CHAPTER 14

# Applications of Linear Optimization

## Learning Objectives

After studying this chapter, you will be able to:

- State the characteristics of some generic types of linear optimization models.
- Describe the different categories of constraints that are typically used in optimization models.
- Build linear optimization models for a variety of applications.
- Use Excel to evaluate scenarios and visualize results for linear optimization models and gain practical insights into the solutions.
- Correctly interpret the *Solver* Sensitivity report for models that have bounded variables.
- Use auxiliary variables to model bound constraints and obtain more complete sensitivity information.
- Ensure that assumptions underlying the use of sensitivity information hold when interpreting *Solver* reports.

**Linear optimization** models are the most ubiquitous of optimization models used in organizations today. Applications abound in operations, finance, marketing, engineering, and many other disciplines. Table 14.1 summarizes some common types of generic linear optimization models. This list represents but a very small sample of the many practical types of linear optimization models that are used in practice throughout business.

Building optimization models is more of an art than a science because there often are several ways of formulating a particular problem. Learning how to build optimization models requires logical thought but can be facilitated by studying examples of different models and observing their characteristics. The Sklenka Ski model we developed and analyzed in Chapter 13 was one example of a simple product-mix model. In this chapter, we illustrate examples of other types of linear optimization models and describe unique issues associated with formulation, implementation on spreadsheets, interpreting results, sensitivity and scenario analysis, using *Premium Solver* and Excel, and gaining insight for making good decisions.

**Table 14.1**

Generic Examples of Linear Optimization Models

| Type of Model | Decision Variables | Objective Function | Typical Constraints |
|---|---|---|---|
| Product mix | Quantities of product to produce and sell | Maximize contribution to profit | Resource limitations (e.g., production time, labor, material); minimum sales requirements; maximum sales potential |
| Process selection | Quantities of product to make using alternative processes | Minimize cost | Demand requirements; resource limitations |
| Blending | Quantity of materials to mix to produce one unit of output | Minimize cost | Specifications on acceptable mixture |
| Portfolio selection | Proportions to invest in different financial instruments | Maximize future return or minimize risk exposure | Limit on available funds; sector requirements or restrictions; proportional relationships on investment mix |
| Transportation | Amount to ship between sources of supply and destinations | Minimize total transportation cost | Limited availability at sources; required demands met at destinations |
| Multiperiod production planning | Quantities of product to produce in each of several time periods; amount of inventory to hold between periods | Minimize total production and inventory costs | Limited production rates; material balance equations |
| Multiperiod financial management | Amounts to invest in short-term instruments | Maximize cash on hand | Cash balance equations; required cash obligations |
| Production/ marketing | Allocation of advertising expenditures; production quantities | Maximize profit | Budget limitation; production limitations; demand requirements |

## Types of Constraints in Optimization Models

The most challenging aspect of model formulation is identifying constraints. Understanding the different types of constraints can help in proper identification and modeling. Constraints generally fall into one of the following categories:

- *Simple Bounds.* **Simple bounds** constrain the value of a single variable. You can recognize simple bounds in problem statements such as no more than $10,000 may be invested in stock ABC or we must produce at least 350 units of product Y to meet customer commitments this month. The mathematical forms for these examples are

$$\text{ABC} \leq 10{,}000$$
$$\text{Y} \geq 350$$

- *Limitations.* **Limitations** usually involve the allocation of scarce resources. Problem statements such as the amount of material used in production cannot exceed the amount available in inventory, minutes used in assembly cannot exceed the available labor hours, or the amount shipped from the Austin plant in July cannot exceed the plant's capacity are typical of these types of constraints.
- *Requirements.* **Requirements** involve the specification of minimum levels of performance. Such statements as enough cash must be available in February to meet financial obligations, production must be sufficient to meet promised customer orders, or the marketing plan should ensure that at least 400 customers are contacted each month are some examples.
- *Proportional Relationships.* **Proportional relationships** are often found in problems involving mixtures or blends of materials or strategies. Examples include the amount invested in aggressive growth stocks cannot be more than twice the amount invested in equity-income funds or the octane rating of gasoline obtained from mixing different crude blends must be at least 89.
- *Balance Constraints.* **Balance constraints** essentially state that input = output and ensure that the flow of material or money is accounted for at locations or between time periods. Examples include production in June plus any available inventory must equal June's demand plus inventory held to July, the total amount shipped to a distribution center from all plants must equal the amount shipped from the distribution center to all customers, or the total amount of money invested or saved in March must equal the amount of money available at the end of February.

Constraints in linear optimization models are generally some combination of constraints from these categories. Problem data or verbal clues in a problem statement often help you identify the appropriate constraint. In some situations, all constraints may not be explicitly stated, but are required for the model to represent the real problem accurately. An example of an implicit constraint is nonnegativity of the decision variables.

In the following sections, we present examples of different types of linear optimization applications. Each of these models has different characteristics, and by studying how they are developed, you will improve your ability to model other problems. We will also use these examples to illustrate how data visualization can be effectively used with optimization modeling, and also provide further insights into using *Solver*. We encourage you to use the process that we illustrated with the Sklenka Ski problem; however, to conserve space in this book, we will go directly to the mathematical model instead of first conceptualizing the constraints and objective functions in verbal terms.

## Process Selection Models

Process selection models generally involve choosing among different types of processes to produce a good. Make-or-buy decisions are examples of process selection models, whereby we must choose whether to make one or more products in-house or subcontract them out to another firm. The following example illustrates these concepts.

### EXAMPLE 14.1 Camm Textiles

Camm Textiles has a mill that produces three types of fabrics on a make-to-order basis. The mill operates on a 24/7 basis. The key decision facing the plant manager is about the type of loom needed to process each fabric during the coming quarter (13 weeks) to meet demands for the three fabrics and not exceed the capacity of the looms in the mill. Two types of looms are used: dobbie and regular. Dobbie looms can be used to make all fabrics and are the only looms that can weave certain fabrics, such as plaids. Demands, variable costs for each fabric, and production rates on the looms are given in Table 14.2. The mill has 15 regular looms and 3 dobbie looms. After weaving, fabrics are sent to the finishing department and then sold. Any fabrics that cannot be woven in the mill because of limited capacity will be purchased from an external supplier, finished at the mill, and sold at the selling price. In addition to determining which looms to use to process the fabrics, the manager also needs to determine which fabrics to buy externally.

To formulate a linear optimization model, define $D_i$ = number of yards of fabric $i$ to produce on dobbie looms, $i = 1, \ldots, 3$. For example, $D_1$ = number of yards of fabric 1 to produce on dobbie looms, $D_2$ = number of yards of fabric 2 to produce on dobbie looms, and $D_3$ = number of yards of fabric 3 to produce on dobbie looms. In a similar fashion, define:

$R_i$ = number of yards of fabric $i$ to produce on regular looms, $i = 2, 3$ only

$P_i$ = number of yards of fabric $i$ to purchase from an outside supplier, $i = 1, \ldots, 3$

Note that we are using *subscripted variables* to simplify their definition rather than defining nine individual variables with unique names.

The objective function is to minimize total cost, found by multiplying the cost per yard based on the mill cost or outsourcing by the number of yards of fabric for each type of decision variable:

$$\begin{aligned} \min\ & 0.65D_1 + 0.61D_2 + 0.50D_3 + 0.61R_2 + 0.50R_3 \\ & + 0.85P_1 + 0.75P_2 + 0.65P_3 \end{aligned}$$

Constraints to ensure meeting production requirements are

$$\text{Fabric 1 demand: } D_1 + P_1 = 45{,}000$$

This constraint states that the amount of fabric 1 produced on dobbie looms or outsourced must equal the total demand of 45,000 yards. The constraints for the other two fabrics are

$$\text{Fabric 2 demand: } D_2 + R_2 + P_2 = 76{,}500$$

$$\text{Fabric 3 demand: } D_3 + R_3 + P_3 = 10{,}000$$

To specify the constraints on loom capacity, we must convert yards per hour into hours per yard. For example, for fabric 1 on a dobbie loom, 4.7 yards/hour = 0.213 hour/yard. Therefore, the term $0.213D_1$ represents the total time required to produce $D_1$ yards of fabric 1 on a dobbie loom (hours/yard × yards). The total capacity for dobbie looms is

$$(24 \text{ hours/day})(7 \text{ days/week})(13 \text{ weeks})(3 \text{ looms}) = 6{,}552 \text{ hours}$$

Thus, the constraint on available production time on dobbie looms is

$$0.213D_1 + 0.192D_2 + 0.227D_3 \le 6{,}552$$

For regular looms we have

$$0.192R_2 + 0.227R_3 \le 32{,}760$$

Finally, all variables must be nonnegative.

The complete model is

$$\min 0.65D_1 + 0.61D_2 + 0.50D_3 + 0.61R_2 + 0.50R_3 + 0.85P_1 + 0.75P_2 + 0.65P_3$$

Fabric 1 demand: $D_1 + P_1 = 45{,}000$

Fabric 2 demand: $D_2 + R_2 + P_2 = 76{,}500$

Fabric 3 demand: $D_3 + R_3 + P_3 = 10{,}000$

Dobbie loom capacity:

$$0.213D_1 + 0.192D_2 + 0.227D_3 \le 6{,}552$$

Regular loom capacity:

$$0.192R_2 + 0.227R_3 \le 32{,}760$$

Nonnegativity: all variables $\ge 0$

**Table 14.2**

Textile Production Data

| Fabric | Demand (yards) | Dobbie Loom Capacity (yards/hour) | Regular Loom Capacity (yards/hour) | Mill Cost ($/yard) | Outsourcing Cost ($/yard) |
|---|---|---|---|---|---|
| 1 | 45,000 | 4.7 | 0.0 | $0.65 | $0.85 |
| 2 | 76,500 | 5.2 | 5.2 | $0.61 | $0.75 |
| 3 | 10,000 | 4.4 | 4.4 | $0.50 | $0.65 |

## Spreadsheet Design and *Solver* Reports

Figure 14.1 shows a spreadsheet implementation (Excel file *Camm Textiles*) with the optimal solution to Example 14.1. Observe the design of the spreadsheet and, in particular, the use of labels in the rows and columns in the model section. Using the principles discussed in the previous chapter, this design makes it easy to read and interpret the Answer and Sensitivity reports. Figure 14.2 shows the *Solver* model. It is easier to define the decision variables as the range B14:D16; however, because we cannot produce fabric 1 on regular looms, we set cell C14 to zero as a constraint. Whenever you restrict a single decision variable to equal a value or set it as a $\ge$ or $\le$ type of constraint, *Solver* considers it as a simple "bound" constraint, which makes the solution process more efficient.

## EXAMPLE 14.2 Interpreting *Solver* Reports for the Camm Textiles Problem

Figures 14.3 and 14.4 show the *Solver* Answer and Sensitivity reports for the Camm Textiles model. In the Answer report, we see that only the regular loom capacity constraint is not binding; the slack value of 15,775.73 hours means that the regular looms have excess capacity, whereas the fact that the dobbie loom constraint is binding means that all capacity is used to meet the demand. Because of the limited dobbie loom capacity and the fact that fabric 1 cannot be made on a regular loom, some of fabric 1 needs to be outsourced, even though the outsourcing cost is high.

The Sensitivity report contains a lot of information, and we highlight only a few pieces of it. Note that the mill cost for fabric 2 is $0.61, whereas the outsourcing cost is $0.75. Therefore, the reduced cost of $0.14 is the difference and is the amount that the outsourcing cost

(*continued*)

would have to be lowered to make it economical to purchase fabric 2 rather than to make it. The shadow price of −\$0.94 for the dobbie loom constraint means that an increase in dobbie loom capacity (up to 3,022 hours) would lower the total cost by 94 cents for each additional hour of capacity. This can help financial managers to justify purchasing or possibly renting new equipment. The shadow prices on the fabric demand constraints explain how much the total cost would increase if demand for the fabric should rise up to the Allowable Increase limits. Producing an extra yard of fabric 1 will cost \$0.85 (the cost of outsourcing, because there is not enough dobbie capacity), whereas producing an extra yard of fabrics 2 and 3 would cost only \$0.61 and \$0.50, respectively (the mill costs), while maintaining the same loom capacities. This information can help the marketing department set prices or promotions with its customers.

**Figure 14.1**

Spreadsheet Model for *Camm Textiles*

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | Camm Textiles | | | | | |
| 2 | | | | | | |
| 3 | Data | | | | | |
| 4 | | Dobbie | Regular | | | |
| 5 | Fabric | Capacity | Capacity | Mill Cost | Outsourcing Cost | Demand |
| 6 | 1 | 4.7 | 0 | \$ 0.65 | \$0.85 | 45000 |
| 7 | 2 | 5.2 | 5.2 | \$ 0.61 | \$0.75 | 76500 |
| 8 | 3 | 4.4 | 4.4 | \$ 0.50 | \$0.65 | 10000 |
| 9 | Hours Available | 6552 | 32760 | | | |
| 10 | | | | | | |
| 11 | Model | | | | | |
| 12 | | | | | | |
| 13 | | on Dobbie | on Regular | Purchased | Total Yards Produced | |
| 14 | Fabric 1 | 30794.4 | 0 | 14205.6 | 45000 | |
| 15 | Fabric 2 | 0 | 76500 | 0 | 76500 | |
| 16 | Fabric 3 | 0 | 10000 | 0 | 10000 | |
| 17 | Hours Used | 6552 | 16984.26573 | | | |
| 18 | | | | | | |
| 19 | | Total | | | | |
| 20 | Cost | \$ 83,756.12 | | | | |

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | Camm Textiles | | | | | |
| 2 | | | | | | |
| 3 | Data | | | | | |
| 4 | | Dobbie | Regular | | | |
| 5 | Fabric | Capacity | Capacity | Mill Cost | Outsourcing Cost | Demand |
| 6 | 1 | 4.7 | 0 | 0.65 | 0.85 | 45000 |
| 7 | 2 | 5.2 | 5.2 | 0.61 | 0.75 | 76500 |
| 8 | 3 | 4.4 | 4.4 | 0.5 | 0.65 | 10000 |
| 9 | Hours Available | =24*7*13*3 | =24*7*13*15 | | | |
| 10 | | | | | | |
| 11 | Model | | | | | |
| 12 | | | | | | |
| 13 | | on Dobbie | on Regular | Purchased | Total Yards Produced | |
| 14 | Fabric 1 | 30794.4 | 0 | 14205.6 | =SUM(B14:D14) | |
| 15 | Fabric 2 | 0 | 76500 | 0 | =SUM(B15:D15) | |
| 16 | Fabric 3 | 0 | 10000 | 0 | =SUM(B16:D16) | |
| 17 | Hours Used | =B14/B6+B15/B7+B16/B8 | =C15/C7+C16/C8 | | | |
| 18 | | | | | | |
| 19 | | Total | | | | |
| 20 | Cost | =SUMPRODUCT(B14:B16,D6:D8)+SUMPRODUCT(C15:C16,D7:D8)+SUMPRODUCT(D14:D16,E6:E8) | | | | |

**Figure 14.2**

*Solver* Model for Camm Textiles

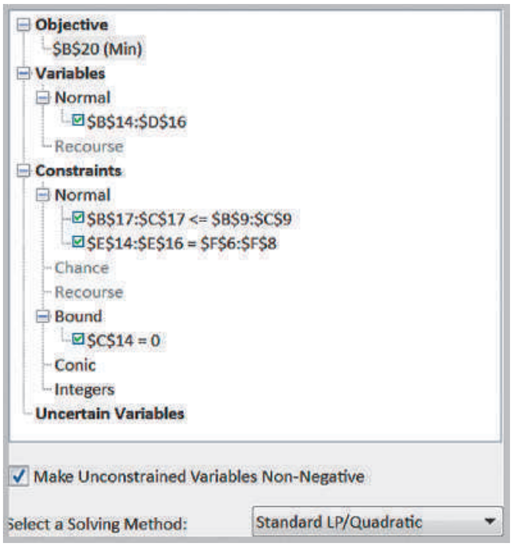



**Figure 14.3**

*Solver* Answer Report for Camm Textiles

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 11 | | | | | | | |
| 12 | Objective Cell (Min) | | | | | | |
| 13 | | Cell | Name | Original Value | Final Value | | |
| 14 | | $B$20 | Cost Total | 0 | 83756.12 | | |
| 15 | | | | | | | |
| 16 | | | | | | | |
| 17 | Decision Variable Cells | | | | | | |
| 18 | | Cell | Name | Original Value | Final Value | Type | |
| 19 | | $B$14 | Fabric 1 on Dobbie | 0 | 30794.4 | Normal | |
| 20 | | $C$14 | Fabric 1 on Regular | 0 | 0 | Normal | |
| 21 | | $D$14 | Fabric 1 Purchased | 0 | 14205.6 | Normal | |
| 22 | | $B$15 | Fabric 2 on Dobbie | 0 | 0 | Normal | |
| 23 | | $C$15 | Fabric 2 on Regular | 0 | 76500 | Normal | |
| 24 | | $D$15 | Fabric 2 Purchased | 0 | 0 | Normal | |
| 25 | | $B$16 | Fabric 3 on Dobbie | 0 | 0 | Normal | |
| 26 | | $C$16 | Fabric 3 on Regular | 0 | 10000 | Normal | |
| 27 | | $D$16 | Fabric 3 Purchased | 0 | 0 | Normal | |
| 28 | | | | | | | |
| 29 | Constraints | | | | | | |
| 30 | | Cell | Name | Cell Value | Formula | Status | Slack |
| 31 | | $B$17 | Hours Used on Dobbie | 6552 | $B$17<=$B$9 | Binding | 0 |
| 32 | | $C$17 | Hours Used on Regular | 16984.26573 | $C$17<=$C$9 | Not Binding | 15775.73427 |
| 33 | | $E$14 | Fabric 1 Total Yards Produced | 45000 | $E$14=$F$6 | Binding | 0 |
| 34 | | $E$15 | Fabric 2 Total Yards Produced | 76500 | $E$15=$F$7 | Binding | 0 |
| 35 | | $E$16 | Fabric 3 Total Yards Produced | 10000 | $E$16=$F$8 | Binding | 0 |
| 36 | | $C$14 | Fabric 1 on Regular | 0 | $C$14=0 | Binding | 0 |

## *Solver* Output and Data Visualization

As you certainly know by now, interpreting the output from *Solver* requires some technical knowledge of linear optimization concepts and terminology, such as reduced costs and shadow prices. Data visualization can help analysts present optimization results in forms that are more understandable and can be easily explained to managers and clients in a report or presentation. We will use the Camm Textiles example to illustrate this.

