MCR3U Final Examination – Answer Key (Jan 2023)

This key corresponds to `final-exam-2023.md`. Exact values are used where appropriate; approximations are indicated.

Part A – Short Answers

1. **If** $f(x) = -2\sqrt{2x-1} + 5$, **determine** $f(5)$.

$2(5) - 1 = 9$, $\sqrt{9} = 3$ \

$f(5) = -2(3) + 5 = -6 + 5 = \boxed{-1}$

2. For $y = -2(x+3)^2 + 7$:

- (a) **Vertex**: already in vertex form $a(x-h)^2 + k$ with $h = -3, k = 7$ \

$\Rightarrow \boxed{(-3, 7)}$

- (b) y**-intercept**: set $x = 0$ \

$y = -2(3)^2 + 7 = -18 + 7 = -11$ \

$\Rightarrow \boxed{(0, -11)}$

- © **Max value**: opens down ($a = -2 < 0$), so max is y-value of vertex \

$\Rightarrow \boxed{7}$

- (d) **Domain**: parabola defined for all real x \

$\Rightarrow \boxed{\mathbb{R}}$

- (e) **Range**: y is at most 7 \

$\Rightarrow \boxed{y \leq 7}$

3. For $y = \dfrac{3}{x-1} + 2$:

- (a) **Parent function**: $\boxed{y = \dfrac{1}{x}}$

- (b) **Vertical stretch factor**: coefficient of parent is 3 \

$\Rightarrow \boxed{3}$

- © **Horizontal translation**: $x - 1$ in denominator means 1 unit right \

$\Rightarrow \boxed{\text{1 unit right}}$

4. **State the inverse of:**

- (a) $f(x) = x - 3$ \

$y = x - 3 \Rightarrow x = y - 3 \Rightarrow y = x + 3$ \

$\boxed{f^{-1}(x) = x + 3}$

- (b) $f(x) = 3x$ \

$y = 3x \Rightarrow x = 3y \Rightarrow y = \dfrac{x}{3}$ \

$\boxed{f^{-1}(x) = \dfrac{x}{3}}$

- © $f(x) = x^2$ \

Inverse relation: $y = \pm\sqrt{x}$; as a function, with domain restricted to $x \geq 0$, $\boxed{f^{-1}(x) = \sqrt{x}}$.

5. **Simplify:**

- (a) $\dfrac{3x}{5x} = \dfrac{3}{5}$ (for $x \neq 0$) \

$\Rightarrow \boxed{\dfrac{3}{5}}$

- (b) $\dfrac{x(x-3)^2}{x^2(x-3)} = \dfrac{(x-3)}{x}$ (cancel one x and one $(x-3)$) \

$\Rightarrow \boxed{\dfrac{x-3}{x}}$

6. **Factor:**

- (a) $5m^2 - 25m = 5m(m-5)$ \

$\boxed{5m(m-5)}$

- (b) $m^2 - 4m - 5 = (m-5)(m+1)$ \

$\boxed{(m-5)(m+1)}$

- © $2x^2 + 3x - 9 = (2x-3)(x+3)$ \

$\boxed{(2x-3)(x+3)}$

7. For $f(x) = -2x^2 - 8x$:

- (a) y**-intercept**: $f(0) = 0$ \

$\boxed{(0,0)}$

- (b) **Direction of opening**: $a = -2 < 0$ \

$\boxed{\text{downward}}$

- © **Zeros**: $-2x^2 - 8x = -2x(x+4) = 0 \Rightarrow x = 0, -4$ \

$\boxed{(0,0),\ (-4,0)}$

- (d) **Vertex**: midpoint of zeros is $x = -2$; $f(-2) = 8$ \

$\boxed{(-2, 8)}$

8. **Reduce** $\sqrt{12}$:

$\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3} \Rightarrow \boxed{2\sqrt{3}}$

9. **Exact value of** $\cos 60^\circ$:

$\boxed{\frac{1}{2}}$

10. **Related acute angle for** 300°:

$360^\circ - 300^\circ = 60^\circ \Rightarrow \boxed{60^\circ}$

11. **Point** $P(-5, -6)$ on terminal arm

$x = -5,\ y = -6,\ r = \sqrt{(-5)^2 + (-6)^2} = \sqrt{61}.$

- (a) $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x/r}{y/r} = \frac{x}{y} = \frac{-5}{-6} = \boxed{\frac{5}{6}}$
- (b) Related acute angle: $\tan\alpha = \left|\frac{y}{x}\right| = \frac{6}{5}$ \

$\Rightarrow \alpha \approx \tan^{-1}\left(\frac{6}{5}\right) \approx \boxed{50.2^\circ}$ (1 d.p.)

