

Problem 1

Use truth tables to show that the following pairs of sentences are equivalent:

- (a) $P \Rightarrow Q, \neg Q \Rightarrow \neg P$
- (b) $P \Leftrightarrow \neg Q, ((P \wedge \neg Q) \vee (\neg P \wedge Q))$

Solution

(a) Let $\Delta_1 = P \Rightarrow Q$ and $\Delta_2 = \neg Q \Rightarrow \neg P$

We then end up with the following truth table:

World	P	Q	Δ_1	Δ_2
1	0	0	1	1
2	0	1	1	1
3	1	0	0	0
4	1	1	1	1

We can thus observe

$$M(\Delta_1) = w_1, w_2, w_4$$

$$M(\Delta_2) = w_1, w_2, w_4$$

$$M(\Delta_1) = M(\Delta_2),$$

$$\Rightarrow \Delta_1 = \Delta_2$$

(b) Let $\Delta_1 = P \Leftrightarrow \neg Q$ and $\Delta_2 = ((P \vee \neg Q) \vee (\neg P \wedge Q))$

We then end up with the following truth table:

World	P	Q	Δ_1	Δ_2
1	0	0	0	0
2	0	1	1	1
3	1	0	1	1
4	1	1	0	0

We can thus observe

$$M(\Delta_1) = w_2, w_3$$

$$M(\Delta_2) = w_2, w_3$$

$$M(\Delta_1) = M(\Delta_2),$$

$$\Rightarrow \Delta_1 = \Delta_2$$

Problem 2

Decide whether each of the following sentences is valid, satisfiable, or neither:

- (a) $(\text{Smoke} \Rightarrow \text{Fire}) \Rightarrow (\neg \text{Smoke} \Rightarrow \neg \text{Fire})$
- (b) $(\text{Smoke} \Rightarrow \text{Fire}) \Rightarrow ((\text{Smoke} \vee \text{Heat}) \Rightarrow \text{Fire})$
- (c) $((\text{Smoke} \wedge \text{Heat}) \Rightarrow \text{Fire}) \Leftrightarrow ((\text{Smoke} \Rightarrow \text{Fire}) \vee (\text{Heat} \Rightarrow \text{Fire}))$

Solution

World	Smoke	Heat	Fire	(a)	(b)	(c)
1	0	0	0	1	1	1
2	0	0	1	1	0	1
3	0	1	0	0	1	1
4	0	1	0	0	1	1
5	1	0	0	1	1	1
6	1	0	1	1	1	1
7	1	1	0	1	1	1
8	1	1	1	1	1	1

This implies (a) is satisfiable, (b) is satisfiable, and (c) is valid.

Problem 3

Consider the following:

If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.

- Represent the information using a propositional logic knowledge base.
- Convert the knowledge base into CNF.
- Can you prove that the unicorn is mythical? magical? horned?

Solution

(a) We will use the properties names: Mythical, Mortal, Mammal, Horned
We can thus represent the knowledge base by:

$$KB = \begin{cases} \text{Mythical} \Rightarrow \neg \text{Mortal} \\ \neg \text{Mythical} \Rightarrow (\text{Mortal} \wedge \text{Mammal}) \\ (\neg \text{Mortal} \vee \text{Mammal}) \Rightarrow \text{Horned} \\ \text{Horned} \Rightarrow \text{Magical} \end{cases}$$

(b) We first convert the knowledge base to a CNF:

$$(\neg \text{Mythical} \vee \neg \text{Mortal}) \wedge (\text{Mythical} \vee \text{Mortal}) \wedge (\text{Mythical} \vee \text{Mammal}) \wedge (\text{Mortal} \vee \text{Horned}) \wedge (\neg \text{Mammal} \vee \text{Horned}) \wedge (\neg \text{Horned} \vee \text{Magical})$$

(c) We can then use this CNF to attempt to resolve each of the statements:

