MA133
Differential Equations
Lecture Notes

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Introduction

"Science is a differential equation. Religion is a boundary condition."
— Alan Turing (1912–1954)

How do you reconstruct a curve given its slope at every point? Can you predict the trajectory of a tennis ball? The basic theory of ordinary differential equations (ODEs) as covered in this module is the cornerstone of all applied mathematics. Indeed, modern applied mathematics essentially began when Newton developed the calculus in order to solve (and to state precisely) the differential equations that followed from his laws of motion.

However, this theory is not only of interest to the applied mathematician: indeed, it is an integral part of any rigorous mathematical training, and is developed here in a systematic way. Just as a 'pure' subject like group theory can be part of the daily armory of the 'applied' mathematician, so ideas from the theory of ODEs prove invaluable in various branches of pure mathematics, such as geometry and topology.

In this module, we will cover only relatively simple examples, first order equations ($\frac{dy}{dx} = f(x,y)$), linear second order equations ($\ddot{x} + p(t)\dot{x} + q(t)x = g(t)$) and coupled first order linear systems with constant coefficients, for most of which we can find an explicit solution. However, even when we can write the solution down it is important to understand what the solution means, i.e. its 'qualitative' properties. This approach is invaluable for equations for which we cannot find an explicit solution.

We also show how the techniques we learned for second order differential equations have natural analogues that can be used to solve difference equations. The course looks at solutions to differential equations in the cases where we are concerned with one- and two-dimensional systems, where the increase in complexity will be followed during the lectures. At the end of the module, in preparation for more advanced modules in this subject, we will discuss why in three dimensions we see new phenomena, and have a first glimpse of chaotic solutions.

The primary text will be J. C. Robinson, *An Introduction to Ordinary Differential Equations*, Cambridge University Press 2003, available from the bookshop or the library. This is invaluable for reference and for the large numbers of examples and exercises to be found within.

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CHAPTER 0

Definitions

Variables

Variables measure things. We have *independent variables* and *dependent variables.* Dependent variables are unknown quantities, for example:

- distance as a function of time – $x(t)$
  – here x is the dependent variable, while t is independent
- velocity as a function of time and position (in 2D) – $v(x, y, t)$
  – this has one dependent variable (v) and three independent variables (x, y, and t).

Dependent variables can (usually) be differentiated with respect to the independent variable(s).

Notation

When it's obvious what we are differentiating with respect to, we use

$$\frac{dy}{dx} = y'.$$

When differentiating with respect to time, we use the following notation:

$$\frac{dx}{dt} = \dot{x}, \quad \frac{d^2x}{dt^2} = \ddot{x}, \text{ etc.}$$

We also use the following:

$$y^{(k)}(x) = \frac{d^k y}{dx^k}$$

Definition 0.1. *A* differential equation *is a function of variables and their derivatives.*

For example:

$$\ddot{x} + a\dot{x} + x(x^2 - 1) = b\cos\omega t \tag{1}$$

is a differential equation known as the *Forced Duffing's Equation.*

If the equation has only one independent variable, it is called an *ordinary differential equation* or *ODE*. If we have a function of several variables, for example $v(x, y, t)$, then we can differentiate with respect to x getting $\frac{\partial v}{\partial x}$, or y getting $\frac{\partial v}{\partial y}$, or t getting $\frac{\partial v}{\partial t}$; in this case the equation is known as a *partial differential equation*. For now we are interested in ODEs, and for the first half of this module, we will only have one dependent variable – "one-dimensional problems".

Order

Definition 0.2. *Assume that we have an ODE that can be written*

$$F(t, x(t), \dot{x}(t), \ldots, x^{(k)}(t)) = 0$$

where $x^{(k)} = \frac{d^k x}{dt^k}$ *and* x *is a function of* t. *We define the* order *of that ODE as the order of its highest derivative, i.e.* k.

Thus the order of Forced Duffing's Equation (equation (1)) is order 2 due to the $\ddot{x}$ term. As a further example, the equation $y''' + 2e^x y'' + yy' = x^4$ is order 3 due to the y''' term.

Definition 0.3. *If the independent variable does not appear explicitly in the ODE, e.g.*

$$F(y, \dot{y}) = 0$$

then it is called autonomous.

For example the equation $y''' + 2e^y y'' + yy' = 0$ is autonomous.

Definition 0.4. *Assuming an ODE can be written as*

$$y^{(n)} = f(t, y, \dot{y}, \ddot{y}, \ldots, y^{(n-1)})$$

then a solution *to this ODE is a function* $\phi(t)$ *so that*

$$\phi^{(n)}(t) = f(t, \phi(t), \ldots, \phi^{(n-1)}(t))$$

on some interval $\alpha \leq t \leq \beta$ *or* $t \in \mathbb{R}$.

Such a solution may not exist, but there are conditions that guarantee existence and uniqueness of solutions. We will discuss this later, but for now assume that solutions to the examples in this module exist and are unique.

Linearity

Definition 0.5. *An* n^{th} *order ODE that can be written in the form*

$$a_n(t)\frac{d^n y}{dt^n} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t) \tag{2}$$

is called linear.

e.g.

$$\frac{d^2 y}{dt^2} + t\frac{dy}{dt} + e^{-t}y = \cos kt$$

Forced Duffing's Equation (equation (1)) is *nonlinear* because of the $x(x^2 - 1)$ term. Furthermore, if, in equation (2), $f(t) = 0$, then it is called a linear *homogeneous* ODE.

Examples

Example 0.6. In Forced Duffing's Equation,

$$\ddot{x} + a\dot{x} + x(x^2 - 1) = b\cos\omega t$$

the independent variable is t and the dependent variable is x. We describe it as a second-order, nonlinear, inhomogeneous, nonautonomous ODE.

Example 0.7. The wave equation

$$\frac{\partial^2 v}{\partial t^2} = c\frac{\partial^2 v}{\partial x^2}$$

used to model a vibrating string, where v is displacement, x is position along the string, and t is time:

- the dependent variable is v;
- the independent variables are x and t;
- it is described as a second-order, linear, homogeneous, PDE (partial differential equation).

Example 0.8. The logistic equation

$$\dot{x} = rx\left(1 - \frac{x}{k}\right)$$

models populations.

- the dependent variable is x;
- the independent variable is t;
- it is a first order, nonlinear, homogeneous, autonomous ODE.

Example 0.9. The Cauchy-Euler equations take the form

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

This is a second-order, linear, homogeneous ODE, with dependent variable y and independent variable x.

CHAPTER 1

First-Order ODEs

1.1 Trivial First-Order ODEs

We start with the easiest case, ODEs of the form

$$\frac{dx}{dt} = f(t) \tag{1.1a}$$

$$\text{or} \quad \frac{dy}{dx} = g(x) \tag{1.1b}$$

Here x is a function of t, and $f(t)$ is what we get when we differentiate x, or y is a function of x and $g(x)$ is what we get when we differentiate y.

Definition 1.1. *A function F that satisfies $F'(t) = f(t)$ is called an* anti-derivative *of $f(t)$.*

Theorem 1.2 (The Fundamental Theorem of Calculus)**.**
Suppose $f : [a, b] \mapsto \mathbb{R}$ is continuous *and for $a \leq x \leq b$ define*

$$G(x) = \int_a^x f(\tilde{x})d\tilde{x}$$

Then

$$\frac{dG}{dx}(x) = f(x)$$

Furthermore,

$$\int_a^b f(x) = F(b) - F(a)$$

for any F with $F'(x) = f(x)$ (i.e. F is the anti-derivative of f).

What theorem 1.2 basically says is that there is a link between differentiation and integration. So if

$$\frac{dy}{dx} = f(x) \tag{1.2}$$

the solution will satisfy

$$\text{“}y(x) = \int f(x)dx\text{”}.$$

Choose a particular anti-derivative $F(x)$ of $f(x)$, then $y(x) = F(x)$ is a solution, but so is

$$y(x) = F(x) + c \tag{1.3}$$

So equation (1.3) is the *general solution* of equation (1.2). We can specify a *particular solution* by saying what $F(x)$ is at a particular value of x (this sort of problem is called an *initial value problem*).

Example 1.3. Let $x(t)$ be position along the M6, measured from Carlisle. Assume a car is driving south along the M6 at constant speed, then

$$\frac{dx}{dt} = a$$

taking the positive direction as south. The solution to this ODE is simply

$$x = at + c$$

where at gives how far the car has travelled, and c represents where the car started.

Newton's Second Law of Motion

Definition 1.4. *The change Δp in the momentum p of an object is equal to F, the force applied, multiplied by Δt, the time over which the force is applied, i.e.*

$$\Delta p = F\Delta t.$$

From this we obtain a differential equation by dividing both sides by Δt and letting Δt tend to zero $\boxed{\text{ANALYSIS II}}$:

$$\lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t} = \frac{dp}{dt} = F(t)$$

Since momentum = mass × velocity, if mass is constant then

$$\frac{dp}{dt} = \frac{d}{dt}(mv) = m\frac{dv}{dt} = F(t)$$

or "force = mass × acceleration".

Example 1.5. A car of mass m is travelling at a speed v_0 when it has to brake. The brakes apply a constant force k until the car stops. How long does it take to stop, and how far does it travel before stopping?

Newton's Second Law tells us that $m\frac{dv}{dt} = -k$ or

$$\frac{dv}{dt} = -\frac{k}{m}.$$

Note that the force is negative since it acts in the opposite direction to the direction of travel. We integrate this to give

$$v(t) = -\frac{kt}{m} + c$$

When $t = 0$, we know that $v(0) = v_0$, so $c = v_0$, giving

$$v(t) = -\frac{kt}{m} + v_0$$

We want to find the time it stops at, call it t'. We know $v(t') = 0$, so $-\frac{kt'}{m} + v_0 = 0$, or

$$t' = \frac{mv_0}{k}$$

Now velocity is the rate of change of distance, i.e.

$$\frac{dx}{dt} = -\frac{kt}{m} + v_0$$

Integrate (by theorem 1.2) to give

$$x(t) = -\frac{kt^2}{2m} + v_0 t + A$$

When $t = 0$, we want $x(0) = 0$, so $x(0) = A$, so $A = 0$. So we have

$$x(t) = -\frac{kt^2}{2m} + v_0 t.$$

To find how far the car travels before coming to a stop, we substitute $t = t'$, i.e. $t = \frac{mv_0}{k}$ into our displacement equation, so

$$\begin{aligned} x\left(\frac{mv_0}{k}\right) &= -\frac{k}{2m}\left(\frac{mv_0}{k}\right)^2 + v_0\frac{mv_0}{k} \\ &= -\frac{mv_0^2}{2k} + \frac{mv_0^2}{k} \\ \text{i.e.} \qquad x(t') &= \frac{mv_0^2}{2k} \end{aligned}$$

Note that we may not always be able to solve these equations, e.g.

$$\frac{dy}{dx} = e^{-x^2}. \tag{1.4}$$

Here we cannot integrate e^{-x^2} over any finite interval. However, it will be shown in GEOMETRY AND MOTION that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

(This integral also has many applications in PROBABILITY and the normal distribution.) This means can write the solution as

$$y(x) = y(0) + \int_0^x e^{-t^2}\,dt$$

and while we cannot solve this we do know what happens as $x \to \infty$. We will discuss qualitative approaches to ODEs later in the module.

1.2 Existence and Uniqueness: A Warning!

We discuss briefly existence and uniqueness of solutions, then we will assume that we always have "nice equations" where solutions exist and are unique.

We consider ODEs of the general form

$$\frac{dx}{dt} = f(x,t) \quad : \quad x(0) = x_0. \tag{1.5}$$

Equations with no solutions (non-existence)

$$x^2 + t^2 \frac{dx}{dt} = 0 \quad : \quad x(0) = c : c \neq 0$$

When $t = 0$, the equation must satisfy $x^2 = 0$, so $x(0) = 0$. But this is impossible given $x(0) \neq 0$.

Equations with lots of solutions (non-uniqueness)

Consider

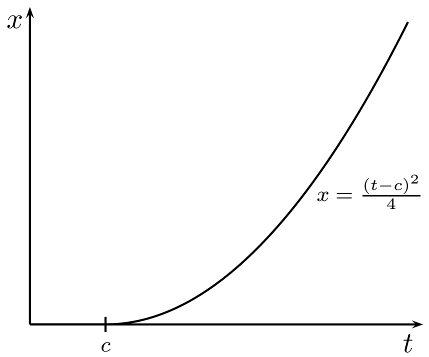
$$\frac{dx}{dt} = \sqrt{x} \quad : \quad x(0) = 0 \tag{1.6}$$

$x(t) = 0$ is an obvious solution. But the function:

$$x(t) = \begin{cases} 0 & \text{if } t \leq c \\ \dfrac{(t-c)^2}{4} & \text{if } t > c \end{cases} \quad : c > 0$$

also satisfies 1.6 since

$$\frac{d}{dt}\left(\frac{(t-c)^2}{4}\right) = \frac{t-c}{2} = \sqrt{\frac{(t-c)^2}{4}} = \sqrt{x}$$



So there is a different solution for every positive value of c.

Note that at university such things are indeed functions: each point $t \geq 0$ is assigned a value $x(t)$. You will come across much weirder functions than this, especially in ANALYSIS III.

Before we state the existence and uniqueness theorem we need some background.

Partial Derivatives

For a function of two or more variables $f(x, y)$, it doesn't make sense to talk about its derivative, as you don't know whether to use x or y. So we call them *partial derivatives* – we specify which variable to use in the differentiation, and treat the other(s) as a constant, e.g.

$$\begin{aligned} f(x,y) &= x^2 \sin y \\ \frac{\partial f}{\partial x} &= 2x \sin y \qquad \text{(treating } y \text{ as a constant)} \\ \frac{\partial f}{\partial y} &= x^2 \cos y \qquad \text{(treating } x \text{ as a constant)} \end{aligned}$$

Note the use of the ∂ symbol instead of a d – this signifies a partial derivative.

Continuous Functions

For the purposes of this course, for "a function $f(x)$ is *continuous*" read "you can draw a graph of $y = f(x)$ against x without taking your pen off the paper". So, for example, x^2, e^{-x} and $\sin x$ are continuous, but $\tan x$ is not, though it is continuous on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. For reference, the formal definition of continuous which you will learn in $\boxed{\text{ANALYSIS II}}$ is as follows:

Definition 1.6. *f(x) is* continuous *if* $\forall\, \epsilon > 0, \exists\, \delta > 0$ *such that*

$$|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$$

The proof of the general existence and uniqueness theorem is beyond the scope of this course, and we will just state the result. But in order to state the theorem properly, we need to have a more precise definition of a 'solution' to an initial value problem:

Definition 1.7 (Solution)**.** *Given an open interval, I, that contains t_0, a* solution *of the* initial value problem *(IVP)*

$$\frac{dx}{dt}(t) = f(x,t) \quad : \quad x(t_0) = x_0 \tag{1.7}$$

on I is a continuous function $x(t)$ with $x(t_0) = x_0$ and

$$\dot{x}(t) = f(x,t) \qquad \forall\, t \in I$$

Essentially this means that a solution does not have to be defined for every value of $t \in \mathbb{R}$, only some interval thereof. (Intervals will be formally defined in $\boxed{\text{ANALYSIS}}$.) But, given this formal definition of a solution, we can state the existence and uniqueness theorem.

Theorem 1.8 (Theorem of Existence and Uniqueness)**.** *If $f(x,t)$ and $\frac{\partial f}{\partial x}(x,t)$ are continuous for $a < x < b$ and for $c < t < d$, then for any $x_0 \in (a,b)$ and $t_0 \in (c,d)$, the initial value problem (1.7) has a unique solution on some open interval I containing t_0.*

This theorem will be proved in $\boxed{\text{ANALYSIS III}}$.

Coming back to our example in equation (1.6), this fails the test of theorem 1.8 because

$$\frac{\partial f}{\partial x} = \tfrac{1}{2}x^{-\frac{1}{2}}$$

This isn't even defined at $x = 0$, so $\frac{\partial f}{\partial x}$ isn't continuous.

1.3 First-Order Linear Homogeneous ODEs with constant coefficients

In this section we will consider equations of the form

$$\frac{dx}{dt} + px = 0 \tag{1.8a}$$

$$\text{or} \qquad \frac{dx}{dt} = -px \tag{1.8b}$$

where p is a constant.

Note that a solution is

$$x(t) = Ae^{-pt}$$

$$\text{since} \qquad \frac{d}{dt}\left[Ae^{-pt}\right] = -Ape^{-pt} = -px$$

Setting $x(0) = x_0$ shows that we must have $A = x_0$ and thus

$$x(t) = x_0 e^{-pt} \tag{1.9}$$

So by uniqueness (theorem 1.8) this is the *only* solution.

Example 1.9. Equations of the form of equation 1.8 arise frequently when looking at half-life decay of radioactive isotopes. In particular ^{14}C exists in all living matter. When a plant or animal dies the amount of radioactive carbon decreases due to decomposition into nitrogen (^{14}N). The rate of decomposition is proportional to the amount of ^{14}C:

$$\frac{dx}{dt} = -kx$$

where $x(t)$ is the amount of ^{14}C (number of atoms).

For ^{14}C, $k \approx 1.216 \times 10^4\,\text{year}^{-1}$. If $N(t)$ is the number of ^{14}C atoms at time t then

$$N(t) = N_0\, e^{-kt}$$

The half-life is the time taken for half of the atoms to have decomposed, i.e. $t_{\frac{1}{2}}$ such that

$$\begin{aligned} N_0\, e^{-kt_{\frac{1}{2}}} &= \frac{N_0}{2} \\ -kt_{\frac{1}{2}} &= -\log 2 \\ t_{\frac{1}{2}} &= \frac{\log 2}{k} \\ t_{\frac{1}{2}} &\approx 5700\,\text{years} \end{aligned}$$

By looking at the amount of stable ^{12}C compared to the amount of radioactive ^{14}C, it can be worked out what proportion of the ^{14}C remains and hence when the sample stopped living.

For example, in carbon extracted from living tissue, there are approximately 6×10^{10} atoms of ^{14}C per gram of carbon. A wood beam from an Egyptian tomb was found to have 3.33×10^{10} atoms of ^{14}C per gram of carbon. Approximately how old is the tomb?

$$x(t) = 6 \times 10^{10}\, e^{-t(1.216 \times 10^4)}$$

We want t such that $x = 3.33 \times 10^{10}$, so:

$$\begin{aligned} 3.33 \times 10^{10} &= 6 \times 10^{10}\, e^{-t(1.216 \times 10^4)} \\ \implies \qquad t &\approx 4800\,\text{years}. \end{aligned}$$

1.4 First-Order Linear Homogeneous ODEs with non-constant coefficients

In this section we will look at equations of the more general form

$$\frac{dx}{dt} + r(t)x = 0 \tag{1.10}$$

Appealing to the constant coefficient case, let us try a solution of the form:

$$\begin{aligned} x(t) &= Ae^{-R(t)} \\ \frac{dx}{dt} &= -A\frac{dR}{dt}e^{-R(t)} \\ \text{so} \qquad \frac{dx}{dt} &= -\frac{dR}{dt}x(t) \end{aligned}$$

So this is a solution if

$$\frac{dR}{dt} = r(t)$$

i.e. if $R(t)$ is an antiderivative of $r(t)$, i.e.

$$\text{“}R(t) = \int r(t)dt\text{”}.$$

Hence the general solution to equations of the form of equation 1.10 is

$$x(t) = Ae^{-\int r(t)dt}. \tag{1.11}$$

Example 1.10. Find a solution to $\dfrac{dy}{dt} + 2ty = 0$, given that $y(10) = 3$.

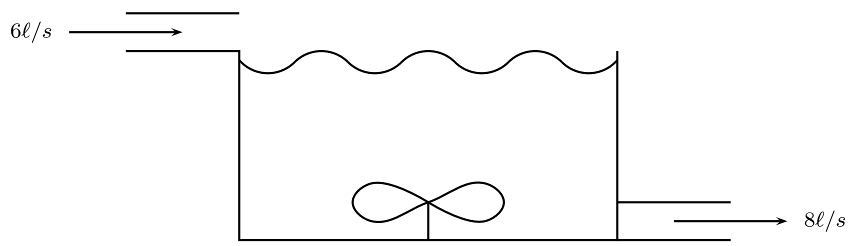
The general solution, from equation 1.11 is

$$\begin{aligned} y(t) &= Ae^{-\int 2tdt} \\ \text{i.e.} \qquad y(t) &= Ae^{-t^2} \end{aligned}$$

When $t = 10$, $y(10) = 3$; substitute these in:

$$\begin{aligned} 3 &= Ae^{-100} \\ \therefore A &= 3e^{100} \\ \therefore y(t) &= 3e^{100-t^2} \end{aligned}$$

Example 1.11 (A Simple Mixing Problem)**.** A 1000 litre tank contains a mixture of water and chlorine. To reduce the amount of chlorine fresh water is pumped into the pool at a rate of $6\ell/s$, the fluid is well-stirred and pumped out at a rate of $8\ell/s$. If the initial concentration of chlorine is $0.02g/\ell$ find the amount of chlorine in the tank as a function of time and the interval of validity of the model



Solution. Let $Q(t)$ denote amount of chlorine and $V(t)$ denote the volume of fluid in the tank. Then:

- $\dfrac{dQ}{dt}$ measures rate of change of chlorine; and
- $8\dfrac{Q(t)}{V(t)}$ is the rate fluid is pumped out times the concentration of chlorine, i.e. the outflow of chlorine.

So $\dfrac{dQ}{dt}$ and $8\dfrac{Q(t)}{V(t)}$ must balance, i.e. $\dfrac{dQ}{dt} + 8\dfrac{Q(t)}{V(t)} = 0$. Since $V(t) = 1000 - 2t$, we have

$$\frac{dQ}{dt} + \frac{8Q(t)}{1000 - 2t} = 0$$

This is of the form in equation 1.10 with $x = Q$ and $r(t) = \frac{8}{1000-2t}$. Integrating $r(t)$ gives

$$R(t) = -4\log(500 - t)$$

so

$$\begin{aligned}
Q(t) &= ce^{-R(t)} = ce^{4\log(500-t)} \\
Q(t) &= c(500 - t)^4 \\
Q(0) = 20 \qquad &\Longrightarrow \qquad 20 = c \times 500^4 \\
\text{so} \qquad Q(t) &= 20\left(\frac{500 - t}{500}\right)^4
\end{aligned}$$

For practical reasons, this is only valid until $t = 500$.