- © Quadrant III, so $\theta = 180^\circ + \alpha \approx 180^\circ + 50.2^\circ = \boxed{230.2^\circ}$ (1 d.p.)

12. For $y = -3\cos(10(x+9)) + 2$:

- (a) **Phase shift**: $x + 9 = x - (-9)$ \

$\Rightarrow$ shift $\boxed{9 \text{ units left}}$

- (b) **Amplitude**: $\boxed{3}$
- © **Period**: $T = \frac{2\pi}{|10|} = \boxed{\frac{\pi}{5}}$
- (d) **Axis (midline)**: $\boxed{y = 2}$

- (e) **Max value**: $2 + 3 = \boxed{5}$
- (f) **Min value**: $2 - 3 = \boxed{-1}$

13. **Simplify with positive exponents:**

- (a) $\dfrac{3x^2}{27x} = \dfrac{3}{27} \cdot \dfrac{x^2}{x} = \dfrac{1}{9}x = \boxed{\dfrac{x}{9}}$
- (b) $\dfrac{(4x)^2}{2x^6} = \dfrac{16x^2}{2x^6} = \dfrac{8}{x^4}$ \

$\boxed{\dfrac{8}{x^4}}$

- © $\left(\dfrac{3x^{-1}}{x^3}\right)^{-2} = (3x^{-4})^{-2} = 3^{-2}x^8 = \boxed{\dfrac{x^8}{9}}$
- (d) $\dfrac{\sqrt[3]{x^4}}{\sqrt{x}} = \dfrac{x^{4/3}}{x^{1/2}} = x^{5/6}$ \

$\boxed{x^{5/6}}$

Part B – Full Solutions

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1. **Simplify** $\dfrac{m^2 - 4m - 5}{5m^2 - 25m} \div \dfrac{m^2 + 3m + 2}{15m^2}$.

Factor:

- $m^2 - 4m - 5 = (m - 5)(m + 1)$
- $5m^2 - 25m = 5m(m - 5)$
- $m^2 + 3m + 2 = (m + 1)(m + 2)$

$\frac{(m-5)(m+1)}{5m(m-5)} \div \frac{(m+1)(m+2)}{15m^2}$

$= \frac{m+1}{5m} \times \frac{15m^2}{(m+1)(m+2)}$

Cancel $(m+1)$ and one m; $15/5 = 3$:

$= \frac{3m}{m+2}$

So $\boxed{\dfrac{3m}{m+2}}$ (restrictions: $m \neq 0, -2, 5, -1$).

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2. **Solve the quadratic equations.**

(a) $2x^2 - 9x - 5 = 0$

Factor by grouping:

$2x^2 - 9x - 5 = 2x^2 - 10x + x - 5$

$= 2x(x-5) + 1(x-5) = (2x+1)(x-5)$

Set each factor to zero:

$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}, \quad x - 5 = 0 \Rightarrow x = 5$

So $\boxed{x = -\dfrac{1}{2},\ 5}$.

(b) $2x^2 + 2.7x - 5.1 = 0$

Use the quadratic formula, $a = 2, b = 2.7, c = -5.1$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2.7 \pm \sqrt{2.7^2 - 4(2)(-5.1)}}{4}$$

$$b^2 - 4ac = 7.29 + 40.8 = 48.09$$

$$x = \frac{-2.7 \pm \sqrt{48.09}}{4}$$

$\sqrt{48.09} \approx 6.93$:

$x_1 \approx \frac{-2.7+6.93}{4} \approx 1.06,$

$x_2 \approx \frac{-2.7-6.93}{4} \approx -2.41$

So $\boxed{x \approx 1.06,\ -2.41}$ (to 2 decimal places).

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3. **Simplify** $(7\sqrt{2}+\sqrt{3})(7\sqrt{2}-\sqrt{3})$.

