1. Proof of: $\tan x + \frac{\cos x}{1+\sin x} = \frac{1}{\cos x}$

Step 1: Convert $\tan x$ to sine and cosine.

$$LHS = \frac{\sin x}{\cos x} + \frac{\cos x}{1 + \sin x}$$

Step 2: Find a common denominator, which is $\cos x(1 + \sin x)$.

$$= \frac{\sin x(1 + \sin x) + \cos x(\cos x)}{\cos x(1 + \sin x)}$$

Step 3: Distribute and simplify the numerator.

$$= \frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1 + \sin x)}$$

Step 4: Apply the Pythagorean identity $\sin^2 x + \cos^2 x = 1$.

$$= \frac{\sin x + 1}{\cos x(1 + \sin x)}$$

Step 5: Cancel the common term $(1 + \sin x)$.

$$= \frac{1}{\cos x}$$

Result: $LHS = RHS$

---

2. Proof of: $\tan^2 x + 1 = \sec^2 x$

Step 1: Convert $\tan^2 x$ to sine and cosine.

$$LHS = \frac{\sin^2 x}{\cos^2 x} + 1$$

Step 2: Find a common denominator.

$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$$

Step 3: Apply the Pythagorean identity $\sin^2 x + \cos^2 x = 1$.

$$= \frac{1}{\cos^2 x}$$

Step 4: Use the definition of secant.

$$= \sec^2 x$$

Result: $LHS = RHS$

---

3. Proof of: $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2 \tan x \sec x$

Step 1: Find a common denominator, which is $(1 - \sin x)(1 + \sin x) = 1 - \sin^2 x$.

$$LHS = \frac{(1 + \sin x) - (1 - \sin x)}{1 - \sin^2 x}$$

Step 2: Simplify the numerator and the denominator.

$$= \frac{2 \sin x}{\cos^2 x}$$

Step 3: Separate the fraction into factors.

$$= 2 \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

Step 4: Convert to tan and sec.

$$= 2 \tan x \sec x$$

Result: $LHS = RHS$

---

4. Proof of: $\tan x + \cot x = \sec x \csc x$

Step 1: Convert to sine and cosine.

$$LHS = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

Step 2: Find a common denominator.

$$= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$

Step 3: Apply the Pythagorean identity.

$$= \frac{1}{\sin x \cos x}$$

Step 4: Separate factors.

$$= \frac{1}{\cos x} \cdot \frac{1}{\sin x} = \sec x \csc x$$

Result: $LHS = RHS$

---

5. Proof of: $\frac{1+\tan^2 x}{1-\tan^2 x} = \frac{1}{\cos^2 x - \sin^2 x}$

Step 1: Substitute $1 + \tan^2 x$ with $\sec^2 x$ (Pythagorean identity).

$$LHS = \frac{\sec^2 x}{1 - \frac{\sin^2 x}{\cos^2 x}}$$

Step 2: Simplify the denominator by finding a common denominator of $\cos^2 x$.

$$= \frac{\sec^2 x}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}$$

Step 3: Rewrite $\sec^2 x$ as $1/\cos^2 x$.

$$= \frac{\frac{1}{\cos^2 x}}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}$$

Step 4: Multiply by the reciprocal.

$$= \frac{1}{\cos^2 x} \cdot \frac{\cos^2 x}{\cos^2 x - \sin^2 x}$$

Step 5: Cancel $\cos^2 x$.

$$= \frac{1}{\cos^2 x - \sin^2 x}$$

Result: $LHS = RHS$

---

6. Proof of: $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$

Step 1: Convert $\tan^2 x$ to sine and cosine.

$$LHS = \frac{\sin^2 x}{\cos^2 x} - \sin^2 x$$

Step 2: Factor out $\sin^2 x$.

$$= \sin^2 x \left(\frac{1}{\cos^2 x} - 1 \right)$$

Step 3: Recognize that $\frac{1}{\cos^2 x} = \sec^2 x$.

$$= \sin^2 x (\sec^2 x - 1)$$

Step 4: Use the identity $\sec^2 x - 1 = \tan^2 x$.

$$= \sin^2 x \tan^2 x$$

Result: $LHS = RHS$

---

7. Proof of: $\frac{1 - \cos x}{\sin x} + \frac{\sin x}{1 - \cos x} = 2 \csc x$

Step 1: Find a common denominator $\sin x (1 - \cos x)$.

$$LHS = \frac{(1 - \cos x)^2 + \sin^2 x}{\sin x(1 - \cos x)}$$

Step 2: Expand the numerator.

$$= \frac{1 - 2\cos x + \cos^2 x + \sin^2 x}{\sin x(1 - \cos x)}$$

Step 3: Use $\cos^2 x + \sin^2 x = 1$.

$$= \frac{1 - 2\cos x + 1}{\sin x(1 - \cos x)} = \frac{2 - 2\cos x}{\sin x(1 - \cos x)}$$