1.5 First-Order Inhomogeneous Linear ODEs

We now turn our attention to equations of the form

$$\frac{dy}{dt} + r(t)y = g(t) \tag{1.12}$$

We have already looked at the cases where $r(t) \equiv 0$ and $g(t) \equiv 0$. We will initially consider equations where $r(t)$ is simply a constant p, i.e. equations of the form

$$\frac{dy}{dt} + py = g(t) \tag{1.13}$$

We note that (clever!)

$$\begin{aligned}
\frac{d}{dt}\left(e^{pt}y(t)\right) &= pe^{pt}y(t) + e^{pt}\frac{dy}{dt} \\
&= e^{pt}\left(\frac{dy}{dt} + py\right) \\
&= e^{pt} \times \text{LHS of equation (1.13)}
\end{aligned}$$

So now multiply both sides of equation (1.13) by e^{pt} to get

$$e^{pt}\left(\frac{dy}{dt} + py\right) = \frac{d}{dt}\left(e^{pt}y(t)\right) = e^{pt}g(t)$$

Integrate both sides with respect to t to get

$$e^{pt}y(t) = \int e^{pt}g(t)dt + A$$

Finally multiply both sides by e^{-pt}:

$$y(t) = \underbrace{e^{-pt}\int_t e^{ps}g(s)ds}_{y_p(t)} + \underbrace{Ae^{-pt}}_{y_h(t)} \tag{1.14}$$

The antiderivative of $e^{pt}g(t)$ may be difficult (or even impossible) to find, but we still have an expression for the solution. The function e^{pt} here is called the *integrating factor.*

Note that $y(t) = y_p(t) + y_h(t)$ where

- $y_p(t)$ is called the *particular solution* to equation (1.13), and
- $y_h(t)$ is called the *complementary solution* that solves the homogeneous case

$$\frac{dy}{dt} + py = 0.$$

Example 1.12. Find the general solution to

$$\frac{dy}{dt} + y = 3t - 5$$

Here $p = 1$, so the integrating factor is e^t, so

$$\frac{d}{dt}\left(e^t y\right) = e^t(3t - 5)$$

An antiderivative of $e^t(3t - 5)$ is just $3te^t - 8e^t$, therefore

$$e^t y = 3te^t - 8e^t + A$$

Multiplying through by e^{-t} gives the solution

$$y = 3t - 8 + Ae^{-t}$$

Example 1.13. Newton's Law of Cooling

> "A body changes temperature at a rate proportional to the difference between the body's temperature and that of the surrounding air (the ambient temperature)."

i.e.

$$\frac{dT}{dt} = -k\left(T - A(t)\right)$$

Here $k > 0$. The minus sign is so that if the ambient temperature is greater than T, then $T - A(t)$ is negative and so the temperature T increases, i.e. $\frac{dT}{dt} > 0$. Assuming $A(t)$ is a constant, we can write this as

$$\frac{dT}{dt} + kt = kA$$

By inspection, the integrating factor is e^{kt} so

$$\frac{d}{dt}\left(e^{kt}T\right) = kAe^{kt}$$

Integrating both sides and multiplying by e^{-kt} giving

$$T(t) = A + ce^{-kt}$$

We now follow Robinson p. 80 onwards.

Let's assume we find a dead body in a room with constant temperature 24°C. At 8am, the body's temperature is 28°C and an hour later it is 26°C. Assuming normal body temperature is 37°C, when was the person killed?

We first need to establish k, so take

$$\begin{aligned} e^{kt_1}T(t_1) &= Ae^{kt_1} + c \\ e^{kt_2}T(t_2) &= Ae^{kt_2} + c \end{aligned}$$

After some algebra, we get

$$T(t_2) = A + (T(t_1) - A)\, e^{-k(t_2-t_1)}$$

Substituting in the given value gives

$$\begin{aligned} 26 &= 24 + (28-24)e^{-k} \\ e^{-k} &= \tfrac{1}{2} \qquad \Longrightarrow \qquad k = \log 2 \end{aligned}$$

We want to find t_0 such that $T(t_0) = 37$, so we want

$$\begin{aligned} T(t_0) &= A + (T(t_1) - A)\, e^{-(\log 2)(t_0-t_1)} \\ 37 &= 24 + (28-24)e^{-(\log 2)(t_0-8)} \\ \log \tfrac{13}{4} &= -(\log 2)(t_0 - 8) \\ t_0 &= \frac{-\log \frac{13}{4}}{\log 2} + 8 \approx 6.3 \text{ or 6:20am.} \end{aligned}$$

For non-constant room-temperature, see Robinson pp. 81-86.

Non-constant coefficients

We now return to the general first-order linear inhomogeneous ODE:

$$\frac{dy}{dt} + r(t)y = g(t) \tag{1.15}$$

We try a similar "integrating factor" approach, i.e. multiply both sides by a function $I(t)$ giving

$$I(t)\frac{dy}{dt} + r(t)I(t)y = I(t)g(t)$$

As before, we want $I(t)$ such that the left-hand side becomes $\frac{d}{dt}\left(I(t)y(t)\right)$. Doing the differentiation gives

$$I(t)\frac{dy}{dt} + \frac{dI}{dt}y$$

So for this to work we need

$$\frac{dI}{dt} = r(t)I(t)$$

By inspection,

$$I(t) = e^{\int r(t)dt}$$

works. So we take

$$\frac{dy}{dt} + r(t)y = g(t)$$

Multiplying by $I(t) = e^{\int r(t)dt}$ gives

$$I(t)\frac{dy}{dt} + r(t)I(t)y = \frac{d}{dt}\left(I(t)y\right) = I(t)g(t)$$

Integrating both sides gives

$$I(t)y = \int I(t)g(t)dt + A$$

Multiplying by $\frac{1}{I(t)} = e^{-\int r(t)dt}$ gives the general solution

$$y = \underbrace{e^{-\int r(t)dt}\int e^{\int r(t)dt}g(t)dt}_{y_p(t)} + \underbrace{Ae^{-\int r(t)dt}}_{y_h(t)} \tag{1.16}$$

since $e^{\int r(t)dt} \neq 0$. And we have a solution, explicit if we can solve the integration. Again $I(t) = e^{\int r(t)dt}$ is called an *integrating factor*.

Note again the general solution consists of:

- a *particular solution*, $y_p(t)$, which is particular to $g(t)$, and
- a *complementary solution*, $y_h(t)$, which is the solution to the homogeneous case

$$\frac{dy}{dt} + r(t)y = 0.$$

Example 1.14. Find the general solution to the equation:

$$\frac{dx}{dt} + \frac{2x}{t} = 8t - 6 \qquad t > 0$$

Here $r(t) = \frac{2}{t}$ so the integrating factor is

$$e^{\int \frac{2}{t}dt} = e^{\log(t^2)} = t^2$$

$$\implies \qquad \frac{d}{dt}\left(t^2x\right) = t^2(8t-6) = 8t^3 - 6t^2$$

Integrate both sides to give

$$t^2x = 2t^4 - 2t^3 + c$$

Since $t \neq 0$, the general solution is

$$x(t) = 2(t^2 - t) + \frac{c}{t^2}$$

1.6 Separable Equations

Another important class of first-order ODEs that we can solve are of the form

$$\frac{dx}{dt} = f(x)g(t)$$

and are called *separable* since we can separate the dependent and independent variables. The idea is that you "divide both sides by $f(x)$, multiply both sides by dt to get $\frac{dx}{f(x)} = g(t)dt$, then integrate both sides to give $\int \frac{dx}{f(x)} = \int g(t)dt$".

Example 1.15. Find the general solution to

$$\frac{dy}{dt} = -e^t y^2$$

Rearranging gives

$$\frac{dy}{y^2} = -e^t dt$$

Then integrate:

$$-\int \frac{dy}{y^2} = \int e^t dt$$
$$\frac{1}{y} = e^t + A$$
$$\text{i.e.} \qquad y(t) = \frac{1}{e^t + A}$$

with the restriction that $e^t + A \neq 0$

Note that if $y = 0$ then $\frac{dy}{dt} = 0$ so y is a constant, so $y \equiv 0$ is also a solution.

Justification

Consider again

$$\frac{dx}{dt} = f(x)g(t) \tag{1.17}$$

where f is sufficiently "nice".

Note that if $x(t)$ is a solution to (1.17) with $f(x(s)) = 0$ for some s, then $x(t) = x(s) \,\forall\, t \in \mathbb{R}$, since if $f(x(s)) = 0$ then $\frac{dx}{dt} = 0$ so x is constant.

So assume that $x(t)$ is a non-constant solution of (1.17) and so $f(x(t))$ is non-zero on an interval I. Now divide both sides of (1.17) by $f(x(t))$:

$$\frac{1}{f(x(t))}\frac{d}{dt}x(t) = g(t)$$

Now note that if $G(t)$ is an antiderivative of $g(t)$ and $F(x)$ is an antiderivative of $\frac{1}{f(x)}$ then we can integrate both sides with respect to t to give

$$F(x(t)) = G(t) + A \tag{1.18}$$

because

$$\frac{d}{dt}\left(F(x(t)\right) = \frac{dF}{dx}\frac{dx}{dt} = \frac{1}{f(x)}\frac{d}{dt}x(t)$$

Note that (1.18) does not give an explicit solution for $x(t)$, but rather a *functional relationship* between $x(t)$ and t, also known as an *implicit solution*. We would like to rearrange this to get an *explicit solution*, i.e. $x(t) = \dots$, but unfortunately this is not always possible.

Example 1.16. Solve the initial value problem

$$\frac{dy}{dt} = 3t^2 e^{-y} \qquad y(0) = 1$$

Since e^{-y} is never zero, we have no constant solutions.

$$e^{y(t)}\frac{d}{dt}y(t) = 3t^2$$

Integrating gives the implicit solution

$$e^{y(t)} = t^3 + c$$

Take logs of both sides gives the explicit solution

$$y(t) = \log(t^3 + c)$$

We want $y(0) = 1$:

$$y(0) = \log c = 1$$

$$\text{so} \qquad c = e$$

Therefore the solution is

$$y(t) = \log(t^3 + e)$$

For this to be valid, we require that $t^3 + e > 0$, so the solution is valid provided

$$t > -e^{\frac{1}{3}} \approx -1.396.$$

Population Dynamics

The equation

$$\frac{dN}{dT} = rN$$

can be thought of as a simple population model where N is the population size. A better population model due to Verhulst is:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{1.19}$$

where r is the birth/death rate, and K limits the growth – $\left(1 - \frac{N}{K}\right)$ becomes small if N gets too large.

This equation is separable:

$$\frac{K}{N(K-N)} dN = r\,dt$$

$$\left[\frac{1}{N} + \frac{1}{K-N}\right] dN = r\,dt$$

where we have used partial fractions. Integrating gives

$$\log|N| - \log|K - N| = rt + A$$

After a little light algebra:

$$N(t) = \frac{cKe^{rt}}{1 + ce^{rt}}.$$

If $N(0) = N_0$ then $N_0 = \frac{cK}{1+c}$, giving $c = \frac{N_0}{K-N_0}$, so the solution is:

$$N(t) = \frac{N_0 K e^{rt}}{K - N_0 + N_0 e^{rt}}$$

Note that there are also two constant solutions. Since $\frac{dN}{dt} = 0$ when $N = 0$ or $N = K$, $N(t) \equiv 0$ and $N(t) \equiv K$ are also solutions.

Thinking about limits $\boxed{\text{ANALYSIS}}$ we can also see by inspection that

$$\lim_{t\to\infty} N(t) = K$$

unless $N_0 = 0$.

1.7 Substitution Methods

Note: there are things called "exact equations" – we will not cover them here, but you should be aware that they exist.

We will look at two types of equation in this section.

Type 1

Equations of the form:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \tag{1.20}$$

We use the substitution $u = \frac{y}{x}$ so $y = ux$ giving

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

so

$$x\frac{du}{dx} = F(u) - u$$

which is separable.

Example 1.17. Solve $xy + y^2 + x^2 - x^2\frac{dy}{dx} = 0$ by using the substitution $u = \frac{y}{x}$.

Solution. Equation is

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2}$$

Using $y = ux$, we get $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$, so

$$\begin{aligned} u + \frac{du}{dx} &= 1 + u + u^2 \\ \frac{du}{dx} &= 1 + u^2 \\ \frac{du}{1+u^2} &= \frac{dx}{x} \end{aligned}$$

Integrating gives

$$\begin{aligned} \arctan u &= \log|x| + c \\ \frac{y}{x} &= \tan\left(\log|x| + c\right) \\ y &= x\tan\left(\log|x| + c\right) \end{aligned}$$

Type 2

Equations of the form:

$$\frac{dy}{dx} + p(x)y = q(x)y^n \tag{1.21}$$

which are known as "Bernoulli Equations".

Note that we have already solved the cases where $n = 0, 1$. For other n, we use the substitution $u = y^{1-n}$ giving us

$$\begin{aligned}\frac{du}{dx} &= (1-n)y^{-n}\frac{dy}{dx} \\ &= (1-n)y^{-n}\left[-p(x)y + q(x)y^n\right] \\ &= (1-n)\left[-p(x)y^{1-n} + q(x)\right]\end{aligned}$$

i.e.

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x)$$

which can be solved by integrating factors.

Example 1.18. Solve $\dfrac{dy}{dx} - 6xy = 2xy^2$.

Solution. Set $u = \frac{1}{y}$ (that is, $n = 2$ in the general case). Then

$$\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} = -\frac{6x}{y} - 2x = -6ux - 2x$$

so

$$\frac{du}{dx} + 6xu = -2x.$$

The integrating factor is thus $e^{\int 6x dx} = e^{3x^2}$ so

$$\begin{aligned}\frac{d}{dx}\left(ue^{3x^2}\right) &= -2xe^{3x^2} \\ u(x)e^{3x^2} &= -\frac{1}{3}e^{3x^2} + A\end{aligned}$$

so

$$y(x) = \frac{3}{Be^{-3x^2} - 1}$$

where $B = 3A$.

If y was a function of time t, i.e. $y(t) = \frac{3}{Be^{-3t^2}-1}$, then assuming $B > 1$, as t increases from 0, $y(t)$ gets larger and larger. In fact, as $t \to \frac{1}{\sqrt{3}}\sqrt{\log B}$, $y(t) \to \infty$, i.e. the solution "becomes infinite" at $t = \frac{1}{\sqrt{3}}\sqrt{\log B}$. This is an example of *"finite time blowup"*; this behaviour is not at all obvious from the equation we started with (we essentially require $y_0 > 0$).

1.8 Direction Fields

The final two sections for first-order ODEs deal with graphical representations.

For an ODE

$$\frac{dx}{dt} = f(x,t) \tag{1.22}$$

the *"enlarged phase space"* is the space of x vs. t, in which every point in the x-t plane has a vector associated with it – each vector having gradient $f(x,t)$; sometimes (though not in these examples) the lengths of the vectors are proportional to $|f(x,t)|$. All such vectors form the *"direction field"*.

For an initial value, if we draw the solution to equation (1.22) through the initial value, it is tangential to every vector. Only one curve goes through any one point due to uniqueness (theorem 1.8). Such curves are called *"integral curves"*.

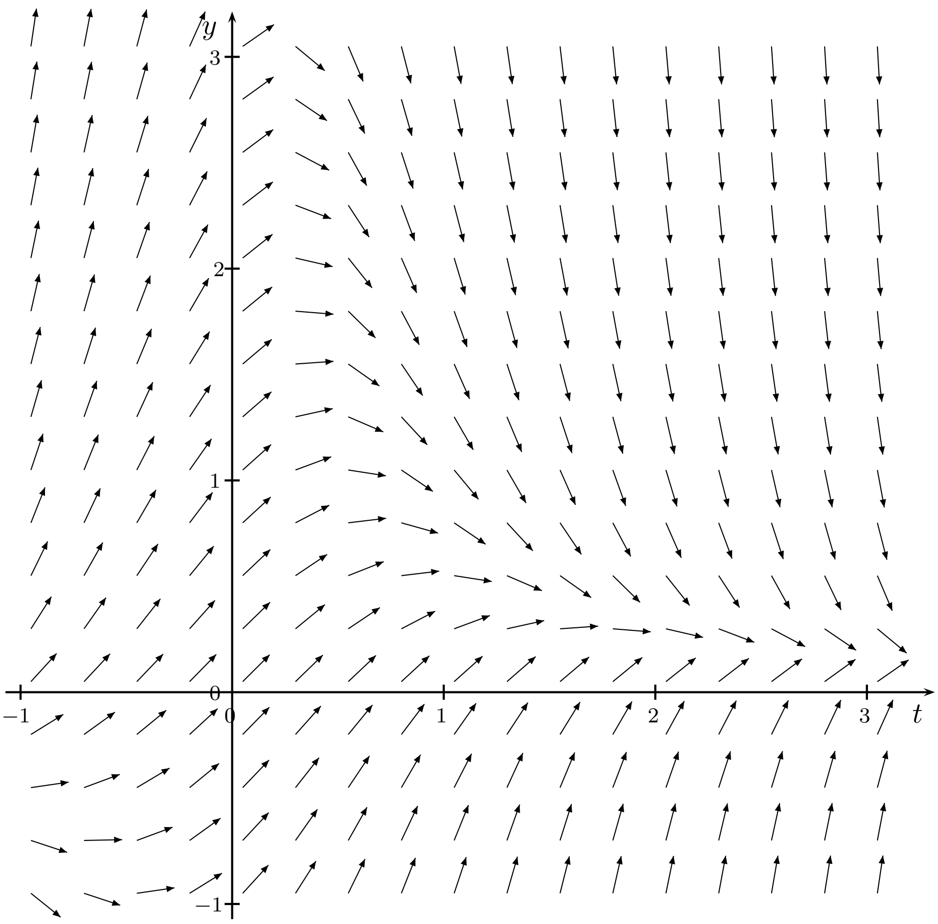


Figure 1.1: The direction field for $\frac{dy}{dt} = 1 - 2ty$.

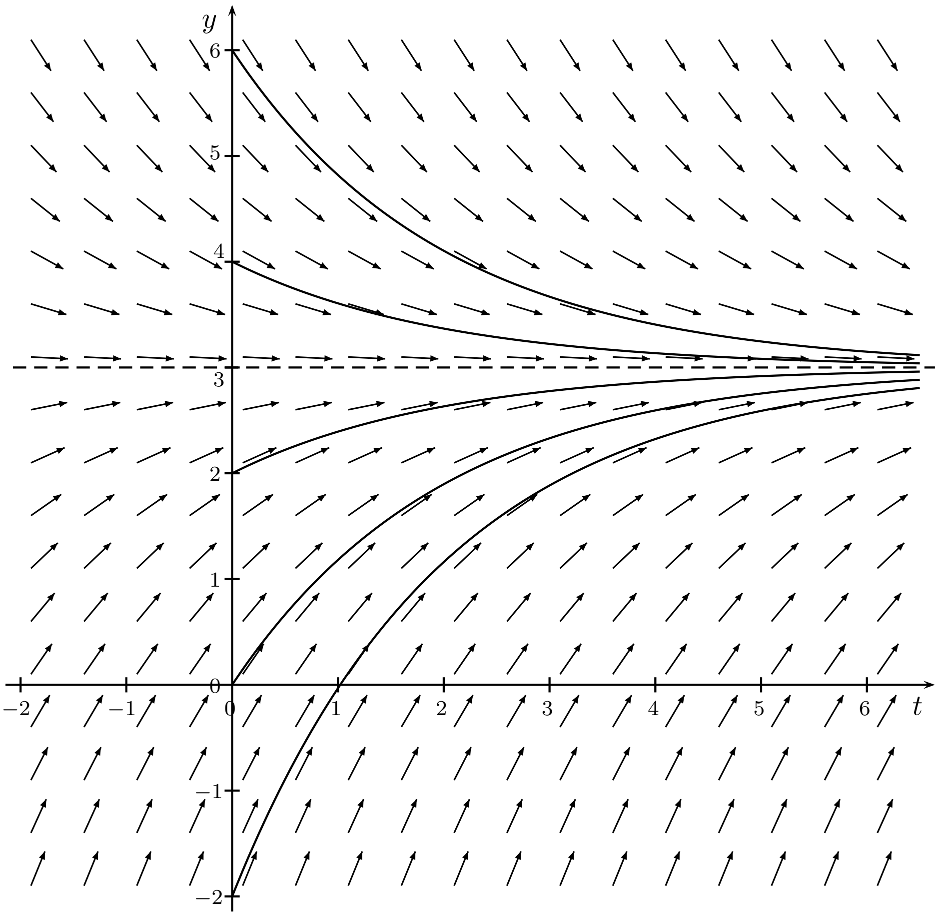


Figure 1.2: The direction field for $\frac{dy}{dt} = \frac{3-y}{2}$ with the integral curves for the initial values $y(0) = -2, 0, 2, 4, 6$. Note that whatever the initial value $y(0)$, as $t \to \infty, y(t) \to 3$.

1.9 Autonomous First-Order ODEs

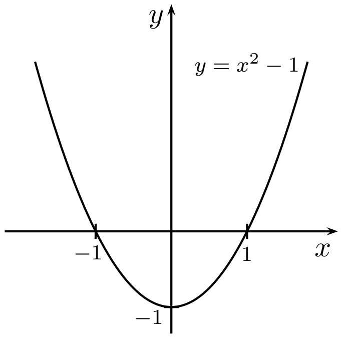
Sometimes we may not be able to find an explicit solution to an ODE, but this may not always be a huge disadvantage. One such class are autonomous ODEs, which are of the form

$$\frac{dx}{dt} = f(x) \tag{1.23}$$

Note that we put no restriction on the linearity of $f(x)$: for example, consider the equation

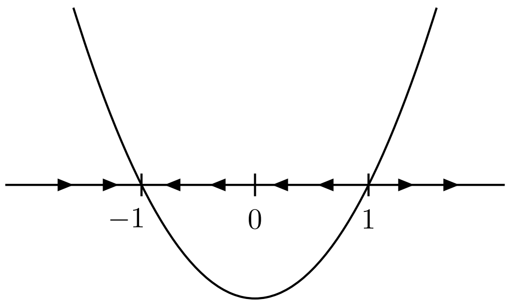
$$\frac{dx}{dt} = f(x) = x^2 - 1$$

and draw the graph of y vs. x, i.e. $y = f(x)$.