The first thing that one might do is to visualize the values of the optimal decision variables and constraints, drawing upon the model output or the information contained in the Answer Report. Figure 14.5 shows a chart of the decision variables, showing the

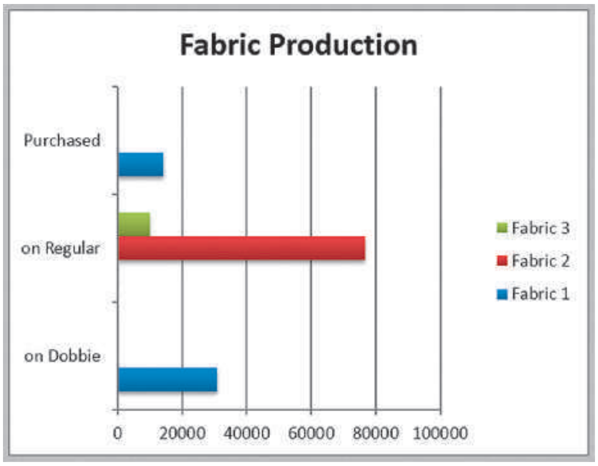
**Figure 14.4**

*Solver* Sensitivity Report for Camm Textile

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 4 | | | | | | | | |
| 5 | Objective Cell (Min) | | | | | | | |
| 6 | | Cell | Name | Final Value | | | | |
| 7 | | $B$20 | Cost Total | 83756.12 | | | | |
| 8 | | | | | | | | |
| 9 | Decision Variable Cells | | | | | | | |
| 10 | | | | Final | Reduced | Objective | Allowable | Allowable |
| 11 | | Cell | Name | Value | Cost | Coefficient | Increase | Decrease |
| 12 | | $B$14 | Fabric 1 on Dobbie | 30794.4 | 0 | 0.65 | 0.200000094 | 1E+30 |
| 13 | | $C$14 | Fabric 1 on Regular | 0 | -0.85 | 0 | 1E+30 | 1E+30 |
| 14 | | $D$14 | Fabric 1 Purchased | 14205.6 | 0 | 0.85 | 1E+30 | 0.200000094 |
| 15 | | $B$15 | Fabric 2 on Dobbie | 0 | 0.180769231 | 0.61 | 1E+30 | 0.180769231 |
| 16 | | $C$15 | Fabric 2 on Regular | 76500 | 0 | 0.61 | 0.1400001 | 1E+30 |
| 17 | | $D$15 | Fabric 2 Purchased | 0 | 0.14 | 0.75 | 1E+30 | 0.14 |
| 18 | | $B$16 | Fabric 3 on Dobbie | 0 | 0.213636364 | 0.5 | 1E+30 | 0.213636364 |
| 19 | | $C$16 | Fabric 3 on Regular | 10000 | 0 | 0.5 | 0.1500001 | 1E+30 |
| 20 | | $D$16 | Fabric 3 Purchased | 0 | 0.15 | 0.65 | 1E+30 | 0.15 |
| 21 | | | | | | | | |
| 22 | Constraints | | | | | | | |
| 23 | | | | Final | Shadow | Constraint | Allowable | Allowable |
| 24 | | Cell | Name | Value | Price | R.H. Side | Increase | Decrease |
| 25 | | $B$17 | Hours Used on Dobbie | 6552 | -0.94 | 6552 | 3022.468085 | 6552 |
| 26 | | $C$17 | Hours Used on Regular | 16984.26573 | 0 | 32760 | 1E+30 | 15775.73427 |
| 27 | | $E$14 | Fabric 1 Total Yards Produced | 45000 | 0.85 | 45000 | 1E+30 | 14205.6 |
| 28 | | $E$15 | Fabric 2 Total Yards Produced | 76500 | 0.61 | 76500 | 82033.81818 | 76500 |
| 29 | | $E$16 | Fabric 3 Total Yards Produced | 10000 | 0.5 | 10000 | 69413.23077 | 10000 |

**Figure 14.5**

Summary of Optimal Solution




amounts of each fabric produced on each type of loom and outsourced. Figure 14.6 shows the capacity utilization of each type of loom. We can easily see that the utilization of regular looms is approximately half the capacity, while dobbie looms are fully utilized, suggesting that the purchase of additional dobbie looms might be useful, at least under the current demand scenario.

The Sensitivity Report is more challenging to visualize effectively. The reduced costs describe how much the unit production or purchasing cost must be changed to force the value of a variable to become positive in the solution. Figure 14.7 shows a visualization of the reduced cost information. The chart displays the unit cost coefficients for each production or outsourcing decision, and for those not currently utilized, the change in cost required to force that variable to become positive in the solution. Note that since fabric 1 cannot be produced on a regular loom, its reduced cost is meaningless and therefore, not displayed.

**Figure 14.6**

Chart of Capacity Utilization

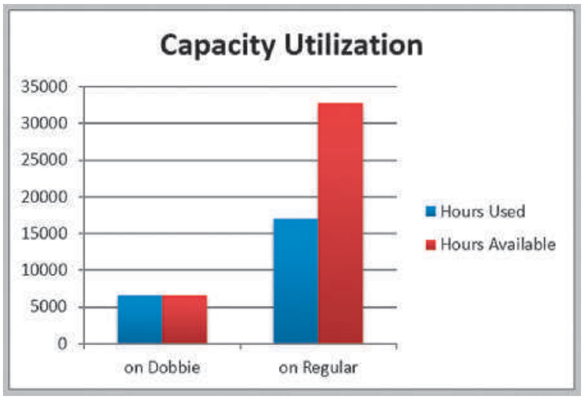
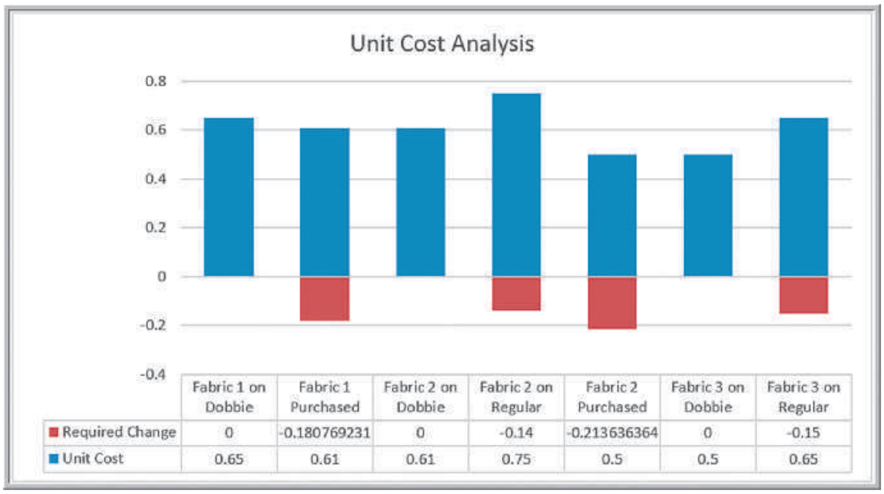



**Figure 14.7**

Summary of Reduced Cost Information




We may also visualize the ranges over which the unit cost coefficients may change without changing the optimal values of the decision variables by using an Excel *Stock Chart*. A stock chart typically shows the "high-low-close" values of daily stock prices; here we can compute the maximum-minimum-current values of the unit cost coefficients. To do this, follow these steps:

1. Create a table in the worksheet by adding the Allowable Increase values and subtracting the Allowable Decrease values from the cost coefficients as shown in Table 14.3. Replace 1E+30 by #N/A in the worksheet so that infinite values are not displayed.
2. Highlight the range of this table and insert an Excel *Stock Chart* and name the series as Maximum, Minimum, and Current.
3. Click the chart, and in the *Format* tab of *Chart Tools*, go to the *Current Selection* group to the left of the ribbon and click on the drop down box (it usually says "Chart Area"). Find the series you wish to format and then click *Format Selection*.

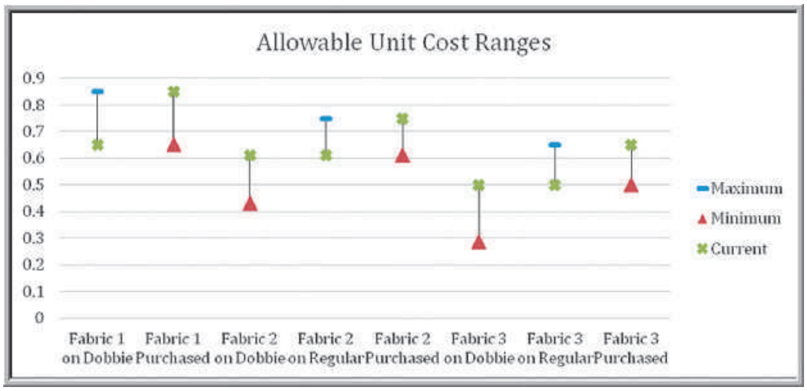
**Table 14.3**

Data Used to Construct Stock Chart for Cost Coefficient Ranges

| | Maximum | Minimum | Current |
|---|---|---|---|
| Fabric 1 on Dobbie | 0.85 | #N/A | 0.65 |
| Fabric 1 Purchased | #N/A | 0.65 | 0.85 |
| Fabric 2 on Dobbie | #N/A | 0.429231 | 0.61 |
| Fabric 2 on Regular | 0.75 | #N/A | 0.61 |
| Fabric 2 Purchased | #N/A | 0.61 | 0.75 |
| Fabric 3 on Dobbie | #N/A | 0.286364 | 0.5 |
| Fabric 3 on Regular | 0.65 | #N/A | 0.5 |
| Fabric 3 Purchased | #N/A | 0.5 | 0.65 |

**Figure 14.8**

Chart of Allowable Unit Cost Ranges




4. In the *Format Data Series* pane that appears in the worksheet, click the paint icon and then *Marker*, making sure to expand the *Marker Options* menu.
5. Choose the type of marker you wish and increase the width of the markers to make them more visible. We chose the green symbol × for the current value, a red triangle for the minimum value, and a blue dash for the maximum value. This results in the chart shown in Figure 14.8.

Now it is easy to visualize the allowable unit cost ranges. For those lines that have no maximum limit (the blue dash) such as with Fabric 1 Purchased, the unit costs can increase to infinity; for those that have no lower limit (the red triangle) such as Fabric 1 on Dobbie, the unit costs can decrease indefinitely.

Shadow prices show the impact of changing the right-hand side of a binding constraint. Because the plant operates on a 24/7 schedule, changes in loom capacity would require in "chunks" (i.e., purchasing an additional loom) rather than incrementally. However, changes in the demand can easily be assessed using the shadow price information. Figures 14.9 and 14.10 show a simple summary of the shadow prices associated with each product, as well as the ranges based on the Allowable Increase and Allowable Decrease values over which these prices are valid, using a similar approach as described earlier for the cost-coefficient ranges.

Figure 14.9

Summary of Shadow Prices

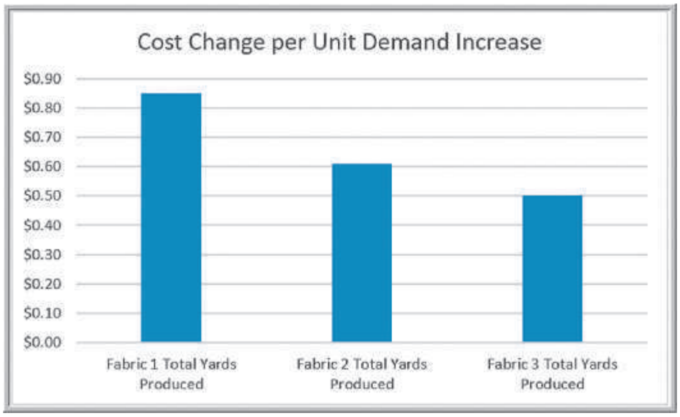



Figure 14.10

Chart of Allowable Demand Ranges for Valid Shadow Prices


## Blending Models

Blending problems involve mixing several raw materials that have different characteristics to make a product that meets certain specifications. Dietary planning, gasoline and oil refining, coal and fertilizer production, and the production of many other types of bulk commodities involve blending. We typically see proportional constraints in blending models.

### EXAMPLE 14.3 BG Seed Company

The BG Seed Company specializes in food products for birds and other household pets. In developing a new birdseed mix, company nutritionists have specified that the mixture should contain at least 13% protein and 15% fat and no more than 14% fiber. The percentages of each of these nutrients in eight types of ingredients that can be used in the mix are given in Table 14.4, along with the wholesale cost per pound. What is the minimum-cost mixture that meets the stated nutritional requirements?

The decisions are the amount of each ingredient to include in a given quantity—for example, 1 pound—of mix. Define $X_i$ = number of pounds of ingredient $i$ to include in 1 pound of the mix, for $i = 1, \ldots, 8$. By defining the variables in this fashion makes the solution easily scalable to any quantity.

The objective is to minimize total cost, obtained by multiplying the cost per pound by the number of pounds used for each ingredient:

$$\text{minimize } 0.22X_1 + 0.19X_2 + 0.10X_3 + 0.10X_4 + 0.07X_5 + 0.05X_6 + 0.26X_7 + 0.11X_8$$

*(continued)*

To ensure that the mix contains the appropriate proportion of ingredients, observe that multiplying the number of pounds of each ingredient by the percentage of nutrient in that ingredient (a dimensionless quantity) specifies the number of pounds of nutrient provided. For example, sunflower seeds contain 16.9% protein; so $0.169X_1$ represents the number of pounds of protein in $X_1$ pounds of sunflower seeds. Therefore, the total number of pounds of protein provided by all ingredients is

$$0.169X_1 + 0.12X_2 + 0.085X_3 + 0.154X_4 + 0.085X_5 + 0.12X_6 + 0.18X_7 + 0.119X_8$$

Because the total number of pounds of ingredients that are mixed together equals $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8$, the proportion of protein in the mix is

$$\frac{0.169X_1 + 0.12X_2 + 0.085X_3 + 0.154X_4 + 0.085X_5 + 0.12X_6 + 0.18X_7 + 0.119X_8}{X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8}$$

This proportion must be at least 0.13 and can be converted to a linear form as discussed in Chapter 13. However, we wish to determine the best amount of ingredients to include in *1 pound* of mix; therefore, we add the constraint

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 = 1$$

Now we can substitute 1 for the denominator in the proportion of protein, simplifying the constraint:

$$0.169X_1 + 0.12X_2 + 0.085X_3 + 0.154X_4 + 0.085X_5 + 0.12X_6 + 0.18X_7 + 0.119X_8 \geq 0.13$$

This ensures that at least 13% of the mixture will be protein. In a similar fashion, the constraints for the fat and fiber requirements are

$$0.26X_1 + 0.014X_2 + 0.038X_3 + 0.063X_4 + 0.038X_5 + 0.017X_6 + 0.179X_7 + 0.04X_8 \geq 0.15$$

$$0.29X_1 + 0.083X_2 + 0.027X_3 + 0.024X_4 + 0.027X_5 + 0.023X_6 + 0.288X_7 + 0.109X_8 \leq 0.14$$

Finally, we have nonnegative constraints:

$$X_i \geq 0, \text{ for } i = 1, 2, \ldots, 8$$

The complete model is:

minimize $0.22X_1 + 0.19X_2 + 0.10X_3 + 0.10X_4 + 0.07X_5 + 0.05X_6 + 0.26X_7 + 0.11X_8$

Mixture: $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 = 1$

Protein: $0.169X_1 + 0.12X_2 + 0.085X_3 + 0.154X_4 + 0.085X_5 + 0.085X_5 + 0.12X_6 + 0.18X_7 + 0.119X_8 \geq 0.13$

Fat: $0.26X_1 + 0.041X_2 + 0.038X_3 + 0.063X_4 + 0.038X_5 + 0.017X_6 + 0.179X_7 + 0.04X_8 \geq 0.15$

Fiber: $0.29X_1 + 0.083X_2 + 0.027X_3 + 0.024X_4 + 0.027X_5 + 0.023X_6 + 0.288X_7 + 0.109X_8 \leq 0.14$

Nonnegativity: $X_i \geq 0, \quad \text{for } i = 1, 2, \ldots, 8$

## Dealing with Infeasibility

Figure 14.11 shows an implementation of this model on a spreadsheet (Excel file *BG Seed Company*) and Figure 14.12 shows the *Solver* model. If we solve the model, however, we find that the problem is infeasible. *Solver* provides a report, called the **Feasibility report**,

**Table 14.4**

Birdseed Nutrition Data

| Ingredient | Protein % | Fat % | Fiber % | Cost/lb |
|---|---|---|---|---|
| Sunflower seeds | 16.9 | 26.0 | 29.0 | $0.22 |
| White millet | 12.0 | 4.1 | 8.3 | $0.19 |
| Kibble corn | 8.5 | 3.8 | 2.7 | $0.10 |
| Oats | 15.4 | 6.3 | 2.4 | $0.10 |
| Cracked corn | 8.5 | 3.8 | 2.7 | $0.07 |
| Wheat | 12.0 | 1.7 | 2.3 | $0.05 |
| Safflower | 18.0 | 17.9 | 28.8 | $0.26 |
| Canary grass seed | 11.9 | 4.0 | 10.9 | $0.11 |

**Figure 14.11**

Spreadsheet Model for *BG Seed Company* Problem

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **BG Seed Company** | | | | | |
| 2 | | | | | | |
| 3 | **Data** | | | | | |
| 4 | | **Ingredient** | **Protein %** | **Fat %** | **Fiber %** | **Cost/lb** |
| 5 | 1 | Sunflower seeds | 16.90% | 26% | 29% | $ 0.22 |
| 6 | 2 | White millet | 12% | 4.10% | 8.30% | $ 0.19 |
| 7 | 3 | Kibble corn | 8.50% | 3.80% | 2.70% | $ 0.10 |
| 8 | 4 | Oats | 15.40% | 6.30% | 2.40% | $ 0.10 |
| 9 | 5 | Cracked corn | 8.50% | 3.80% | 2.70% | $ 0.07 |
| 10 | 6 | Wheat | 12% | 1.70% | 2.30% | $ 0.05 |
| 11 | 7 | Safflower | 18% | 17.90% | 28.80% | $ 0.26 |
| 12 | 8 | Canary grass seed | 11.90% | 4% | 10.90% | $ 0.11 |
| 13 | | Requirement | 13% | 15% | | |
| 14 | | Limitation | | | 14% | |
| 15 | | | | | | |
| 16 | **Model** | | | | | |
| 17 | | **Ingredient** | **Pounds** | | | |
| 18 | 1 | Sunflower seeds | 0 | | | **Total** |
| 19 | 2 | White millet | 0 | | **Cost/lb.** | $ - |
| 20 | 3 | Kibble corn | 0 | | **Protein** | 0.00% |
| 21 | 4 | Oats | 0 | | **Fat** | 0.00% |
| 22 | 5 | Cracked corn | 0 | | **Fiber** | 0.00% |
| 23 | 6 | Wheat | 0 | | | |
| 24 | 7 | Safflower | 0 | | | |
| 25 | 8 | Canary grass seed | 0 | | | |
| 26 | | **Total** | 0 | | | |

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **BG Seed Company** | | | | | |
| 2 | | | | | | |
| 3 | **Data** | | | | | |
| 4 | | **Ingredient** | **Protein %** | **Fat %** | **Fiber %** | **Cost/lb** |
| 5 | 1 | Sunflower seeds | 0.169 | 0.26 | 0.29 | 0.22 |
| 6 | 2 | White millet | 0.12 | 0.041 | 0.083 | 0.19 |
| 7 | 3 | Kibble corn | 0.085 | 0.038 | 0.027 | 0.1 |
| 8 | 4 | Oats | 0.154 | 0.063 | 0.024 | 0.1 |
| 9 | 5 | Cracked corn | 0.085 | 0.038 | 0.027 | 0.07 |
| 10 | 6 | Wheat | 0.12 | 0.017 | 0.023 | 0.05 |
| 11 | 7 | Safflower | 0.18 | 0.179 | 0.288 | 0.26 |
| 12 | 8 | Canary grass seed | 0.119 | 0.04 | 0.109 | 0.11 |
| 13 | | Requirement | 0.13 | 0.15 | | |
| 14 | | Limitation | | | 0.14 | |
| 15 | | | | | | |
| 16 | **Model** | | | | | |
| 17 | | **Ingredient** | **Pounds** | | | |
| 18 | 1 | Sunflower seeds | 0 | | | **Total** |
| 19 | 2 | White millet | 0 | | **Cost/lb.** | =SUMPRODUCT(F5:F12,C18:C25) |
| 20 | 3 | Kibble corn | 0 | | **Protein** | =SUMPRODUCT(C5:C12,C18:C25) |
| 21 | 4 | Oats | 0 | | **Fat** | =SUMPRODUCT(D5:D12,C18:C25) |
| 22 | 5 | Cracked corn | 0 | | **Fiber** | =SUMPRODUCT(E5:E12,C18:C25) |
| 23 | 6 | Wheat | 0 | | | |
| 24 | 7 | Safflower | 0 | | | |
| 25 | 8 | Canary grass seed | 0 | | | |
| 26 | | **Total** | =SUM(C18:C25) | | | |

that can help in understanding why. This is shown in Figure 14.13. From this report it appears that a conflict exists in trying to meet both the fat and fiber constraints. If you look closely at the data, you can see that only sunflower seeds and safflower seeds have the high-enough amounts of fat needed to meet the 15% requirement; however, they also have very high amounts of fiber, so including them in the mixture makes it impossible to meet the fiber limitation.