Step 4: Factor the numerator.

$$= \frac{2(1 - \cos x)}{\sin x(1 - \cos x)}$$

Step 5: Cancel common terms.

$$= \frac{2}{\sin x} = 2\csc x$$

Result: $LHS = RHS$

---

8. Proof of: $\frac{\sec x - 1}{\sec x + 1} = \frac{1 - \cos x}{1 + \cos x}$

Step 1: Rewrite $\sec x$ as $1/\cos x$.

$$LHS = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1}$$

Step 2: Multiply numerator and denominator by $\cos x$ to clear fractions.

$$= \frac{\cos x\left(\frac{1}{\cos x} - 1\right)}{\cos x\left(\frac{1}{\cos x} + 1\right)}$$

Step 3: Distribute.

$$= \frac{1 - \cos x}{1 + \cos x}$$

Result: $LHS = RHS$

---

9. Proof of: $1 + \cot^2 x = \csc^2 x$

Step 1: Convert $\cot^2 x$ to cosine and sine.

$$LHS = 1 + \frac{\cos^2 x}{\sin^2 x}$$

Step 2: Find a common denominator.

$$= \frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$

Step 3: Apply Pythagorean identity.

$$= \frac{1}{\sin^2 x}$$

Step 4: Definition of cosecant.

$$= \csc^2 x$$

Result: $LHS = RHS$

---

10. Proof of: $\frac{\csc^2 x - 1}{\csc^2 x} = \cos^2 x$

Step 1: Use the identity $\csc^2 x - 1 = \cot^2 x$.

$$LHS = \frac{\cot^2 x}{\csc^2 x}$$

Step 2: Convert to sine and cosine.

$$= \frac{\frac{\cos^2 x}{\sin^2 x}}{\frac{1}{\sin^2 x}}$$

Step 3: Multiply by the reciprocal.

$$= \frac{\cos^2 x}{\sin^2 x} \cdot \frac{\sin^2 x}{1}$$

Step 4: Cancel $\sin^2 x$.

$$= \cos^2 x$$

Result: $LHS = RHS$

---

11. Proof of: $\frac{\cot x - 1}{\cot x + 1} = \frac{1 - \tan x}{1 + \tan x}$

Step 1: Rewrite $\cot x$ as $1/\tan x$.

$$LHS = \frac{\frac{1}{\tan x} - 1}{\frac{1}{\tan x} + 1}$$

Step 2: Multiply numerator and denominator by $\tan x$.

$$= \frac{\tan x \left(\frac{1}{\tan x} - 1 \right)}{\tan x \left(\frac{1}{\tan x} + 1 \right)}$$

Step 3: Distribute.

$$= \frac{1 - \tan x}{1 + \tan x}$$

Result: $LHS = RHS$

---

12. Proof of: $(\sin x + \cos x)(\tan x + \cot x) = \sec x + \csc x$

Step 1: Convert the second bracket to sine and cosine.

$$LHS = (\sin x + \cos x)\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)$$

Step 2: Combine fractions in the second bracket.

$$= (\sin x + \cos x)\left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right)$$

Step 3: Simplify the numerator to 1.

$$= (\sin x + \cos x)\left(\frac{1}{\sin x \cos x}\right)$$

Step 4: Distribute the fraction.

$$= \frac{\sin x}{\sin x \cos x} + \frac{\cos x}{\sin x \cos x}$$

Step 5: Cancel terms.

$$= \frac{1}{\cos x} + \frac{1}{\sin x} = \sec x + \csc x$$

Result: $LHS = RHS$

---

13. Proof of: $\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$

Step 1: Factor the numerator as a sum of cubes $(a^3 + b^3 = (a + b)(a^2 - ab + b^2))$.

$$LHS = \frac{(\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)}{\sin x + \cos x}$$

Step 2: Cancel the $(\sin x + \cos x)$ term.

$$= \sin^2 x - \sin x \cos x + \cos^2 x$$

Step 3: Group $\sin^2 x + \cos^2 x = 1$.

$$= 1 - \sin x \cos x$$

Result: $LHS = RHS$

---

14. Proof of: $\frac{\cos x + 1}{\sin^3 x} = \frac{\csc x}{1 - \cos x}$

Step 1: Rewrite LHS denominator $\sin^3 x$ as $\sin x \cdot \sin^2 x$.

$$LHS = \frac{1 + \cos x}{\sin x(1 - \cos^2 x)}$$

Step 2: Factor $(1 - \cos^2 x)$ as difference of squares.

$$= \frac{1 + \cos x}{\sin x(1 - \cos x)(1 + \cos x)}$$

Step 3: Cancel $(1 + \cos x)$.

$$= \frac{1}{\sin x(1 - \cos x)}$$

Step 4: Separate terms.