Let $x(t)$ denote the position of a particle on the x-axis at time t, then $\frac{dx}{dt}$ denotes velocity of the particle at time t.
Now look at some starting positions $x(0)$. If $x(0) = 0$ then initially $\frac{dx}{dt} = -1 < 0$ so particle moves to left. In fact, this is true for all $-1 < x < 1$.
If $x(0) > 1$ or $x(0) < -1$ then $\frac{dx}{dt} > 0$ therefore the particle moves to the right.

We represent this as follows:

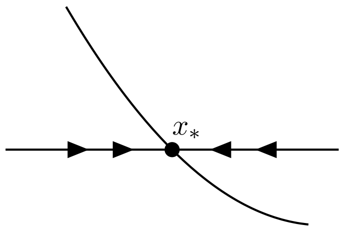


Finally we have two special points, $x(0) = \pm 1$ where $\frac{dx}{dt} = 0$ – the particle just sits there. These points are called *fixed points* (stationary points, equilibrium points).

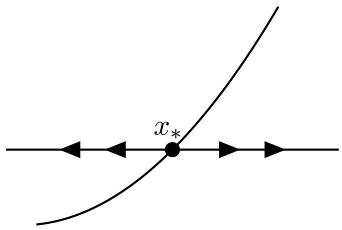
Without having solved the ODE we know the qualitative behaviour of the solutions: a particle starting between $-1 < x < 1$ will move towards -1, as will a particle starting at $x < -1$, while particles starting off with $x > 1$ will move to the right indefinitely. The behaviour of a solution as $t \to \infty$ is called the *asymptotic behaviour* (compare with the equation $\frac{dx}{dt} = e^{-t^2}$ on page 7).

Points such as $x = -1$ are called *stable fixed points* – "if you start near it, you get pulled towards it." Points such as $x = 1$ are called *unstable fixed points* – "points nearby get pushed away".

It should be clear that the stability of a fixed point depends on the gradient of the graph of f at the fixed point:



stable
$f'(x_*) < 0$



unstable
$f'(x_*) > 0$

Finally, if $f'(x) = 0$ at a F.P., the equation is called *structurally unstable* – a small change to the equation, e.g.

$$\frac{dx}{dt} = f(x) + \epsilon(x) \quad : \quad |\epsilon(x)| \ll 1$$

can make the fixed point 'stable' or 'unstable' by making $f'(x)$ non-zero at the fixed point (which may also move a little).

Justification

Consider the Taylor series expansion $\boxed{\text{ANALYSIS II}}$ of $f(x)$ near the fixed point x_0. This says that for small h

$$f(x+h) \approx f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots \tag{1.24}$$

Then

$$f(x_0+y) \approx f(x_0) + yf'(x_0) + \frac{y^2}{2!}f''(x_0) + \ldots$$

Since x_0 is a fixed point, $f(x_0) = 0$, so if y is small, i.e. $|y| \ll 1$, then

$$f(x_0+y) \approx yf'(x_0)$$

But

$$f(x_0+y) = \frac{d}{dt}(x_0+y) = \left.\frac{dx}{dt}\right|_{x_0} + \frac{dy}{dt}$$

And $f(x_0) = \frac{dx}{dt}\big|_{x_0} = 0$, so

$$\frac{dy}{dt} = yf'(x_0) \tag{1.25}$$

This is called the *linear approximation* of $\frac{dx}{dt} = f(x)$ at x_0.

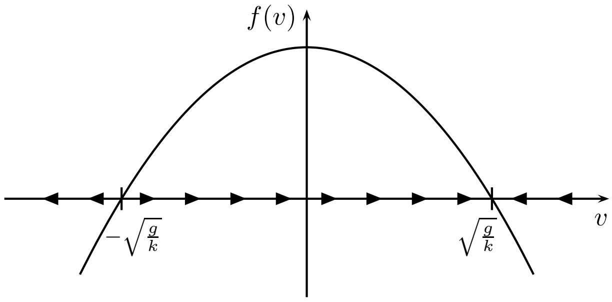
We can solve this to give

$$y(t) = y_0 e^{f'(x_0)t}$$

So if $f'(x_0) < 0$, then as $t \to \infty$, $y(t) \to 0$. Similarly if $f'(x_0) > 0$, as $t \to \infty$, $y(t) \to \pm\infty$.

Example 1.19 (Terminal Velocity)**.** Find the fixed points of the equation $\frac{dv}{dt} = g - kv^2$. (This balances the downward force due to gravity with the upward force due to air resistance in a falling body.)

The fixed points are where $\frac{dv}{dt} = 0$ i.e. $v^2 = \frac{g}{k}$, so the fixed points lie at $v = \pm\sqrt{\frac{g}{k}}$. So we draw a graph of $f(v) = g - kv^2$ against v.



As a check, we find $f'(v) = -2kv$.
At $v = +\sqrt{\frac{g}{k}}$, $f'(v) = -2k\sqrt{\frac{g}{k}} < 0$, so the fixed point is stable.
At $v = -\sqrt{\frac{g}{k}}$, $f'(v) = +2k\sqrt{\frac{g}{k}} < 0$, so the fixed point is unstable.

Example 1.20 (Verhulst)**.** Find the fixed points of the equation $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$ where $r > 0$ (see discussion on page 17).

The fixed points are where $rN\left(1 - \dfrac{N}{K}\right) = 0$, i.e. $N_0 = 0$ or K.
As a check, we find $f'(N) = r\left(1 - \dfrac{2N}{K}\right)$ so:
at $N = 0$, $f'(0) = r > 0$, so the fixed point is unstable;
at $N = K$, $f'(K) = -r < 0$, so the fixed point is stable.

So solutions tend towards K and away from zero (compare with the actual solution on page 17).

CHAPTER 2

Second-Order Linear ODEs

2.1 General Homogeneous Equations

In this section we will study ODEs of the general form

$$a(t)\frac{d^2x}{dt^2} + b(t)\frac{dx}{dt} + c(t)x = 0 \tag{2.1}$$

Observations

Observation 1 (Initial Value Problem) For such equations, to make an initial value problem we require *two* initial values, e.g. $x(0)$ and $\dot{x}(0)$. For example,

$$\frac{d^2x}{dt^2} = 0$$

Integrating gives $$\frac{dx}{dt} = A$$

Integrating again gives $$x(t) = At + B$$

This is a general solution: to specify a particular solution, we require A and B.

Observation 2 (Linearity) The underlying assumption of linearity means that if $x_1(t)$ and $x_2(t)$ are both solutions to (2.1) then so is

$$\alpha x_1(t) + \beta x_2(t)$$

Proof. Substitute $\alpha x_1(t) + \beta x_2(t)$ in the general equation

$$a(t)\frac{d^2x}{dt^2} + b(t)\frac{dx}{dt} + c(t)x$$

to give

$$\begin{aligned}
&a(t)\frac{d^2}{dt^2}[\alpha x_1 + \beta x_2] + b(t)\frac{d}{dt}[\alpha x_1 + \beta x_2] + c(t)[\alpha x_1 + \beta x_2] \\
&= \alpha[a(t)\ddot{x}_1 + b(t)\dot{x}_1 + c(t)x_1] + \beta[a(t)\ddot{x}_2 + b(t)\dot{x}_2 + c(t)x_2] \\
&= \alpha \cdot 0 + \beta \cdot 0 = 0
\end{aligned}$$

since x_1 and x_2 are both solutions. $\square$

Equation (2.1) defines something called a "linear differential operator". $\boxed{\text{LINEAR ALGEBRA}}$

In a similar vein to first order ODEs there are conditions on a second order ODEs that will be satisfied by the above such that the solution to the IVP, plus two initial values, exist and are unique:

Theorem 2.1. *For "nice" second-order ODEs with two initial values $x(t_0) = x_0$, and $\dot{x}(t_0) = v_0$ there exists a unique solution to the initial value problem*

$$a(t)\frac{d^2x}{dt^2} + b(t)\frac{dx}{dt} + c(t)x = 0 \quad : \quad x(t_0) = x_0, \quad \dot{x}(t_0) = v_0 \tag{2.2}$$

We will now show that we need precisely two such solutions for our ODE to form a general solution (i.e. to produce all possible solutions by choosing suitable initial values).

There is a notion of linear independence $\boxed{\text{LINEAR ALGEBRA}}$; for functions this can be stated as

Definition 2.2. *Two functions $x_1(t)$ and $x_2(t)$ defined on an interval I are* linearly independent *if the only solution to*

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) = 0$$

is $\alpha_1 = \alpha_2 = 0$.

For two functions, this is the same as saying that $x_1(t)$ is *not* a scalar multiple of $x_2(t)$, since if x_1 is a scalar multiple of x_2 then $x_1 = cx_2$ so $x_1 - cx_2 = 0$ on I, so x_1, x_2 are *linearly dependent.*

Similarly if $\alpha_1 x_1 + \alpha_2 x_2 = 0$ $(\forall\, t \in I)$ for some $\alpha_1, \alpha_2 \neq 0$ then

$$x_1 = \frac{-\alpha_2}{\alpha_1} x_2 = cx_2$$

Having defined linear independence of solutions, we now show that it is not possible to obtain all solutions to 2.2 as multiples of a single special equation.

Suppose $x_1(t)$ is a solution to 2.2 satisfying $x_1(0) = 1$ and $\dot{x}_1(0) = 0$ and $x_2(t)$ is another solution satisfying $x_2(0) = 0$ and $\dot{x}_2(0) = 1$. By theorem 2.1, both exist and are unique. It should be clear that $x_1(t)$ cannot be a multiple of $x_2(t)$, since if it were, $\dot{x}_1(t)$ would have to be the same multiple of $\dot{x}_2(t)$, and it isn't.

Now we show that two linearly independent solutions $x_1(t)$ and $x_2(t)$ are sufficient. Assume that

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$$

is a solution to the IVP in equation 2.2 solving $x(0) = x_0$ and $\dot{x}(0) = v_0$. The correct values α_1, α_2 can be obtained by solving:

$$\begin{aligned} x_0 &= \alpha_1 x_1(0) + \alpha_2 x_2(0) \\ v_0 &= \alpha_1 \dot{x}_1(0) + \alpha_2 \dot{x}_2(0) \end{aligned}$$

As a matrix equation this is

$$\begin{pmatrix} x_1(0) & x_2(0) \\ \dot{x}_1(0) & \dot{x}_2(0) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

We can solve this provided the matrix has a non-zero determinant, i.e.

$$x_1(0)\dot{x}_2(0) - x_2(0)\dot{x}_1(0) \neq 0$$

If the determinant is zero,

$$\begin{aligned} x_1(0)\dot{x}_2(0) &= x_2(0)\dot{x}_1(0) \\ \Longrightarrow \quad \frac{x_1(0)}{x_2(0)} &= \frac{\dot{x}_1(0)}{\dot{x}_2(0)} = c \end{aligned}$$

But then, $x_1(0) = cx_2(0)$ and $\dot{x}_1(0) = c\dot{x}_2(0)$. Since $x_2(t)$ is a solution, $y(t) = cx_2(t)$ is a solution with

$$\begin{aligned} y(0) &= cx_2(0) = x_1(0) \\ \dot{y}(0) &= c\dot{x}_2(0) = \dot{x}_1(0) \end{aligned}$$

i.e. another solution with the same initial values as $x_1(t)$. By uniqueness, this implies that $x_1(t) = cx_2(t)\,\forall\, t$, i.e. $x_1(t)$, $x_2(t)$ are linearly dependent. But this is a contradiction, since we assumed that $x_1(t)$, $x_2(t)$ are linearly independent.

(Of course the same argument works if we take a different initial value at $t_0 \neq 0$.)

2.2 Homogeneous Linear Second-Order ODEs with constant coefficients

In this section we will look at equations of the form

$$a\frac{d^2x}{dx^2} + b\frac{dx}{dt} + cx = 0 \tag{2.3}$$

where a, b, c are constant.

We expect solutions of the form

$$x(t) = Ax_1(t) + Bx_2(t) \tag{2.4}$$

where $x_1(t)$, $x_2(t)$ are linearly independent.

Appealing to first order ODEs, let's look for solutions of the form $x(t) = e^{kt}$. This gives

$$ak^2e^{kt} + bke^{kt} + ce^{kt} = 0$$

Since $e^{kt} \neq 0$, we have

$$ak^2 + bk + c = 0 \tag{2.5}$$

This is called the *auxiliary equation*, or sometimes the characteristic equation.

We now have three cases:

1. two distinct real roots, k_1, k_2
2. one repeated root, k
3. a complex conjugate pair of roots, $p \pm iq$

Case 1 – Two real roots k_1, k_2

In this case, $x_1(t) = Ae^{k_1 t}$ and $x_2(t) = Be^{k_2 t}$ are both solutions. Obviously $x_1(t)$ is not a scalar multiple of $x_2(t)$, so we have two linearly independent solutions. So by uniqueness (theorem 2.1), the general solution is

$$x(t) = Ae^{k_1 t} + Be^{k_2 t} \tag{2.6}$$

Example 2.3. Find the solution to

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

subject to $y(0) = 5$, $y'(0) = 0$.

Letting $y(x) = e^{kx}$ yields the auxiliary equation

$$\begin{aligned} k^2 + k - 6 &= 0 \\ (k+3)(k-2) &= 0 \end{aligned}$$

giving the roots $k = -3$ or $k = 2$, so

$$y(x) = Ae^{-3x} + Be^{2x}$$

is the general solution.

$$\left.\begin{aligned} y(0) = 5 &\implies & A + B &= 5 \\ y'(0) = 0 &\implies & -3A + 2B &= 0 \end{aligned}\right\} \implies A = 2, B = 3$$

so the solution is

$$y(x) = 2e^{-3x} + 3e^{2x}$$

Case 2 – One repeated root, k

Now there is only one exponential solution, $x(t) = Ae^{kt}$, where $k = -\frac{b}{2a}$.

The second solution can be found by a process known as "reduction of order". We try a trial solution of the form $y(t) = u(t)e^{kt}$, where $u(t)$ is a nonconstant function.

$$\begin{aligned} y &= ue^{kt} \\ \frac{dy}{dt} &= e^{kt}\frac{du}{dt} + kue^{kt} \\ \frac{d^2y}{dt^2} &= e^{kt}\frac{d^2u}{dt^2} + ke^{kt}\frac{du}{dt} + ke^{kt}\frac{du}{dt} + k^2ue^{kt} \end{aligned}$$

Substituting these into our general equation (2.3) gives

$$a\frac{d^2u}{dt^2} + (2ak+b)\frac{du}{dt} + (ak^2 + bk + c)u = 0$$

but $ak^2 + bk + c = 0$ by construction, and since $k = -\frac{b}{2a}$ we have that $2ak + b = 0$, so if $u(t)e^{kt}$ is a solution, then we must have

$$\frac{d^2u}{dt^2} = 0.$$

Integrating twice yields $u(t) = A + Bt$, so

$$y(t) = u(t)e^{kt} = (A + Bt)e^{kt}$$

is a solution to equation 2.3.

Putting $x_1 = e^{kt}$ and $x_2 = te^{kt}$, we note that the determinant

$$\begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} = \begin{vmatrix} e^{kt} & te^{kt} \\ ke^{kt} & (1+kt)e^{kt} \end{vmatrix} = e^{2kt} \neq 0$$

and so $x_1(t)$ and $x_2(t)$ are linearly independent solutions.

Thus the general solution to equation 2.3 if the auxiliary equation (2.5) has one repeated root k is

$$x(t) = (A + Bt)e^{kt} \tag{2.7}$$

Note that if $k < 0$, then $x(t) \to 0$ as $t \to \infty$.

Example 2.4. Find the solution to the IVP

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

subject to $y(0) = 2$, $y'(0) = -5$.

The auxiliary equation is

$$\begin{aligned} k^2 + 6x + 9 &= 0 \\ \text{or} \quad (k+3)^2 &= 0 \\ \text{so repeated root} \quad k &= -3 \end{aligned}$$

giving the general solution

$$y(x) = (A + Bx)e^{-3x}$$

$$\begin{aligned} y(0) = 2 &\implies A = 2 \\ y'(0) = -5 &\implies B - 3A = -5 \implies B = 1 \end{aligned}$$

so

$$y(x) = (2 + x)e^{-3x}$$

Case 3 – Complex roots, $k = p \pm iq$

As before, the general solution is

$$y(t) = Ae^{(p+iq)t} + Be^{(p-iq)t} \tag{2.8}$$

since $\frac{d}{dt}e^{kt} = ke^{kt}$ regardless of whether k is real or complex $\boxed{\text{ANALYSIS III}}$.

Sometimes complex solutions are useful, e.g. electrical circuits, but we can retrieve real solutions as follows:

$$\begin{aligned} y(t) &= Ae^{pt}e^{iqt} + Be^{pt}e^{-iqt} \\ &= e^{pt}(Ae^{iqt} + Be^{-iqt}) \end{aligned}$$

Now using Euler's formula:

$$\begin{aligned} & e^{iqt} = \cos qt + i\sin qt \\ & e^{-iqt} = \cos qt - i\sin qt \\ \implies \quad & y(t) = e^{pt}\left[(A+B)\cos qt + (A-B)i\sin qt\right] \end{aligned}$$

Letting $C = A + B$, $D = (A - B)i$ gives

$$y(t) = e^{pt}\left(C\cos qt + D\sin qt\right) \tag{2.9}$$

which is a linear combination of two real functions.

Exercise Verify that

$$\begin{aligned} & y_1(t) = Ce^{pt}\cos qt \\ \text{and} \quad & y_2(t) = De^{pt}\sin qt \end{aligned}$$

are linearly independent solutions.

Note. Using trigonometric identities it is sometimes more informative to write

$$y(t) = Ee^{pt}\cos(qt - \phi) \tag{2.10}$$

which we obtain as follows:

$$Ce^{pt}\cos qt + De^{pt}\sin qt = \sqrt{C^2+D^2}\,e^{pt}\left(\frac{C}{\sqrt{C^2+D^2}}\cos qt + \frac{D}{\sqrt{C^2+D^2}}\sin qt\right)$$

Setting $\cos\phi = \frac{C}{\sqrt{C^2+D^2}}$, $\sin\phi = \frac{D}{\sqrt{C^2+D^2}}$ (and hence $\tan\phi = \frac{D}{C}$) and $E = \sqrt{C^2+D^2}$, we obtain

$$y(t) = Ee^{pt}\left(\cos\phi\cos qt + \sin\phi\sin qt\right)$$

By standard trigonometric identites, this gives

$$y(t) = Ee^{pt}\cos(qt - \phi).$$

Example 2.5. Solve

$$\ddot{y} - 2\dot{y} + 26y = 0 \qquad : \qquad y(0) = \frac{3}{\sqrt{2}}, \quad \dot{y}(0) = -\frac{12}{\sqrt{2}}$$

The auxiliary equation is $k^2 - 2k + 26 = 0$, giving roots $k = 1 \pm 5i$ and $p = 1$, $q = 5$. So the solution is

$$x(t) = e^{t}\left(A\cos 5t + B\sin 5t\right)$$

$$\begin{aligned} y(0) &= A = \frac{3}{\sqrt{2}} \\ y'(0) &= A + 5B = -\frac{12}{\sqrt{2}} \implies B = -\frac{3}{\sqrt{2}} \end{aligned}$$

so

$$x(t) = \frac{3}{\sqrt{2}}e^{t}\left(\cos 5t - \sin 5t\right)$$

We can also write this in the alternative form:

$$A = \frac{3}{\sqrt{2}} = C\cos\phi \qquad B = -\frac{3}{\sqrt{2}} = C\sin\phi$$

This gives us $C = \sqrt{A^2+B^2} = 3$ and $\phi = \arctan\left(\frac{B}{A}\right) = \arctan(-1) = -\frac{\pi}{4}$, so

$$y(t) = 3e^{t}\cos(5t + \tfrac{\pi}{4})$$

2.3 Mass/Spring Systems

Recall Newton's Second Law of Motion:

$$m\frac{dv}{dt} = F(t)$$

$$\text{or} \qquad m\frac{d^2x}{dt^2} = F(t)$$

that is, "force = mass $\times$ acceleration".

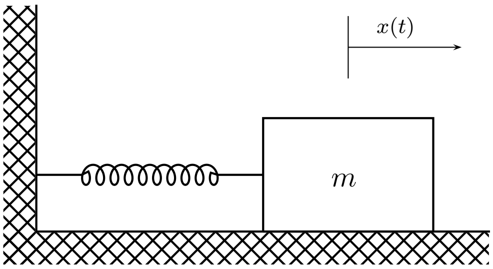
Consider a (linear) spring, then Hooke's Law states

"the restoring force is proportional to the extension" or

$$F = kx$$

where x is the displacement of the spring from rest (compression is negative and extension is positive).

Consider the following:



Pull the mass and let go, then by Newton's Second Law, if there is no friction,

$$m\frac{d^2x}{dt^2} = -kx$$

since the force is a "restoring force"

The simplest model for *friction* is that it is proportional to the velocity of the object. This force acts in the opposite direction to the velocity, i.e. against the direction of travel.

So this gives

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

or

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{2.11}$$

In the "ideal case" where there is no friction (i.e. $c = 0$), we get *simple harmonic motion* governed by

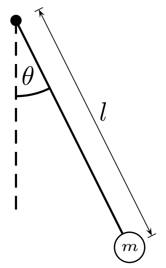
$$m\frac{d^2x}{dt^2} + kx = 0$$

which we can solve by the methods in section 2.2. The auxiliary equation is $m\lambda^2 + k = 0$ which has roots $\lambda = \pm i\sqrt{\frac{k}{m}}$ so the solution is

$$x(t) = A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t$$

$$\text{or} \qquad x(t) = C\cos\left(\sqrt{\frac{k}{m}}\,t - \phi\right)$$

Note that the time taken for one oscillation is dependent only on k and m, and not the initial position of the mass relative to the equilibrium position.



For a simple pendulum with small θ, it can be shown the motion satisfies

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

(neglecting air resistance) which has solution

$$\theta(t) = C\cos\left(\sqrt{\frac{g}{l}}\,t - \phi\right)$$

Adding friction or air resistance we retrieve equation (2.11) which, as we have seen, the solution of which depends on whether the discriminant $(b^2 - 4ac)$ is positive, negative or zero. In this case the discriminant is $c^2 - 4mk$, where $c, m, k \geq 0$.