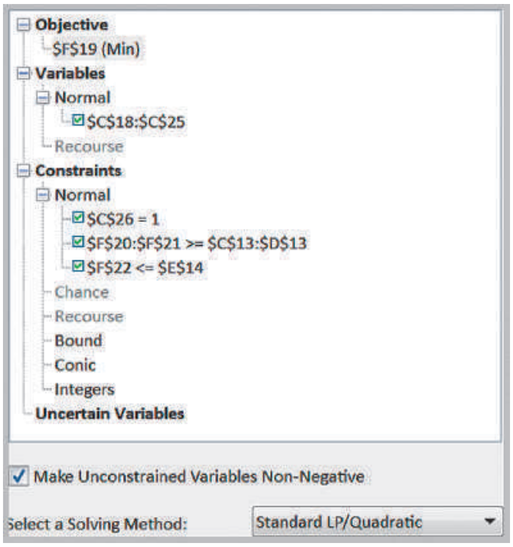



**Figure 14.12**

*Solver* Model for BG Seed Company Problem

**Figure 14.13**

Feasibility Report for BG Seed Model

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 5 | | | | | | | |
| 6 | Constraints that Make the Problem Infeasible | | | | | | |
| 7 | | Cell | Name | Cell Value | Formula | Status | Slack |
| 8 | | $C$26 | Total Pounds | 1 | $C$26=1 | Binding | 0 |
| 9 | | $F$21 | Fat Total | 15.00% | $F$21>=$D$13 | Binding | 0 |
| 10 | | $F$22 | Fiber Total | 14.00% | $F$22<=$E$14 | Binding | 0 |

**Figure 14.14**

Model Scenarios for BG Seed Company Problem

| | H | I | J | K |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | Scenario | 14.5% Fat | 14.5% Fiber |
| 3 | | Ingredient | Pounds | Pounds |
| 4 | 1 | Sunflower seeds | 0.434 | 0.454 |
| 5 | 2 | White millet | 0.000 | 0.000 |
| 6 | 3 | Kibble corn | 0.000 | 0.000 |
| 7 | 4 | Oats | 0.422 | 0.450 |
| 8 | 5 | Cracked corn | 0.144 | 0.096 |
| 9 | 6 | Wheat | 0.000 | 0.000 |
| 10 | 7 | Safflower | 0.000 | 0.000 |
| 11 | 8 | Canary grass seed | 0.000 | 0.000 |
| 12 | | | | |
| 13 | | Cost/lb. | $0.148 | $ 0.152 |
| 14 | | Protein | 15.06% | 15.42% |
| 15 | | Fat | 14.50% | 15.00% |
| 16 | | Fiber | 14.00% | 14.50% |

So what should the company owner do? One option is to investigate other potential ingredients to use in the mixture that have different nutritional characteristics and see if a feasible solution can be found. The second option is to either lower the fat requirement or raise the fiber limitation, recognizing that these are not ironclad constraints, but simply nutritional goals that can probably be modified in consultation with the company nutritionists. Figure 14.14 shows *Solver* solutions to two what-if scenarios, where the fat requirement is lowered to 14.5%, and the fiber limitation is raised to 14.5%, with all other data remaining the same in each case. Feasible solutions were found for both cases, and there is little difference in the results.

## Portfolio Investment Models

Many types of financial investment problems are modeled and solved using linear optimization. Such portfolio investment models problems have the basic characteristics of blending models.

### EXAMPLE 14.4 Innis Investments

Innis Investments is a small, family-owned business that manages personal financial portfolios. The company manages six mutual funds and has a client that has acquired \$500,000 from an inheritance. Characteristics of the funds are given in Table 14.5.

Innis Investments uses a proprietary algorithm to establish a measure of risk for its funds based on the historical volatility of the investments. The higher the volatility, the greater the risk. The company recommends that no more than \$200,000 be invested in any individual fund, that at least \$50,000 be invested in each of the multinational and balanced funds, and that the total amount invested in income equity and balanced funds be at least 40% of the total investment, or \$200,000. The client would like to have an average return of at least 5% but would like to minimize risk. What portfolio would achieve this?

Let $X_1$ through $X_6$ represent the dollar amount invested in funds 1 through 6, respectively. The total risk would be measured by the weighted risk of the portfolio, where the weights are the proportion of the total investment in any fund ($X_j$/500,000). Thus, the objective function is

$$\text{minimize total risk} = \frac{10.57X_1 + 13.22X_2 + 14.02X_3 + 2.39X_4 + 9.30X_5 + 7.61X_6}{500{,}000}$$

The first constraint ensures that \$500,000 is invested:

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 500{,}000$$

The next constraint ensures that the weighted return is at least 5%:

$$\frac{8.13X_1 + 9.02X_2 + 7.56X_3 + 3.62X_4 + 7.79X_5 + 4.40X_6}{500{,}000} \geq 5.00$$

The next constraint ensures that at least 40% be invested in the income equity and balanced funds:

$$X_5 + X_6 \geq 0.4(500{,}000)$$

The following constraints specify that at least \$50,000 be invested in each of the multinational and balanced funds:

$$X_2 \geq 50{,}000$$

$$X_6 \geq 50{,}000$$

Finally, we restrict each investment to a maximum of \$200,000 and include nonnegativity:

$$X_j \leq 200{,}000 \quad \text{for } j = 1, \ldots, 6$$

$$X_j \geq 0 \quad \text{for } j = 1, \ldots, 6$$

**Table 14.5**

Mutual Fund Data

| Fund | Expected Annual Return | Risk Measure |
|---|---|---|
| 1. Innis Low-priced Stock Fund | 8.13% | 10.57 |
| 2. Innis Multinational Fund | 9.02% | 13.22 |
| 3. Innis Mid-cap Stock Fund | 7.56% | 14.02 |
| 4. Innis Mortgage Fund | 3.62% | 2.39 |
| 5. Innis Income Equity Fund | 7.79% | 9.30 |
| 6. Innis Balanced Fund | 4.40% | 7.61 |

**Figure 14.15**

Spreadsheet Model for *Innis Investments*

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | Innis Investments | | | | | |
| 2 | | | | | | |
| 3 | Data | | | | | |
| 4 | | | Expected | | | |
| 5 | | Fund | Return | Risk Measure | Maximum | Minimum |
| 6 | 1 | Low Priced Stock | 8.13% | 10.57 | $ 200,000 | |
| 7 | 2 | Multinational | 9.02% | 13.22 | $ 200,000 | $ 50,000 |
| 8 | 3 | Mid Cap | 7.56% | 14.02 | $ 200,000 | |
| 9 | 4 | Mortgage | 3.62% | 2.39 | $ 200,000 | |
| 10 | 5 | Income Equity | 7.79% | 9.3 | $ 200,000 | |
| 11 | 6 | Balanced | 4.40% | 7.61 | $ 200,000 | $ 50,000 |
| 12 | | | | | | |
| 13 | | Investment = | $ 500,000 | | | |
| 14 | | Target return ≥ | 5% | | | |
| 15 | | Inc. Eq. + Balanced ≥ | $200,000 | | | |
| 16 | | | | | | |
| 17 | Model | | | | | |
| 18 | | | | | | |
| 19 | | Fund | Amount Invested | | | |
| 20 | 1 | Low Priced Stock | $ - | | | |
| 21 | 2 | Multinational | $ 50,000.00 | | | |
| 22 | 3 | Mid Cap | $ - | | | |
| 23 | 4 | Mortgage | $ 200,000.00 | | | |
| 24 | 5 | Income Equity | $ 66,371.68 | | | |
| 25 | 6 | Balanced | $ 183,628.32 | | | |
| 26 | | Total | $ 500,000.00 | | | |
| 27 | | | | | | |
| 28 | | | | | | |
| 29 | | | Total | | | |
| 30 | | Risk | 6.3073 | | | |
| 31 | | Weighted Return | 5.00% | | | |
| 32 | | Inc Eq + Balanced | $250,000 | | | |

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | Innis Investments | | | | | |
| 2 | | | | | | |
| 3 | Data | | | | | |
| 4 | | | Expected | | | |
| 5 | | Fund | Return | Risk Measure | Maximum | Minimum |
| 6 | 1 | Low Priced Stock | 0.0813 | 10.57 | 200000 | |
| 7 | 2 | Multinational | 0.0902 | 13.22 | 200000 | 50000 |
| 8 | 3 | Mid Cap | 0.0756 | 14.02 | 200000 | |
| 9 | 4 | Mortgage | 0.0362 | 2.39 | 200000 | |
| 10 | 5 | Income Equity | 0.0779 | 9.3 | 200000 | |
| 11 | 6 | Balanced | 0.044 | 7.61 | 200000 | 50000 |
| 12 | | | | | | |
| 13 | | Investment = | 500000 | | | |
| 14 | | Target return ≥ | 0.05 | | | |
| 15 | | Inc. Eq. + Balanced ≥ | =0.4*C13 | | | |
| 16 | | | | | | |
| 17 | Model | | | | | |
| 18 | | | | | | |
| 19 | | Fund | Amount Invested | | | |
| 20 | 1 | Low Priced Stock | 0 | | | |
| 21 | 2 | Multinational | 50000 | | | |
| 22 | 3 | Mid Cap | 0 | | | |
| 23 | 4 | Mortgage | 200000 | | | |
| 24 | 5 | Income Equity | 66371.6814159293 | | | |
| 25 | 6 | Balanced | 183628.318584071 | | | |
| 26 | | Total | =SUM(C20:C25) | | | |
| 27 | | | | | | |
| 28 | | | | | | |
| 29 | | | Total | | | |
| 30 | | Risk | =SUMPRODUCT(D6:D11,C20:C25)/C13 | | | |
| 31 | | Weighted Return | =SUMPRODUCT(C6:C11,C20:C25)/C13 | | | |
| 32 | | Inc Eq + Balanced | =C24+C25 | | | |

**Figure 14.16**

*Solver* Model for *Innis Investments*

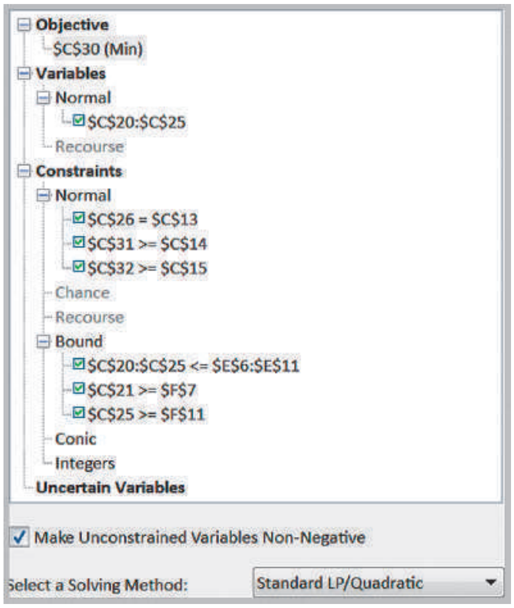



Figure 14.15 shows a spreadsheet implementation of this model (Excel file *Innis Investments*) with the optimal solution. The *Solver* model is given in Figure 14.16. All constraints are met with a minimum risk measure of 6.3073.

## Evaluating Risk versus Reward

In financial decisions such as these, it is often useful to compare risk versus reward to make an informed decision, particularly since the target return is subjective.

### EXAMPLE 14.5 Risk versus Reward

If you examine the Sensitivity report (not shown) in the Excel file for the Innis investment problem, you will find that the Allowable Increase and Allowable Decrease values for the weighted return are very small, 0.00906 and 0.00111, respectively; so any changes in the target return will require re-solving the model. Figure 14.17 shows such an analysis for target returns between 4% and 7%. We see that below 5%, we can obtain a return of 4.89% with a minimum risk. The chart on the right shows that as the target return increases, the risk increases, and at 6%, begins to increase at a faster rate. As the target return increases, the investment mix begins to change toward a higher percentage of low-price stock, which is a riskier investment, as shown in the chart on the left. A more conservative client might be willing to take a small amount of additional risk to achieve a 6% return but not venture beyond that value. We will discuss this further in Chapter 16 when we address decision analysis. This example clearly shows the value of using optimization models in a predictive analytics context, as we discussed at the end of the previous chapter.

**Figure 14.17**

Scenario Analysis for *Innis Investments*

| | Target Return | 4.0% | 4.5% | 5.0% | 5.5% | 6.0% | 6.5% | 7.0% |
|---|---|---|---|---|---|---|---|---|
| | Fund | $ Invested | $ Invested | $ Invested | $ Invested | $ Invested | $ Invested | $ Invested |
| 1 | Low Priced Stock | $ - | $ - | $ - | $ - | $ 10,421.29 | $ 65,853.66 | $121,286.03 |
| 2 | Multinational | $ 50,000.00 | $ 50,000.00 | $ 50,000.00 | $ 50,000.00 | $ 50,000.00 | $ 50,000.00 | $ 50,000.00 |
| 3 | Mid Cap | $ - | $ - | $ - | $ - | $ - | $ - | $ - |
| 4 | Mortgage | $200,000.00 | $200,000.00 | $200,000.00 | $200,000.00 | $189,578.71 | $134,146.34 | $ 78,713.97 |
| 5 | Income Equity | $ 50,000.00 | $ 50,000.00 | $ 66,371.68 | $140,117.99 | $200,000.00 | $200,000.00 | $200,000.00 |
| 6 | Balanced | $200,000.00 | $200,000.00 | $183,628.32 | $109,882.01 | $ 50,000.00 | $ 50,000.00 | $ 50,000.00 |
| | Weighted Return | 4.89% | 4.89% | 5.00% | 5.50% | 6.00% | 6.50% | 7.00% |
| | Risk | 6.2520 | 6.2520 | 6.3073 | 6.5566 | 6.9295 | 7.8364 | 8.7432 |



## Scaling Issues in Using *Solver*

A *poorly scaled* model is one that computes values of the objective, constraints, or intermediate results that differ by several orders of magnitude. Because of the finite precision of computer arithmetic, when these values of very different magnitudes (or others derived from them) are added, subtracted, or compared—in the user's model or in the *Solver*'s own calculations—the result will be accurate to only a few significant digits. As a result, *Solver* may detect or suffer from "numerical instability." The effects of poor scaling in an optimization model can be among the most difficult problems to identify and resolve. It can cause *Solver* engines to return messages such as "Solver could not find a feasible solution," "Solver could not improve the current solution," or even "The linearity conditions required by this Solver engine are not satisfied," or it may return results that are suboptimal or otherwise very different from your expectations.

In the *Solver* options, you can check the box *Use Automatic Scaling*. When this option is selected, the *Solver* rescales the values of the objective and constraint functions internally to minimize the effects of poor scaling. But this can only help with the *Solver*'s own calculations—it cannot help with poorly scaled results that arise *in the middle of your Excel formulas*. The best way to avoid scaling problems is to carefully choose the "units" implicitly used in your model so that all computed results are within a few orders of magnitude of each other. For example, if you express dollar amounts in units of (say) millions, the actual numbers computed on your worksheet may range from perhaps 1 to 1,000.

## EXAMPLE 14.6 Little Investment Advisors

Little Investment Advisors is working with a client on determining an optimal portfolio of bond funds. The firm suggests six different funds, each with different expected returns and risk measures (based on historical data):

| Bond Portfolio | Expected Return | Risk Measure |
|---|---|---|
| 1. Ohio National Bond Portfolio | 6.11% | 4.62 |
| 2. PIMCO Global Bond Unhedged Portfolio | 7.61% | 7.22 |
| 3. Federated High Income Bond Portfolio | 5.29% | 9.75 |
| 4. Morgan Stanley UIF Core Plus Fixed Income Portfolio | 2.79% | 3.95 |
| 5. PIMCO Real Return Portfolio | 7.37% | 6.04 |
| 6. PIMCO Total Return Portfolio | 5.65% | 5.17 |

The client wants to invest \$350,000. Find the optimal investment strategy to achieve the largest weighted percentage return while keeping the weighted risk measure no greater than 5.00.

The model is simple. Let $X_1$ through $X_6$ be the amount invested in each of the six funds.

Maximize

$$(6.11X_1 + 7.61X_2 + 5.29X_3 + 2.79X_4 + 7.37X_5 + 5.65X_6)/350{,}000$$

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 350{,}000$$

$$(4.62X_1 + 7.22X_2 + 9.75X_3 + 3.95X_4 + 6.04X_5 + 5.17X_6)/350{,}000 \le 5.00$$

$$X_1, \ldots, X_6 \ge 0$$

Figure 14.18 shows the solution using *Premium Solver* without scaling the variables. *Solver* displayed no messages, but the answer is incorrect! This occurs because the objective function (in percent) is several orders of magnitude smaller than the decision

**Figure 14.18**

Solution without Scaling

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Little Investment Advisors** | | | |
| 2 | | | | |
| 3 | Bond Fund | | **Expected Returr** | **Risk Measure** |
| 4 | **1** | Ohio National Bond | 6.11% | 4.62 |
| 5 | **2** | PIMCO Global Bond Unhedged | 7.61% | 7.22 |
| 6 | **3** | Federated High Income Bond | 5.29% | 9.75 |
| 7 | **4** | Morgan Stanley UIF Core Plus Fixed Inco | 2.79% | 3.95 |
| 8 | **5** | PIMCO Real Return | 7.37% | 6.04 |
| 9 | **6** | PIMCO Total Return | 5.65% | 5.17 |
| 10 | | | | |
| 11 | | Investment | \$350,000.00 | |
| 12 | | Target risk <= | 5.00 | |
| 13 | | | | |
| 14 | Model | | | |
| 15 | | | | |
| 16 | Bond Fund | | **Amount Invested** | |
| 17 | **1** | Ohio National Bond | \$0.00 | |
| 18 | **2** | PIMCO Global Bond Unhedged | \$0.00 | |
| 19 | **3** | Federated High Income Bond | \$0.00 | |
| 20 | **4** | Morgan Stanley UIF Core Plus Fixed Inco | \$48,770.49 | |
| 21 | **5** | PIMCO Real Return | \$0.00 | |
| 22 | **6** | PIMCO Total Return | \$301,229.51 | |
| 23 | | Total | \$350,000.00 | |
| 24 | | | | |
| 25 | | Risk | 5.00 | |
| 26 | | Percent Return | 5.25% | |

**Figure 14.19**

Solution after Scaling the Model

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Little Investment Advisors** | | | |
| 2 | | | | |
| 3 | Bond Fund | | **Expected Return** | **Risk Measure** |
| 4 | **1** | Ohio National Bond | 6.11% | 4.62 |
| 5 | **2** | PIMCO Global Bond Unhedged | 7.61% | 7.22 |
| 6 | **3** | Federated High Income Bond | 5.29% | 9.75 |
| 7 | **4** | Morgan Stanley UIF Core Plus Fixed Inco | 2.79% | 3.95 |
| 8 | **5** | PIMCO Real Return | 7.37% | 6.04 |
| 9 | **6** | PIMCO Total Return | 5.65% | 5.17 |
| 10 | | | | |
| 11 | | Investment | $350 | (in thousands) |
| 12 | | Target risk <= | 5.00 | |
| 13 | | | | |
| 14 | Model | | | |
| 15 | | | | |
| 16 | Bond Fund | | **Amount Invested** | (in thousands) |
| 17 | **1** | Ohio National Bond | $256.34 | |
| 18 | **2** | PIMCO Global Bond Unhedged | $0.00 | |
| 19 | **3** | Federated High Income Bond | $0.00 | |
| 20 | **4** | Morgan Stanley UIF Core Plus Fixed Inco | $0.00 | |
| 21 | **5** | PIMCO Real Return | $93.66 | |
| 22 | **6** | PIMCO Total Return | $0.00 | |
| 23 | | Total | $350.00 | |
| 24 | | | | |
| 25 | | Risk | 5.00 | |
| 26 | | Percent Return | 6.45% | |

variables and investment constraint (in hundreds of thousands of dollars). Figure 14.19 shows the result after the data have been scaled by expressing the decision variables and investment amount to thousands of dollars. This is the correct answer. So check your models carefully for possible scaling issues!

## Transportation Models

Many practical models in supply chain optimization stem from a very simple model called the **transportation problem**. This involves determining how much to ship from a set of sources of supply (factories, warehouses, etc.) to a set of demand locations (warehouses, customers, etc.) at minimum cost.

### EXAMPLE 14.7 General Appliance Corporation

General Appliance Corporation (GAC) produces refrigerators at two plants: Marietta, Georgia, and Minneapolis, Minnesota. They ship them to major distribution centers in Cleveland, Baltimore, Chicago, and Phoenix. The accounting, production, and marketing departments have provided the information in Table 14.6, which shows the unit cost of shipping between any plant and distribution center, plant capacities over the next planning period, and distribution center demands. GAC's supply chain manager faces the problem of determining how much to ship between each plant and distribution center to minimize the total transportation cost, not exceed available capacity, and meet customer demand.

To develop a linear optimization model, we first define the decision variables as the amount to ship between each plant and distribution center. In this model, we use *double-subscripted variables* to simplify the formulation. Define $X_{ij}$ = amount shipped from plant $i$ to distribution center $j$, where $i = 1$ represents Marietta, $i = 2$ represents Minneapolis, $j = 1$ represents Cleveland, and so on.