Recognize a difference of squares, $(a+b)(a-b) = a^2 - b^2$ with $a = 7\sqrt{2}, b = \sqrt{3}$:

$(7\sqrt{2})^2 - (\sqrt{3})^2 = 49 \cdot 2 - 3 = 98 - 3 = 95$

So $\boxed{95}$.

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4. **Projectile problem** – $h(t) = -5t^2 + 20t + 0.8$

(a) **Initial height** (when the ball is hit): set $t = 0$.

$h(0) = -5(0)^2 + 20(0) + 0.8 = 0.8$

So the ball is $\boxed{0.8 \text{ m}}$ above the ground when hit.

(b) **Maximum height**: vertex of the parabola.

For $h(t) = at^2 + bt + c$ with $a = -5, b = 20$:

$t_{\max} = -\frac{b}{2a} = -\frac{20}{2(-5)} = 2 \text{ s}$

$h(2) = -5(2)^2 + 20(2) + 0.8 = -20 + 40 + 0.8 = 20.8$

So the maximum height is $\boxed{20.8 \text{ m}}$.

© **Time when the ball hits the ground**: solve $h(t) = 0$.

$$-5t^2 + 20t + 0.8 = 0$$

Quadratic formula with $a = -5, b = 20, c = 0.8$:

$$t = \frac{-20 \pm \sqrt{20^2 - 4(-5)(0.8)}}{2(-5)}$$

$$= \frac{-20 \pm \sqrt{400 + 16}}{-10}$$

$$= \frac{-20 \pm \sqrt{416}}{-10}$$

$\sqrt{416} \approx 20.4$ gives one negative and one positive root; the physical solution is the positive time:

$$t \approx 4.04 \text{ s}$$

So the ball hits the ground at about $\boxed{4.0 \text{ s}}$ (to 1 d.p.).

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5. **Diagram problem (length x)**

Using labels: bottom left A, middle bottom B, bottom right C, top point D.

Given:

- $AD = 37\,\text{m}$
- $\angle DAB = 100^\circ$
- $\angle D = 22^\circ$ (between AD and DB)
- $BC = 25\,\text{m}$
- Need $DC = x$.

Step 1: Triangle $\triangle DAB$

Angles in triangle:

$\angle DAB = 100^\circ,\ \angle D = 22^\circ$

$\Rightarrow \angle B = 180^\circ - 100^\circ - 22^\circ = 58^\circ$

Apply the sine law:

$\frac{AD}{\sin \angle B} = \frac{AB}{\sin \angle D}$

$\Rightarrow$

$\frac{37}{\sin 58^\circ} = \frac{AB}{\sin 22^\circ}$

$AB = 37 \cdot \frac{\sin 22^\circ}{\sin 58^\circ} \approx 16.3\,\text{m}$

So

$AC = AB + BC \approx 16.3 + 25 = 41.3\,\text{m}.$

Step 2: Triangle $\triangle DAC$

Now $AD = 37$, $AC \approx 41.3$, and the included angle at A is still 100°.

Apply the cosine law:

$DC^2 = AD^2 + AC^2 - 2(AD)(AC)\cos 100^\circ$

$DC^2 \approx 37^2 + 41.3^2 - 2(37)(41.3)\cos 100^\circ \approx 3609$

$DC \approx \sqrt{3609} \approx 60.1\,\text{m}$

So $\boxed{x \approx 60.1\,\text{m}}$ (to one decimal place).

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6. **For** $\cos(A) = -\dfrac{2}{3}, 0^\circ \leq A \leq 360^\circ.$

(a) **Sketch**: cosine negative in quadrants II and III, so terminal arms lie in QII and QIII.

(b) **Exact values of** $\tan(A)$

$\cos^2 A = \left(-\frac{2}{3}\right)^2 = \frac{4}{9},\ \sin^2 A = 1 - \frac{4}{9} = \frac{5}{9}$

So $|\sin A| = \dfrac{\sqrt{5}}{3}$.

- In QII: $\sin A = \dfrac{\sqrt{5}}{3}, \cos A = -\dfrac{2}{3}$ \

$$\tan A = \frac{\sin A}{\cos A} = \frac{\sqrt{5}/3}{-2/3} = -\frac{\sqrt{5}}{2}$$

- In QIII: $\sin A = -\dfrac{\sqrt{5}}{3}, \cos A = -\dfrac{2}{3}$ \

$$\tan A = \frac{-\sqrt{5}/3}{-2/3} = \frac{\sqrt{5}}{2}$$

So $\boxed{\tan A = -\dfrac{\sqrt{5}}{2}\ (\text{QII}),\ \tan A = \dfrac{\sqrt{5}}{2}\ (\text{QIII})}$.

