Problem 1

Use truth tables to show that the following pairs of sentences are equivalent:
(a) $P \Rightarrow Q, \neg Q \Rightarrow \neg P$
(b) $P \Leftrightarrow \neg Q, ((P \land \neg Q) \lor (\neg P \land Q))$

Solution

(a) Let $\Delta_1 = P \Rightarrow Q$ and $\Delta_2 = \neg Q \Rightarrow \neg P$
We then end up with the following truth table:

World	P	Q	Δ_1	Δ_2
1	0	0	1	1
2	0	1	1	1
3	1	0	0	0
4	1	1	1	1

We can thus observe

$$\begin{aligned} M(\Delta_1) &= w_1, w_2, w_4 \\ M(\Delta_2) &= w_1, w_2, w_4 \\ M(\Delta_1) &= M(\Delta_2), \\ &\implies \Delta_1 = \Delta_2 \end{aligned}$$

(b) Let $\Delta_1 = P \Leftrightarrow \neg Q$ and $\Delta_2 = ((P \lor \neg Q) \lor (\neg P \land Q))$
We then end up with the following truth table:

World	P	Q	Δ_1	Δ_2
1	0	0	0	0
2	0	1	1	1
3	1	0	1	1
4	1	1	0	0

We can thus observe

$$\begin{aligned} M(\Delta_1) &= w_2, w_3 \\ M(\Delta_2) &= w_2, w_3 \\ M(\Delta_1) &= M(\Delta_2), \\ &\implies \Delta_1 = \Delta_2 \end{aligned}$$

Problem 2

Decide whether each of the following sentences is valid, satisfiable, or neither:

(a) (Smoke $\Rightarrow$ Fire) $\Rightarrow$ ($\neg$ Smoke $\Rightarrow$ $\neg$ Fire)

(b) (Smoke $\Rightarrow$ Fire) $\Rightarrow$ ((Smoke $\vee$ Heat) $\Rightarrow$ Fire)

(c) ((Smoke $\wedge$ Heat) $\Rightarrow$ Fire) $\Leftrightarrow$ ((Smoke $\Rightarrow$ Fire) $\vee$ (Heat $\Rightarrow$ Fire))

Solution

World	Smoke	Heat	Fire	(a)	(b)	(c)
1	0	0	0	1	1	1
2	0	0	1	1	0	1
3	0	1	0	0	1	1
4	0	1	0	0	1	1
5	1	0	0	1	1	1
6	1	0	1	1	1	1
7	1	1	0	1	1	1
8	1	1	1	1	1	1

This implies (a) is satisfiable, (b) is satisfiable, and (c) is valid.

Problem 3

Consider the following:

If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.

(a) Represent the information using a propositional logic knowledge base.
(b) Convert the knowledge base into CNF.
(c) Can you prove that the unicorn is mythical? magical? horned?

Solution

(a) We will use the properties names: Mythical, Mortal, Mammal, Horned
We can thus represent the knowledge base by:

$$KB = \begin{cases} \text{Mythical} \ \Rightarrow \neg\text{Mortal} \\ \neg\text{Mythical} \Rightarrow (\text{Mortal} \wedge \text{Mammal}) \\ (\neg\text{Mortal} \vee \text{Mammal}) \Rightarrow \text{Horned} \\ \text{Horned} \Rightarrow \text{Magical} \end{cases}$$

(b) We first convert the knowledge base to a CNF:

$$(\neg\text{Mythical} \vee \neg\text{Mortal}) \wedge (\text{Mythical} \vee \text{Mortal}) \wedge (\text{Mythical} \vee \text{Mammal}) \wedge \\ (\text{Mortal} \vee \text{Horned}) \wedge (\neg\text{Mammal} \vee \text{Horned}) \wedge (\neg\text{Horned} \vee \text{Magical})$$

(c) We can then use this CNF to attempt to resolve each of the statements:

	Mythical	Magical	Horned
1.	$\neg$Mythical $\vee$ $\neg$Mortal	$\neg$Mythical $\vee$ $\neg$Mortal	$\neg$Mythical $\vee$ $\neg$Mortal
2.	Mythical $\vee$ Mortal	Mythical $\vee$ Mortal	Mythical $\vee$ Mortal
3.	Mythical $\vee$ Mammal	Mythical $\vee$ Mammal	Mythical $\vee$ Mammal
4.	Mortal $\vee$ Horned	Mortal $\vee$ Horned	Mortal $\vee$ Horned
5.	$\neg$Mammal $\vee$ Horned	$\neg$Mammal $\vee$ Horned	$\neg$Mammal $\vee$ Horned
6.	$\neg$Horned $\vee$ Magical	$\neg$Horned $\vee$ Magical	$\neg$Horned $\vee$ Magical
7.	$\neg$Mythical	$\neg$Magical	$\neg$Horned
8.	Horned $< 3, 7 >$	$\neg$Horned $< 6, 7 >$	Mortal $< 4, 7 >$
9.	Mortal $< 2, 7 >$	Mortal $< 4, 8 >$	Mythical $< 3, 7 >$
10.	Magical $< 6, 8 >$	$\neg$Mortal $< 1, 9 >$	$\neg$Mythical $< 1, 8 >$
11.	...	$\Phi < 9, 10 >$	$\Phi < 9, 10 >$

Thus we can prove that the unicorn must be Magical and Horned.

Problem 4

Consider the two NNF circuits in Figure 1 and Figure 2.
Identify whether they are decomposable, deterministic, smooth and why.

Solution

Decomposability: any inputs of an AND gate have no shared inputs.
Determinism: inputs to OR gates are mutually exclusive.
Smoothness: all input atoms to OR gates are identical.
Thus Figure 1 is decomposable and deterministic but not smooth
(second OR on the second OR level: $\{\text{C}\}$ vs $\{\neg C, \neg D\}$)
And Figure 2 is decomposable and smooth but non-deterministic
(first OR on the second OR level: $\{\neg A, B\}$)

Problem 5

The weight of a truth assignment is defined as the product of its literals weights.
The WMC of a formula is the added weight of its satisfying assignments.
If we assign the weights of literals to all the leaf nodes on a circuit,

(i) the count of each $\wedge$ node is the product of the counts of its children
(ii) the count of each $\vee$ node is the sum of the counts of its children.

Suppose we have the following literal weights:

w(A)=0.2, w(B)=0.4, w(C)=0.6, w(D)=0.8
w(¬A)=0.8, w(¬B)=0.6, w(¬C)=0.4 w(¬D)=0.2.

Use the above values to answer the following:

(a) Compute the WMC for formula (¬A ∧ B) ∨ (¬ B ∧ A).
(b) Consider the smooth d-DNNF circuit in Figure 3.
What is the relation between the root count and the WMC?
(c) Compute the WMC for the smooth d-DNNF circuit in Figure 4.

Solution

(a)

$$\begin{aligned}\text{Weight}(\neg A \wedge B) \vee (\neg B \wedge A) &= (0.8)(0.4) + (0.2)(0.6) \\ &= (0.32) + (0.12) = (0.44)\end{aligned}$$

(b) The root count is (0.44); they match!
(c) Let w(x) denote the weight of the literal x.

$$\begin{aligned}\text{Count(Figure 4)} &= [(w(\neg A))(w(B)) + (w(A))(w(\neg B))] \cdot [(w(C))(w(D)) + (w(\neg C))(w(\neg D))] \\ &\quad + [w(\neg A)w(\neg B) + w(A)w(B)] \cdot [w(C)w(\neg D) + w(\neg C)w(D)] \\ &= [(0.8)(0.4) + (0.6)(0.2)] \cdot [(0.6)(0.8) + (0.2)(0.4)] \\ &\quad + [(0.8)(0.6) + (0.2)(0.4)] \cdot [(0.2)(0.6) + (0.4)(0.8)] \\ &= 2 \cdot [(0.32) + (0.12)] \cdot [(0.48) + (0.08)] = (0.4928)\end{aligned}$$