**Table 14.6**

GAC Cost, Capacity, and Demand Data

| | Distribution Center | | | | |
|---|---|---|---|---|---|
| **Plant** | **Cleveland** | **Baltimore** | **Chicago** | **Phoenix** | **Capacity** |
| Marietta | \$12.60 | \$14.35 | \$11.52 | \$17.58 | 1,200 |
| Minneapolis | \$9.75 | \$16.26 | \$8.11 | \$17.92 | 800 |
| **Demand** | 150 | 350 | 500 | 1,000 | |

Using the unit-cost data in Table 14.5, the total cost of shipping is equal to the unit cost multiplied by the amount shipped, summed over all combinations of plants and distribution centers. Therefore, the objective function is to minimize total cost:

$$\begin{aligned}\text{minimize } & 12.60X_{11} + 14.35X_{12} + 11.52X_{13} + 17.58X_{14} \\ & + 9.75X_{21} + 16.26X_{22} + 8.11X_{23} + 17.92X_{24}\end{aligned}$$

Because capacity is limited, the amount shipped from each plant cannot exceed its capacity. The total amount shipped from Marietta, for example, is $X_{11} + X_{12} + X_{13} + X_{14}$. Therefore, we have the constraint

$$X_{11} + X_{12} + X_{13} + X_{14} \le 1{,}200$$

Similarly, the capacity limitation at Minneapolis leads to the constraint

$$X_{21} + X_{22} + X_{23} + X_{24} \le 800$$

Next, we must ensure that the demand at each distribution center is met. This means that the total amount shipped to any distribution center from both plants must equal the demand. For instance, at Cleveland, we must have:

$$X_{11} + X_{21} = 150$$

For the remaining three distribution centers, the constraints are

$$X_{12} + X_{22} = 350$$

$$X_{13} + X_{23} = 500$$

$$X_{14} + X_{24} = 1{,}000$$

Last, we need nonnegativity, $X_{ij} \ge 0$, for all $i$ and $j$. The complete model is

$$\begin{aligned}\text{minimize } & 12.60X_{11} + 14.35X_{12} + 11.52X_{13} + 17.58X_{14} \\ & + 9.75X_{21} + 16.26X_{22} + 8.11X_{23} + 17.92X_{24}\end{aligned}$$

$$X_{11} + X_{12} + X_{13} + X_{14} \le 1200$$

$$X_{21} + X_{22} + X_{23} + X_{24} \le 800$$

$$X_{11} + X_{21} = 150$$

$$X_{12} + X_{22} = 350$$

$$X_{13} + X_{23} = 500$$

$$X_{14} + X_{24} = 1000$$

$$X_{ij} \ge 0, \quad \text{for all } i \text{ and } j$$

---

Figure 14.20 shows a spreadsheet implementation for the GAC transportation problem with the optimal solution (Excel file *General Appliance Corporation*), and Figure 14.21 shows the *Solver* model. The Excel model is very simple. In the model section, the decision

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | General Appliance Corporation | | | | | |
| 2 | | | | | | |
| 3 | Data | | | | | |
| 4 | | | Distribution Center | | | |
| 5 | Plant | Cleveland | Baltimore | Chicago | Phoenix | Capacity |
| 6 | Marietta | \$ 12.60 | \$ 14.35 | \$ 11.52 | \$ 17.58 | 1200 |
| 7 | Minneapolis | \$ 9.75 | \$ 16.26 | \$ 8.11 | \$ 17.92 | 800 |
| 8 | Demand | 150 | 350 | 500 | 1000 | |
| 9 | | | | | | |
| 10 | Model | | | | | |
| 11 | | | Distribution Center | | | |
| 12 | Plant | Cleveland | Baltimore | Chicago | Phoenix | Total shipped |
| 13 | Marietta | 0 | 350 | 0 | 850 | 1200 |
| 14 | Minneapolis | 150 | 0 | 500 | 150 | 800 |
| 15 | Demand met | 150 | 350 | 500 | 1000 | |
| 16 | | | | | | |
| 17 | | Cost | | | | |
| 18 | Total | \$ 28,171 | | | | |

**Figure 14.20**

*General Appliance Corporation* Model Spreadsheet Implementation and Solution

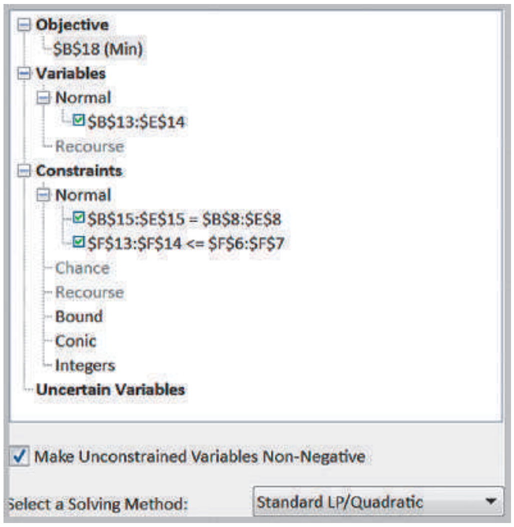



**Figure 14.21**

*General Appliance Corporation Solver* Model

variables are stored in the plant-distribution-center matrix. The objective function in cell B18 can be stated as

$$\text{total cost} = \text{B6} \times \text{B13} + \text{C6} \times \text{C13} + \text{D6} \times \text{D13} + \text{E6} \times \text{E13} + \text{B7} \times \text{B14} + \text{C7} \times \text{C14} + \text{D7} \times \text{D14} + \text{E7} \times \text{E14}$$

However, the SUMPRODUCT function is particularly useful for such large expressions; so it is more convenient to express the total cost as

SUMPRODUCT(B6:E7,B13:E14)

The SUMPRODUCT function can be used for any two arrays as long as the dimensions are the same. Here, the function multiplies pairwise the cost coefficients in the range B6:E7 by the amounts shipped in the range B13:E14 and then adds the terms. In the model, we also use the SUM function in cells F13 and F14 to sum the amount shipped from each plant, and also in cells B15 to E15 to sum the total amount shipped to each distribution center.

## Formatting the Sensitivity Report

Depending on how cells in your spreadsheet model are formatted, the Sensitivity report produced by *Solver* may not reflect the accurate values of reduced costs or shadow prices because an insufficient number of decimal places may be displayed. For example, Figure 14.22 shows the Sensitivity report created by *Solver*. Note that the data in columns headed reduced cost and shadow price are formatted as whole numbers. More-accurate values are shown in Figure 14.23 (obtained by simply formatting the data to have two decimal places). Thus, we *highly recommend* that after you save the Sensitivity report to your workbook, you select the reduced cost and shadow price ranges and format them to have at least two or three decimal places.

**Figure 14.22**

Original Sensitivity Report for GAC Transportation Model

Objective Cell (Min)

| Cell | Name | Final Value |
|---|---|---|
| $B$18 | Total Cost | 28171 |

Decision Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$13 | Marietta Cleveland | 0 | 3 | 12.6 | 1E+30 | 3.19 |
| $C$13 | Marietta Baltimore | 350 | 0 | 14.35 | 1.5700001 | 1E+30 |
| $D$13 | Marietta Chicago | 0 | 4 | 11.52 | 1E+30 | 3.75 |
| $E$13 | Marietta Phoenix | 850 | 0 | 17.58 | 0.3400001 | 1.5700001 |
| $B$14 | Minneapolis Cleveland | 150 | 0 | 9.75 | 3.1900001 | 1E+30 |
| $C$14 | Minneapolis Baltimore | 0 | 2 | 16.26 | 1E+30 | 1.57 |
| $D$14 | Minneapolis Chicago | 500 | 0 | 8.11 | 3.7500001 | 1E+30 |
| $E$14 | Minneapolis Phoenix | 150 | 0 | 17.92 | 1.5700001 | 0.3400001 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$15 | Demand met Cleveland | 150 | 10 | 150 | 0 | 150 |
| $C$15 | Demand met Baltimore | 350 | 15 | 350 | 0 | 150 |
| $D$15 | Demand met Chicago | 500 | 8 | 500 | 0 | 500 |
| $E$15 | Demand met Phoenix | 1000 | 18 | 1000 | 0 | 150 |
| $F$13 | Marietta Total shipped | 1200 | 0 | 1200 | 150 | 0 |
| $F$14 | Minneapolis Total shipped | 800 | 0 | 800 | 1E+30 | 0 |

**Figure 14.23**

Accurate Sensitivity Report for GAC Transportation Model

Objective Cell (Min)

| Cell | Name | Final Value |
|---|---|---|
| $B$18 | Total Cost | 28171 |

Decision Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$13 | Marietta Cleveland | 0 | 3.19 | 12.6 | 1E+30 | 3.19 |
| $C$13 | Marietta Baltimore | 350 | 0.00 | 14.35 | 1.5700001 | 1E+30 |
| $D$13 | Marietta Chicago | 0 | 3.75 | 11.52 | 1E+30 | 3.75 |
| $E$13 | Marietta Phoenix | 850 | 0.00 | 17.58 | 0.3400001 | 1.5700001 |
| $B$14 | Minneapolis Cleveland | 150 | 0.00 | 9.75 | 3.1900001 | 1E+30 |
| $C$14 | Minneapolis Baltimore | 0 | 1.57 | 16.26 | 1E+30 | 1.57 |
| $D$14 | Minneapolis Chicago | 500 | 0.00 | 8.11 | 3.7500001 | 1E+30 |
| $E$14 | Minneapolis Phoenix | 150 | 0.00 | 17.92 | 1.5700001 | 0.3400001 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$15 | Demand met Cleveland | 150 | 9.75 | 150 | 0 | 150 |
| $C$15 | Demand met Baltimore | 350 | 14.69 | 350 | 0 | 150 |
| $D$15 | Demand met Chicago | 500 | 8.11 | 500 | 0 | 500 |
| $E$15 | Demand met Phoenix | 1000 | 17.92 | 1000 | 0 | 150 |
| $F$13 | Marietta Total shipped | 1200 | -0.34 | 1200 | 150 | 0 |
| $F$14 | Minneapolis Total shipped | 800 | 0.00 | 800 | 1E+30 | 0 |

## EXAMPLE 14.8 Interpreting Sensitivity Information for the GAC Model

The transportation model is a good example to use to discuss the interpretation of reduced costs. First, note that the reduced costs are zero for all variables that are positive in the solution. Now examine the reduced cost, 3.19, associated with shipping from Marietta to Cleveland. A question to ask is, Why does the optimal solution ship nothing between these cities? The answer is simple: It is not economical to do so. In other words, it costs too much to ship from Marietta to Cleveland; the demand can be met less expensively by shipping from Minneapolis. The next logical question to ask is, What would the unit shipping cost have to be to make it attractive to ship from Marietta instead of Minneapolis? The answer is given by the reduced cost. If the unit cost can be reduced by at least \$3.19, then the optimal solution will change and would include a positive value for the Marietta–Cleveland variable. Again, this is true only if all other data are held constant. A supply chain manager might use this information to identify alternative transportation carriers or negotiate freight rates.

To interpret the shadow prices, you need to look at the information closely. For instance, the Allowable Increase for all the demand constraints is zero. This is because the total capacity equals the total demand; therefore, we cannot increase the demand at any distribution center without creating an infeasible problem. Nevertheless, the shadow prices do reflect the cost savings that would occur for a unit decrease in demand at one of the distribution centers. For example, the shadow price for the demand constraint at Cleveland is \$9.75, which is exactly equal to the unit cost of shipping from Minneapolis. If the demand at Cleveland decreases by 1, we simply ship one less unit from Minneapolis. However, note that the shadow price for the Baltimore constraint, \$14.69, is not equal to the unit cost of shipping from either Marietta or Minneapolis. If we ship one less unit from Marietta, we save only \$14.35. We could save more by adjusting other shipping decisions. If you change the Baltimore demand to 349 and re-solve the model, you will find that the optimal solution ships 349 units from Marietta to Cleveland at a cost savings of \$14.35 but now also ships 851 units from Marietta to Phoenix at a cost increase of \$17.58 and 149 units from Minneapolis to Phoenix at a cost savings of \$17.92. The net change in cost is $-\$14.35 + \$17.58 - \$17.92 = \$14.69$, which is the value of the shadow price.

Finally, the shadow price of $-0.34$ for the Marietta constraint states that if the capacity at Marietta can be increased (up to 150 units), the total cost can be reduced by \$0.34 per unit by reallocating the shipping decisions. However, the shadow price of 0 for Minneapolis means that even if the capacity is increased, no cost savings will occur because the optimal solution will not change.

### Degeneracy

Example 14.8 also exhibits a phenomenon called *degeneracy*. A solution is a **degenerate solution** if the right-hand-side value of any constraint has a zero Allowable Increase or Allowable Decrease, as we see in Figure 14.23. A full discussion of the implications of degeneracy is beyond the scope of this book; however, it is important to know that degeneracy can impact the interpretation of sensitivity analysis information. For example, reduced costs and shadow prices may not be unique, and you may have to change objective function coefficients beyond their allowable increases or decreases before the optimal solution will change. Thus, some caution should be exercised when interpreting the information. When in doubt, consult a business analytics expert.

## Multiperiod Production Planning Models

Many linear optimization problems involve planning production over multiple time periods. The basic decisions are how much to produce in each time period to meet anticipated demand over each period. Although it might seem obvious to simply produce to the anticipated level of sales, it may be advantageous to produce more than needed in earlier time periods when production costs may be lower and store the excess production as inventory for use in later time periods, thereby letting lower production costs offset the costs of holding the inventory. So the best decision is often not obvious.

## EXAMPLE 14.9 K&L Designs

K&L Designs is a home-based company that makes hand-painted jewelry boxes for teenage girls. Forecasts of sales for the next year are 150 in the autumn, 400 in the winter, and 50 in the spring. Plain jewelry boxes are purchased from a supplier for \$20. The cost of capital is estimated to be 24% per year (or 6% per quarter); thus, the holding cost per item is 0.06(\$20) = \$1.20 per quarter. The company hires art students part-time to craft designs during the autumn, and they earn \$5.50 per hour. Because of the high demand for part-time help during the winter holiday season, labor rates are higher in the winter, and workers earn \$7.00 per hour. In the spring, labor is more difficult to keep, and the owner must pay \$6.25 per hour to retain qualified help. Each jewelry box takes 2 hours to complete. How should production be planned over the three quarters to minimize the combined production and inventory-holding costs?

The principal decision variables are the number of jewelry boxes to produce during each of the three quarters. However, since we have the option of carrying inventory to other time periods, we must also define decision variables for the number of units to hold in inventory at the end of each quarter. The decision variables are

$P_A$ = amount to produce in autumn

$P_W$ = amount to produce in winter

$P_S$ = amount to produce in spring

$I_A$ = inventory held at the end of autumn

$I_W$ = inventory held at the end of winter

$I_S$ = inventory held at the end of spring

The production cost per unit is computed by multiplying the labor rate by the number of hours required to produce one. Thus, the unit cost in the autumn is (\$5.50)(2) = \$11.00; in the winter, (\$7.00)(2) = \$14.00; and in the spring, (\$6.25)(2) = \$12.50. The objective function is to minimize the total cost of production and inventory. (Because the cost of the boxes themselves is constant, it is not relevant to the problem we are addressing.) The objective function is, therefore,

$$\text{minimize } 11P_A + 14P_W + 12.50P_S + 1.20I_A + 1.20I_W + 1.20I_S$$

The only explicit constraint is that demand must be satisfied. Note that both the production in a quarter as well as the inventory held from the *previous* time quarter can be used to satisfy demand. In addition, any amount in excess of the demand is held to the next quarter. Therefore, the constraints take the form of *inventory balance equations* that essentially say that what is available in any time period must be accounted for somewhere. More formally,

production + inventory from the previous quarter
= demand + inventory held to the next quarter

This can be represented visually using the diagram in Figure 14.24. For each quarter, the sum of the variables coming in must equal the sum of the variables going out. Drawing such a figure is very useful for any type of multiple time period planning model. This results in the constraint set

$$P_A + 0 = 150 + I_A$$

$$P_W + I_A = 400 + I_W$$

$$P_S + I_W = 50 + I_S$$

Moving all variables to the left-side results in the model

$$\text{minimize } 11P_A + 14P_W + 12.50P_S + 1.20I_A + 1.20I_W + 1.20I_S$$

subject to

$$P_A - I_A = 150$$

$$P_W + I_A - I_W = 400$$

$$P_S + I_W - I_S = 50$$

$$P_i \geq 0, \quad \text{for all } i$$

$$I_j \geq 0, \quad \text{for all } j$$

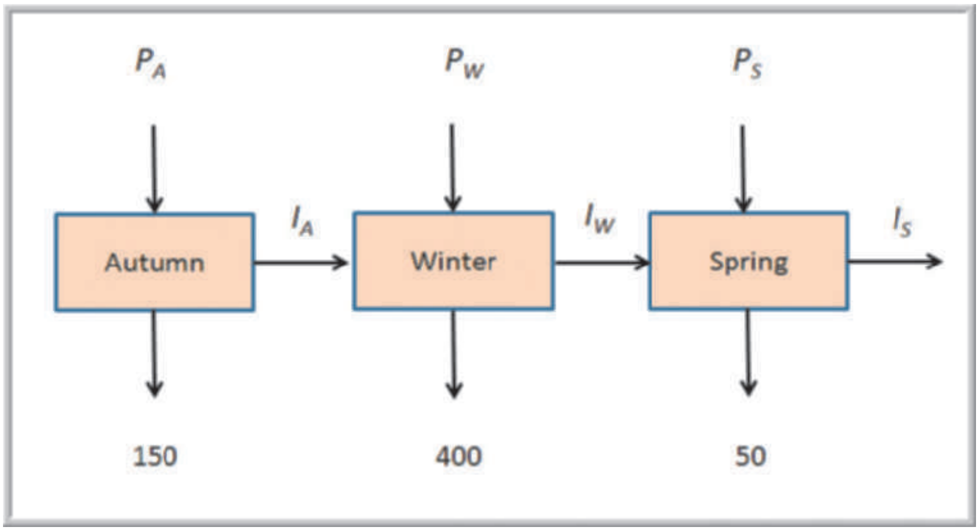



**Figure 14.24**

Material Balance Constraint Structure

Figure 14.25 shows a spreadsheet implementation for the K&L Designs model (Excel file *K&L Designs*); Figure 14.26 shows the associated *Solver* model. For the optimal solution, we produce the demand for the autumn and winter quarters in the autumn and store the excess inventory until the winter. This takes advantage of the lower production cost in the autumn. However, it is not economical to pay the inventory holding cost to carry the spring demand for two quarters.

## Building Alternative Models

As we have seen, developing models is more of an art than a science; consequently, there is often more than one way to model a particular problem. Using the ideas presented in the K&L Designs example, we may construct an alternative model involving only the production variables.

## EXAMPLE 14.10 An Alternative Optimization Model for K&L Designs

In the *K&L Designs* problem, we simply have to ensure that demand is satisfied. We can do this by guaranteeing that the cumulative production in each quarter is at least as great as the cumulative demand. This is expressed by the following constraints:

$$P_A \geq 150$$

$$P_A + P_W \geq 550$$

$$P_A + P_W + P_S \geq 600$$

$$P_A, P_W, P_S \geq 0$$

The differences between the left- and right-hand sides of these constraints are the ending inventories for each period (and we need to keep track of these amounts because inventory has a cost associated with it). Thus, we use the following objective function:

$$\text{minimize } 11P_A + 14P_W + 12.50P_S + 1.20(P_A - 150) + 1.20(P_A + P_W - 550) + 1.20(P_A + P_W + P_S - 600)$$

Of course, this function can be simplified algebraically by combining like terms. Although these two models look very different, they are mathematically equivalent and will produce the same solution.