There are thus four cases to be considered:

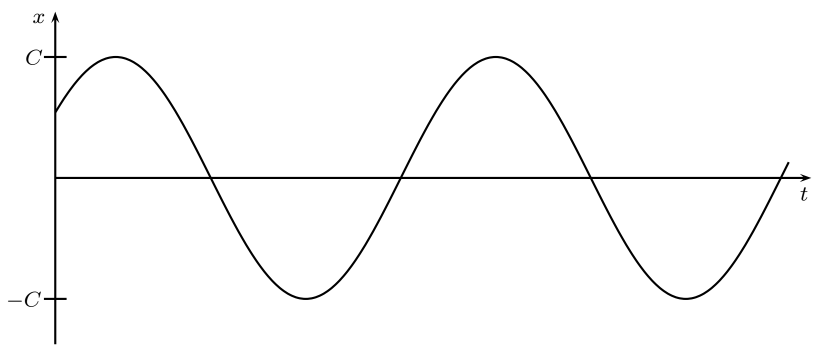
- undamped, $c = 0$
- underdamped, $c^2 - 4mk < 0$
- critically damped, $c^2 - 4mk = 0$
- overdamped, $c^2 - 4mk > 0$

and we now deal with each of these cases in turn.

Undamped, $c = 0$

We have already seen that in this case equation (2.11) gives the solution

$$x(t) = C\cos\left(\sqrt{\frac{k}{m}}\,t - \phi\right)$$



In this case, there is no friction and so the mass oscillates back and forth ad infinitum.

Underdamped, $c^2 - 4mk < 0$

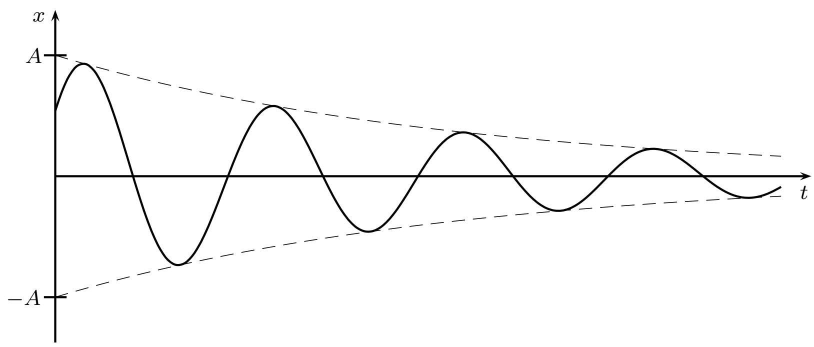
In this case, the auxiliary equation has roots

$$-\frac{c}{2m} \pm i\frac{\sqrt{4mk - c^2}}{2m}$$

so the solution is

$$x(t) = Ae^{-\frac{c}{2m}t}\cos(\omega t - \phi)$$

where $\omega = \frac{\sqrt{4mk-c^2}}{2m}$.



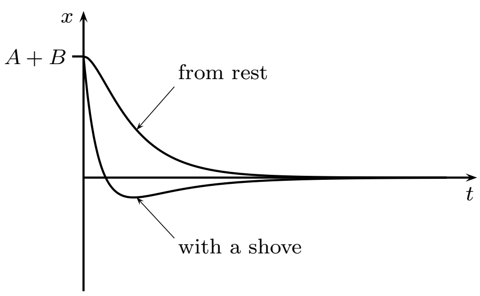
In this case, the system still oscillates, but the amplitude decreases gradually over time.

Critically damped, $c^2 - 4mk = 0$

In this case, the roots are $-\frac{c}{2m}$ repeated, with solution

$$x(t) = (A + Bt)e^{-\frac{c}{2m}t}$$

Now if the mass is released from rest (i.e. $\dot{x}(0) = 0$) then its distance from equilibrium decreases monotonically to zero. Sufficient initial momentum (velocity) will make the mass overshoot, and then increase to zero.



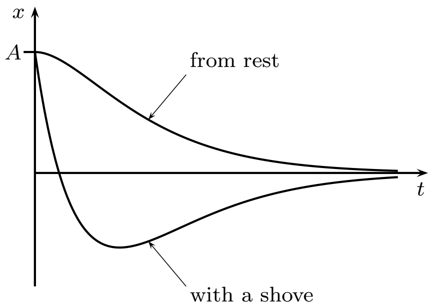
This is called critical damping since a small decrease in c will allow oscillations.

Overdamped, $c^2 - 4mk > 0$

In this case, the roots are $\frac{-c\pm\sqrt{c^2-4mk}}{2m}$, both negative since $\sqrt{c^2-4mk} < \sqrt{c^2} = c$. This gives the solution

$$x(t) = Ae^{-k_1 t} + Be^{-k_2 t}$$

where $k_i = -\left(\frac{-c\pm\sqrt{c^2-4mk}}{2m}\right)$.



This gives a similar result to critical damping, but the decay is less severe.

2.4 Inhomogeneous Linear Second-Order ODEs

In this section we turn our attention to equations of the form

$$a(t)\frac{d^2x}{dt^2} + b(t)\frac{dx}{dt} + c(t)x = f(t) \tag{2.12}$$

Note that $f(t)$ is sometimes known as a "forcing term".

We observe that if $x_p(t)$ is a solution to (2.12) and $Ax_1(t) + Bx_2(t)$ is a solution to the homogeneous case

$$a(t)\frac{d^2x}{dt^2} + b(t)\frac{dx}{dt} + c(t)x = 0 \tag{2.3}$$

then $Ax_1(t) + Bx_2(t) + x_p(t)$ is also a (general) solution to equation (2.12) because of linearity. This should be obvious; if not, then look back at section 2.1, observation 2, on page 25.

Compare this with first order ODEs, where the solution to inhomogeneous first order ODEs is in two bits (see equation (1.14) on page 12, and equation (1.16) on page 15).

Here the solution $Ax_1(t) + Bx_2(t)$ to the homogeneous case is called the *complementary function*, and the solution $x_p(t)$ is called the *particular integral*.

So solving equation (2.12) is a two-part process:

1. Solve the homogeneous case.

2. Find a particular integral.

3. Add the two parts together.

(You undoubtedly know the old joke about there being three kinds of mathematician in the world: those who can count and those who can't.)

We've already seen part 1 for constant coefficients (section 2.2, pp. 27 onwards). Part 2, however, is more tricky, and is usually done by the method of "inspired guesswork".

2.5 Inhomogeneous Linear Second-Order ODEs with constant coefficients

Case 1 – $f(t)$ is a polynomial

If $f(t)$ is an n^{th} degree polynomial then $x(t)$ must also be n^{th} degree, so $\frac{dx}{dt}$ is a $(n-1)^{\text{th}}$ degree polynomial and $\frac{d^2x}{dt^2}$ is an $(n-2)^{\text{th}}$ degree polynomial.

Example 2.6. Find the general solution of

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = t^2 \tag{2.13}$$

The complementary function, or C.F., is the solution to the equivalent homogeneous case $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 0$, which is just $y(t) = \alpha e^t + \beta e^{-2t}$.

For the particular integral, or P.I., we try the most general possible quadratic:

$$\begin{aligned} y(t) &= Ct^2 + Dt + E \\ \frac{dy}{dt} &= 2Ct + D \\ \frac{d^2y}{dt^2} &= 2C \end{aligned}$$

Substituting all that in gives

$$\begin{aligned} \frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 2C + 2Ct + D - 2Ct^2 - 2Dt - 2E &= t^2 \\ -2Ct^2 + (2C - 2D)t + (2C + D - 2E) &= t^2 \end{aligned}$$

Equating coefficients gives

$$-2C = 1 \qquad 2C - 2D = 0 \qquad 2C + D - 2E = 0$$

The first gives $C = -\frac{1}{2}$, substituting this into the second gives $D = -\frac{1}{2}$ as well, and finally substituting these into the third yields $E = -\frac{3}{4}$.

Thus the P.I. $= -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}$, and so the general solution to equation 2.13 is thus:

$$y(t) = \alpha e^t + \beta e^{-2t} - \tfrac{1}{2}t^2 - \tfrac{1}{2}t - \tfrac{3}{4} \tag{2.14}$$

Case 2 – Exponentials, $f(t) = e^{kt}$

In this case we try $y(t) = Ae^{kt}$.

Example 2.7.

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = e^{-t} \tag{2.15}$$

Since the left-hand side is the same as equation 2.13, the C.F. is $y(t) = \alpha e^t + \beta e^{-2t}$ as before. For the P.I., we try the most general exponential function:

$$\begin{aligned} y(t) &= Ae^{-t} \\ \frac{dy}{dt} &= -Ae^{-t} \\ \frac{d^2y}{dt^2} &= Ae^{-t} \end{aligned}$$

Substituting those in gives

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = Ae^{-t} - Ae^{-t} - 2Ae^{-t} = e^{-t}$$
$$-2Ae^{-t} = e^{-t}$$

Equating coefficients gives $A = -\frac{1}{2}$ and hence the general solution is

$$y(t) = \alpha e^t + \beta e^{-2t} - \tfrac{1}{2}e^{-t} \tag{2.16}$$

There is, however, a potential problem with this method – what if k is a root of the auxiliary equation? For example:

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 3e^t \tag{2.17}$$

Here trying Ae^t on the LHS will give 0, not e^t, on the RHS.

Instead, we try Ate^t, since differentiating this will yield multiples of e^t. For (2.17):

$$y(t) = Ate^t$$
$$\frac{dy}{dt} = Ae^t + Ate^t$$
$$\frac{d^2y}{dt^2} = 2Ae^t + Ate^t$$

Substituting those in gives

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 2Ae^t + Ate^t + Ae^t + Ate^t - 2Ate^t = 3e^t$$
$$3Ae^t = 3e^t$$

giving $A = 1$ and general solution

$$y(t) = \alpha e^t + \beta e^{-2t} + te^{-t} \tag{2.18}$$

Similarly, in the case where the RHS is of the form e^{kt} and the LHS has repeated root k, where the C.F. has terms in both e^{kt} and te^{kt}, we try a P.I. of the form

$$y(t) = At^2e^{kt}$$

Case 3 – Trigonometric functions, $f(t) = \cos\omega t$ or $\sin\omega t$

In this case we must try a P.I. of the form

$$y(t) = A\cos\omega t + B\sin\omega t$$

and then follow previous methods ... but the best way is to practice!

Functions to "guess"

$f(t)$	Try solution $x_p(t) =$
ae^{kt} (k not a root)	Ae^{kt}
ae^{kt} (k a root)	Ate^{kt} or At^2e^{kt}
$a\sin(\omega t)$ or $a\cos(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
at^n where $n \in N$	$P(t)$ general polynomial of degree n
at^ne^{kt}	$P(t)e^{kt}$, $P(t)$ general polynomial of degree n
$t^n(a\sin(\omega t) + b\cos(\omega t)$	$P_1(t)\sin(\omega t) + P_2(t)\cos(\omega t)$ where $P_i(t)$ general polynomial of degree n
$e^{kt}(a\sin(\omega t) + b\cos(\omega t))$	$e^{kt}(A\sin(\omega t) + B\cos(\omega t))$

2.6 Mass/Spring Systems with Forcing

Consider again the mass/spring system we saw in section 2.3. If we now force this oscillating mass/spring system, the solutions become much more interesting. If we can push the mass every second, say, it may reduce the oscillations, or, if we hit a certain frequency, the oscillations may become much larger very quickly. We can approximate the force as $F \cos \Omega t$, giving us our general equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + \omega^2 x = F \cos \Omega t \tag{2.19}$$

representing a mass/spring system being forced periodically.

Assume that the discriminant $c^2 - 4m\omega^2 < 0$, i.e. that the auxiliary equation has complex roots. Simplifying the problem slightly by taking $m = 1$, it can be shown (exercise!) that equation 2.19 has the solution:

$$x(t) = A\cos(\Omega t - \phi) + Be^{-\frac{ct}{2}} \cos(\alpha t + \delta) \tag{2.20}$$

where

$$\alpha^2 = \omega^2 - \frac{c^2}{4} \qquad A = \frac{F}{\sqrt{(\omega^2 - \Omega^2)^2 + c^2\Omega^2}} \qquad \phi = \arccos\left(\frac{\omega^2 - \Omega^2}{\sqrt{(\omega^2 - \Omega^2)^2 + c^2\Omega^2}}\right)$$

Comments

1. As $t \to \infty$, the second term "vanishes". This second term is called the *"transient behaviour"*, while the first term is known as the *"steady state solution"*.

2. With no forcing ($F = 0$) and no friction ($c = 0$) we have $\alpha = \omega$ and so the system oscillates with frequency $\frac{\omega}{2\pi}$ – the *"natural frequency"*.

3. If there is little or no friction and $F \neq 0$, then we look at what happens as we change Ω – in particular, what happens as Ω gets close to ω.

   As $\Omega \to \omega$, the amplitude A gets larger and larger. In fact, if $c = 0$, then as $\Omega \to \omega$, $A \to \infty$. This is known as *"resonance"*.

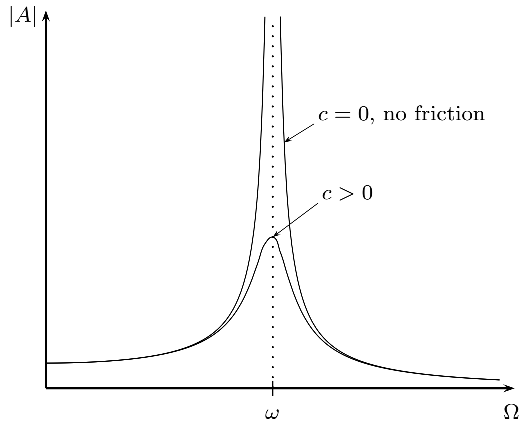


Figure 2.1: A graph of $|A|$ against Ω for the solution of equation (2.20).

CHAPTER 3

Introduction to Difference Equations

3.1 Terminology

In the previous two chapters, when we were finding a solution to an ODE, we found an expression for $x(t)$ which gave us a value of x for all times t. While we sometimes came across integrals that were not solvable analytically, we were still able to obtain an expression for the solution. However, the cases in which it is possible to obtain an explicit expression for the solution $x(t)$ to an ODE are in fact extremely limited, and it becomes next to impossible for ODEs of higher orders. For equations we cannot solve explicitly, we can turn to numerical solution methods which involve increasing the time t by very small increments, and generating a numerical approximation to the solution; one such scheme is known as *Euler's method*, and we'll come back to this notion in example 3.1 below.

Moreover, quite often we're interested not in obtaining a continuous solution, but rather a model relating one day/hour/year to the next, e.g. stock prices at the end of trading each day, or population models in which we're given the number of, say, rabbits this year, and want to figure out the population next year. In both examples we want a *discrete* solution, rather than a continuous solution. For example, a population model for discrete values of time is:

$$N_{t+1} = rN_t\left(1 - \frac{N_t}{k}\right) \tag{3.1}$$

where N_t is the number of rabbits in year t; compare this with equation (1.19) on page 17.

This kind of equation, and similar types relating one value to the previous value(s) is known as a *difference equation*, or sometimes as a *"recurrence equation"*, in with the values $N_0, N_1, N_2, N_3, \ldots$ (where N_0 is the initial value supplied in the question) form a sequence.

ANALYSIS

Example 3.1 (Euler's Method)**.** Consider an ODE of the form

$$\frac{dx}{dt} = f(t, x) \quad : \quad x(0) = x_0 \tag{3.2}$$

that we cannot solve. In this case, we must give up the idea of finding a solution for all values of the independent variable ($x(t)$ for any $t \in \mathbb{R}$) and instead try to find an approximation to the solution at a discrete set of values of t.

A computer drawing a graph of a first-order ODE *with* a solution essentially draws a series of dots representing the solution. That series of dots looks like a line if it is dense enough, but it's still a series of dots. So for a computer plotting a graph of equation (3.2), it assumes that a solution $x(t)$ exists and that it is unique (i.e. that theorem 1.8 applies) and then all it needs is a series of dots which approximate the solution $x(t)$. The simplest way to do this is by Euler's method.

For this method, we choose a small time step h and make the assumption that over that interval h, the derivative $\frac{dx}{dt}$ is constant, and so by the Taylor expansion (equation (1.24) on page 23):

$$x(t+h) = x(t) + h\dot{x}(t) = x(t) + hf(t, x(t)) \tag{3.3}$$

(We can ignore subsequent terms in the Taylor expansion since we are assuming that $\frac{dx}{dt}$ is constant over the small timestep h, and thus $\frac{d^2x}{dt^2} = 0$.[1])

Implementing this to solve (3.2) yields the following:

$$\begin{aligned}
x_0 &= x(0) \\
x_1 &= x(h) = x(0+h) = x(0) + h\dot{x}(0) &&= x_0 + hf(0, x_0) \\
x_2 &= x(2h) = x(h+h) = x(h) + hf(x(h)) &&= x_1 + hf(h, x_1) \\
&\vdots
\end{aligned}$$

$$x_{k+1} = x_k + hf(kh, x_k) \tag{3.4}$$

At each step we have everything we need *without* knowing the explicit solution $x(t)$.

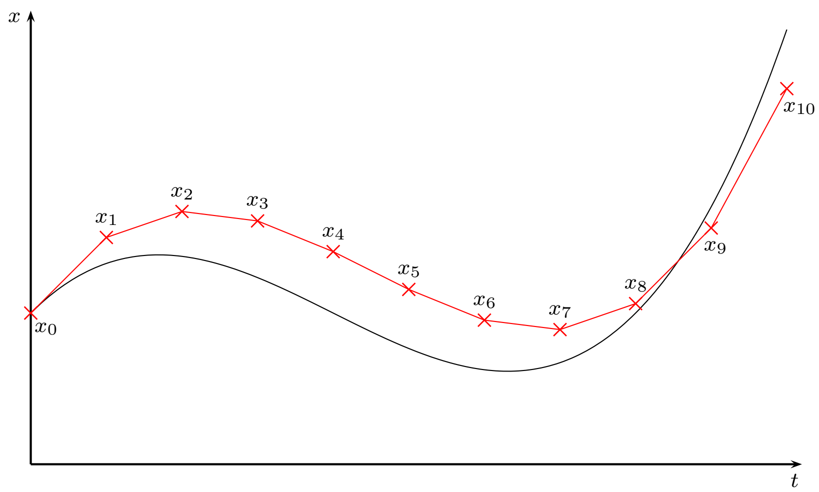


Figure 3.1: The exact solution curve and a series of approximations to it with $h = 1$.

As you can see, while Euler's method yields a solution, the errors grow the longer we continue in this iterative fashion.

[1]In reality, $\frac{dx}{dt}$ is not constant, and this assumption can in fact lead to very poor approximations. The *Improved Euler Method*, surprisingly enough, improves this method by assuming not that $\frac{dx}{dt}$ is constant, but that $\frac{d^2x}{dt^2}$ is constant, and hence acheives a better approximation by using two terms of the Taylor expansion. The *Runge-Kutta Method* takes this further by assuming that the fourth derivative, $\frac{d^4x}{dt^4}$, is constant, and in most cases this method is indistinguishable from the true solution. However, these methods are very much more complicated to put in to practice, and we will not study them here: those interested should consider taking MA228 Numerical Analysis next year.

Example 3.2 (Fibonacci's Rabbits)**.** Consider the Fibonacci sequence, given by the difference equation

$$x_{n+2} = x_{n+1} + x_n \quad : \quad x_0 = x_1 = 1 \tag{3.5}$$

We can think of this as a simplified population model of rabbits in which:

- we start with one baby rabbit;
- a rabbit can't give birth until it's 2 years old; and
- a single unisex adult rabbit gives birth to one baby per year.

Using these modelling facts, we generate the following table:

Table 3.1: Table of Fibonacci's Rabbits.

Year	0	1	2	3	4	5	6...
Adult	0	0	1 →	1 →	2 →	3 →	5...
One year	0	1	0	1	1	2	3...
Newborns	1	0	1	1	2	3	5...
TOTAL	1	1	2	3	5	8	13...
number in year n	x_0	x_1	x_2	x_3	x_4	x_5	$x_6 \ldots$

From the table, we can see that the number of rabbits in year n is equal to the number of rabbits in year $n-1$ plus the number of babies born in year n. But the number of babies in year n is just equal to the number of adults in year n, which is itself the number of rabbits in year $n-2$, since two years later all the rabbits will be adults. This gives us the formula:

$$\underbrace{x_k}_{\text{rabbits in year } k} = \underbrace{x_{k-1}}_{\text{rabbits in year } k-1} + \underbrace{x_{k-2}}_{\text{rabbits in year } k-2 \,=\, \text{adults in year } k}$$

which is the same as equation 3.5.

There are, in fact, strong similarities between difference equations and ODEs; for example,

$$x_{n+2} - 5x_{n+1} + 3x_n = \cos \frac{n}{63}$$

is described as a second-order inhomogeneous linear difference equation.

Definition 3.3. *The* order *of a difference equation is the difference between the highest index of x and the lowest.*

For example,

$$x_{n+7} - \cos x_{n+3} = Ae^{n^2}$$

has order 4. Euler's method (example 3.1) is order 1, while the Fibonacci series (example 3.2) is order 2.

3.2 First-Order Homogeneous Linear Difference Equations

First-order equations, or "next-generation models" are equations of the form

$$x_{n+1} = f(x_n, n)$$

Linear[2] homogeneous first-order equations are simply equations such as

$$x_{n+1} = ax_n \tag{3.6}$$

where a is a constant and, as with ODEs, we also specify an initial value x_0.

We have a rule, and an initial value, so we just plug everthing into our sausage machine.

$$\begin{aligned} x_1 &= ax_0 \\ x_2 &= ax_1 = a^2 x_0 \\ x_3 &= ax_2 = a^3 x_0 \\ &\vdots \\ x_n &= a^n x_0 \end{aligned} \tag{3.7}$$

and lo and behold, we have a solution.

If $x_0 = 0$ then $x_n = a^n.0 = 0$. In this case x_0 is called a *fixed point*, since if we start there we stay there, i.e. $x_n = x_0 \,\forall\, n \in \mathbb{N}$.

If $|a| < 1$, then for any x_0,

$$\lim_{n\to\infty} x_n = 0$$

so $x_0 = 0$ is a *stable fixed point*.

Conversely if $|a| > 1$ then $x_0 = 0$ is an *unstable fixed point*. The case where $|a| = 1$ is an example of *structural instability* (compare with section 1.9).

