E.D. de var. separáveis

$\Longleftrightarrow \int \frac{1}{\sqrt{1-y^2}}\,dy = \int \frac{1}{x}\,dx$

$\Rightarrow \arcsin y = \ln|x| + c, \; c \in \mathbb{R}$

$\Rightarrow y = \sin(\ln|x| + c), \; c \in \mathbb{R}$

b) m.v: $\boxed{v = y \cdot x} \longrightarrow v' = y'x + y$

$v + \sqrt{x^2 - v^2} = x\,v' \xrightarrow[m.v]{} y \cdot x + \sqrt{x^2 - x^2y^2} = x \cdot (y'x + y)$

$\Rightarrow yx + \sqrt{x^2(1-y^2)} = x(y'x + y)$

$\Rightarrow yx + \underbrace{|x|}_{=x}\sqrt{1-y^2} = x(y'x + y)$

$\Rightarrow \cancel{y} + \sqrt{1-y^2} = y'x + \cancel{y}$

c.q.d.

⑦

a) $y' - 3y = e^t$

b) F.I: $e^{\int -3\,dt} = e^{-3t + \cancel{c}}$

$\xrightarrow[\times F.I]{} e^{-3t}y' - 3e^{-3t}y = e^{-3t}e^t \Rightarrow (e^{-3t} \cdot y)' = e^{-2t} \Rightarrow e^{-3t} \cdot y = -\frac{1}{2}\int -2e^{-2t}$

$\Rightarrow e^{-3t} \cdot y = -\frac{1}{2}e^{-2t} + c$

$\Rightarrow y = -\frac{1}{2}e^t + ce^{3t}, \; c \in \mathbb{R}.$