**Figure 14.25**

Spreadsheet Model and Optimal Solution for *K&L Designs*

| | A | B | C | D |
|---|---|---|---|---|
| 1 | K&L Designs | | | |
| 2 | | | | |
| 3 | Data | | | |
| 4 | | | | |
| 5 | | Autumn | Winter | Spring |
| 6 | Unit Production Cost | $ 11.00 | $ 14.00 | $ 12.50 |
| 7 | Unit Inventory Holding Cost | $ 1.20 | $ 1.20 | $ 1.20 |
| 8 | Demand | 150 | 400 | 50 |
| 9 | | | | |
| 10 | Model | | | |
| 11 | | Autumn | Winter | Spring |
| 12 | Production | 550 | 0 | 50 |
| 13 | Inventory | 400 | 0 | 0 |
| 14 | | | | |
| 15 | Net production | 150 | 400 | 50 |
| 16 | | | | |
| 17 | | Cost | | |
| 18 | Total | $7,155.00 | | |

| | A | B | C | D |
|---|---|---|---|---|
| 1 | K&L Designs | | | |
| 2 | | | | |
| 3 | Data | | | |
| 4 | | | | |
| 5 | | Autumn | Winter | Spring |
| 6 | Unit Production Cost | 11 | 14 | 12.5 |
| 7 | Unit Inventory Holding Cost | 1.2 | 1.2 | 1.2 |
| 8 | Demand | 150 | 400 | 50 |
| 9 | | | | |
| 10 | Model | | | |
| 11 | | Autumn | Winter | Spring |
| 12 | Production | 550 | 0 | 50 |
| 13 | Inventory | 400 | 0 | 0 |
| 14 | | | | |
| 15 | Net production | =B12-B13 | =C12-C13+B13 | =D12-D13+C13 |
| 16 | | | | |
| 17 | | Cost | | |
| 18 | Total | =SUMPRODUCT(B6:D7,B12:D13) | | |

**Figure 14.26**

*Solver* Model for *K&L Designs*

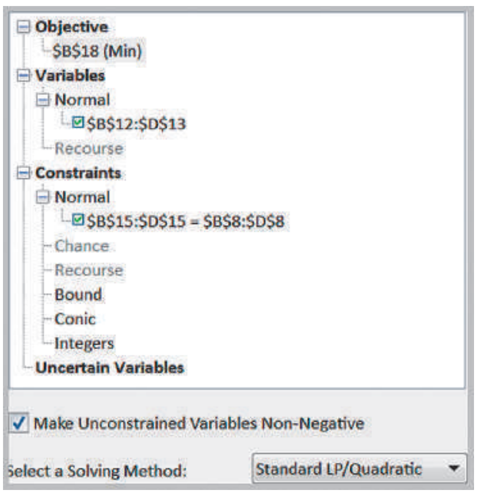


Figure 14.27 shows a spreadsheet implementation of this alternate model (available on a separate worksheet in the *K&L Designs* workbook), and Figure 14.28 shows the *Solver* model. Both have the same optimal solution; however, significant differences exist in the Sensitivity reports. Figure 14.29 shows a comparison of the Sensitivity reports.

**Figure 14.27**

Alternative Spreadsheet Model for *K&L Designs*

| | A | B | C | D |
|---|---|---|---|---|
| 1 | K&L Designs Alternate Model | | | |
| 2 | | | | |
| 3 | Data | | | |
| 4 | | | | |
| 5 | | Autumn | Winter | Spring |
| 6 | Unit Production Cost | $ 11.00 | $ 14.00 | $ 12.50 |
| 7 | Unit Inventory Holding Cost | $ 1.20 | $ 1.20 | $ 1.20 |
| 8 | Demand | 150 | 400 | 50 |
| 9 | Cumulative Demand | 150 | 550 | 600 |
| 10 | | | | |
| 11 | Model | | | |
| 12 | | Autumn | Winter | Spring |
| 13 | Production | 550 | 0 | 50 |
| 14 | Cumulative Production | 550 | 550 | 600 |
| 15 | Inventory | 400 | 0 | 0 |
| 16 | | | | |
| 17 | | Cost | | |
| 18 | Total | $ 7,155.00 | | |

| | A | B | C | D |
|---|---|---|---|---|
| 1 | K&L Designs Alternate Model | | | |
| 2 | | | | |
| 3 | Data | | | |
| 4 | | | | |
| 5 | | Autumn | Winter | Spring |
| 6 | Unit Production Cost | 11 | 14 | 12.5 |
| 7 | Unit Inventory Holding Cost | 1.2 | 1.2 | 1.2 |
| 8 | Demand | 150 | 400 | 50 |
| 9 | Cumulative Demand | =B8 | =B8+C8 | =B8+C8+D8 |
| 10 | | | | |
| 11 | Model | | | |
| 12 | | Autumn | Winter | Spring |
| 13 | Production | 550 | 0 | 50 |
| 14 | Cumulative Production | =B13 | =B13+C13 | =B13+C13+D13 |
| 15 | Inventory | =B14-B9 | =C14-C9 | =D14-D9 |
| 16 | | | | |
| 17 | | Cost | | |
| 18 | Total | =SUMPRODUCT(B13:D13,B6:D6)+SUMPRODUCT(B15:D15,B7:D7) | | |

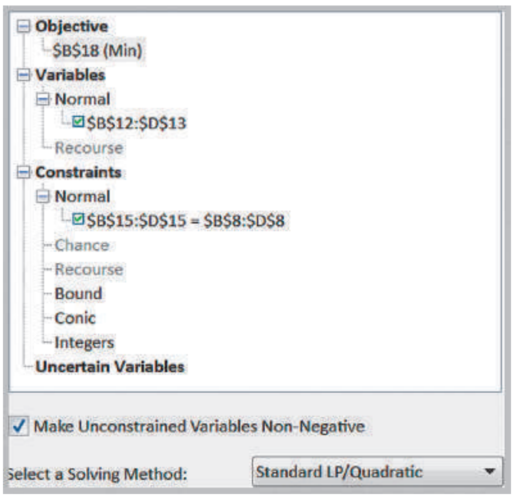



**Figure 14.28**

*Solver* Model for Alternative *K&L Designs* Model

**Original Model**

Objective Cell (Min)

| Cell | Name | Final Value |
|---|---|---|
| $B$18 | Total Cost | 7155 |

Decision Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$12 | Production Autumn | 550 | 0 | 11 | 1.8000001 | 0.9000001 |
| $C$12 | Production Winter | 0 | 1.8 | 14 | 1E+30 | 1.8 |
| $D$12 | Production Spring | 50 | 0 | 12.5 | 0.9000001 | 13.7000001 |
| $B$13 | Inventory Autumn | 400 | 0 | 1.2 | 1.8000001 | 0.9000001 |
| $C$13 | Inventory Winter | 0 | 0.9 | 1.2 | 1E+30 | 0.9 |
| $D$13 | Inventory Spring | 0 | 13.7 | 1.2 | 1E+30 | 13.7 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$15 | Net production Autumn | 150 | 11 | 150 | 1E+30 | 550 |
| $C$15 | Net production Winter | 400 | 12.2 | 400 | 1E+30 | 400 |
| $D$15 | Net production Spring | 50 | 12.5 | 50 | 1E+30 | 50 |

**Alternate Model**

Objective Cell (Min)

| Cell | Name | Final Value |
|---|---|---|
| $B$18 | Total Cost | 7155 |

Decision Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$13 | Production Autumn | 550 | 0 | 14.6 | 1.8000001 | 0.9000001 |
| $C$13 | Production Winter | 0 | 1.8 | 16.4 | 1E+30 | 1.8 |
| $D$13 | Production Spring | 50 | 0 | 13.7 | 0.9000001 | 13.7000001 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$14 | Cumulative Production Autumn | 550 | 0 | 150 | 400 | 1E+30 |
| $C$14 | Cumulative Production Winter | 550 | 0.9 | 550 | 50 | 400 |
| $D$14 | Cumulative Production Spring | 600 | 13.7 | 600 | 1E+30 | 50 |

**Figure 14.29**

Comparison of Sensitivity Reports for *K&L Designs* Models

Although the alternative model is more streamlined, the Sensitivity report provides less information of use to managers. For example, the alternative model does not provide the capability to study the effect of changing inventory costs or demands for each quarter individually. Therefore, it is important to consider the practical implications of generating good information from optimization models when building them.

## Multiperiod Financial Planning Models

Financial planning often occurs over an extended time horizon. Financial planning models have similar characteristics to multiperiod production planning and can be formulated as multiperiod optimization models.

### EXAMPLE 14.11 D. A. Branch & Sons

The financial manager at D. A. Branch & Sons must ensure that funds are available to pay company expenditures in the future but would also like to maximize investment income. Three short-term investment options are available over the next 6 months: *A*, a 1-month CD that pays 0.25%, available each month; *B*, a 3-month CD that pays 1.00% (at maturity), available at the beginning of the first 4 months; and *C*, a 6-month CD that pays 2.3% (at maturity), available in the first month. The net expenditures for the next 6 months are forecast as $50,000, ($12,000), $23,000, ($20,000), $41,000, and ($13,000). Amounts in parentheses indicate a net inflow of cash. The company must maintain a cash balance of at least $10,000 at the end of each month. The company currently has $200,000 in cash.

At the beginning of each month, the manager must decide how much to invest in each alternative that may be available. Define the following:

$A_i$ = amount ($) to invest in a 1-month CD at the start of month $i$

$B_i$ = amount ($) to invest in a 3-month CD at the start of month $i$

$C_i$ = amount ($) to invest in a 6-month CD at the start of month $i$

*(continued)*

Because the time horizons on these alternatives vary, it is helpful to draw a picture to represent the investments and returns for each year as shown in Figure 14.30. Each circle represents the beginning of a month. Arrows represent the investments and cash flows. For example, investing in a 3-month CD at the start of month 1 ($B_1$) matures at the beginning of month 4. It is reasonable to assume that all funds available would be invested.

From Figure 14.30, we see that investments $A_6$, $B_4$, and $C_1$ will mature at the end of month 6—that is, at the beginning of month 7. To maximize the amount of cash on hand at the end of the planning period, we have the objective function

$$\text{maximize } 1.0025A_6 + 1.00B_4 + 1.023C_1$$

The only constraints necessary are minimum cash balance equations. For each month, the net cash available, which is equal to the cash in less cash out, must be at least \$10,000. These follow directly from Figure 14.28. The complete model is

$$\text{maximize } 1.0025A_6 + 1.01B_4 + 1.023C_1$$

subject to

$$200{,}000 - (A_1 + B_1 + C_1 + 50{,}000) \geq 10{,}000 \text{ (month 1)}$$
$$1.0025A_1 + 12{,}000 - (A_2 + B_2) \geq 10{,}000 \text{ (month 2)}$$
$$1.0025A_2 - (A_3 + B_3 + 23{,}000) \geq 10{,}000 \text{ (month 3)}$$
$$1.0025A_3 + 1.01B_1 + 20{,}000 - (A_4 + B_4) \geq 10{,}000 \text{ (month 4)}$$
$$1.0025A_4 + 1.01B_2 - (A_5 + 41{,}000) \geq 10{,}000 \text{ (month 5)}$$
$$1.0025A_5 + 1.01B_3 + 13{,}000 - A_6 \geq 10{,}000 \text{ (month 6)}$$
$$A_i, B_i, C_i \geq 0, \quad \text{for all } i$$

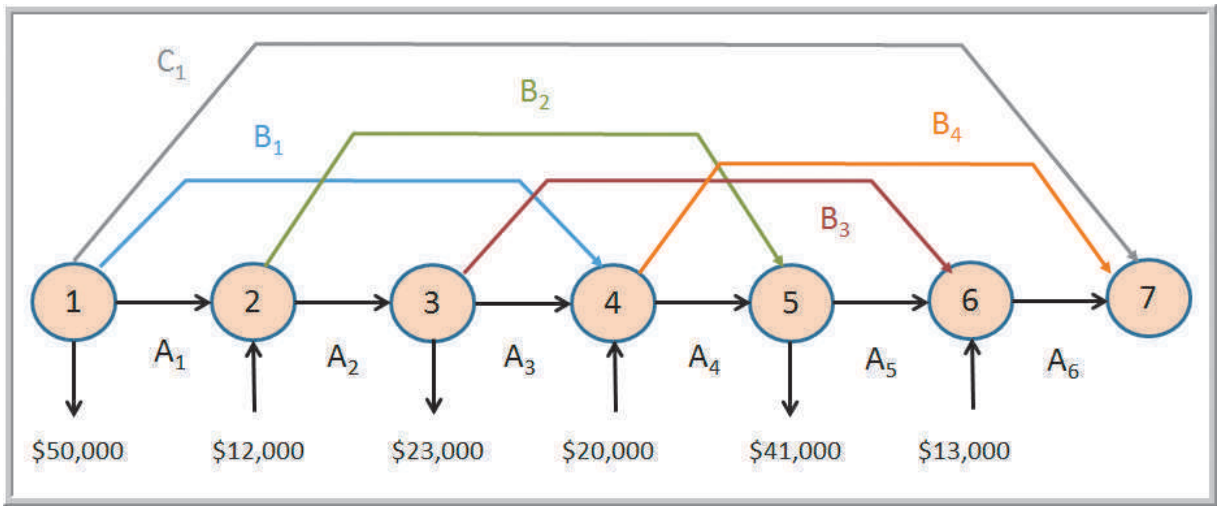



**Figure 14.30**

Cash Balance Constraint Structure

Figure 14.31 shows a spreadsheet model for this problem (Excel file *D. A. Branch & Sons*); the *Solver* model is shown in Figure 14.32. The spreadsheet model may look somewhat complicated; however, it has similar characteristics of a typical financial spreadsheet. The key to constructing the *Solver* model is the summary section. Here we calculate the monthly balance based on the amount of cash available (previous balance plus any investment returns), the net expenditures (remember that a negative expenditure is a cash inflow), and the amount invested as reflected by the decision variables. These balances are a practical interpretation of the constraint functions for each month in the model. In the *Solver* model, these balances simply need to be greater than or equal to the \$10,000 cash-balance requirement for each month.

**Figure 14.31**

Spreadsheet Model for *D. A. Branch & Sons*

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | D.A. Branch & Sons | | | | | | | |
| 2 | | | | | | | | |
| 3 | Data | | | | | | | |
| 4 | | | | | | | | |
| 5 | Month | 1 | 2 | 3 | 4 | 5 | 6 | |
| 6 | Net expenditures | $ 50,000 | $(12,000) | $23,000 | $(20,000) | $41,000 | $(13,000) | |
| 7 | Cash balance requirement | $ 10,000 | $ 10,000 | $10,000 | $ 10,000 | $10,000 | $ 10,000 | |
| 8 | | | | | | | | |
| 9 | Current balance | $200,000 | | | | | | |
| 10 | | | | | | | | |
| 11 | Model | | | | | | | |
| 12 | | | | | | | | |
| 13 | Investment | 1 | 2 | 3 | 4 | 5 | 6 | Rate of Return |
| 14 | A | $ 31,606 | $ 22,943 | $ - | $ 20,000 | $ - | $ 13,000 | 0.25% |
| 15 | B | $ - | $ 20,743 | $ - | $ - | | | 1.00% |
| 16 | C | $108,394 | | | | | | 2.30% |
| 17 | Total | $140,000 | $ 43,685 | $ - | $ 20,000 | $ - | $ 13,000 | |
| 18 | | | | | | | | |
| 19 | Returns | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 20 | A | | $ 31,685 | $23,000 | $ - | $20,050 | $ - | $ 13,033 |
| 21 | B | | | | $ - | $20,950 | $ - | $ - |
| 22 | C | | | | | | | $ 110,887 |
| 23 | Total | | $ 31,685 | $23,000 | $ - | $41,000 | $ - | $ 123,919 |
| 24 | | | | | | | | |
| 25 | Summary | | | | | | | |
| 26 | Amount available | $200,000 | $ 41,685 | $33,000 | $ 10,000 | $51,000 | $ 10,000 | |
| 27 | Net expenditures | $ 50,000 | $(12,000) | $23,000 | $(20,000) | $41,000 | $(13,000) | |
| 28 | Amount invested | $140,000 | $ 43,685 | $ - | $ 20,000 | $ - | $ 13,000 | |
| 29 | Balance | $ 10,000 | $ 10,000 | $10,000 | $ 10,000 | $10,000 | $ 10,000 | |

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | D.A. Branch & Sons | | | | | | | |
| 2 | | | | | | | | |
| 3 | Data | | | | | | | |
| 4 | | | | | | | | |
| 5 | Month | 1 | 2 | 3 | 4 | 5 | 6 | |
| 6 | Net expenditures | 50000 | -12000 | 23000 | -20000 | 41000 | -13000 | |
| 7 | Cash balance requirement | 10000 | 10000 | 10000 | 10000 | 10000 | 10000 | |
| 8 | | | | | | | | |
| 9 | Current balance | 200000 | | | | | | |
| 10 | | | | | | | | |
| 11 | Model | | | | | | | |
| 12 | | | | | | | | |
| 13 | Investment | 1 | 2 | 3 | 4 | 5 | 6 | Rate of Return |
| 14 | A | 31606.202143588 | 22942.6433915212 | 0 | 20000 | 0 | 13000 | 0.0025 |
| 15 | B | 0 | 20742.5742574257 | 0 | 0 | | | 0.01 |
| 16 | C | 108393.797856412 | | | | | | 0.023 |
| 17 | Total | =SUM(B14:B16) | =SUM(C14:C16) | =SUM(D14:D16) | =SUM(E14:E16) | =SUM(F14:F16) | =SUM(G14:G16) | |
| 18 | | | | | | | | |
| 19 | Returns | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 20 | A | | =(1+$H$14)*B14 | =(1+$H$14)*C14 | =(1+$H$14)*D14 | =(1+$H$14)*E14 | =(1+$H$14)*F14 | =(1+$H$14)*G14 |
| 21 | B | | | | =(1+$H$15)*B15 | =(1+$H$15)*C15 | =(1+$H$15)*D15 | =(1+$H$15)*E15 |
| 22 | C | | | | | | | =(1+H16)*B16 |
| 23 | Total | | =SUM(C20:C22) | =SUM(D20:D22) | =SUM(E20:E22) | =SUM(F20:F22) | =SUM(G20:G22) | =SUM(H20:H22) |
| 24 | | | | | | | | |
| 25 | Summary | | | | | | | |
| 26 | Amount available | =B9 | =B29+C23 | =C29+D23 | =D29+E23 | =E29+F23 | =F29+G23 | |
| 27 | Net expenditures | 50000 | -12000 | 23000 | -20000 | 41000 | -13000 | |
| 28 | Amount invested | =B17 | =C17 | =D17 | =E17 | =F17 | =G17 | |
| 29 | Balance | =B26-B27-B28 | =C26-C27-C28 | =D26-D27-D28 | =E26-E27-E28 | =F26-F27-F28 | =G26-G27-G28 | |

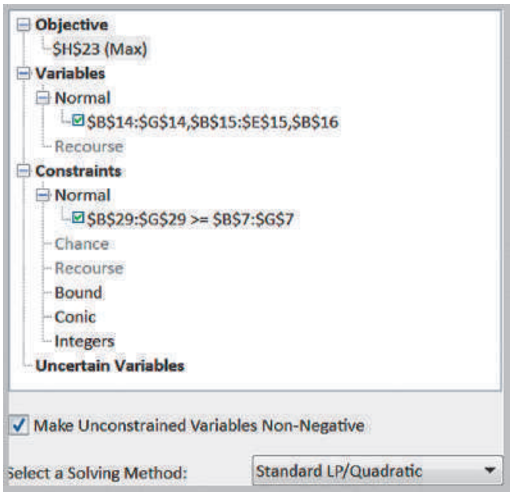



**Figure 14.32**

*Solver* Model for *D. A. Branch & Sons*

## Analytics in Practice: Linear Optimization in Bank Financial Planning

One of the first applications of linear optimization in banking was developed by Central Carolina Bank and Trust Company (CCB).[1] The bank's management became increasingly concerned with coordinating the activities of the bank to maximize interest rate differentials between sources and uses of funds. To address these concerns, the bank established a financial planning committee comprising all senior bank officers. The committee was charged with the responsibility of integrating the following functions; (1) interest rate forecasting, (2) forecasting demand for bank services, (3) liquidity management policy, and (4) funds allocation. At the same time, CCB's executive committee authorized the development of a balance sheet optimization model using linear programming.

The initial stage in the model's development involved a series of meetings with the financial planning committee to determine how complex the model needed to be. After a thorough discussion of the available options, the group settled on a 1-year single-period model, containing 66 asset and 32 liability and equity categories. Even though a single-period planning model ignores many important time-related linkages, it was felt that a single-period framework would result in a model structure whose output could be readily internalized by management. An integral part of these discussions involved an attempt to assure senior managers that the resulting model would capture their perceptions of the banking environment.

Next, the model was formulated and its data requirements were clearly identified. The major data inputs needed to implement the model were

- expected yields on all securities and loan categories,
- expected interest rates on deposits and money market liabilities,
- administrative and/or processing costs on major loan and deposit categories,
- expected loan losses, by loan type, as a percentage of outstanding loans,
- maturity structure of all asset and liability categories,
- forecasts of demand for bank services.

The bank's financial records served as a useful database for the required inputs. In those instances where meaningful data did not exist, studies were initiated to fill the gap.

[1] Based on Sheldon D. Balbirer and David Shaw, "An Application of Linear Programming to Bank Financial Planning," *Interfaces* 11, 5 (October 1981).