© Related acute angle and all values of A

Related acute angle:

$$\alpha = \cos^{-1}\left(\tfrac{2}{3}\right) \approx 48.2^\circ$$

Cosine is negative in quadrants II and III:

$$A_1 = 180^\circ - \alpha \approx 180^\circ - 48.2^\circ = 131.8^\circ$$

$$A_2 = 180^\circ + \alpha \approx 180^\circ + 48.2^\circ = 228.2^\circ$$

So the related acute angle is $\boxed{48.2^\circ}$, and \

$\boxed{A \approx 131.8^\circ,\ 228.2^\circ}$ (1 d.p.).

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7. **Exponential transformation** – $y = 2 \cdot 3^{-2(x+1)} - 3$

Let the parent be $f(x) = 3^x$.

(a) **Pointwise formula and 3 key points**

We can write the transformation as

$$y = 2f(-2(x+1)) - 3.$$

Let (x, y) lie on $y = 3^x$ (so $y = 3^x$). For the image point (X, Y):

$$x = -2(X+1) \Rightarrow X = -\tfrac{x}{2} - 1, \quad Y = 2y - 3.$$

So the pointwise rule is

$$\boxed{(x, y) \mapsto \left(-\frac{x}{2} - 1,\ 2y - 3\right)}.$$

Use three key points of $y = 3^x$:

- $(-1, \frac{1}{3}) \mapsto \left(-\frac{1}{2},\ -\frac{7}{3}\right)$
- $(0, 1) \mapsto (-1, -1)$
- $(1, 3) \mapsto \left(-\frac{3}{2},\ 3\right)$

(b) **Graph of** $y = 2 \cdot 3^{-2(x+1)} - 3$

Important features to label:

- Horizontal asymptote as $x \to \infty$: $y \to -3$ \

$\Rightarrow$ asymptote $\boxed{y = -3}$

- Example key points: $(-1, -1)$ and $\left(-\frac{3}{2}, 3\right)$ and $\left(-\frac{1}{2}, -\frac{7}{3}\right)$
- y-intercept ($x = 0$): \

$$y = 2 \cdot 3^{-2} - 3 = \frac{2}{9} - 3 = -\frac{25}{9}.$$

Plot these points and the asymptote to complete the sketch.

8. **Cosine transformation** – $y = -3\cos(10(x+9)) + 2$

Parent: $y = \cos x$.

(a) **Pointwise formula and 3 key points**

Write

$$y = -3f(10(x+9)) + 2, \quad \text{where } f(x) = \cos x.$$

If (x, y) lies on $y = \cos x$, then for the image point (X, Y):

$$x = 10(X+9) \Rightarrow X = \tfrac{x}{10} - 9, \quad Y = -3y + 2.$$

So

$$\boxed{(x, y) \mapsto \left(\frac{x}{10} - 9,\ -3y + 2\right)}.$$

Use three standard key points of $y = \cos x$ over $[0, 2\pi]$:

- $(0, 1) \mapsto (-9, -1)$
- $(\pi, -1) \mapsto \left(-9 + \frac{\pi}{10},\ 5\right)$
- $(2\pi, 1) \mapsto \left(-9 + \frac{\pi}{5},\ -1\right)$

(b) **Sketch one period of** $y = -3\cos(10(x+9)) + 2$

Key characteristics:

- Amplitude: $\boxed{3}$
- Period: $T = \dfrac{2\pi}{10} = \boxed{\dfrac{\pi}{5}}$
- Phase shift: $x + 9 = x - (-9) \backslash$

$\Rightarrow$ shift $\boxed{\text{9 units left}}$

- Vertical shift / axis: $\boxed{y = 2}$

- Max value: $2+3=5$; min value: $2-3=-1$.

Over one period from $x=-9$ to $x=-9+\frac{\pi}{5}$, mark:

- $(-9,-1)$ (minimum)
- $\left(-9+\frac{\pi}{10},5\right)$ (maximum)
- $\left(-9+\frac{\pi}{5},-1\right)$ (minimum)

Draw a cosine wave through these points around the axis $y=2$ to complete the sketch.