	Mythical	Magical	Horned
1.	$\neg \text{Mythical} \vee \neg \text{Mortal}$	$\neg \text{Mythical} \vee \neg \text{Mortal}$	$\neg \text{Mythical} \vee \neg \text{Mortal}$
2.	$\text{Mythical} \vee \text{Mortal}$	$\text{Mythical} \vee \text{Mortal}$	$\text{Mythical} \vee \text{Mortal}$
3.	$\text{Mythical} \vee \text{Mammal}$	$\text{Mythical} \vee \text{Mammal}$	$\text{Mythical} \vee \text{Mammal}$
4.	$\text{Mortal} \vee \text{Horned}$	$\text{Mortal} \vee \text{Horned}$	$\text{Mortal} \vee \text{Horned}$
5.	$\neg \text{Mammal} \vee \text{Horned}$	$\neg \text{Mammal} \vee \text{Horned}$	$\neg \text{Mammal} \vee \text{Horned}$
6.	$\neg \text{Horned} \vee \text{Magical}$	$\neg \text{Horned} \vee \text{Magical}$	$\neg \text{Horned} \vee \text{Magical}$
7.	$\neg \text{Mythical}$	$\neg \text{Magical}$	$\neg \text{Horned}$
8.	$\text{Mortal} < 2, 7 >$	$\neg \text{Horned} < 6, 7 >$	$\text{Mortal} < 4, 7 >$
9.	$\text{Mammal} < 3, 7 >$	$\text{Mortal} < 4, 8 >$	$\text{Mythical} < 3, 7 >$
10.	$\text{Horned} < 6, 7 >$	$\neg \text{Mortal} < 1, 9 >$	$\neg \text{Mythical} < 1, 8 >$
11.	$\text{Magical} < 6, 10 >$	$\Phi < 9, 10 >$	$\Phi < 9, 10 >$

Thus we can prove that the unicorn must be Magical and Horned.

Problem 4

Consider the two NNF circuits in Figure 1 and Figure 2.
Identify whether they are decomposable, deterministic, smooth and why.

Solution

Decomposability: any inputs of an AND gate have no shared inputs.

Determinism: inputs to OR gates are mutually exclusive.

Smoothness: all input atoms to OR gates are identical.

Thus Figure 1 is decomposable and deterministic but not smooth

(second OR on the second OR level: $\{C\}$ vs $\{\neg C, \neg D\}$)

And Figure 2 is decomposable and smooth but non-deterministic

(first OR on the second OR level: $\{\neg A, B\}$)

Problem 5

The weight of a truth assignment is defined as the product of its literals weights.

The WMC of a formula is the added weight of its satisfying assignments.

If we assign the weights of literals to all the leaf nodes on a circuit,

- (i) the count of each \wedge node is the product of the counts of its children
- (ii) the count of each \vee node is the sum of the counts of its children.

Suppose we have the following literal weights:

$$w(A)=0.2, w(B)=0.4, w(C)=0.6, w(D)=0.8$$

$$w(\neg A)=0.8, w(\neg B)=0.6, w(\neg C)=0.4, w(\neg D)=0.2.$$

Use the above values to answer the following:

- (a) Compute the WMC for formula $(\neg A \wedge B) \vee (\neg B \wedge A)$.
- (b) Consider the smooth d-DNNF circuit in Figure 3.

What is the relation between the root count and the WMC?

- (c) Compute the WMC for the smooth d-DNNF circuit in Figure 4.

Solution

(a)

$$\begin{aligned} \text{Weight}(\neg A \wedge B) \vee (\neg B \wedge A) &= (0.8)(0.4) + (0.2)(0.6) \\ &= (0.32) + (0.12) = (0.44) \end{aligned}$$

(b) The root count is (0.44); they match!

(c) Let $w(x)$ denote the weight of the literal x .

$$\begin{aligned} \text{Count(Figure 4)} &= [(w(\neg A))(w(B)) + (w(A))(w(\neg B))] \cdot [(w(C))(w(D)) + (w(\neg C))(w(\neg D))] \\ &\quad + [w(\neg A)w(\neg B) + w(A)w(B)] \cdot [w(C)w(\neg D) + w(\neg C)w(D)] \\ &= [(0.8)(0.4) + (0.6)(0.2)] \cdot [(0.6)(0.8) + (0.2)(0.4)] \\ &\quad + [(0.8)(0.6) + (0.2)(0.4)] \cdot [(0.2)(0.6) + (0.4)(0.8)] \\ &= 2 \cdot [(0.32) + (0.12)] \cdot [(0.48) + (0.08)] = (0.4928) \end{aligned}$$