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The decision variables in the model represented different asset categories, such as cash, treasury securities, consumer loans, and commercial loans, among others; other variables represented liabilities and equities such as savings accounts, money market certificates, and certificates of deposit. The objective function was to maximize profits, equaling the difference between net yields and costs. Constraints reflected various operational, legal, and policy considerations, including bounds on various asset or liability categories that represent forecasts of demand for bank services; minimum values of turnover for assets and liabilities; policy constraints that influence the allocation of funds among earning assets or the mix of funds used to finance assets; legal and regulatory constraints; and constraints that prevent the allocation of short-term sources of funds to long-term uses, which gave the model a multiperiod dimension by considering the funds flow characteristics of the target balance sheet beyond the immediate planning horizon. Using the model, CCB successfully structured its assets and liabilities to better determine the bank's future position under different sets of assumptions.

## Models with Bounded Variables

*Solver* handles simple lower bounds (e.g., $C \geq 500$) and upper bounds (e.g., $D \leq 1{,}000$) quite differently from ordinary constraints in the Sensitivity report. In *Solver*, lower and upper bounds are treated in a manner similar to nonnegativity constraints, which also do not appear explicitly as constraints in the model. *Solver* does this to increase the efficiency of the solution procedure used; for large models this can represent significant savings in computer-processing time. However, it makes it more difficult to interpret the sensitivity information, because we no longer have the shadow prices and allowable increases and decreases associated with these constraints. Actually, this isn't quite true; the shadow prices are there but in a different form. The following example explains these concepts.

### EXAMPLE 14.12 J&M Manufacturing

Suppose that J&M Manufacturing makes four models of gas grills, A, B, C, and D. Each grill must flow through five departments, stamping, painting, assembly, inspection, and packaging. Table 14.7 shows the relevant data. Production rates are shown in units/hour. (Grill *A* uses imported parts and does not require painting). J&M wants to determine how many grills to make to maximize monthly profit.

To formulate this as a linear optimization model, let:

$A$, $B$, $C$, and $D$ = number of units of models A, B, C, and D to produce, respectively

The objective function is to maximize the total net profit:

$$\text{maximize } (250-210)A + (300-240)B + (400-300)C + (650-520)D$$

$$= 40A + 60B + 100C + 130D$$

The constraints include limitations on the amount of production hours available in each department, the minimum sales requirements, and maximum sales potential limits. Here is an example of where you must carefully look at the dimensions of the data. The production rates are given in units/hour, so if you multiply these values by the

*(continued)*

number of units produced, you will have an expression that makes no sense. Therefore, you must divide the decision variables by units per hour—or, equivalently, convert these data to hours/unit—and then multiply by the decision variables:

$$A/40 + B/30 + C/10 + D/10 \leq 320 \text{ (stamping)}$$

$$B/20 + C/10 + D/10 \leq 320 \text{ (painting)}$$

$$A/25 + B/15 + C/15 + D/12 \leq 320 \text{ (assembly)}$$

$$A/20 + B/20 + C/25 + D/15 \leq 320 \text{ (inspection)}$$

$$A/50 + B/40 + C/40 + D/30 \leq 320 \text{ (packaging)}$$

The sales constraints are simple upper and lower bounds on the variables:

$$A \geq 0$$

$$B \geq 0$$

$$C \geq 500$$

$$D \geq 500$$

$$A \leq 4{,}000$$

$$B \leq 3{,}000$$

$$C \leq 2{,}000$$

$$D \leq 1{,}000$$

Nonnegativity constraints are implied by the lower bounds on the variables and, therefore, do not need to be explicitly stated.

**Table 14.7**

*J&M Manufacturing* Data

| Grill Model | Selling Price/Unit | Variable Cost/Unit | Minimum Monthly Sales Requirements | Maximum Monthly Sales Potential |
|---|---|---|---|---|
| A | \$250 | \$210 | 0 | 4,000 |
| B | \$300 | \$240 | 0 | 3,000 |
| C | \$400 | \$300 | 500 | 2,000 |
| D | \$650 | \$520 | 500 | 1,000 |

| Department | A | B | C | D | Hours Available |
|---|---|---|---|---|---|
| Stamping | 40 | 30 | 10 | 10 | 320 |
| Painting | | 20 | 10 | 10 | 320 |
| Assembly | 25 | 15 | 15 | 12 | 320 |
| Inspection | 20 | 20 | 25 | 15 | 320 |
| Packaging | 50 | 40 | 40 | 30 | 320 |

Figure 14.33 shows a spreadsheet implementation (Excel file *J&M Manufacturing*) with the optimal solution and Figure 14.34 shows the *Solver* model used to find it. Examine the Answer and Sensitivity reports for the J&M Manufacturing model in Figures 14.35 and 14.36. In the Answer report, all constraints are listed along with their status. For example, we see that the upper bound on model D and lower bound on model B are binding. However, none of the bound constraints appear in the Constraints section of the Sensitivity report.

First, let us interpret the reduced costs. Recall that in an ordinary model with only nonnegativity constraints and no other simple bounds, the reduced cost tells how much the objective coefficient needs to be reduced for a variable to become positive in an optimal solution. For product B, we have the lower bound constraint $B \geq 0$. Note that the

**Figure 14.33**

Spreadsheet Implementation for *J&M Manufacturing*

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | J&M Manufacturing | | | | | |
| 2 | | | | | | |
| 3 | Data | | | | | |
| 4 | Grill model | Selling price | Variable cost | Min Sales | Max Sales | |
| 5 | A | $ 250.00 | $ 210.00 | 0 | 4000 | |
| 6 | B | $ 300.00 | $ 240.00 | 0 | 3000 | |
| 7 | C | $ 400.00 | $ 300.00 | 500 | 2000 | |
| 8 | D | $ 650.00 | $ 520.00 | 500 | 1000 | |
| 9 | | | | | | |
| 10 | Production rates (hours/unit) | A | B | C | D | Hours Available |
| 11 | Stamping | 40 | 30 | 10 | 10 | 320 |
| 12 | Painting | | 20 | 10 | 10 | 320 |
| 13 | Assembly | 25 | 15 | 15 | 12 | 320 |
| 14 | Inspection | 20 | 20 | 25 | 15 | 320 |
| 15 | Packaging | 50 | 40 | 40 | 30 | 320 |
| 16 | | | | | | |
| 17 | Model | | | | | |
| 18 | Department | A | B | C | D | Hours Used |
| 19 | Stamping | 96.429 | 0.000 | 123.571 | 100.000 | 320.000 |
| 20 | Painting | | 0.000 | 123.571 | 100.000 | 223.571 |
| 21 | Assembly | 154.286 | 0.000 | 82.381 | 83.333 | 320.000 |
| 22 | Inspection | 192.857 | 0.000 | 49.429 | 66.667 | 308.952 |
| 23 | Packaging | 77.143 | 0.000 | 30.893 | 33.333 | 141.369 |
| 24 | | | | | | |
| 25 | Number produced | 3857.142857 | 0 | 1235.714286 | 1000 | |
| 26 | Net profit/unit | $ 40.00 | $ 60.00 | $ 100.00 | $ 130.00 | Total Profit |
| 27 | Profit contribution | $ 154,285.71 | $ - | $ 123,571.43 | $ 130,000.00 | $ 407,857.14 |

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | J&M Manufacturing | | | | | |
| 2 | | | | | | |
| 3 | Data | | | | | |
| 4 | Grill model | Selling price | Variable cost | Min Sales | Max Sales | |
| 5 | A | 250 | 210 | 0 | 4000 | |
| 6 | B | 300 | 240 | 0 | 3000 | |
| 7 | C | 400 | 300 | 500 | 2000 | |
| 8 | D | 650 | 520 | 500 | 1000 | |
| 9 | | | | | | |
| 10 | Production rates (hours/unit) | A | B | C | D | Hours Available |
| 11 | Stamping | 40 | 30 | 10 | 10 | 320 |
| 12 | Painting | | 20 | 10 | 10 | 320 |
| 13 | Assembly | 25 | 15 | 15 | 12 | 320 |
| 14 | Inspection | 20 | 20 | 25 | 15 | 320 |
| 15 | Packaging | 50 | 40 | 40 | 30 | 320 |
| 16 | | | | | | |
| 17 | Model | | | | | |
| 18 | Department | A | B | C | D | Hours Used |
| 19 | Stamping | =B$25/B11 | =C$25/C11 | =D$25/D11 | =E$25/E11 | =SUM(B19:E19) |
| 20 | Painting | | =C$25/C12 | =D$25/D12 | =E$25/E12 | =SUM(B20:E20) |
| 21 | Assembly | =B$25/B13 | =C$25/C13 | =D$25/D13 | =E$25/E13 | =SUM(B21:E21) |
| 22 | Inspection | =B$25/B14 | =C$25/C14 | =D$25/D14 | =E$25/E14 | =SUM(B22:E22) |
| 23 | Packaging | =B$25/B15 | =C$25/C15 | =D$25/D15 | =E$25/E15 | =SUM(B23:E23) |
| 24 | | | | | | |
| 25 | Number produced | 3857.14285714286 | 0 | 1235.71428571429 | 1000 | |
| 26 | Net profit/unit | =B5-C5 | =B6-C6 | =B7-C7 | =B8-C8 | Total Profit |
| 27 | Profit contribution | =B25*B26 | =C25*C26 | =D25*D26 | =E25*E26 | =SUM(B27:E27) |

optimal solution specifies that we produce only the minimum amount required. Why? It is simply not economical to produce more because the profit contribution of B is too low relative to the other products. How much more would the profit on B have to be for it to be economical to produce anything other than the minimum amount required? The

**Figure 14.34**

*Solver* Model for *J&M Manufacturing*

- Objective
  - $F$27 (Max)
- Variables
  - Normal
    - ☑ $B$25:$E$25
  - Recourse
- Constraints
  - Normal
    - ☑ $F$19:$F$23 <= $F$11:$F$15
  - Chance
  - Recourse
  - Bound
    - ☑ $B$25:$E$25 <= $E$5:$E$8
    - ☑ $B$25:$E$25 >= $D$5:$D$8
  - Conic
  - Integers
- Uncertain Variables

☑ Make Unconstrained Variables Non-Negative

Select a Solving Method: Standard LP/Quadratic

**Figure 14.35**

*J&M Manufacturing Solver* Answer Report

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 12 | Objective Cell (Max) | | | | | | |
| 13 | | Cell | Name | Original Value | Final Value | | |
| 14 | | $F$27 | Profit contribution Total Profit | 0 | 407857.1429 | | |
| 15 | | | | | | | |
| 16 | | | | | | | |
| 17 | Decision Variable Cells | | | | | | |
| 18 | | Cell | Name | Original Value | Final Value | Type | |
| 19 | | $B$25 | Number produced A | 0 | 3857.142857 | Normal | |
| 20 | | $C$25 | Number produced B | 0 | 0 | Normal | |
| 21 | | $D$25 | Number produced C | 0 | 1235.714286 | Normal | |
| 22 | | $E$25 | Number produced D | 0 | 1000 | Normal | |
| 23 | | | | | | | |
| 24 | Constraints | | | | | | |
| 25 | | Cell | Name | Cell Value | Formula | Status | Slack |
| 26 | | $F$19 | Stamping Hours Used | 320.000 | $F$19<=$F$11 | Binding | 0 |
| 27 | | $F$20 | Painting Hours Used | 223.571 | $F$20<=$F$12 | Not Binding | 96.42857143 |
| 28 | | $F$21 | Assembly Hours Used | 320.000 | $F$21<=$F$13 | Binding | 0 |
| 29 | | $F$22 | Inspection Hours Used | 308.952 | $F$22<=$F$14 | Not Binding | 11.04761905 |
| 30 | | $F$23 | Packaging Hours Used | 141.369 | $F$23<=$F$15 | Not Binding | 178.6309524 |
| 31 | | $B$25 | Number produced A | 3857.142857 | $B$25<=$E$5 | Not Binding | 142.8571429 |
| 32 | | $C$25 | Number produced B | 0 | $C$25<=$E$6 | Not Binding | 3000 |
| 33 | | $D$25 | Number produced C | 1235.714286 | $D$25<=$E$7 | Not Binding | 764.2857143 |
| 34 | | $E$25 | Number produced D | 1000 | $E$25<=$E$8 | Binding | 0 |
| 35 | | $B$25 | Number produced A | 3857.142857 | $B$25>=$D$5 | Not Binding | 3857.142857 |
| 36 | | $C$25 | Number produced B | 0 | $C$25>=$D$6 | Binding | 0 |
| 37 | | $D$25 | Number produced C | 1235.714286 | $D$25>=$D$7 | Not Binding | 735.7142857 |
| 38 | | $E$25 | Number produced D | 1000 | $E$25>=$D$8 | Not Binding | 500 |

answer is given by the reduced cost. The unit profit on B would have to be reduced by at least −\$1.905 (i.e., *increased* by at least +\$1.905). If a nonzero lower-bound constraint is binding, the interpretation is similar; the reduced cost is the amount the unit profit would have to be reduced to produce more than the minimum amount.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 5 | Objective Cell (Max) | | | | | | | |
| 6 | | Cell | Name | Final Value | | | | |
| 7 | | $F$27 | Profit contribution Total Profit | 407857.1429 | | | | |
| 8 | | | | | | | | |
| 9 | Decision Variable Cells | | | | | | | |
| 10 | | | | Final | Reduced | Objective | Allowable | Allowable |
| 11 | | Cell | Name | Value | Cost | Coefficient | Increase | Decrease |
| 12 | | $B$25 | Number produced A | 3857.142857 | 0 | 40 | 20.00000004 | 1.000000042 |
| 13 | | $C$25 | Number produced B | 0 | -1.904761905 | 60 | 1.904761905 | 1E+30 |
| 14 | | $D$25 | Number produced C | 1235.714286 | 0 | 100 | 13.33333389 | 33.33333339 |
| 15 | | $E$25 | Number produced D | 1000 | 19.28571429 | 130 | 1E+30 | 19.28571429 |
| 16 | | | | | | | | |
| 17 | Constraints | | | | | | | |
| 18 | | | | Final | Shadow | Constraint | Allowable | Allowable |
| 19 | | Cell | Name | Value | Price | R.H. Side | Increase | Decrease |
| 20 | | $F$19 | Stamping Hours Used | 320.000 | 571.429 | 320 | 44.58333333 | 5 |
| 21 | | $F$20 | Painting Hours Used | 223.571 | 0.000 | 320 | 1E+30 | 96.42857143 |
| 22 | | $F$21 | Assembly Hours Used | 320.000 | 642.857 | 320 | 3.333333333 | 71.33333333 |
| 23 | | $F$22 | Inspection Hours Used | 308.952 | 0.000 | 320 | 1E+30 | 11.04761905 |
| 24 | | $F$23 | Packaging Hours Used | 141.369 | 0.000 | 320 | 1E+30 | 178.6309524 |

**Figure 14.36**

*J&M Manufacturing Solver* Sensitivity Report

For product D, the reduced cost is $19.29. Note that D is at its upper bound, 1,000. We want to produce as much of D as possible because it generates a large profit. How much would the unit profit have to be *lowered* before it is no longer economical to produce the maximum amount? Again, the answer is the reduced cost, $19.29.

Now, let's ask these questions in a different way. For product B, what would the effect be of increasing the right-hand-side value of the bound constraint, $B \geq 0$, by 1 unit? If we increase the right-hand side of a lower-bound constraint by 1, we are essentially forcing the solution to produce one more than the minimum requirement. How would the objective function change if we do this? It would have to decrease because we would lose money by producing an extra unit of a nonprofitable product. How much? The answer again is the reduced cost. Producing an additional unit of product B will result in a profit reduction of $1.905. Similarly, increasing the right-hand side of the constraint $D \leq 1{,}000$ by 1 will increase the profit by $19.29. Thus, *the reduced cost associated with a bounded variable is the same as the shadow price of the bound constraint.* However, we no longer have the allowable range over which we can change the constraint values. (*Important*: The Allowable Increase and Allowable Decrease values in the Sensitivity report refer to the objective coefficients, not the reduced costs.)

## Auxiliary Variables for Bound Constraints

Interpreting reduced costs as shadow prices for bounded variables can be a bit confusing. Fortunately, there is a neat little trick that you can use to eliminate this issue. In your spreadsheet model, define **auxiliary variables**—a new set of cells for any decision variables that have upper- or lower-bound constraints by referencing (not copying) the original changing cells. Then in the *Solver* model, use these auxiliary variable cells—*not* the changing variable cells as defined—to define the bound constraints.

## EXAMPLE 14.13 Using Auxiliary Variable Cells

Figure 14.37 shows a portion of the J&M Manufacturing model with the inclusion of auxiliary variables in row 29. The formula in cell B29, for example, is =B25. The *Solver* is modified as shown in Figure 14.38 by changing the decision variable cells in the bound constraints to the auxiliary variable cells. The Sensitivity report for this model is shown in Figure 14.39. We now see that the *Constraints* section has rows corresponding to the bound constraints and that the shadow prices are the same as the reduced costs in the original sensitivity report. Moreover, we now know the allowable increases and decreases for each shadow price, which we did not have before. Thus, we recommend that you use this approach unless solution efficiency is an important issue.

**Figure 14.37**

Auxiliary Variable Cells in *J&M Manufacturing* Model

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 24 | | | | | | |
| 25 | Number produced | 0 | 0 | 0 | 0 | |
| 26 | Net profit/unit | $ 40.00 | $ 60.00 | $ 100.00 | $ 130.00 | Total Profit |
| 27 | Profit contribution | $ - | $ - | $ - | $ - | $ - |
| 28 | | | | | | |
| 29 | Auxiliary variable | 0 | 0 | 0 | 0 | |

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 24 | | | | | | |
| 25 | Number produced | 0 | 0 | 0 | 0 | |
| 26 | Net profit/unit | =B5-C5 | =B6-C6 | =B7-C7 | =B8-C8 | Total Profit |
| 27 | Profit contribution | =B25*B26 | =C25*C26 | =D25*D26 | =E25*E26 | =SUM(B27:E27) |
| 28 | | | | | | |
| 29 | Auxiliary variable | =B25 | =C25 | =D25 | =E25 | |

**Figure 14.38**

*Solver* Model for *J&M Manufacturing* with Auxiliary Variables

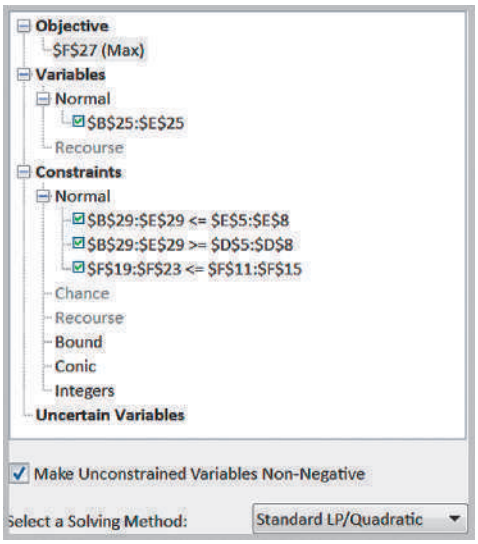

**Figure 14.39**

*J&M Manufacturing* Sensitivity Report with Auxiliary Variables

Objective Cell (Max)

| Cell | Name | Final Value |
|---|---|---|
| $F$27 | Profit contribution Total Profit | 407857.1429 |

Decision Variable Cells

| Cell | Name | Final Value | Reduced Cost | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$25 | Number produced A | 3857.142857 | 0 | 40 | 20.00000004 | 1.000000053 |
| $C$25 | Number produced B | 0 | -1.904761905 | 60 | 1.904761905 | 1E+30 |
| $D$25 | Number produced C | 1235.714286 | 0 | 100 | 13.33333403 | 33.33333339 |
| $E$25 | Number produced D | 1000 | 0 | 130 | 1E+30 | 19.28571439 |

Constraints

| Cell | Name | Final Value | Shadow Price | Constraint R.H. Side | Allowable Increase | Allowable Decrease |
|---|---|---|---|---|---|---|
| $B$29 | Auxiliary variable A | 3857.142857 | 0 | 4000 | 1E+30 | 142.8571429 |
| $C$29 | Auxiliary variable B | 0 | 0 | 3000 | 1E+30 | 3000 |
| $D$29 | Auxiliary variable C | 1235.714286 | 0 | 2000 | 1E+30 | 764.2857143 |
| $E$29 | Auxiliary variable D | 1000 | 19.28571429 | 1000 | 895.6521739 | 200 |
| $B$29 | Auxiliary variable A | 3857.142857 | 0 | 0 | 3857.142857 | 1E+30 |
| $C$29 | Auxiliary variable B | 0 | 0 | 0 | 0 | 1E+30 |
| $D$29 | Auxiliary variable C | 1235.714286 | 0 | 500 | 735.7142857 | 1E+30 |
| $E$29 | Auxiliary variable D | 1000 | 0 | 500 | 500 | 1E+30 |
| $F$19 | Stamping Hours Used | 320.000 | 571.429 | 320 | 44.58333333 | 5 |
| $F$20 | Painting Hours Used | 223.571 | 0.000 | 320 | 1E+30 | 96.42857143 |
| $F$21 | Assembly Hours Used | 320.000 | 642.857 | 320 | 3.333333333 | 71.33333333 |
| $F$22 | Inspection Hours Used | 308.952 | 0.000 | 320 | 1E+30 | 11.04761905 |
| $F$23 | Packaging Hours Used | 141.369 | 0.000 | 320 | 1E+30 | 178.6309524 |

## A Production/Marketing Allocation Model

Many problems involve allocation of marketing effort, such as advertising dollars. The following is an example of combining elements of a product-mix model with marketing budget allocation decisions based on demand elasticity. This example also illustrates some important issues of properly interpreting sensitivity results and the influence that modeling approaches can have.