3.3 Second-Order Linear Difference Equations with constant coefficients

Now we turn our attention to difference equations of the form

$$x_{n+2} + ax_{n+1} + bx_n = f_n \tag{3.8}$$

For first-order difference equations, we saw solutions of the form a^n (in place of e^{at} for first-order ODEs of the form $\frac{dx}{dt} = ax$). So, we try analogous solutions in the second-order case.

3.3.1 Homogeneous Case

In this case, we consider the restricted case where $f_n = 0$, i.e. equations of the form

$$x_{n+2} + ax_{n+1} + bx_n = 0 \tag{3.9}$$

Appealing to previous solutions, we "guess" a solution $x_n = Ak^n$, giving us

$$Ak^{n+2} + Aak^{n+1} + Abk^n = 0$$

[2]Nonlinear difference equations can have extremely complicated solutions, and we will only consider them briefly later, in section 3.4.

which is true if

$$Ak^n(k^2 + ak + b) = 0$$

So, either $k = 0$ and $x_n \equiv 0$, which isn't particularly interesting, or alternatively, k is a root of the *auxiliary equation*

$$k^2 + ak + b = 0 \tag{3.10}$$

so

$$k = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \tag{3.11}$$

which, as with the ODE case, gives us three distinct cases detailed below.

1. Distinct real roots, k_1, k_2

In this case both $x_n = Ak_1^n$ and $x_n = Bk_2^n$ are solutions so by linearity the general solution is

$$x_n = Ak_1^n + Bk_2^n \tag{3.12}$$

Example 3.4 (Fibonacci)**.** As in example 3.2, we have the equation

$$x_{n+2} - x_{n+1} - x_n = 0 \tag{3.13}$$

This gives us the auxilary equation $k^2 - k - 1 = 0$, with roots $k = \frac{1 \pm \sqrt{5}}{2}$ (where $k = \frac{1+\sqrt{5}}{2}$ is known as the "golden ratio"). Plugging this into our general solution gives

$$x_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

When $x_0 = x_1 = 1$ as in the classic Fibonacci sequence, then we can substitute these in and get two simultaneous equations in A and B:

$$\begin{aligned} x_0 &= A + B = 1 \\ x_1 &= A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{aligned}$$

We can solve these $\boxed{\text{Exercise}}$ to give $A = \dfrac{1+\sqrt{5}}{2\sqrt{5}}$, $B = \dfrac{\sqrt{5}-1}{2\sqrt{5}}$ which gives us our solution

$$x_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \tag{3.14}$$

which, despite the proliferance of $\sqrt{5}$ does, in fact, give an integer for every $n \in \mathbb{N}$ as required.

2. Repeated real roots, $k = -\frac{a}{2}$ ($b = \frac{a^2}{4}$)

Once again, we look to ODEs for inspriation, and try a solution of the form

$$x_n = Ak^n + Bnk^n \tag{3.15}$$

Since $x_n = Ak^n$ yields one solution, our question is whether $x_n = Bnk^n$ another solution? In order to find out, we substitute into the difference equation:

$$B(n+2)k^{n+2} + aB(n+1)k^{n+1} + bBnk^n$$

which, rearranged, gives

$$Bnk^n \underbrace{\left(k^2 + ak + b\right)}_{=0,\text{ aux eqn}} + Bk^n \underbrace{\left(2k^2 + ak\right)}_{=0,\ k=-\frac{a}{2}} = 0$$

as required. So nk^n is also a solution when k is a repeated root. Therefore by linearity, equation 3.15 is indeed the general solution when k is a repeated root.

Example 3.5. Find a general solution of

$$x_n - 6x_{n-1} + 9x_{n-2} = 0 \tag{3.16}$$

The auxiliary equation is just

$$\begin{aligned} \lambda^2 - 6\lambda + 9 &= 0 \\ (\lambda - 3)^2 &= 0 \\ \lambda &= 3 \text{ repeated} \end{aligned}$$

so from equation 3.15, the general solution is just

$$x_n = (A + Bn)3^n \tag{3.17}$$

3. Complex roots, $k = p \pm iq$

In order to make all our lives easier, we write the roots in the form $k_\pm = re^{\pm i\theta}$, where $r^2 = p^2 + q^2$ and $\theta = \arctan \frac{q}{p}$.

In this case, the general solution is

$$x_n = r^n \left(A \cos n\theta + B \sin n\theta\right) \tag{3.18}$$

This follows by considering

$$C\left(re^{i\theta}\right)^n + \overline{C}\left(re^{-i\theta}\right)^n$$

where $\overline{C}$ is the complex conjugate of C (with C chosen so that the solution is real), and by noting that $re^{i\theta} = r(\cos\theta + i\sin\theta)$ and that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ (de Moivre's theorem).

Example 3.6. Solve

$$x_{n+2} - 2x_{n+1} + 2x_n = 0 \tag{3.19}$$

The auxiliary equation is

$$\begin{aligned} \lambda^2 - 2\lambda + 2 &= 0 \\ \lambda &= 1 \pm i = \sqrt{2}e^{\pm\frac{i\pi}{4}} \end{aligned}$$

so, plugging this into our general equation 3.18, we get

$$x_n = (\sqrt{2})^n \left(A\cos\left(\frac{n\pi}{4}\right) + B\sin\left(\frac{n\pi}{4}\right)\right) \tag{3.20}$$

3.3.2 Inhomogeneous Case

In this subsection we will consider difference equations of the form

$$x_{n+2} + ax_{n+1} + bx_n = f_n \tag{3.21}$$

Since 3.21 is linear, we start by finding the solution to the homogeneous case, the *complementary function*, e.g. $x_n = Ak_1^n + Bk_2^n$.

We then find a *particular solution*[3] that gives f_n on the right-hand side. Then, by linearity,

[3] Obviously *particular integral* would make no sense in this context since there are no differentials or integrals.

the general solution is simply the complementary function and the particular solution added together.

In this section we will look at two forms for f_n; other forms follow in a similar

1. f_n is a polynomial in n

We will first look at a simple first-order difference equation,

$$x_{n+1} = ax_n + b \tag{3.22}$$

The homogeneous case gives a complementary function $x_n = Aa^n$. For the particular solution, we try $x_n = B$. Then

$$\begin{aligned} x_{n+1} &= ax_n + b \\ B &= aB + b \\ B &= \frac{b}{1-a} \end{aligned}$$

giving the general solution

$$x_n = Aa^n + \frac{b}{1-a} \tag{3.23}$$

For higher-degree polynomials and higher-order difference equations, the method is the same as ODEs, so for example if $f_n = n^2$, we try $An^2 + Bn + C$.

Example 3.7. Solve

$$x_n - x_{n-1} - 6x_{n-2} = -36n \tag{3.24}$$

Solving the homogeneous case $x_n - x_{n-1} - 6x_{n-2} = 0$ gives us our complementary function, so we find the auxiliary equation:

$$\begin{aligned} k^2 - k - 6 &= 0 \\ (k-3)(k+2) &= 0 \\ k &= 3, -2 \end{aligned}$$

which gives our complementary function:

$$x_n = A \cdot 3^n + B(-2)^n \tag{3.25}$$

It doesn't matter in this case that the leading term is x_n rather than x_{n+2} – you *could* argue that the solution is $x_n = A \cdot 3^{n-2} + B(-2)^{n-2}$, but then $x_n = \frac{A}{9} \cdot 3^n + \frac{B}{4} \cdot (-2)^n$, and then we can choose $A' = \frac{A}{9}, B' = \frac{B}{4}$ since A and B are just arbitrary constants, giving $x_n = A' \cdot 3^n + B'(-2)^n$.

For the particular solution, since f_n is a polynomial of degree 1, we try the most general 1-degree polynomial possible, i.e. $x_n = Cn + D$, and substitute this into (3.24):

$$\begin{aligned} x_n - x_{n-1} - 6x_{n-2} &= \underbrace{Cn + D}_{x_n} \underbrace{-C(n-1) - D}_{-x_{n-1}} \underbrace{-6C(n-2) - 6D}_{-6x_{n-2}} = -36n \\ \text{or} \qquad & n(-6C) + (13C - 6D) = -36n \end{aligned}$$

giving $C = 6, D = 13$ and our particular solution of $x_n = 6n + 13$.

Combining this with (3.25) gives us the general solution

$$x_n = A \cdot 3^n + B(-2)^n + 6n + 13 \tag{3.26}$$

2. f_n **is an exponential,** a^n

This is the "equivalent" case to that of e^{at} in ODEs. Once again, we have to be careful when a^n is also part of the complementary function - but once again, the ideas from ODEs are carried over. So, for the particular solution when $f_n = a^n$, try:

$x_n = Ca^n$	if a is not a root of the auxiliary equation
$x_n = Cna^n$	if a is a root
$x_n = Cn^2a^n$	if a is a repeated root

We will now look at two examples.

Example 3.8. Find the general solution of

$$x_{n+2} + x_{n+1} - 6x_n = 12(-2)^n \tag{3.27}$$

First we find the complementary function. Our auxiliary equation is $k^2+k-6=0$, giving roots $k = 2, -3$ and so we have our complementary function

$$x_n = A \cdot 2^n + B(-3)^n \tag{3.28}$$

$(-2)^n$ is not a solution of the homogeneous case, so we try

$$x_n = C(-2)^n$$

as a particular solution. Substituting this in gives

$$\begin{aligned} x_{n+2} + x_{n+1} - 6x_n = C(-2)^{n+2} + C(-2)^{n+1} - 6C(-2)^n &= 12(-2)^n \\ \implies \qquad 4C - 2C - 6C &= 12 \\ C &= -3 \end{aligned}$$

Bringing everything together gives us our general solution:

$$x_n = A \cdot 2^n + B(-3)^n - 3(-2)^n \tag{3.29}$$

Example 3.9. Solve

$$x_{n+2} + x_{n+1} - 6x_n = 40 \cdot 2^n \tag{3.30}$$

As in example 3.8, the complementary function is $x_n = A \cdot 2^n + B(-3)^n$. However, this time, 2^n *is* a solution of the homogeneous case, so we try

$$x_n = Cn2^n$$

as a particular solution. Substituting this in gives

$$\begin{aligned} x_{n+2} + x_{n+1} - 6x_n = C(n+2)2^{n+2} + C(n+1)2^{n+1} - 6Cn2^n &= 40 \cdot 2^n \\ \implies \qquad 4C(n+2) + 2C(n+1) - 6Cn &= 40 \\ 8C + 2C &= 40 \\ C &= 4 \end{aligned}$$

giving us our general solution

$$x_n = A \cdot 2^n + B(-3)^n + 4n \cdot 2^n \tag{3.31}$$

3.4 First-Order (Autonomous) Nonlinear Difference Equations

Solving nonlinear difference equations, even first-order ones, can be extremely difficult. We will approach this topic largely through one example, the "logistic equation":

$$x_{n+1} = \lambda x_n (1 - x_n) \tag{3.32}$$

This is related to the Verhulst ODE, and can be thought of as a population model where x_n is population size as a proportion of a maximum size K. Typically $0 \leq x_n \leq 1$, and if so, then we must have the restriction $0 \leq \lambda \leq 4$, or else the values would leave the range $0 \leq x_n \leq 1$.

For small x_n, then $x_{n+1} \approx \lambda x_n$; λ can be thought of as the "birth rate" when $\lambda > 1$. However, be warned: this equation looks deceptively simple.

In general, we are thinking of nonlinear autonomous difference equations of the form

$$x_{n+1} = f(x_n) \tag{3.33}$$

given some x_0. This is *autonomous* because f is a function of x_n, but not of n, i.e. it doesn't involve terms like n^2 or $e^n x_n$ etc.

Running our sausage machine (3.33) gives:

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) = f(f(x_0)) = f^2(x_0) \\ &\vdots \\ x_n &= f^n(x_0) \end{aligned}$$

Here for shorthand purposes we have used the notation $f^2(x) = f(f(x))$, or more generally:

$$\underbrace{f(f(\ldots f(f(}_{n}x))\ldots)) = f^n(x)$$

Note that $f^n(x) \neq [f(x)]^n$, and also that $f^n(x) \neq f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$.

If $f(x_n)$ is *nonlinear*, then (most of the time) you *cannot* write an explicit solution of the form

$$x_n = \text{some function of } x$$

and, moreover, we can get very strange things happening, but first some theory.

We met *fixed points* of difference equations in section 3.2, but we now give a more rigorous definition:

Definition 3.10. *A* fixed point *of the difference equation*

$$x_{n+1} = f(x_n) \tag{3.34}$$

is a point x_* such that $f(x_*) = x_*$, i.e. if $x_n = x_*$ then $x_{n+1} = f(x_n) = x_n \, \forall n$.

From section 3.2, we know that the linear equation $x_{n+1} = a x_n$ has solution $x_n = a^n x_0$, and that $x_0 = 0$ is a fixed point, stable if $|a| < 1$ and unstable if $|a| > 1$. The upshot of this and the definition above is that limits of sequences and stability are related. $\boxed{\text{ANALYSIS}}$

The \$64,000 question is: does this generalise? Fortunately the answer is yes.

Say x_* is a fixed point of $x_{n+1} = f(x_n)$, i.e. $f(x_*) = x_*$. (Note the difference with ODEs: a common mistake in exams is to set $f(x) = 0$ instead of $f(x) = x$.) Consider a nearby point $x_0 = x_* + h$ with $|h| \ll 1$, then

$$\begin{aligned} x_1 &= f(x_0) \\ &= f(x_* + h) \\ &= f(x_*) + hf'(x_*) + \frac{h^2}{2} f''(x_*) + \dots \end{aligned}$$

For sufficiently small h, we can say

$$\begin{aligned} x_1 &\approx f(x_*) + hf'(x_*) \\ x_1 &\approx x_* + hf'(x_*) \end{aligned} \tag{3.35}$$

If $|f'(x_*)| < 1$, then $|x_0 - x_*| > |x_1 - x_*|$, i.e. we have moved closer to x_*. (More formally, if $|f'(x_*)| < 1$, then $f^n(x_* + h) \to x_*$ as $n \to \infty$.) This means that if $|f'(x_*)| < 1$, x_* is a *stable* fixed point. Conversely if $|f'(x_*)| > 1$ then x_* is *unstable*.

Note that the sequence $x_{n+1} = f(x_n)$ need *not* be monotonic (unlike the ODE equivalent); it can converge to a fixed point via an alternating (i.e. oscillating) sequence.

Returning to the example of the logistic equation (3.32), we can now look for its fixed points, i.e. x_* such that

$$x_* = \lambda x_*(1 - x_*)$$

i.e.

$$\lambda x_*^2 + (1 - \lambda)x_* = 0 \tag{3.36}$$

giving the two fixed points $x_* = 0$ and $x_* = \frac{\lambda - 1}{\lambda}$ (with $\lambda > 1$).

Since $f(x) = \lambda x(1 - x)$, $f'(x) = \lambda(1 - 2x)$, so:

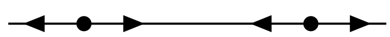
- for $x_* = 0$, $f'(0) = \lambda$, so $x_* = 0$ is an unstable fixed point if $\lambda > 1$, while if $\lambda < 1$ then our growth rate for small x_n will kill off the population and $x_* = 0$ is stable;

- for $x_* = \frac{\lambda-1}{\lambda}$, $f'(\frac{\lambda-1}{\lambda}) = 2 - \lambda$, so provided $1 < \lambda < 3$, $x_* = \frac{\lambda-1}{\lambda}$ is a stable fixed point.

So that would seem to be it, yes? We have a stable fixed point at $x_* = \frac{\lambda-1}{\lambda}$ to which all x_n with $0 < x_0 < 1$ converge.

No. We quietly slipped in the fact that $x_* = \frac{\lambda-1}{\lambda}$ was only stable if $1 < \lambda < 3$. What happens if $3 \le \lambda \le 4$? Well, in that case, $f'(\frac{\lambda-1}{\lambda}) = 2 - \lambda \le -1$ and the fixed point at $x_* = \frac{\lambda-1}{\lambda}$ is now unstable[4]. That means that both fixed points are unstable, and there are no other fixed points left.

Think about the equivalent ODE case. The phase diagram (see section 1.9) would look like this:



[4]Note that $\lambda = 3$ is an example of structural instability - the sequence does converge to $x_* = \frac{\lambda-1}{\lambda}$, but *very, very* slowly indeed.

If this were an ODE, there would have to be a fixed point between those two where $\frac{dx}{dt} = 0$. But since difference equations need not be monotonic, this is not a problem, but rather a critical feature – difference equations allow more complicated behaviour, which it is not easy to pick up simply from looking at fixed points.

We thus need a graphical way of studying the solutions.

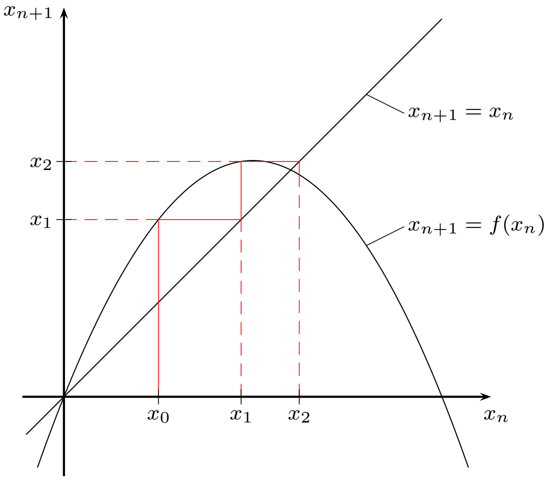


Figure 3.2: The cobweb diagram of the logistic map for $\lambda = 2.5$.

As in figure 3.2, we plot the graphs of $y = x$ and $y = f(x)$ on one set of axes. The two graphs cross where $x_{n+1} = f(x_n) = x_n$, i.e. at the fixed points. The procedure for drawing our diagram is as follows:

1. Start at x_0 on the x_n axis.

2. Go up to the graph of f and across to the x_{n+1} axis to get x_1.

3. Draw a line from x_1 to graph of $x_n = x_{n+1}$, then down to x_n axis.

4. Repeat from item 2.

These diagrams are known as *cobweb diagrams*. While they can be drawn by hand, as we shall see accuracy is vital, and so the drawings are best created by computer software such as Matlab. $\boxed{\text{MATHS BY COMPUTER}}$

To see the behaviour of the logistic equation in detail, we will consider a number of examples.

$\lambda = 2.5$

In this case, we simply spiral in towards the fixed point $x_* = \frac{\lambda-1}{\lambda}$. The cobweb diagram is plotted in figure 3.3.

$\lambda = 3.2$

Now the fixed point $x_* = \frac{\lambda-1}{\lambda}$ is *unstable*. The cobweb diagram is plotted in figure 3.4.

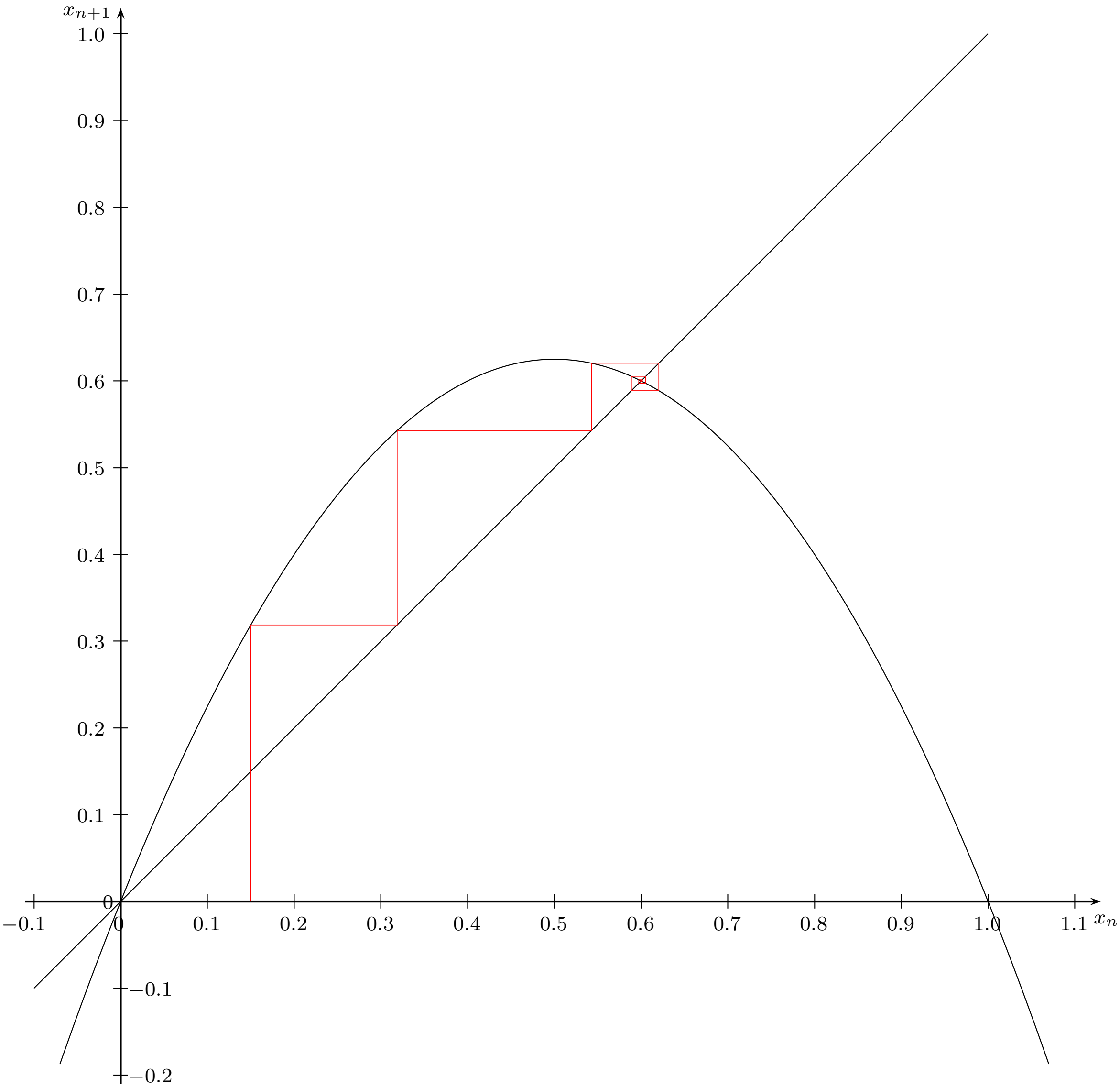


Figure 3.3: The logistic equation with $\lambda = 2.5$, with initial value $x_0 = 0.15$, iterated 50 times. As we can see, the solution simply spirals into the fixed point $x_* = \frac{\lambda - 1}{\lambda}$.