$$= \frac{1}{\sin x} \cdot \frac{1}{1 - \cos x} = \frac{\csc x}{1 - \cos x}$$

Result: $LHS = RHS$

---

15. Proof of: $\frac{1 + \sin x}{1 - \sin x} - \frac{1 - \sin x}{1 + \sin x} = 4 \tan x \sec x$

Step 1: Find a common denominator $(1 - \sin^2 x) = \cos^2 x$.

$$LHS = \frac{(1 + \sin x)^2 - (1 - \sin x)^2}{\cos^2 x}$$

Step 2: Expand the numerators.

$$= \frac{(1 + 2\sin x + \sin^2 x) - (1 - 2\sin x + \sin^2 x)}{\cos^2 x}$$

Step 3: Simplify numerator.

$$= \frac{4\sin x}{\cos^2 x}$$

Step 4: Separate terms.

$$= 4 \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = 4\tan x \sec x$$

Result: $LHS = RHS$

---

16. Proof of: $\csc^4 x - \cot^4 x = \csc^2 x + \cot^2 x$

Step 1: Factor LHS as a difference of squares.

$$LHS = (\csc^2 x - \cot^2 x)(\csc^2 x + \cot^2 x)$$

Step 2: Apply identity $\csc^2 x - \cot^2 x = 1$.

$$= 1 \cdot (\csc^2 x + \cot^2 x)$$

Step 3: Simplify.

$$= \csc^2 x + \cot^2 x$$

Result: $LHS = RHS$

---

17. Proof of: $\frac{\sin^2 x}{\cos^2 x + 3\cos x + 2} = \frac{1 - \cos x}{2 + \cos x}$

Step 1: Replace $\sin^2 x$ with $1 - \cos^2 x$.

$$LHS = \frac{1 - \cos^2 x}{\cos^2 x + 3\cos x + 2}$$

Step 2: Factor numerator and denominator.

Numerator (Diff of squares): $(1 - \cos x)(1 + \cos x)$

Denominator (Quadratic factoring): $(\cos x + 1)(\cos x + 2)$

Step 3: Substitute back into fraction.

$$= \frac{(1 - \cos x)(1 + \cos x)}{(1 + \cos x)(2 + \cos x)}$$

Step 4: Cancel $(1 + \cos x)$.

$$= \frac{1 - \cos x}{2 + \cos x}$$

Result: $LHS = RHS$

---

18. Proof of: $\frac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$

Step 1: Rewrite denominator in terms of tangent.

$$LHS = \frac{\tan x + \tan y}{\frac{1}{\tan x} + \frac{1}{\tan y}}$$

Step 2: Combine fractions in the denominator.

$$= \frac{\tan x + \tan y}{\frac{\tan y + \tan x}{\tan x \tan y}}$$

Step 3: Multiply by reciprocal.

$$= (\tan x + \tan y) \cdot \frac{\tan x \tan y}{\tan x + \tan y}$$

Step 4: Cancel terms.

$$= \tan x \tan y$$

Result: $LHS = RHS$

---

19. Proof of: $\frac{1+\tan x}{1-\tan x} = \frac{\cos x+\sin x}{\cos x-\sin x}$

Step 1: Convert $\tan x$ to $\sin x / \cos x$.

$$LHS = \frac{1 + \frac{\sin x}{\cos x}}{1 - \frac{\sin x}{\cos x}}$$

Step 2: Multiply numerator and denominator by $\cos x$.

$$= \frac{\cos x\left(1 + \frac{\sin x}{\cos x}\right)}{\cos x\left(1 - \frac{\sin x}{\cos x}\right)}$$

Step 3: Distribute.

$$= \frac{\cos x + \sin x}{\cos x - \sin x}$$

Result: $LHS = RHS$

---

20. Proof of: $(\sin x - \tan x)(\cos x - \cot x) = (\sin x - 1)(\cos x - 1)$

Step 1: Factor out $\sin x$ from first term and $\cos x$ from second term.

First term: $\sin x - \frac{\sin x}{\cos x} = \sin x\left(1 - \frac{1}{\cos x}\right)$

Second term: $\cos x - \frac{\cos x}{\sin x} = \cos x\left(1 - \frac{1}{\sin x}\right)$

Step 2: Multiply them together.

$$LHS = \sin x \cos x \left(1 - \frac{1}{\cos x}\right)\left(1 - \frac{1}{\sin x}\right)$$

Step 3: Expand the brackets.

$$= \sin x \cos x \left(1 - \frac{1}{\sin x} - \frac{1}{\cos x} + \frac{1}{\sin x \cos x} \right)$$

Step 4: Distribute $\sin x \cos x$.

$$= \sin x \cos x - \cos x - \sin x + 1$$

Step 5: Factor by grouping.

$$= \sin x (\cos x - 1) - 1(\cos x - 1)$$

$$= (\sin x - 1)(\cos x - 1)$$

Result: $LHS = RHS$