### EXAMPLE 14.14 Walker Wines

A small winery, Walker Wines, buys grapes from local growers and blends the pressings to make two types of wine: Shiraz and merlot.[2] It costs \$1.60 to purchase the grapes needed to make a bottle of Shiraz and \$1.40 to purchase the grapes needed to make a bottle of merlot. The contract requires that they provide at least 40% but not more than 70% Shiraz. Based on market research related to it, it is estimated that the base demand for Shiraz is 1,000 bottles, but demand increases by 5 bottles for each \$1 spent on advertising; the base demand for merlot is 2,000 bottles and increases by 8 bottles for each \$1 spent on advertising. Production should not exceed demand. Shiraz sells to retail stores for \$6.25 per bottle and merlot is sold for \$5.25 per bottle. Walker Wines has \$50,000 available to purchase grapes and advertise its products, with an objective of maximizing profit contribution.

To formulate this model, let

$S$ = number of bottles of Shiraz produced
$M$ = number of bottles of merlot produced
$A_s$ = dollar amount spent on advertising Shiraz
$A_m$ = dollar amount spent on advertising merlot

The objective is to maximize profit:

(revenue minus costs)

$$= (\$6.25S + \$5.25M) - (\$1.60S + \$1.40M + A_s + A_m)$$
$$= 4.65S + 3.85M - A_s - A_m$$

Constraints are defined as follows:

1. Budget cannot be exceeded:

$$\$1.60S + \$1.40M + A_s + A_m \le \$50{,}000$$

*(continued)*

[2]Based on an example in Roger D. Eck, *Operations Research for Business* (Belmont, CA: Wadsworth, 1976): 129–131.

2. Contractual requirements must be met:

$$0.4 \le S/(S + M) \le 0.7$$

Expressed in linear form,

$$0.6S - 0.4M \ge 0 \text{ and } 0.3S - 0.7M \le 0$$

3. Production must not exceed demand:

$$S \le 1{,}000 + 5A_s$$
$$M \le 2{,}000 + 8A_m$$

4. Nonnegativity

**Figure 14.40**

*Walker Wines* Spreadsheet Model

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | Walker Wines | | | | |
| 2 | | | | | |
| 3 | Data | | | | |
| 4 | | Shiraz | Merlot | | |
| 5 | Cost/bottle | $ 1.60 | $ 1.40 | | |
| 6 | Price/bottle | $ 6.25 | $ 5.25 | | |
| 7 | | | | | |
| 8 | Base demand | 1,000.00 | 2,000.00 | | |
| 9 | Increase/$1 Adv. | 5 | 8 | | |
| 10 | Min. percent requirement | 40% | | | |
| 11 | Max. percent limitation | 70% | | | |
| 12 | | | | | |
| 13 | Total Budget | $ 50,000.00 | | | |
| 14 | | | | | |
| 15 | Model | | | | |
| 16 | | | | | |
| 17 | | Shiraz | Merlot | Total | |
| 18 | Unit profit | $ 4.65 | $ 3.85 | | |
| 19 | Advertising dollars | $ 3,912.37 | $ 851.53 | $ 4,763.90 | |
| 20 | Demand | 20,561.86 | 8,812.23 | 29,374.09 | |
| 21 | Quantity produced | 20,561.86 | 8,812.23 | 29,374.09 | |
| 22 | | | | | |
| 23 | Min. percent requirement | 8812.227074 | >= | 0 | |
| 24 | Max. percent limitation | 0 | <= | 0 | |
| 25 | | | | | |
| 26 | | | | Used | Unused |
| 27 | Budget | $ 36,811.35 | $ 13,188.65 | $ 50,000.00 | $ - |
| 28 | | | | | |
| 29 | | Total | | | |
| 30 | Profit | $ 124,775.84 | | | |

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | Walker Wines | | | | |
| 2 | | | | | |
| 3 | Data | | | | |
| 4 | | Shiraz | Merlot | | |
| 5 | Cost/bottle | 1.6 | 1.4 | | |
| 6 | Price/bottle | 6.25 | 5.25 | | |
| 7 | | | | | |
| 8 | Base demand | 1000 | 2000 | | |
| 9 | Increase/$1 Adv. | 5 | 8 | | |
| 10 | Min. percent requirement | 0.4 | | | |
| 11 | Max. percent limitation | 0.7 | | | |
| 12 | | | | | |
| 13 | Total Budget | 50000 | | | |
| 14 | | | | | |
| 15 | Model | | | | |
| 16 | | | | | |
| 17 | | Shiraz | Merlot | Total | |
| 18 | Unit profit | =B6-B5 | =C6-C5 | | |
| 19 | Advertising dollars | 3912.37263464338 | 851.528384279476 | =SUM(B19:C19) | |
| 20 | Demand | =B8+(B9*B19) | =C8+(C9*C19) | =SUM(B20:C20) | |
| 21 | Quantity produced | 20561.8631732169 | 8812.22707423581 | =SUM(B21:C21) | |
| 22 | | | | | |
| 23 | Min. percent requirement | =(1-B10)*B21-B10*C21 | >= | 0 | |
| 24 | Max. percent limitation | =(1-B11)*B21-B11*C21 | <= | 0 | |
| 25 | | | | | |
| 26 | | | | Used | Unused |
| 27 | Budget | =B19+(B21*B5) | =C19+(C21*C5) | =SUM(B27:C27) | =B13-D27 |
| 28 | | | | | |
| 29 | | Total | | | |
| 30 | Profit | =(B18*B21)+(C18*C21)-B19-C19 | | | |

Figure 14.41

*Walker Wines Solver* Model

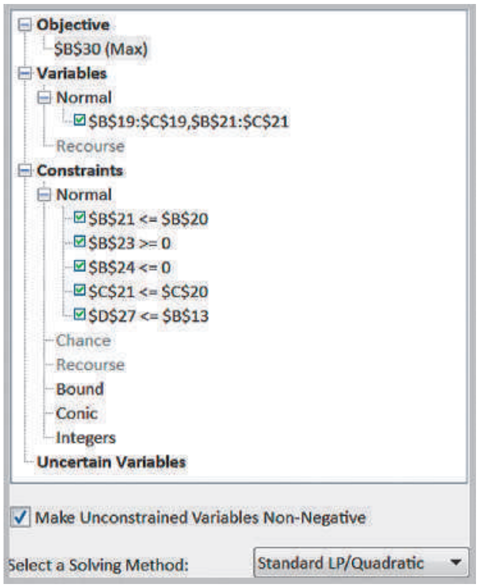


Figure 14.40 shows a spreadsheet implementation of this model (Excel file *Walker Wines*) along with the optimal solution. Figure 14.41 shows the *Solver* model.

## Using Sensitivity Information Correctly

One crucial assumption in interpreting sensitivity analysis information for changes in model parameters is that *all other model parameters are held constant*. It is easy to fall into a trap of ignoring this assumption and blindly crunching through the numbers. This is particularly true when using spreadsheet models. The following example illustrates this.

## EXAMPLE 14.15 Evaluating a Cost Increase for Walker Wines

Figure 14.42 shows the *Solver* sensitivity report. A variety of practical questions can be posed around the sensitivity report. For example, suppose that the accountant noticed a small error in computing the profit contribution for Shiraz. The cost of Shiraz grapes should have been $1.65 instead of $1.60. How will this affect the solution?

In the model formulation, you can see that a $0.05 increase in cost results in a drop in the unit profit of Shiraz from $4.65 to $4.60. In the Sensitivity report, however, the change in the profit coefficient is within the allowable decrease of 0.05328, thus concluding that no change in the optimal solution will result. However, this is *not* the correct interpretation. If the model is re-solved using the new cost parameter, the solution changes dramatically, as shown in Figure 14.43.

Why did this happen? In this case, the unit cost is also reflected in the binding budget constraint. When we change the cost parameter, the constraint also changes. This violates the assumption that all other model parameters are held constant. The change causes the budget constraint to become infeasible, and the solution must be adjusted to maintain feasibility.

**Figure 14.42**

*Walker Wines Solver* Sensitivity Report

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 6 | Objective Cell (Max) | | | | | | | |
| 7 | | Cell | Name | Final Value | | | | |
| 8 | | $B$30 | Profit Total | 124775.837 | | | | |
| 9 | | | | | | | | |
| 10 | Decision Variable Cells | | | | | | | |
| 11 | | | | Final | Reduced | Objective | Allowable | Allowable |
| 12 | | Cell | Name | Value | Cost | Coefficient | Increase | Decrease |
| 13 | | $B$19 | Advertising dollars Shiraz | $ 3,912.37 | $ - | -1 | 3.771791052 | 0.266394356 |
| 14 | | $C$19 | Advertising dollars Merlot | $ 851.53 | $ - | -1 | 0.36111235 | 112.8666705 |
| 15 | | $B$21 | Quantity produced Shiraz | 20,561.86 | 0.00 | 4.65 | 1E+30 | 0.053278871 |
| 16 | | $C$21 | Quantity produced Merlot | 8,812.23 | 0.00 | 3.85 | 0.045139044 | 14.10833381 |
| 17 | | | | | | | | |
| 18 | Constraints | | | | | | | |
| 19 | | | | Final | Shadow | Constraint | Allowable | Allowable |
| 20 | | Cell | Name | Value | Price | R.H. Side | Increase | Decrease |
| 21 | | $B$21 | Quantity produced Shiraz | 20,561.86 | 0.69 | 1000 | 21297.93978 | 195000 |
| 22 | | $B$23 | Min. percent requirement Shiraz | 8812.227074 | 0 | 0 | 8812.227074 | 1E+30 |
| 23 | | $B$24 | Max. percent limitation Shiraz | 0 | 0.047307132 | 0 | 6500 | 9256.880734 |
| 24 | | $C$21 | Quantity produced Merlot | 8,812.23 | 0.43 | 2000 | 6964.285714 | 383971.4286 |
| 25 | | $D$27 | Budget Used | $ 50,000.00 | $ 2.46 | 50000 | 1E+30 | 39000 |

**Figure 14.43**

*Walker Wines Solver* Solution after Cost Increase

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | Walker Wines | | | | |
| 2 | | | | | |
| 3 | Data | | | | |
| 4 | | Shiraz | Merlot | | |
| 5 | Cost/bottle | $ 1.65 | $ 1.40 | | |
| 6 | Price/bottle | $ 6.25 | $ 5.25 | | |
| 7 | | | | | |
| 8 | Base demand | 1,000.00 | 2,000.00 | | |
| 9 | Increase/$1 Adv. | 5 | 8 | | |
| 10 | Min. percent requirement | 40% | | | |
| 11 | Max. percent limitation | 70% | | | |
| 12 | | | | | |
| 13 | Total Budget | $ 50,000.00 | | | |
| 14 | | | | | |
| 15 | Model | | | | |
| 16 | | | | | |
| 17 | | Shiraz | Merlot | Total | |
| 18 | Unit profit | $ 4.60 | $ 3.85 | | |
| 19 | Advertising dollars | $ 2,238.67 | $ 2,036.25 | $ 4,274.92 | |
| 20 | Demand | 12,193.35 | 18,290.03 | 30,483.38 | |
| 21 | Quantity produced | 12,193.35 | 18,290.03 | 30,483.38 | |
| 22 | | | | | |
| 23 | Min. percent requirement | 0 | >= | 0 | |
| 24 | Max. percent limitation | -9145.01511 | <= | 0 | |
| 25 | | | | | |
| 26 | | | | Used | Unused |
| 27 | Budget | $ 22,357.70 | $27,642.30 | $50,000.00 | $ - |
| 28 | | | | | |
| 29 | | Total | | | |
| 30 | Profit | $122,231.12 | | | |

This example points out the importance of fully understanding the mathematical model when analyzing sensitivity information. One suggestion to ensure that sensitivity analysis information is interpreted properly in spreadsheet models is to use Excel's formula-auditing capability. If you select the cost of Shiraz (cell B5) and apply the "Trace Dependents" command from the *Formula Auditing* menu, you will see that the unit cost influences both the unit profit (cell B30) *and* the budget constraint function (cell B27).

## Key Terms

Auxiliary variables
Balance constraints
Degenerate solution
Feasibility report
Limitations
Proportional relationships
Requirements
Simple bounds
Transportation problem

## Problems and Exercises

*Note: Data for most of these problems can be found in the Excel file* Chapter 14 Problem Data *to facilitate model development and Excel implementation. Tab names correspond to the problem scenario names.*

**1.** Classify the following descriptions of constraints as bounds, limitations, requirements, proportional relationships, or balance constraints:

**a.** Each serving of chili should contain a quarter pound of beef.

**b.** Customer demand for a cereal is not expected to exceed 800 boxes during the next month.

**c.** The amount of cash available to invest in March is equal to the accounts receivable in February plus investment yields due on February 28.

**d.** A can of premium nuts should have at least twice as many cashews as peanuts.

**e.** A warehouse has 3,500 units available to ship to customers.

**f.** A call center needs at least 15 service representatives on Monday morning.

**g.** An ice cream manufacturer has 40 dozen fresh eggs at the start of the production shift.

**2.** An airlines corporation is considering the purchase of jet passenger planes so as to increase their passenger service. The type A plane costs $450 million each, the type B costs $400 million each, and the type C costs $250 million each. The corporation has budgeted $50 billion for the purchase of these planes in the forthcoming financial year. The three types of planes, if purchased, would be utilized at essentially maximum capacity. It is estimated that the net annual profit resulting from utilization of these planes would be $15 million for type A, $10.5 million for type B, and $7.5 million for type C. It is estimated that 25 trained pilots will be available, and if only C type planes were purchased, the maintenance facilities would be able to handle 30 new planes. However, each type B plane is equivalent to 4/3 type C plane and each type A plane is equivalent to 5/3 type C planes in terms of their use of maintenance facilities.

**a.** Develop a linear optimization model to determine how many of each type of plane should be purchased.

**b.** Implement your model on a spreadsheet and find an optimal solution.

**3.** Korey is a business student at State U. She has just completed a course in decision models, which had a midterm exam, a final exam, individual assignments, and class participation. She earned an 86% on the midterm, 94% on the final, 93% on the individual assignments, and 85% on participation. The benevolent instructor is allowing his students to determine their own weights for each of the four grade components—of course, with some restrictions:

- The participation weight can be no more than 15%.
- The midterm weight must be at least twice as much as the individual assignment weight.
- The final exam weight must be at least three times as much as the individual assignment weight.
- The weights of the four components must be at least 10%.
- The weights must sum to 1.0 and be nonnegative.

**a.** Develop a model that will yield a valid set of weights to maximize Korey's score for the course.

**b.** Implement your model on a spreadsheet and find a good solution using only your intuition.

**c.** Find an optimal solution using *Solver*.

**4.** The Martinez Model Car Company produces four different radio-controlled model cars based on exotic production models: Ferrari, BMW, Lotus, and Tesla. Each model requires production in five departments: molding, sanding, polishing, painting, and finishing.

The number of minutes required for each product in each department, the selling price per unit, and the minutes available in each department each day are shown below.

| | Ferrari | BMW | Lotus | Tesla | Minutes Available |
|---|---|---|---|---|---|
| Molding | 5.00 | 3.50 | 1.00 | 3.00 | 600 |
| Sanding | 4.00 | 3.20 | 2.00 | 3.65 | 600 |
| Polishing | 3.50 | 2.00 | 3.00 | 1.00 | 480 |
| Painting | 3.75 | 3.25 | 1.75 | 2.00 | 480 |
| Finishing | 4.00 | 1.00 | 2.00 | 3.00 | 480 |
| Price | $350.00 | $330.00 | $270.00 | $255.00 | |

**a.** How many of each type of car should be produced to maximize profit?

**b.** If marketing requires that at least 25 units of each be produced each day, what is the optimal production plan and profit? Before you solve this, how would you expect the profit to compare with your answer to part (a)?

**5.** An international toy manufacturing company manufactures three types of stuffed animals – dog, lion and giraffe, each made from different combination of red, yellow and blue cloth and cotton. The data (cloth requirement in yards) is as below:

| Toy | Dog | Lion | Giraffe |
|---|---|---|---|
| Red | 1/2 | 1/2 | 1.5 |
| Yellow | 2 | 1.5 | 1.5 |
| Blue | 1 | 1.5 | 1.5 |
| Cotton (in kg.) | 2 | 3 | 4 |

The dog sells at $47.5, lion at $60, and giraffe at $75. The company has 1500 yards of red, 2000 yards of blue and 200 yards of yellow cloth, while 50,000 kg of cotton is available. Develop and solve a linear optimization model to determine the optimal mix to maximize turnover, and write a short explanation of the sensitivity information.

**6.** Young Energy operates a power plant that includes a coal-fired boiler to produce steam to drive a generator. The company can purchase different types of coals and blend them to meet the requirements for burning in the boiler. The following table shows the characteristics of the different types of coals:

| Type | BTU/lb | % Ash | % Moisture | Cost ($/lb) |
|---|---|---|---|---|
| A | 11,500 | 13% | 10% | $2.49 |
| B | 11,800 | 10% | 8% | $3.04 |
| C | 12,200 | 12% | 8% | $2.99 |
| D | 12,100 | 12% | 8% | $2.61 |

The required BTU/pound must be at least 11,900. In addition, the ash content can be at most 12.2% and the moisture content, at most 9.4%. Develop and solve a linear optimization model to find the best coal blend for Young Energy. Explain how the company might reduce its costs by changing the blending restrictions.

**7.** Holcomb Candles, Inc., manufactures decorative candles and has contracted with a national retailer to supply a set of special holiday candles to its 8,500 stores. These include large jars, small jars, large pillars, small pillars, and a package of four votive candles. In negotiating the contract for the display, the manufacturer and retailer agreed that 8 feet would be designated for the display in each store, but that at least 2 feet would be dedicated to large jars and large pillars, and at least 1 foot, to the votive candle packages. At least as many jars as pillars must be provided. The manufacturer has obtained 200,000 pounds of wax, 250,000 feet of wick, and 100,000 ounces of holiday fragrance. The amount of materials and display size required for each product are shown in the following table:

How many of each product should be made to maximize the profit? Interpret the shadow prices in the Sensitivity report.

| | Large Jar | Small Jar | Large Pillar | Small Pillar | Votive Pack |
|---|---|---|---|---|---|
| Wax | 0.5 | 0.25 | 0.5 | 0.25 | 0.3125 |
| Fragrance | 0.24 | 0.12 | 0.24 | 0.12 | 0.15 |
| Wick | 0.43 | 0.22 | 0.58 | 0.33 | 0.8 |
| Display feet | 0.48 | 0.24 | 0.23 | 0.23 | 0.26 |
| Profit/unit | $0.25 | $0.20 | $0.24 | $0.21 | $0.16 |

**8.** The Children's Theater Company is a nonprofit corporation managed by Shannon Board. The theater performs in two venues: Kristin Marie Hall and the Lauren Elizabeth Theater. For the upcoming season, seven shows have been chosen. The question Shannon faces is how many performances of each of the seven shows should be scheduled. A financial analysis has estimated revenues for each performance of the seven shows, and Shannon has set the minimum number of performances of each show based on union agreements with Actor's Equity Association and the popularity of the shows in other markets. These data are shown in the table at the right.