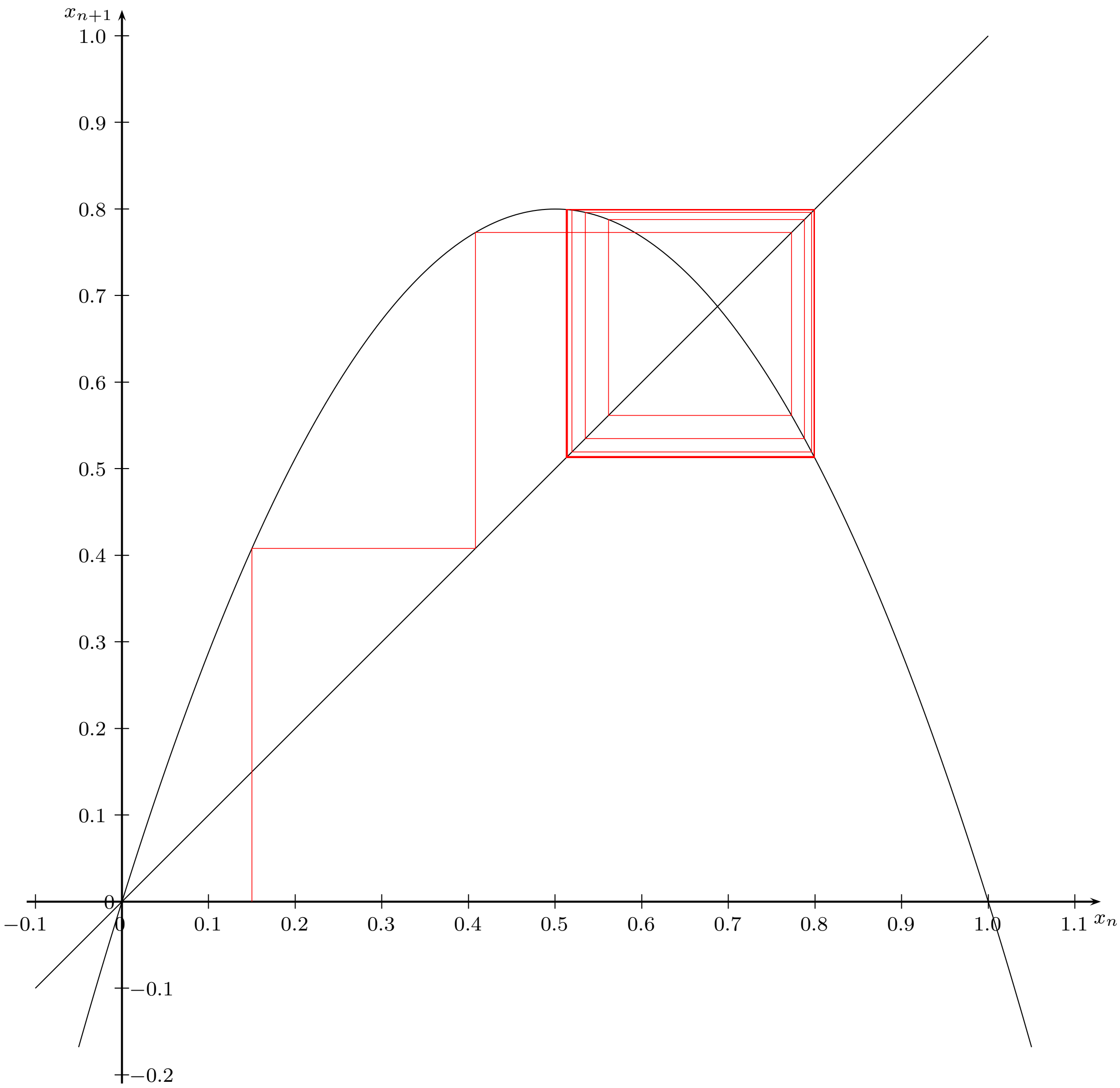


Figure 3.4: The logistic equation with $\lambda = 3.2$, with initial value $x_0 = 0.15$, iterated 50 times. Here the solution settles down to oscillate with period 2.

As we can see, when $\lambda = 3.2$, we are attracted to a "period two orbit", i.e. two alternating points; we repeat every $t = 2$. In some sense, this is stable – we get attracted towards the orbit.

Each point of the orbit is a fixed point of $f^2(x)$ (i.e. $f(f(x))$), i.e. a point x_* s.t. $f^2(x_*) = x_*$.

For $f(x) = \lambda x_n(1 - x_n)$, we have

$$f^2(x_n) = \lambda \left[\lambda x_n(1 - x_n)\right] \left(1 - \left[\lambda x_n(1 - x_n)\right]\right) \tag{3.37}$$

We now want x_* s.t. $f^2(x_*) = x_*$, i.e. we need to solve a quartic equation. While there is a formula for this, it is hugely complicated.

However, we already know two of the factors – we know that $x_* = 0$ and $x_* = \frac{\lambda-1}{\lambda}$ will solve the quartic, since if x_* solves $f(x_*) = x_*$, then

$$f^2(x_*) = f(f(x_*)) = f(x_*) = x_*$$

So we divide the quartic by $x_n(x_n - \frac{\lambda-1}{\lambda})$, giving: FOUNDATIONS

$$\lambda x_n^2 - (1 + \lambda)x_n + \left(1 + \frac{1}{\lambda}\right) = 0 \tag{3.38}$$

with roots

$$x_\pm = \frac{(1 + \lambda) \pm \sqrt{(\lambda + 1)(\lambda - 3)}}{2\lambda} \tag{3.39}$$

These exist provided $\lambda > 3$, which is consistent with the lack of stable fixed points of $f(x)$ beyond $\lambda > 3$.

By increasing λ through $\lambda = 3$, we change the stability of the fixed point $x_* = \frac{\lambda-1}{\lambda}$ and create new solutions to $f^2(x) = x$. This is an example of *bifurcation*. These points are stable fixed points of $f^2(x_n)$ if $3 < \lambda < 1 + \sqrt{6} \approx 3.45$. If we call the two points x_+ and x_-, then we find that $f(x_+) = x_-$ and $f(x_-) = x_+$, and we have a period two orbit.

Once again, however, we hit the same problem: beyond $\lambda \approx 3.45$, these fixed points become unstable. So, we investigate what happens in this case.

$\lambda = 3.5$

The cobweb diagram appears in figure 3.5.

In this case, the oscillations settle down, but we no longer have a period two orbit, but rather a period four orbit, in which $f^4(x_*) = x_*$.

If we then increase λ further in small increments, we get periods of 8, 16, 32 etc. occurring; this is called a period-doubling cascade. Beyond a certain point[5], the solutions we observe are apparently random – in fact it is an example of chaos.

[5] This occurs at roughly $\lambda \approx 3.57$, and is known as the accumulation point or Feigenbaum point.

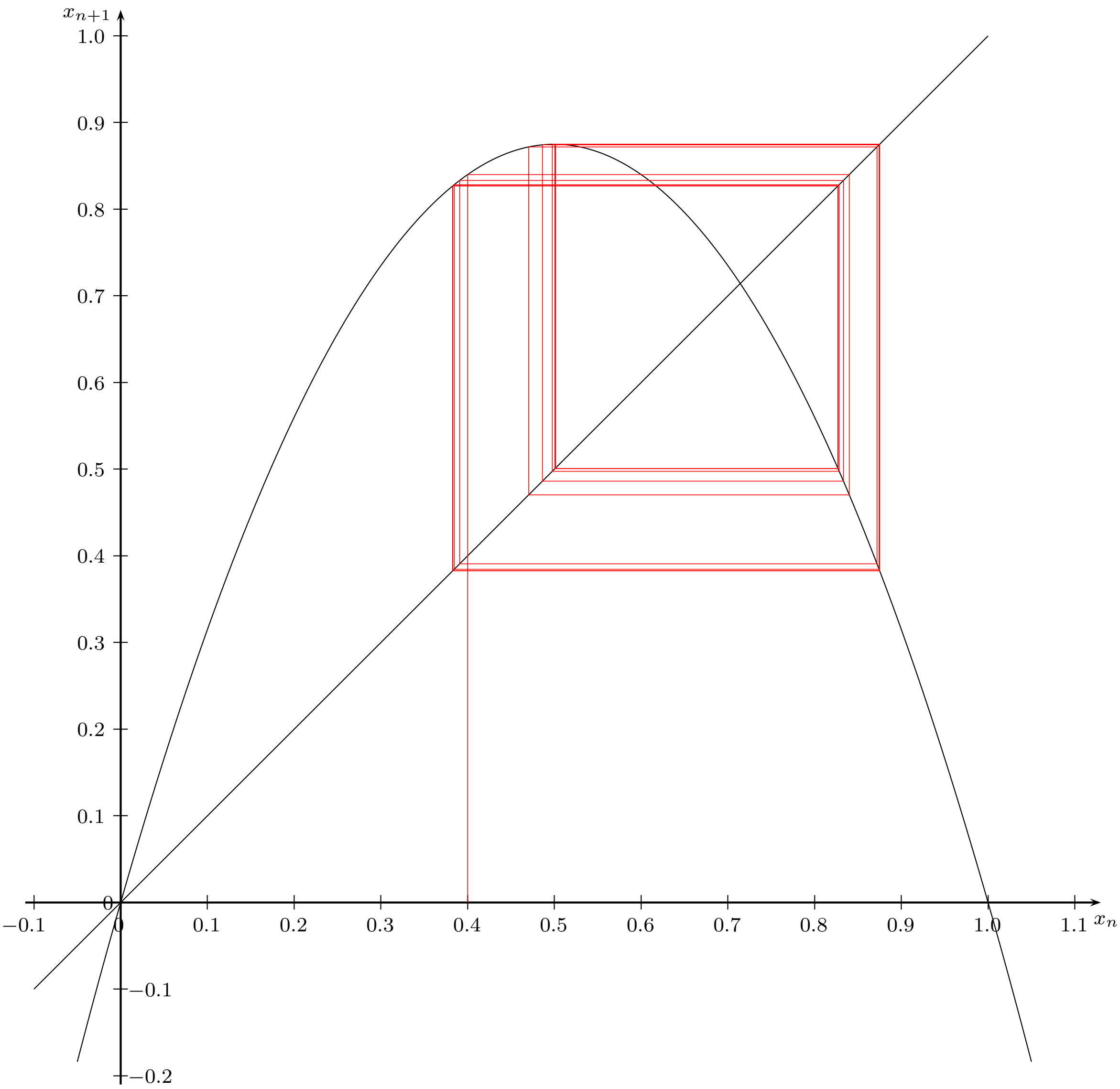


Figure 3.5: The logistic equation with $\lambda = 3.5$, with initial value $x_0 = 0.4$, iterated 50 times. We can see that the oscillations settle down, but that the period is now 4.

CHAPTER 4

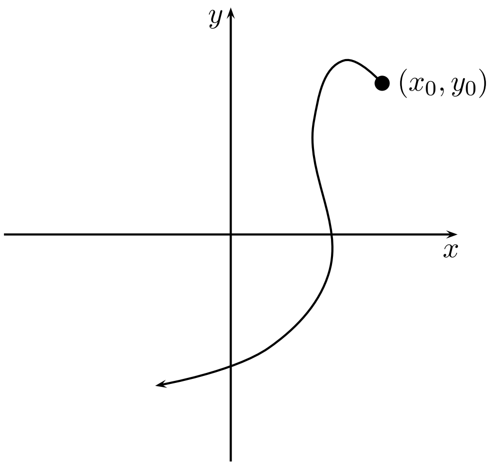
Systems of Linear First-Order ODEs

So far we have concentrated on *one-dimensional* systems, i.e. one dependent variable. The "phase plane" was a line, e.g.

We now increase the dimension to two, by considering two dependent variables $x(t)$, $y(t)$. Some uses for such equations include:

- competing populations, e.g. rabbits and foxes
- chemical reactions
- more complicated mixing problems (compare with example 1.11 on page 11)

Now the phase plane is two-dimensional:



Every point gives a value for x and y at a time t.

4.1 In General

An $n \times n$ system of first-order ODEs is:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \dots, x_n, t) \\
\dot{x}_2 &= f_2(x_1, x_2, \dots, x_n, t) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \dots, x_n, t)
\end{aligned} \tag{4.1}$$

such that $x_i \in \mathbb{R}$. We can write this in *vector form*:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \qquad : \qquad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{f} : \mathbb{R}^{n+1} \to \mathbb{R}^n \tag{4.2}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} \qquad \mathbf{f}(\mathbf{x}, t) = \mathbf{f}(x_1, x_2, \dots, x_n, t) = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n, t) \\ f_2(x_1, x_2, \dots, x_n, t) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n, t) \end{pmatrix}$$

Having established our basic notation, we can now proceed to cover the technicalities. This is very much a generalisation of first-order one-dimensional equations.

Definition 4.1. *A* solution *of the IVP*

$$\frac{d}{dt}(\mathbf{x}(t)) = \mathbf{f}(\mathbf{x}, t) \qquad : \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x} \in \mathbb{R}^n \tag{4.3}$$

on an open interval I that contains t_0 is a continuous function $\boxed{\text{ANALYSIS II}}$ *i.e. "nice" function $\mathbf{x} : I \to \mathbb{R}^n$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t)\ \forall t \in I$.*

Recall that for a function $g(u, v)$, $\frac{\partial g}{\partial u}$ is the partial derivative of g with respect to u, i.e. we treat v as a constant and differentiate with respect to u alone. Armed with this knowledge, we make the following definition:

Definition 4.2. *The* Jacobian matrix *of a function*[1] $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ *is the matrix of partial derivatives as follows:*

$$D\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \tag{4.4}$$

We are now ready for our existence and uniqueness theorem.

Theorem 4.3 (Existence and Uniqueness)**.** *If $\mathbf{f}(\mathbf{x}, t)$ and $D\mathbf{f}(\mathbf{x}, t)$ are continuous functions ("nice") for $\mathbf{x} \in$ some set $U \subseteq \mathbb{R}^n$, $a < t < b$, then for* any *$\mathbf{x}_0 \in U$ and $t_0 \in (a, b)$, there exists a unique solution to (4.3) on some open interval containing t_0.*

So in order for there to be a unique solution, we need the partial derivatives $\frac{\partial f_i}{\partial x_j}$ to be "nice".

[1] Here we are considering t as a part of the functions $f_1, f_2, \dots, f_n$, which is why it is not $\mathbf{f}(x) : \mathbb{R}^{n+1} \to \mathbb{R}^n$.

Important Special Case

When we discussed second-order ODEs, we assumed existence and uniqueness. Consider

$$\frac{d^2x}{dt^2} = f\left(\frac{dx}{dt}, x, t\right) \tag{4.5}$$

where $x \in \mathbb{R}$. By introducing a new variable $y = \dot{x}$ we can rewrite this as:

$$\begin{aligned}\frac{dx}{dt} &= y = f_1(\mathbf{x}, t) \\ \frac{dy}{dt} &= \ddot{x} = f(y, x, t) = f_2(\mathbf{x}, t)\end{aligned}$$

In this case, we have that the Jacobian matrix is

$$D\mathbf{f} = \begin{pmatrix} 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \tag{4.6}$$

so to guarantee unique solutions for second order ODEs we require that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are "nice", and we also require an initial condition (x_0, y_0), i.e. $x(t_0) = x_0$, $\dot{x}(t_0) = y_0$.

Example 4.4. Consider the ODE

$$\ddot{x} + \cos t \dot{x} - x^2 = 0$$

In this example, $\dot{x} = y$, so $\dot{y} = f(y, x, t) = x^2 - (\cos t)y$, giving

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x \\ \frac{\partial f}{\partial y} &= -\cos t\end{aligned}$$

This gives us our Jacobian

$$D\mathbf{f} = \begin{pmatrix} 0 & 1 \\ 2x & -\cos t \end{pmatrix}$$

It should be clear that $2x$ and $-\cos t$ are "nice".

4.2 Coupled 2×2 Linear Systems with Constant Coefficients

A general linear coupled 2×2 system may look like

$$\begin{aligned}\frac{dx}{dt} &= 5x - 2y + \cos t \\ \frac{dy}{dt} &= e^t x + y\end{aligned}$$

Here, however, as in ODEs we consider constant coefficients. One approach is to see if we can transform the equations into a second-order ODE, and then solve that using our techniques from chapter 2.

Example 4.5. Find an explicit solution to

$$\frac{dx}{dt} = x + y \tag{4.7a}$$

$$\frac{dy}{dt} = 4x - 2y + 4e^{-2t} \tag{4.7b}$$

We first rearrange (4.7a) to get:

$$\begin{aligned} y &= \dot{x} - x \\ \implies \quad \dot{y} &= \ddot{x} - \dot{x} \\ &= 4x \underbrace{-2(\dot{x} - x)}_{-2y} + 4e^{-2t} \end{aligned}$$

$$\implies \quad \ddot{x} + \dot{x} - 6x = 4e^{-2t} \tag{4.8}$$

with solution

$$x(t) = Ae^{2t} + Be^{-3t} - e^{-2t} \tag{4.9}$$

However, this method is not easy to generalise to $n \times n$ equations. Is there a better way?

4.3 Homogeneous Linear 2×2 Systems with Constant Coeffs.

Consider systems of the form

$$\frac{dx}{dt} = px + qy \tag{4.10a}$$

$$\frac{dy}{dt} = rx + sy \tag{4.10b}$$

or, if $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then we can write this as

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{4.11}$$

Using knowledge of previous equations we try a solution of the form

$$\mathbf{x} = e^{\lambda t} \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{or } \mathbf{x} = e^{\lambda t}\mathbf{v}$$

Substituting gives:

$$\dot{\mathbf{x}} = \frac{d}{dt}(\mathbf{x}) = A\mathbf{x} = \lambda e^{\lambda t}\mathbf{v}$$

but since $\mathbf{x} = e^{\lambda t}\mathbf{v}$, $A\mathbf{x} = e^{\lambda t}A\mathbf{v}$, so we require

$$e^{\lambda t}A\mathbf{v} = \lambda e^{\lambda t}\mathbf{v}$$

and since $e^{\lambda t} \neq 0$, this is equivalent to seeking

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{4.12}$$

So finding a solution $\mathbf{x} = e^{\lambda t}\mathbf{v}$ is equivalent to finding the eigenvalues (λ) and eigenvectors ($\mathbf{v}$) of A. $\boxed{\text{LINEAR ALGEBRA}}$

If we have two such solutions, then by linearity

$$\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t} \tag{4.13}$$

is also a solution to (4.11).

We will discuss repeated (equal) eigenvalues later. If, however, the eigenvalues are distinct, be they real or complex, then so are the eigenvectors, and so the two solutions will be linearly independent. $\boxed{\text{LINEAR ALGEBRA}}$

Example 4.6. Find the general solution to

$$\dot{\mathbf{x}} = \begin{pmatrix} 2 & 6 \\ -2 & -5 \end{pmatrix} \mathbf{x} \tag{4.14}$$

The eigenvalues of $\left(\begin{smallmatrix} 2 & 6 \\ -2 & -5 \end{smallmatrix}\right)$ are -1 and -2, with corresponding eigenvectors $\left(\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -3 \\ 2 \end{smallmatrix}\right)$ respectively. $\boxed{\text{Exercise}}$ So, the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = ae^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + be^{-2t} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \tag{4.15}$$

Note that for any initial value (x_0, y_0), the solution will head towards $(0, 0)$ as $t \to \infty$.

As with second order ODEs, there will be three cases of eigenvalues. (This should come as no surprise when we consider that second order ODEs with constant coefficients can be written as a 2×2 system.)

1. Distinct Real Eigenvalues As we've already seen, if A has two distinct real eigenvalues λ_1, λ_2 with eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, then the solution is $\mathbf{x}(t) = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t}$.

2. Complex Eigenvalues, $p \pm iq$ or λ, $\overline{\lambda}$ It can be shown that if λ and $\mathbf{v}$ are an eigenvalue and eigenvector of A, respectively, then so are $\overline{\lambda}$ and $\overline{\mathbf{v}}$. Splitting $\mathbf{v}$ into real and imaginary parts, we can write $\mathbf{v} = \mathbf{v}_1 + i\mathbf{v}_2$, giving that $\overline{\mathbf{v}} = \mathbf{v}_1 - i\mathbf{v}_2$. The solution now looks like:

$$\mathbf{x}(t) = c\mathbf{v}e^{\lambda t} + \overline{c}\overline{\mathbf{v}}e^{\overline{\lambda} t} \tag{4.16}$$

where the choice of c, $\overline{c}$ is such that such that $\mathbf{x}(t)$ is real.

Noting that for $z \in \mathbb{C}$, $z + \overline{z} = 2\Re[z]$ (where $\Re[z]$ represents the real part of z), then we have that

$$\mathbf{x}(t) = 2\Re\left[ce^{\lambda t}\mathbf{v}\right]$$

Letting $\lambda = p + iq$ and $c = \alpha + i\beta$ gives

$$\mathbf{x}(t) = 2\Re\left[(\alpha + i\beta)e^{pt}(\cos qt + i \sin qt)(\mathbf{v}_1 + i\mathbf{v}_2)\right]$$

since $e^{\lambda t} = e^{pt}e^{iqt} = e^{pt}(\cos qt + i \sin qt)$, so

$$\begin{aligned} \mathbf{x}(t) &= 2e^{pt}\Re\left[((\alpha \cos qt - \beta \sin qt) + i(\beta \cos qt + \alpha \sin qt))\,(\mathbf{v}_1 + i\mathbf{v}_2)\right] \\ \mathbf{x}(t) &= 2e^{pt}\left[(\alpha \cos qt - \beta \sin qt)\mathbf{v}_1 - (\beta \cos qt + \alpha \sin qt)\mathbf{v}_2\right] \end{aligned}$$

giving the solution

$$\mathbf{x}(t) = e^{pt}\left[(a \cos qt + b \sin qt)\mathbf{v}_1 + (b \cos qt - a \sin qt)\mathbf{v}_2\right] \tag{4.17}$$

where $\mathbf{v} = \mathbf{v}_1 + i\mathbf{v}_2$ and a, b are constants determined by the initial conditions.

Example 4.7. Solve

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix} \mathbf{x} \tag{4.18}$$

The matrix $\left(\begin{smallmatrix} 1 & 5 \\ -1 & -3 \end{smallmatrix}\right)$ has eigenvalues $\lambda_+ = -1 + i$, $\lambda_- = -1 - i$, with corresponding eigenvectors $\boxed{\text{Exercise}}$

$$\mathbf{v}_+ = \begin{pmatrix} 5 \\ -2 + i \end{pmatrix}, \quad \mathbf{v}_- = \begin{pmatrix} 5 \\ -2 - i \end{pmatrix}$$

This gives us the solution:

$$\mathbf{x}(t) = a \begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix} e^{-t} + b \begin{pmatrix} 5\sin t \\ \cos t - 2\sin t \end{pmatrix} e^{-t} \tag{4.19}$$

Note again that whatever the initial value (a, b), $\mathbf{x}(t) \to 0$ as $t \to \infty$.

3. Repeated Real Eigenvalues We will return to this case later.

4.4 Phase Portraits and Change of Variable

4.4.1 Distinct Real Eigenvalues

Consider the system

$$\dot{\mathbf{x}} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \mathbf{x} \tag{4.20}$$

This equation has eigenvalues 2 and -3, with eigenvectors $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 \\ -4 \end{smallmatrix}\right)$. $\boxed{\text{Exercise}}$
(See Example 28.1 in Robinson, pp. 270-271 for further details.)
Equation (4.20) therefore has solution

$$\mathbf{x}(t) = ae^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} \tag{4.21}$$

Note that $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ is a "fixed point", i.e. $\dot{\mathbf{x}} = \mathbf{0}$. Therefore:

- Any solution with initial value $a = 0$, $b \neq 0$, i.e. $\mathbf{x}(0) = \left(\begin{smallmatrix} k \\ -4k \end{smallmatrix}\right)$ will give $\mathbf{x}(t) \to \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ as $t \to \infty$. In the language of the 1-D case, $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ is "stable".

- Any solution with initial value $a \neq 0$, $b = 0$, i.e. $\mathbf{x}(0) = \left(\begin{smallmatrix} k \\ k \end{smallmatrix}\right)$ will give $\mathbf{x}(t) \to \infty$ as $t \to \infty$. For such points, $\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ is "unstable".

- By linearity, if $a \neq 0$, $b \neq 0$ then the solutions will be a combination of these two cases.

All this gives us the phase diagram in figure 4.1 on page 61.

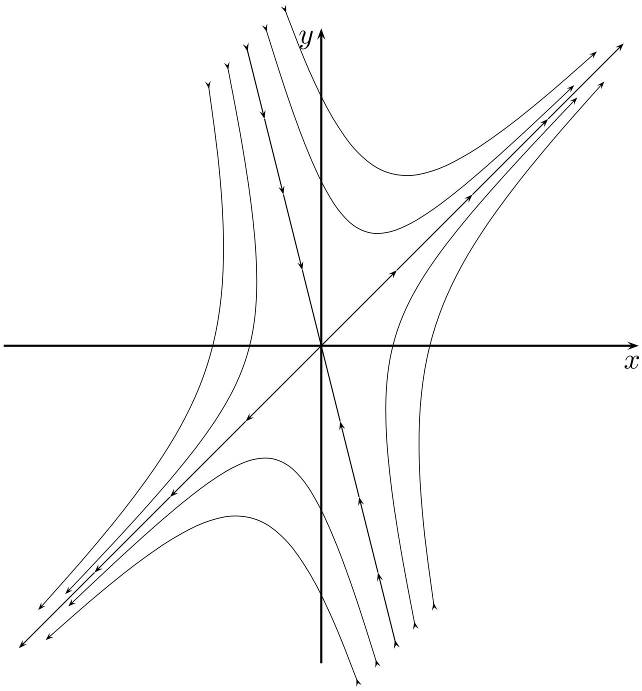


Figure 4.1: The phase diagram for equation (4.20).

Diagonalisation and Uncoupling

Consider the system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{4.22}$$

where A has distinct eigenvalues. Can we change coordinates to do something clever?

Recall from $\boxed{\text{LINEAR ALGEBRA}}$ that for a matrix A with distinct eigenvalues λ_1, λ_2 and corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, we can call the matrix $P = (\mathbf{v}_1|\mathbf{v}_2)$ and then

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

So for our system (4.22), we change the coordinates by taking $\mathbf{y} = P^{-1}\mathbf{x}$, so

$$\begin{aligned} \dot{\mathbf{y}} &= P^{-1}\dot{\mathbf{x}} && \qquad (4.23) \\ &= P^{-1}A\mathbf{x} && \text{since } \dot{\mathbf{x}} = A\mathbf{x} \\ &= P^{-1}AP\mathbf{y} && \text{since } \mathbf{x} = P\mathbf{y} \\ \text{i.e.} \quad \dot{\mathbf{y}} &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{y} && \qquad (4.24) \end{aligned}$$

Call $\mathbf{y} = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$. Then we have

$$\dot{y}_1 = \lambda_1 y_1 \quad \Longrightarrow \quad y_1 = ae^{\lambda_1 t} \tag{4.25a}$$

$$\dot{y}_2 = \lambda_2 y_2 \quad \Longrightarrow \quad y_2 = be^{\lambda_2 t} \tag{4.25b}$$

i.e. the equations have been *"uncoupled"*.

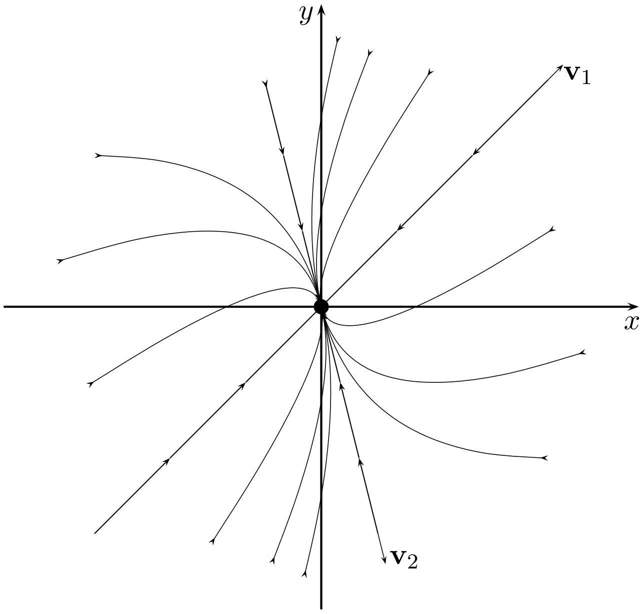


Figure 4.2: The phase diagram when $\lambda_1 < \lambda_2 < 0$. Here $(0, 0)$ is a stable fixed point, also known as a "sink".

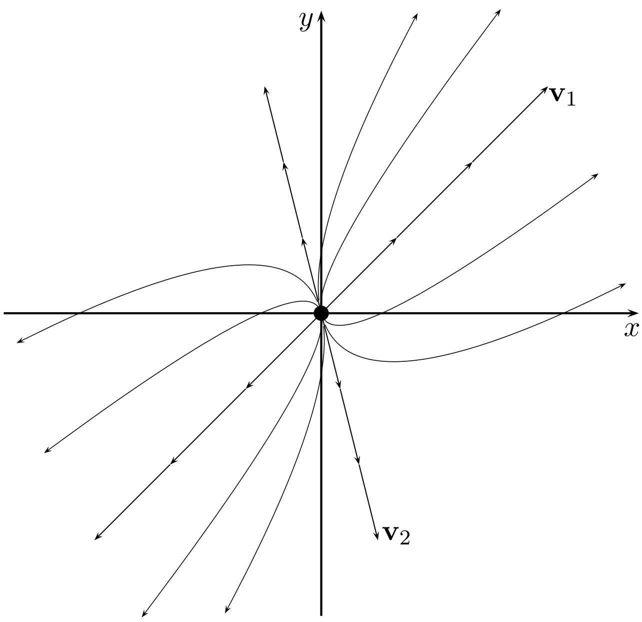


Figure 4.3: The phase diagram when $\lambda_1 > \lambda_2 > 0$. Here $(0, 0)$ is an unstable fixed point, also known as a "source".

More Phase Diagrams

When we have distinct, real eigenvalues, we have to take into account whether the eigenvalues are positive or negative. This gives rise to three cases:

- We have already seen the case where $\lambda_1 < 0 < \lambda_2$. Here $(0,0)$ is called a "saddle point".
- If $\lambda_1 < \lambda_2 < 0$, all solutions tend to $(0,0)$ – but they will more faster in the $\mathbf{v}_1$ direction ($e^{\lambda_1 t}$ instead of $e^{\lambda_2 t}$), so the phase portrait will look like figure 4.2 on page 62. Now $(0,0)$ is a stable fixed point or a "sink".
- Alternatively, if $\lambda_1 > \lambda_2 > 0$, we will get something like figure 4.3 on page 62. Now $(0,0)$ is an unstable fixed point or a "source".

4.4.2 Complex Eigenvalues

As we have seen on page 59, the solution to a general system $\dot{\mathbf{x}} = A\mathbf{x}$ when A has complex eigenvalues is

$$\mathbf{x}(t) = e^{pt}\left[(a\cos qt + b\sin qt)\mathbf{v}_1 + (b\cos qt - a\sin qt)\mathbf{v}_2\right] \tag{4.26}$$

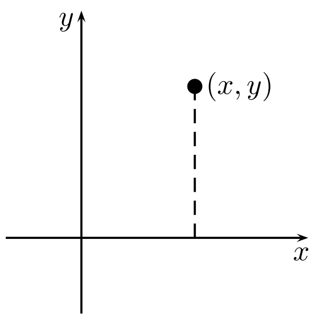
where $\lambda = p + iq$ and $\mathbf{v} = \mathbf{v}_1 + i\mathbf{v}_2$ are an eigenvalue and corresponding eigenvector (the others are the complex conjugates $\overline{\lambda}$ and $\overline{\mathbf{v}}$).

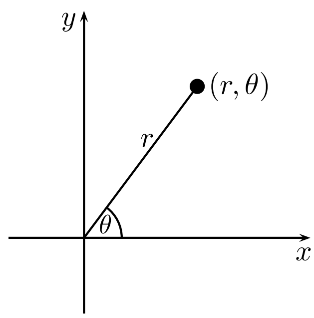
In this case, the distance from the origin depends on the e^{pt} part, i.e.

- if $p = \Re[\lambda] < 0$, then $(0,0)$ is stable, but
- if $p = \Re[\lambda] > 0$, then $(0,0)$ is unstable.

The imaginary part, which gives rise to the $[(a\cos qt + b\sin qt)\mathbf{v}_1 + (b\cos qt - a\sin qt)\mathbf{v}_2]$ expression, makes the solutions spiral; the direction we rotate depends on the sign of q.

Now, as you know, any point in $\mathbb{R}^2$ can be uniquely represented as a pair of Cartesian coordinates, (x, y):





However, we can also represent any point "uniquely" by a distance $r > 0$ from the origin, and an angle $0 \le \theta < 2\pi$; the coordinates are now (r, θ). By convention we denote the origin as $(0,0)$. Compare this with complex numbers:

$$x + iy \mapsto r(\cos\theta + i\sin\theta)$$

To swap between the two:

$$x = r\cos\theta \qquad\qquad r = \pm\sqrt{x^2 + y^2}$$

$$y = r\sin\theta \qquad\qquad \theta = \arctan\frac{y}{x}$$

Returning to our system $\dot{\mathbf{x}} = A\mathbf{x}$ when A has complex eigenvalues, changing coordinates using eigenvectors yields the system

$$\dot{\mathbf{y}} = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \mathbf{y}.$$

We now convert this into polar form. Differentiating the relationship $r^2 = y_1^2 + y_2^2$ yields

$$\begin{aligned} 2r\dot{r} &= 2y_1\dot{y_1} + 2y_2\dot{y_2} \\ &= 2y_1(py_1 + qy_2) + 2y_2(-qy_1 + py_2) \\ &= 2p(y_1^2 + y_2^2) = 2pr^2 \\ \implies \dot{r} &= pr. \end{aligned}$$

Similarly differentiating $\theta = \arctan \frac{y_2}{y_1}$ gives

$$\begin{aligned} \dot{\theta} &= \frac{1}{1 + \left(\frac{y_2}{y_1}\right)^2} \cdot \frac{y_1\dot{y_2} - y_2\dot{y_1}}{y_1^2} \\ &= \frac{y_1^2}{y_1^2 + y_2^2} \cdot \frac{y_1(-qy_1 + py_2) - y_2(py_1 + qy_2)}{y_1^2} = -q. \end{aligned}$$

So we have succeeded in separating variables, and our uncoupled system is

$$\dot{r} = pr \tag{4.27a}$$
$$\dot{\theta} = -q \tag{4.27b}$$

with solution

$$r(t) = ae^{pt} \tag{4.28a}$$
$$\theta(t) = -qt(+c) \tag{4.28b}$$

(See Robinson chapter 29, pp. 285 ff. for a further discussion of this topic.)

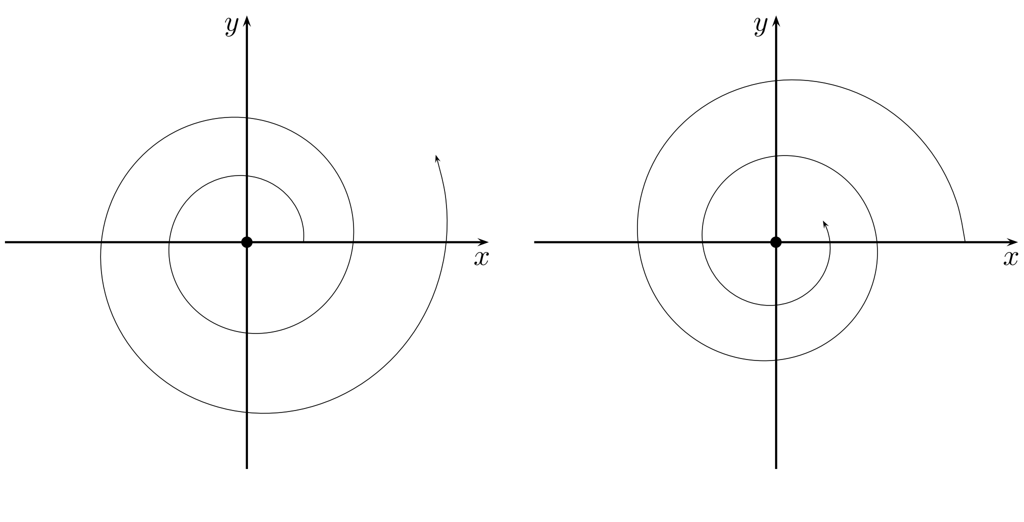


Figure 4.4: The phase diagrams when the eigenvalues are complex; the example on the left is when $\Re[\lambda] > 0$, while on the right $\Re[\lambda] < 0$.

4.4.3 Repeated Real Eigenvalues

Finally, we return to the case of a repeated eigenvalue λ and corresponding eigenvector $\mathbf{v}$. The problem in this case is that we only have one solution, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$. Thinking of previous work, we try a solution of the form

$$\mathbf{x}(t) = te^{\lambda t}\mathbf{a}$$

where $\mathbf{a}$ is some vector. Substituting this into $\dot{\mathbf{x}} = A\mathbf{x}$ gives:

$$\underbrace{\mathbf{a}e^{\lambda t} + \mathbf{a}\lambda te^{\lambda t}}_{\dot{\mathbf{x}}} = \underbrace{A\mathbf{a}te^{\lambda t}}_{A\mathbf{x}}$$

Collecting terms gives $\lambda\mathbf{a} = A\mathbf{a}$, but also $\mathbf{a} = \mathbf{0}$, which means there are no non-zero solutions of the form $\mathbf{x}(t) = te^{\lambda t}\mathbf{a}$. So, what do we try instead? Let's try a more general solution:

$$\mathbf{x}(t) = \mathbf{a}e^{\lambda t} + \mathbf{b}te^{\lambda t}$$

where $\mathbf{a}$, $\mathbf{b}$ are vectors, not necessarily eigenvectors. Then we get

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{a}\lambda e^{\lambda t} + \mathbf{b}e^{\lambda t} + \mathbf{b}\lambda te^{\lambda t} \\ \therefore \quad \underbrace{\mathbf{a}\lambda e^{\lambda t} + \mathbf{b}e^{\lambda t} + \mathbf{b}\lambda te^{\lambda t}}_{\dot{\mathbf{x}}} &= \underbrace{A\mathbf{a}e^{\lambda t} + A\mathbf{b}te^{\lambda t}}_{A\mathbf{x}} \end{aligned}$$

Equating coefficients gives

$$\mathbf{a}\lambda + \mathbf{b} = A\mathbf{a} \qquad \text{or} \qquad (A - \lambda I)\mathbf{a} = \mathbf{b} \tag{4.29}$$

$$\mathbf{b}\lambda = A\mathbf{b} \qquad \text{or} \qquad (A - \lambda I)\mathbf{b} = \mathbf{0} \tag{4.30}$$

From (4.30) we see that $\mathbf{b}$ is an eigenvector of A. So, take $\mathbf{b} = \mathbf{v}$, and we can now find $\mathbf{a}$. And so we have our solution:

$$\mathbf{x}(t) = Be^{\lambda t}\mathbf{v} + Ce^{\lambda t}(\mathbf{a} + t\mathbf{v}) \tag{4.31}$$

where p, q are arbitrary constants.

Example 4.8. Solve

$$\dot{\mathbf{x}} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x} \tag{4.32}$$

This has eigenvalue $\lambda = 1$ repeated, with eigenvector $\mathbf{v} = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$. So

$$\mathbf{x}(t) = ae^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is one solution.

We now need a vector $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{v}$, i.e.

$$\begin{aligned} \left(\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \mathbf{u} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \mathbf{u} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

If $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, then we require $\left\{ \begin{array}{c} 2u_1 - 4u_2 = 2 \\ u_1 - 2u_2 = 1 \end{array} \right\}$; one solution is $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

This gives the second solution to (4.32) as

$$be^t \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = be^t \begin{pmatrix} 1 + 2t \\ t \end{pmatrix}$$

Thus the general solution to (4.32) is

$$\mathbf{x}(t) = ae^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + be^t \begin{pmatrix} 1 + 2t \\ t \end{pmatrix} \tag{4.33}$$

The phase diagram therefore looks like figure 4.5.

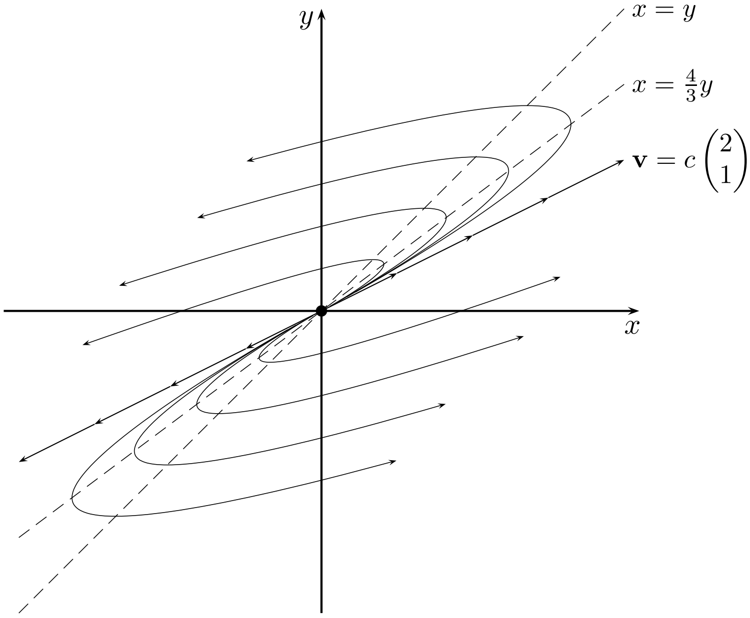


Figure 4.5: The phase diagram for equation 4.32.

The system is $\left\{ \begin{array}{l} \dot{x} = 3x - 4y \\ \dot{y} = x - y \end{array} \right\}$. Note that $\dot{x} = 0$ when $x = \frac{4}{3}y$, and that $\dot{y} = 0$ when $x = y$; this means that points on the lines $x = \frac{4}{3}y$ and $x = y$ are where the solutions "turn".

Second-Order Differential Equations

Finally, we return to second-order ODEs of the form

$$a\ddot{x} + b\dot{x} + cx = 0 \tag{4.34}$$

By setting $\dot{x} = y$, we can rewrite this as a coupled first-order system of two equations in two unknowns:

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = -\frac{c}{a}x - \frac{b}{a}y \end{array} \right.$$

or, in matrix form:

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \tag{4.35}$$

Now the eigenvalues of (4.35) solve the equation

$$\begin{aligned} \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} &= 0 \\ \lambda\left(\lambda + \frac{b}{a}\right) + \frac{c}{a} &= 0 \\ a\lambda^2 + b\lambda + c &= 0 \end{aligned} \tag{4.36}$$

i.e. the auxiliary equation!

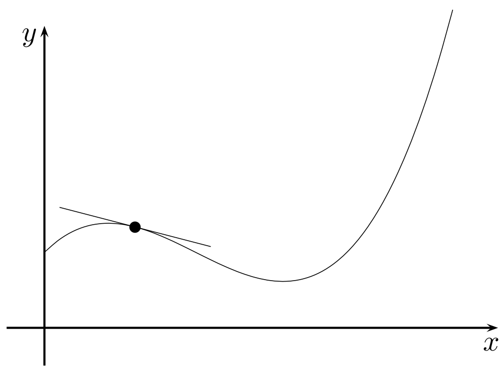
4.5 Functions of Two Variables

We have already seen functions of the form

$$\dot{x}_1 = f_1(x_1, x_2) \tag{4.37a}$$
$$\dot{x}_2 = f_2(x_1, x_2) \tag{4.37b}$$

However, so far, we have only considered f_1, f_2 when they are linear. What if they are no longer linear? Here we discuss derivatives of functions of two variables. Throughout we assume that we have "nice" functions.

For a function of one variable, $f(x)$, we can think of the graph $y = f(x)$ as a line in $\mathbb{R}^2$:



At any point x, $f'(x)$ gives the *gradient* of f at x, i.e. the gradient of the tangent at x. The derivative is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

assuming $f(x)$ is "nice". $\boxed{\text{ANALYSIS II}}$

For a function $f(x, y)$, the "graph" of $f(x, y)$ against (x, y) is a surface in $\mathbb{R}^3$, $z = f(x, y)$:

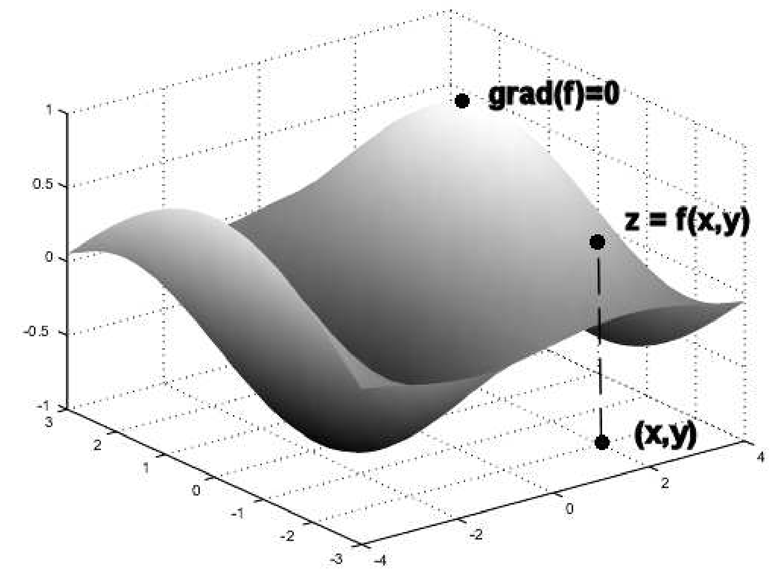


Figure 4.6: A surface $z = f(x, y)$.

Now the gradient depends on the direction we wish to move. Taking $\mathbf{x} = \binom{x}{y}$, the *directional derivative* is defined as

$$D_{\mathbf{v}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}$$

Equivalently this is

$$D_{\mathbf{v}} f(\mathbf{x}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \cdot \hat{\mathbf{v}}$$

where $\hat{\mathbf{v}}$ is a unit vector in direction $\mathbf{v}$. For $\mathbf{v} = \binom{1}{0}$ or $\binom{0}{1}$, this is just $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, where, for example,

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

For shorthand purposes, $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$ is denoted $\nabla f(x, y)$, or grad(f), the *gradient* of f.

The chain rule for functions of two variables

Say we have $f(x, y)$, where x, y are themselves functions of some variable t. What is $\frac{df}{dt}$? This depends on how x and y change with time. We have

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

For example, if $f = x^2 + y$ with $x = 3t^2 + 1$, $y = 2t - 4$, then

$$\begin{aligned} \frac{df}{dt} &= \underbrace{2(3t^2 + 1)}_{\frac{\partial f}{\partial x}} \cdot \underbrace{6t}_{\frac{dx}{dt}} + \underbrace{1}_{\frac{\partial f}{\partial y}} \cdot \underbrace{(2)}_{\frac{dy}{dt}} \\ &= 12t(3t^2 + 1) + 2. \end{aligned}$$

Level curves

For a function $f(x, y)$, we can ask ourselves at which points $f(x, y) = k$, where k is some constant. For "nice" functions, these points (x, y) lie on smooth curves, called *level curves* (or sometimes level sets). For our graph they will look something like this (think of contours of a map):

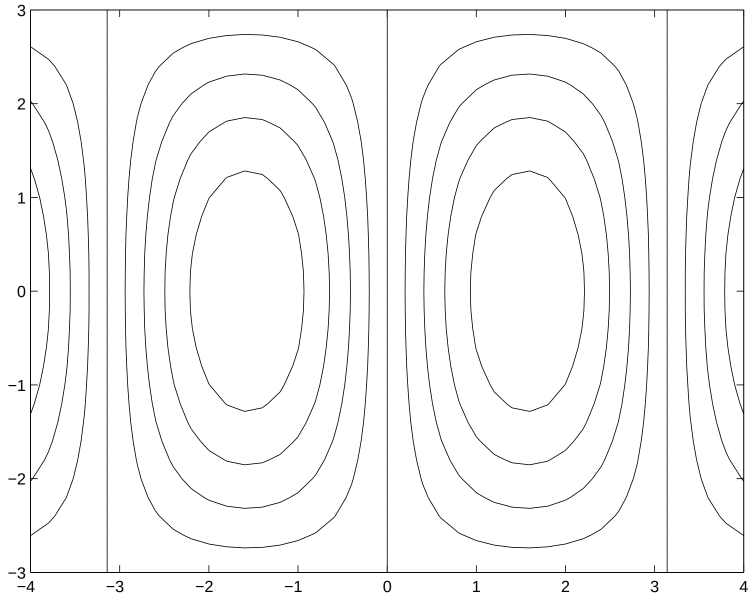


Figure 4.7: Contour plot (level sets) for previous surface.

The question is how can we walk around the hill keeping at constant height.

Parametric Curves

A smooth curve in $\mathbb{R}^2$ is parametric if it can be written as a function of one variable, say t, i.e. $(x(t), y(t))$; t now says "how far along the line we are". GEOMETRY AND MOTION

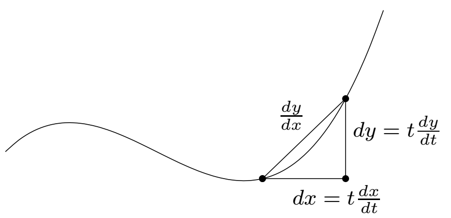
For example:

$$(x(t), y(t)) = (t^2 + 5, \tfrac{1}{t}) \qquad : \qquad 1 \leq t \leq 35\tfrac{1}{2}$$

or

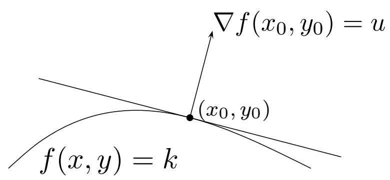
$$(x(t), y(t)) = (\cos t, \sin t) \qquad : \qquad 0 \leq t < 2\pi$$

For such curves, the vector $(\frac{dx}{dt}, \frac{dy}{dt})$ is tangent vector to the line at (x, y) (think of $\frac{dy}{dx}$ for $y = f(x)$ in 1-D):



so $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Theorem 4.9. *At a point* (x_0, y_0)*, the vector* $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Big|_{(x_0,y_0)}$*, i.e.* $\nabla f(x_0, y_0)$*, is* normal *to the level curve through* (x_0, y_0)*.*



Proof. On a level curve parametrised by t, i.e. given by $(x(t), y(t))$, we must have $\frac{df}{dt} = 0$, as the value of f does not change. By the chain rule,

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \nabla f \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \end{aligned}$$

So if $\nabla f \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = 0$, then ∇f is perpendicular to the tangent and hence to the curve. $\square$

Theorem 4.10. *The maximum value of $D_{\mathbf{v}} f(x, y)$ ("gradient in direction of $\mathbf{v}$") occurs in the direction of ∇f, with maximum value $|\nabla f(x, y)|$.*

Proof. Let θ denote the angle between $\nabla f(x, y)$ and $\mathbf{v}$. Then

$$\begin{aligned} D_{\mathbf{v}} f(x, y) &= \nabla f(x, y) \cdot \hat{\mathbf{v}} \\ &= |\nabla f(x, y)|\, |\hat{\mathbf{v}}| \cos\theta \\ &= |\nabla f(x, y)| \cos\theta \end{aligned}$$

The maximum occurs when $\cos\theta = 1 \implies \theta = 0$, i.e. $\mathbf{v}$ in the direction of $\nabla f(x, y)$. $\square$

So $\nabla f(x, y)$ is the direction of steepest ascent (think of walking up hills).

Approximation of a function of two variables

Recall that for one variable,

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \dots$$

For two variables there is a similar result: GEOMETRY AND MOTION

$$\begin{aligned} f(\mathbf{x} + \mathbf{h}) = f(x + h_1, y + h_2) \approx f(x, y) &+ h_1 \frac{\partial f}{\partial x}(x, y) + h_2 \frac{\partial f}{\partial y}(x, y) \\ &+ \frac{h_1^2}{2}\frac{\partial^2 f}{\partial x^2} + h_1 h_2 \frac{\partial^2 f}{\partial x \partial y} + \frac{h_2^2}{2}\frac{\partial^2 f}{\partial y^2} + \dots \end{aligned} \tag{4.38}$$

If $\mathbf{h}$ is small, then

$$f(\mathbf{x} + \mathbf{h}) \approx f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h}$$

If $f(\mathbf{x}) = 0$ then

$$f(\mathbf{x} + \mathbf{h}) \approx \nabla f(\mathbf{x}) \cdot \mathbf{h}$$

Approximating a solution of a 2×2 system near fixed points

We can now finally return to general, nonlinear 2×2 system of ODEs. Consider the system

$$\frac{dx}{dt} = f_1(x, y) \tag{4.39a}$$

$$\frac{dy}{dt} = f_2(x, y) \tag{4.39b}$$

Or, letting $\mathbf{x} = (x, y)$ and $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$, we have

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

Assume we have fixed point, that is an $\mathbf{x}_* = (x_*, y_*)$ such that $\mathbf{f}(\mathbf{x}_*) = \mathbf{0}$ and consider points near $\mathbf{x}_*$. We consider a small change $\mathbf{u} = (u, v)$ in $\mathbf{x}$ near $\mathbf{x}_*$, such that if $(u, v) = (0, 0)$ then we are at $\mathbf{x}_*$, and if u and v are small then we are nearby, given by $x = x_* + u$, $y = y_* + v$.

In essence then, we have changed coordinates, with $(u, v) = (0, 0)$ being the fixed point. For points near $\mathbf{x}_*$, we can write the system as follows. Consider equation (4.39a):

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}(x_* + u) \\ &= \frac{d}{dt}(x_*) + \frac{du}{dt} = \frac{du}{dt} \quad \text{since } x_* \text{ is a constant} \\ &= f_1(\mathbf{x}_* + \mathbf{u}) \\ &\approx f_1(\mathbf{x}_*) + \nabla f_1 \cdot \mathbf{u} \\ \text{so} \qquad \frac{du}{dt} &= \nabla f_1 \cdot \mathbf{u} \end{aligned}$$

Similarly, considering equation (4.39b):

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}(y_* + v) \\ &= \frac{d}{dt}(y_*) + \frac{dv}{dt} = \frac{dv}{dt} \quad \text{since } y_* \text{ is a constant} \\ &= f_2(\mathbf{x}_* + \mathbf{u}) \\ &\approx f_2(\mathbf{x}_*) + \nabla f_2 \cdot \mathbf{u} \\ \text{so} \qquad \frac{dv}{dt} &= \nabla f_2 \cdot \mathbf{u} \end{aligned}$$

Putting this all together, we get the following:

$$\dot{\mathbf{u}} = \begin{pmatrix} \frac{\partial f_1}{\partial x}(\mathbf{x}_*) & \frac{\partial f_1}{\partial y}(\mathbf{x}_*) \\ \frac{\partial f_2}{\partial x}(\mathbf{x}_*) & \frac{\partial f_2}{\partial y}(\mathbf{x}_*) \end{pmatrix} \mathbf{u} \tag{4.40}$$

That is, closed to the fixed point $\mathbf{x}_*$, the solutions behave like

$$\dot{\mathbf{x}} = D\mathbf{f}(\mathbf{x}_*)\mathbf{x}$$

i.e. a linear system where the stability of the fixed point $\mathbf{x}_*$ is given by the eigenvalues of $D\mathbf{f}(\mathbf{x}_*)$, the Jacobian matrix of $\mathbf{f}$ evaluated at the fixed point. In other words, the behaviour of solutions to nonlinear systems can be approximated by a linear system near fixed points (there is actually a theorem justifying this more rigorously, the Hartman-Grobman Theorem). All this now allows us to compute the stability of fixed points.

Example 4.11 (Lotka-Volterra)**.** Consider the predator-prey model

$$\frac{dN}{dt} = N(a - bP) \tag{4.41a}$$

$$\frac{dP}{dt} = P(cN - d) \tag{4.41b}$$

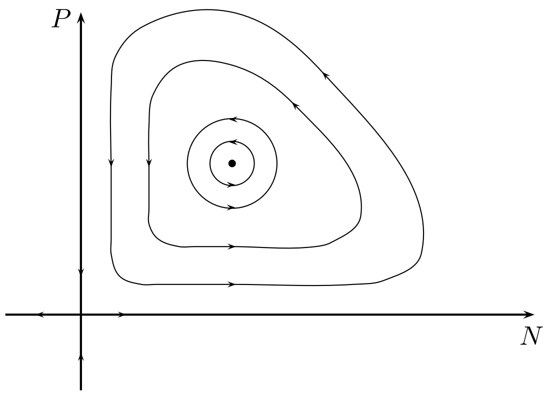
with $a, b, c, d > 0$.

The fixed points occur at $(0, 0)$ and $(\frac{d}{c}, \frac{a}{b})$, and the Jacobian matrix is

$$D\mathbf{f} = \begin{pmatrix} a - bP & -bN \\ cP & cN - d \end{pmatrix}$$

So, at $(0, 0)$, this is $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, with eigenvalues a, $-d$ and eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

At $(\frac{d}{c}, \frac{a}{b})$, we have $\begin{pmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{pmatrix}$ with eigenvalues $\pm i\sqrt{ad}$, a *centre*:



"Physically", the size of both populations oscillates.

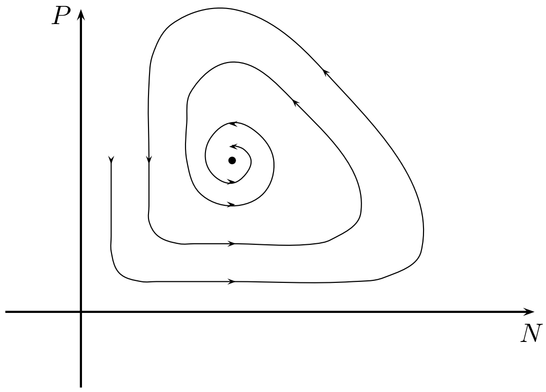
If we give the system a "bump", the Jacobian may change to

$$D\mathbf{f} = \begin{pmatrix} -\epsilon & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{pmatrix} \quad : \quad \epsilon \ll 1$$

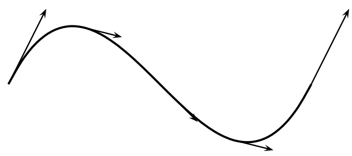
and the eigenvalues are now

$$\lambda_{\pm} = \frac{-\epsilon \pm \sqrt{\epsilon^2 - 4ad}}{2}$$

Now, $\Re(\lambda_{\pm}) < 0$ and so the long term behaviour is a fixed point:



Note: At every point on a solution $\mathbf{x}(t)$, $(\frac{dx}{dt}, \frac{dy}{dt})$ points in the direction of the solution:



The collection of all such arrows is called the *vector field.* For any (x, y), we can draw the vector $(\frac{dx}{dt}, \frac{dy}{dt})$ without solving anything, meaning that we can once again find integral curves without actually obtaining an explicit solution. This is exhibited in figure 4.8.

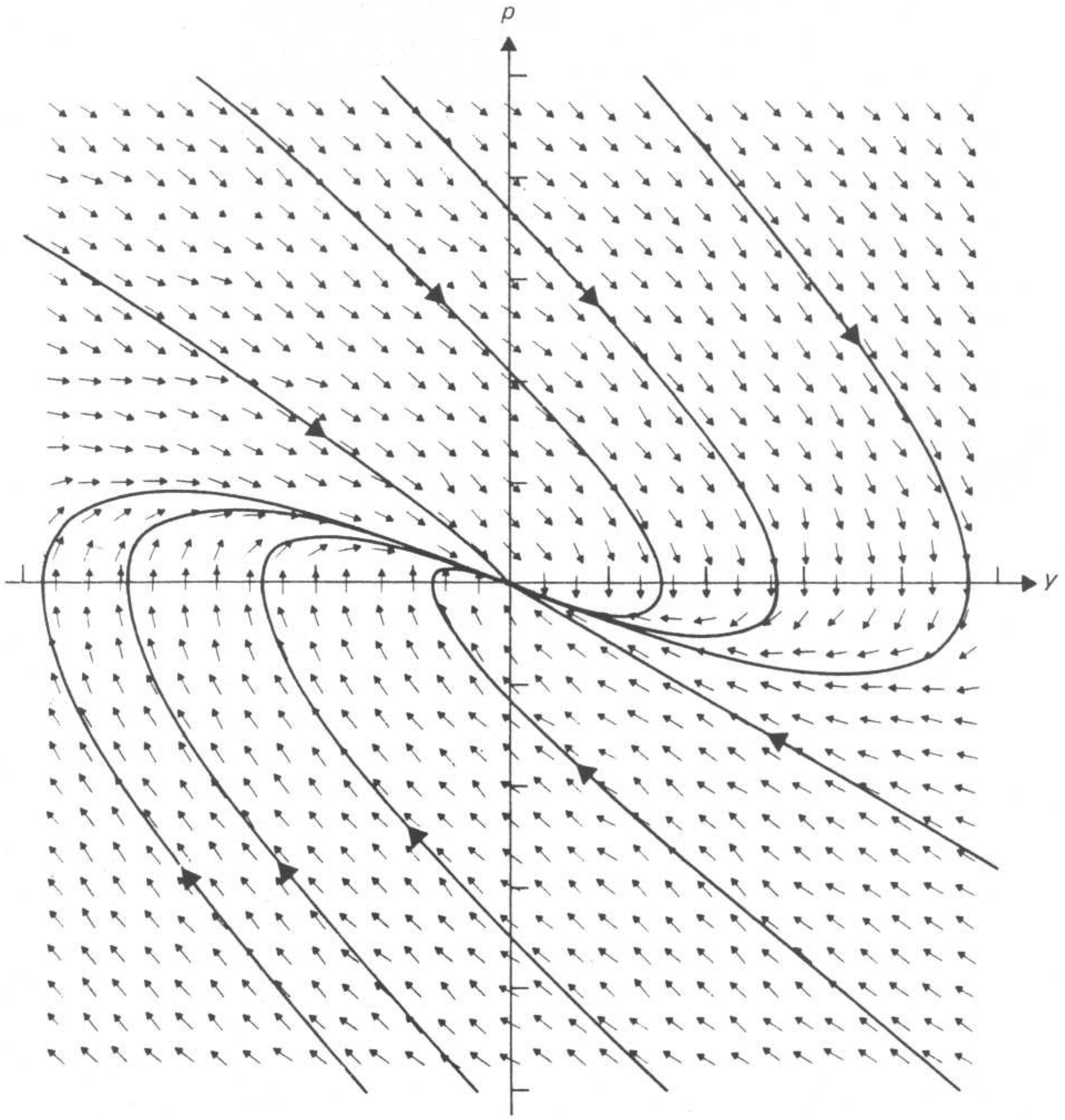


Figure 4.8: A vector field.

APPENDIX A

Ready Reference

This is a summary sheet for the first half of the module, solving first and second order ordinary differential equations. We assume conditions for FTC hold. You should learn all these techniques by heart, and practice, practice, practice!

First Order Differential Equations We consider the main scenarios

Trivial Case (Section 1.1)

$$\frac{dx}{dt} = f(t)$$

By Fundamental Theorem of Calculus simply integrate both sides with respect to t

$$x(t) = \int f(t)\ dt$$

Linear Non-homogeneous (Sections 1.3/1.4/1.5)

$$\frac{dx}{dt} + p(t)x = q(t)$$

Multiply both sides by an *Integrating Factor* $P(t) = \exp\left(\int p(t)dt\right)$ so that

$$\frac{d}{dt}(P(t)x(t)) = P(t)q(t)$$

Then integrate so that

$$x(t) = P(t)^{-1}\int_t P(s)q(s)ds + AP(t)^{-1}$$

Separable Equations (Section 1.6)

$$\frac{dx}{dt} = f(x)g(t)$$

First look for constant solutions, i.e. where $f(x) = 0$. Then look for non-constant solutions (so $f(x)$ never zero) and "divide both sides by $f(x)$, multiply both sides by dt and integrate".

$$\int \frac{dx}{f(x)} = \int g(t)dt$$

Autonomous First Order ODEs (Section 1.9)

$$\frac{dx}{dt} = f(x)$$

Look for *fixed points* x_*, which satisfy $f(x_*) = 0$, i.e. are points where $\frac{dx}{dt} = 0$. A fixed point x_* is *stable* if $f'(x_*) < 0$ and *unstable* if $f'(x_*) > 0$.

Second Order Ordinary Differential Equations With Constant Coefficients

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

The solution consists of $x(t) = x_c(t) + x_p(t)$ where $x_c(t)$, the *complementary solution*, solves the homogeneous case $f(t) = 0$ and $x_p(t)$, the *particular integral*, gives the $f(t)$.

The Complementary Solution Solves

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

Find the roots to the *auxiliary equations*

$$a\lambda^2 + b\lambda + c = 0$$

i.e. $\lambda_\pm = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ then we have

- Real roots k_1, k_2 complementary solution is

$$Ae^{k_1 t} + Be^{k_2 t}$$

- Repeated real root k complementary solution is

$$Ae^{kt} + Bte^{kt}$$

- Complex roots $p \pm iq$ complementary solution is

$$e^{pt}(A\sin(qt) + B\cos(qt))$$

or

$$Ae^{pt}\cos(qt - \phi)$$

The Particular Integral Functions to "guess":

$f(t)$	Try solution $x_p(t) =$
ae^{kt} (k not a root)	Ae^{kt}
ae^{kt} (k a root)	Ate^{kt} or At^2e^{kt}
$a\sin(\omega t)$ or $a\cos(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
at^n where $n \in N$	$P(t)$ general polynomial of degree n
$at^n e^{kt}$	$P(t)e^{kt}$, $P(t)$ general polynomial of degree n
$t^n(a\sin(\omega t) + b\cos(\omega t)$	$P_1(t)\sin(\omega t) + P_2(t)\cos(\omega t)$
	where $P_i(t)$ general polynomial of degree n
$e^{kt}(a\sin(\omega t) + b\cos(\omega t))$	$e^{kt}(A\sin(\omega t) + B\cos(\omega t))$