Kristin Marie Hall is available for 60 performances during the season, whereas Lauren Elizabeth Theater is available for 150 performances. Shows 3 and 7 must be performed in Kristin Marie Hall, and the other shows are performed in either venue. The company wants to achieve revenues of at least $550,000 while minimizing its production costs. Develop and solve a linear optimization model to determine the best way to schedule the shows. Is it possible to achieve revenues of $600,000? What is the highest amount of revenue that can be achieved?

| Show | Revenue | Cost | Minimum Number of Performances |
|---|---|---|---|
| 1 | $2,217 | $ 968 | 32 |
| 2 | $2,330 | $1,568 | 13 |
| 3 | $1,993 | $ 755 | 23 |
| 4 | $3,364 | $1,148 | 34 |
| 5 | $2,868 | $1,180 | 35 |
| 6 | $3,851 | $1,541 | 16 |
| 7 | $1,836 | $1,359 | 21 |

**9.** Jaycee's department store chain is planning to open a new store. It needs to decide how to allocate the 100,000 square feet of available floor space among seven departments. Data on expected performance of each department per month, in terms of square feet (sf), are shown next.

| Department | Investment/sf | Risk as a % of $ Invested | Minimum sf | Maximum sf | Expected Profit per sf |
|---|---|---|---|---|---|
| Electronics | $100 | 24 | 6,000 | 30,000 | $12.00 |
| Furniture | $50 | 12 | 10,000 | 30,000 | $6.00 |
| Men's Clothing | $30 | 5 | 2,000 | 5,000 | $2.00 |
| Clothing | $600 | 10 | 3,000 | 40,000 | $30.00 |
| Jewelry | $900 | 14 | 1,000 | 10,000 | $20.00 |
| Books | $50 | 2 | 1,000 | 5,000 | $1.00 |
| Appliances | $400 | 3 | 12,000 | 40,000 | $13.00 |

The company has gathered $20 million to invest in floor stock. The risk column is a measure of risk associated with investment in floor stock based on past data from other stores and accounts for outdated inventory, pilferage, breakage, and so on. For instance, electronics loses 24% of its total investment, furniture loses 12% of its total investment, and so on. The amount of risk should be no more than 10% of the total investment.

**a.** Develop a linear optimization model to maximize profit.

**b.** If the chain obtains another $1 million of investment capital for stock, what would the new solution be?

**10.** A recent MBA graduate, Dara, has gained control over custodial accounts that her parents had established. Currently, her money is invested in four funds, but she has identified several other funds as options for investment. She has $100,000 to invest with the following restrictions:

- Keep at least $5,000 in savings.
- Invest at least 14% in the money market fund.

- Invest at least 16% in international funds.
- Keep 35% of funds in current holdings.
- Do not allocate more than 20% of funds to any one investment except for the money market and savings account.
- Allocate at least 30% into new investments.

| | Average Return | Expenses | |
|---|---|---|---|
| 1. Large cap blend | 17.2% | 0.93% | (current holding) |
| 2. Small cap growth | 20.4% | 0.56% | (current holding) |
| 3. Green fund | 26.3% | 0.70% | (current holding) |
| 4. Growth and income | 15.6% | 0.92% | (current holding) |
| 5. Multicap growth | 19.8% | 0.92% | |
| 6. Midcap index | 22.1% | 0.22% | |
| 7. Multicap core | 27.9% | 0.98% | |
| 8. Small cap international | 35.0% | 0.54% | |
| 9. Emerging international | 36.1% | 1.17% | |
| 10. Money market fund | 4.75% | 0 | |
| 11. Savings account | 1.0% | 0 | |

**a.** Develop a linear optimization model to maximize the net return.

**b.** Interpret the Sensitivity report.

**c.** Use *Solver*'s parameter-analysis method to investigate different assumptions about the portfolio constraints.

**d.** Summarize your results and write a short memo in nontechnical language to Dara.

**11.** Janette Douglas is coordinating a bake sale for a nonprofit organization. The organization has acquired $2,200 in donations to hold the sale. The following table shows the amounts and costs of ingredients used per batch of each baked good.

| Ingredient | Brownies | Cupcakes | Peanut Butter Cups | Shortbread Cookies | Cost/Unit |
|---|---|---|---|---|---|
| Butter (cups) | 0.67 | 0.33 | 1 | 0.75 | $1.44 |
| Flour (cups) | 1.5 | 1.5 | 1.25 | 2 | $0.09 |
| Sugar (cups) | 1.75 | 1 | 2 | 0.25 | $0.16 |
| Vanilla (tsp) | 2 | 0.5 | 0 | 0 | $0.06 |
| Eggs | 3 | 2 | 1 | 0 | $0.12 |
| Walnuts (cups) | 2 | 0 | 0 | 0 | $0.31 |
| Milk (cups) | 0.5 | 1 | 2 | 0 | $0.05 |
| Chocolate (oz) | 8 | 2.5 | 9 | 0 | $0.10 |
| Baking soda (tsp) | 2 | 1 | 0 | 0 | $0.07 |
| Frosting (cups) | 0.5 | 1.5 | 0 | 1 | $2.74 |
| Peanut butter (cups) | 0 | 0 | 2.5 | 0 | $2.04 |

One batch of each results in 10 brownies, 12 cupcakes, 8 peanut butter cups, and 12 shortbread cookies. Each batch of brownies can be sold for $6.00, cupcakes for $10.00, peanut butter cups for $12.00, and shortbread cookies for $7.50. The organization anticipates that a total of at least 4,000 baked goods must be made. For adequate variety, at least 30 batches of each baked good are required, except for the popular brownies, which require at least 100 batches. In addition, no more than 40 batches of shortbread cookies should be made. How can the organization best use its budget and make the largest amount of money?

**12.** Example 14.6 described the Little Investment Advisors problem and illustrated scaling issues. In answering the following questions, be sure to scale the model appropriately.

**a.** How would the results in Figure 14.19 change if there is a limit of $100,000 in each fund?

**b.** What if, in addition to the limitation in part (a), the client wants to invest at least $50,000 in the Federated High Income Bond fund?

**c.** What would be the optimal investment strategy if the client wants to minimize risk and achieve a return of at least 6% (with no additional limitations or requirements)?

**d.** How would your results to part (c) change if there is a limit of $100,000 in each fund?

**e.** What if, in addition to the limitation in part (d), the client wants to invest at least $50,000 in the Federated High Income Bond fund?

**f.** Use parameter analysis to analyze the solution to the base case by varying the risk limitation and return requirement, respectively, and visualize the results.

**13.** Kelly Foods has two plants and ships canned vegetables to customers in four cities. The cost of shipping one case from a plant to a customer is given in the following table.

| Plant/Customer | Chicago | Cincinnati | Indianapolis | Pittsburgh |
|---|---|---|---|---|
| Akron | $1.70 | $2.30 | $2.50 | $2.15 |
| Evansville | $1.95 | $2.35 | $1.65 | $2.95 |

The plant in Akron has a capacity of 2,800 cases per week, and the Evansville plant can produce 4,500 cases per week. Customer orders for the next week are

Chicago: 2,000 cases
Cincinnati: 1,200 cases
Indianapolis: 2,500 cases
Pittsburgh: 1,400 cases

Find the minimum-cost shipping plan. Interpret the Sensitivity report and write a short memo to the VP of Operations explaining your results.

**14.** Liquid Gold, Inc., transports radioactive waste from nuclear power plants to disposal sites around the country. Each plant has an amount of material that must be moved each period. Each site has a limited capacity per period. The cost of transporting between sites is given in the accompanying table (some combinations of plants and storage sites are not to be used, and no figure is given). Develop and solve a transportation model for this problem.

| | | Cost to Site | | | | | |
|---|---|---|---|---|---|---|---|
| Plant | Material | S1 | S2 | S3 | S4 | Site | Capacity |
| P1 | 20,876 | $105 | $86 | — | $23 | S1 | 285,922 |
| P2 | 50,870 | $86 | $58 | $41 | — | S2 | 308,578 |
| P3 | 38,652 | $93 | $46 | $65 | $38 | S3 | 111,955 |
| P4 | 28,951 | $116 | $27 | $94 | — | S4 | 208,555 |
| P5 | 87,423 | $88 | $56 | $82 | $89 | | |
| P6 | 76,190 | $111 | $36 | $72 | — | | |
| P7 | 58,237 | $169 | $65 | $48 | — | | |

**15.** Shafer Office Supplies has four distribution centers, located in Atlanta, Lexington, Milwaukee, and Salt Lake City, and ships to 12 retail stores, located in Seattle, San Francisco, Las Vegas, Tuscon, Denver, Charlotte, Minneapolis, Fayetteville, Birmingham, Orlando, Cleveland, and Philadelphia. The company wants to minimize the transportation costs of shipping one of its higher-volume products, boxes of standard copy paper. The per-unit shipping cost from each distribution center to each retail location and the amounts currently in inventory and ordered at each retail location are shown in the following table. Develop and solve an optimization model to minimize the total transportation cost and answer the following questions. Use the sensitivity report to answer parts c and d.

**a.** What is the minimum cost of shipping?

**b.** Which distribution centers will operate at capacity in this solution?

**c.** Suppose that 500 units of extra supply are available (and that the cost of this extra capacity is a sunk cost). To which distribution center should this extra supply be allocated, and why?

**d.** Suppose that the cost of shipping from Atlanta to Birmingham increased to $0.45 per unit. What would happen to the optimal solution?

| | Seattle | San Francisco | Las Vegas | Tuscon | Denver | Charlotte | Minneapolis |
|---|---|---|---|---|---|---|---|
| Atlanta | $2.15 | $2.10 | $1.75 | $1.50 | $1.20 | $0.65 | $0.90 |
| Lexington | $1.95 | $2.00 | $1.70 | $1.53 | $1.10 | $0.55 | $0.60 |
| Milwaukee | $1.70 | $1.85 | $1.50 | $1.41 | $0.95 | $0.40 | $0.40 |
| Salt Lake City | $0.60 | $0.55 | $0.35 | $0.60 | $0.40 | $0.95 | $1.00 |
| Demand | 5,000 | 16,000 | 4,200 | 3,700 | 4,500 | 7,500 | 3,000 |

| | Fayetteville | Birmingham | Orlando | Cleveland | Philadelphia | Supply |
|---|---|---|---|---|---|---|
| Atlanta | $0.80 | $0.35 | $0.15 | $0.60 | $0.50 | 40,000 |
| Lexington | $1.05 | $0.60 | $0.50 | $0.25 | $0.30 | 35,000 |
| Milwaukee | $0.95 | $0.70 | $0.70 | $0.35 | $0.40 | 15,000 |
| Salt Lake City | $1.10 | $1.35 | $1.60 | $1.60 | $1.70 | 16,000 |
| Demand | 9,000 | 3,300 | 12,000 | 9,500 | 16,000 | |

**16.** Roberto's Honey Farm in Chile makes five types of honey: cream, filtered, pasteurized, mélange (a mixture of several types), and strained, which are sold in 1-kilogram or 0.5-kilogram glass containers, 1-kilogram and 0.75-kilogram plastic containers, or in bulk. Key data are shown in the following tables.

| | Selling Prices (Chilean pesos) | | | | |
|---|---|---|---|---|---|
| | 0.75-kg Plastic | 1-kg Plastic | 0.5-kg Glass | 1-kg Glass | Bulk/kg |
| Cream | 744 | 880 | 760 | 990 | 616 |
| Filtered | 635 | 744 | 678 | 840 | 521 |
| Pasteurized | 696 | 821 | 711 | 930 | 575 |
| Mélange | 669 | 787 | 683 | 890 | 551 |
| Strained | 683 | 804 | 697 | 910 | 563 |

| | Minimum Demand | | | |
|---|---|---|---|---|
| | 0.75-kg Plastic | 1-kg Plastic | 0.5-kg Glass | 1-kg Glass |
| Cream | 300 | 250 | 350 | 200 |
| Filtered | 250 | 240 | 300 | 180 |
| Pasteurized | 230 | 230 | 350 | 300 |
| Mélange | 350 | 300 | 250 | 350 |
| Strained | 360 | 350 | 250 | 380 |

| | Maximum Demand | | | |
|---|---|---|---|---|
| | 0.75-kg Plastic | 1-kg Plastic | 0.5-kg Glass | 1-kg Glass |
| Cream | 550 | 350 | 470 | 310 |
| Filtered | 400 | 380 | 440 | 300 |
| Pasteurized | 360 | 390 | 490 | 400 |
| Mélange | 530 | 410 | 390 | 430 |
| Strained | 480 | 420 | 380 | 500 |
| | Package Costs (Chilean pesos) | | | |
| | 0.75-kg Plastic | 1-kg Plastic | 0.5-kg Glass | 1-kg Glass |
| | 91 | 112 | 276 | 351 |

Harvesting and production costs (in pesos) for each product per kilogram are

Cream: 322
Filtered: 255
Pasteurized: 305
Mélange: 272
Strained: 287

Develop a linear optimization model to maximize profit if a total of 10,000 kilograms of honey are available.

**17.** Sandford Tile Company makes ceramic and porcelain tile for residential and commercial use. They produce three different grades of tile (for walls, residential flooring, and commercial flooring), each of which requires different amounts of materials and production time, and generates different contributions to profit. The following information shows the percentage of materials needed for each grade and the profit per square foot.

| | Grade I | Grade II | Grade III |
|---|---|---|---|
| Profit/square foot | $2.50 | $4.00 | $5.00 |
| Clay | 50% | 30% | 25% |
| Silica | 5% | 15% | 10% |
| Sand | 20% | 15% | 15% |
| Feldspar | 25% | 40% | 50% |

Each week, Sanford Tile receives raw-material shipments, and the operations manager must schedule the plant to efficiently use the materials to maximize profitability. Currently, inventory consists of 6,000 pounds of clay, 3,000 pounds of silica, 5,000 pounds of sand, and 8,000 pounds of feldspar. Because demand varies for the different grades, marketing estimates that at most 8,000 square feet of Grade III tile should be produced, and that at least 1,500 square feet of Grade I tiles are required. Each square foot of tile weighs approximately 2 pounds.

**a.** Develop a linear optimization model to determine how many of each grade of tile the company should make next week to maximize profit contribution.

**b.** Implement your model on a spreadsheet and find an optimal solution.

**c.** Explain the sensitivity information for the objective coefficients. What happens if the profit on Grade I is increased by $0.05?

**d.** If an additional 500 pounds of feldspar is available, how will the optimal solution be affected?

**e.** Suppose that 1,000 pounds of clay are found to be of inferior quality. What should the company do?

**f.** Use the auxiliary variable cells technique to handle the bound constraints and generate all shadow prices.

**18.** The Hansel Corporation, located in Bangalore, India, makes plastics materials that are mixed with various additives and reinforcing materials before being melted, extruded, and cut into small pellets for sale to other manufacturers. Four grades of plastic are made, each of which might include up to four different additives. The following table shows the number of pounds of additive per pound of each grade of final product, the weekly availability of the additives, and cost and profitability information.

| | Grade 1 | Grade 2 | Grade 3 | Grade 4 | Availability |
|---|---|---|---|---|---|
| Additive A | 0.40 | 0.37 | 0.34 | 0.90 | 100,000 |
| Additive B | 0.30 | 0.33 | 0.33 | | 90,000 |
| Additive C | 0.20 | 0.25 | 0.33 | | 40,000 |
| Additive D | 0.10 | 0.05 | | 0.10 | 10,000 |
| Profit/lb | $2.00 | $1.70 | $1.50 | $2.80 | |

Because of marketing considerations, the total amount of grades 1 and 2 should not exceed 65% of the total of all grades produced, and at least 25% of the total product mix should be grade 4.

**a.** How much of each grade should be produced to maximize profit? Develop and solve a linear optimization model.

**b.** A labor strike in India leads to a shortage of 20,000 units of additive C. What should the production manager do?

**c.** Management is considering raising the price on grade 2 to $2.00 per pound. How will the solution be changed?

**19.** Mirza Manufacturing makes four electronic products, each of which comprises three main materials: magnet, wiring, and casing. The products are shipped to three distribution centers in North America, Europe, and Asia. Marketing has specified that no location should receive more than the maximum demand and should receive at least the minimum demand. The material costs/unit are magnet—$0.59, wire—$0.29, and casing—$0.31. The following table shows the number of units of each material required in each unit of end product and the production cost per unit.

| Product | Production Cost/Unit | Magnets | Wire | Casing |
|---|---|---|---|---|
| A | $0.25 | 4 | 2 | 2 |
| B | $0.35 | 3 | 1 | 3 |
| C | $0.15 | 2 | 2 | 1 |
| D | $0.10 | 8 | 3 | 2 |

Additional information is provided next.

| Min Demand | | | |
|---|---|---|---|
| Product | NA | EU | Asia |
| A | 850 | 900 | 100 |
| B | 700 | 200 | 500 |
| C | 1,100 | 800 | 600 |
| D | 1,500 | 3,500 | 2,000 |
| **Max Demand** | | | |
| Product | NA | EU | Asia |
| A | 2,550 | 2,700 | 300 |
| B | 2,100 | 600 | 1,500 |
| C | 3,300 | 2,400 | 1,800 |
| D | 4,500 | 10,500 | 6,000 |
| **Packaging and Shipping Cost/Unit** | | | |
| Product | NA | EU | Asia |
| A | $0.20 | $0.25 | $0.35 |
| B | $0.18 | $0.22 | $0.30 |
| C | $0.18 | $0.22 | $0.30 |
| D | $0.17 | $0.20 | $0.25 |

| Unit Sales Revenue | | | |
|---|---|---|---|
| Product | NA | EU | Asia |
| A | $4.00 | $4.50 | $4.55 |
| B | $3.70 | $3.90 | $3.95 |
| C | $2.70 | $2.90 | $2.40 |
| D | $6.80 | $6.50 | $6.90 |
| **Available Raw Material** | | | |
| Magnet | 120,000 | | |
| Wire | 50,000 | | |
| Casing | 40,000 | | |

Develop an appropriate linear optimization model to maximize net profit.

**20.** A furniture manufacturing company planes to make three products – chairs, tables and desks from its available weekly resources, which consist of 400 cubic feet of timber and 500 man-hours of labor. To make a chair, it requires 5 cubic feet of timber and 10 man-hours of labor and yields a profit of $25. A table uses 20 cubic feet of timber and 15 man-hours of labor and yields a profit of $40. A desk needs 25 cubic feet of timber and 20 man-hours of labor and yields a profit of $50. Develop a linear optimization model and find optimal product mix.

**21.** Reddy & Rao (R&R) is a small company in India that makes handmade artistic chairs for commercial businesses. The company makes four models. The time required to make each of the models and cost per chair is given below.

| | Model A | Model B | Model C | Model D |
|---|---|---|---|---|
| Cost per Unit | $900.00 | $650.00 | $500.00 | $750.00 |
| Hours Required per unit | 40 | 22 | 12 | 34 |

R&R employs four people. Each of them works 8 hour shifts, 5 days a week (assume 4 weeks/ month). The demand for the next 3 months is estimated to be:

| Demand (Units) | Model A | Model B | Model C | Model D |
|---|---|---|---|---|
| Month 1 | 7 | 4 | 4 | 9 |
| Month 2 | 7 | 4 | 5 | 4 |
| Month 3 | 6 | 8 | 8 | 6 |

R&R keeps at most two of each model in inventory each month but wants to have at least one of Model D in inventory at all times. The current inventory of each model is 2. The cost to hold these finished chairs is 10% of the production cost. Develop and solve an optimization model to determine the optimal number of chairs to produce each month and the monthly inventories to minimize total cost and meet the expected demand.

**22.** An international graduate student will receive a $28,000 foundation scholarship and reduced tuition. She must pay $1,500 in tuition for each of the autumn, winter, and spring quarters, and $500 in the summer. Payments are due on the first day of September, December, March, and May, respectively. Living expenses are estimated to be $1,500 per month, payable on the first day of the month. The foundation will pay her $18,000 on August 1 and the remainder on May 1. To earn as much interest as possible, the student wishes to invest the money. Three types of investments are available at her bank: a 3-month CD, earning 0.75% (net 3-month rate); a 6-month CD, earning 1.9%; and a 12-month CD, earning 4.2%. Develop a linear optimization model to determine how she can best invest the money and meet her financial obligations.

**23.** Jason Wright is a part-time business student who would like to optimize his financial decisions. Currently, he has $16,000 in his savings account. Based on an analysis of his take-home pay, expected bonuses, and anticipated tax refund, he has estimated his income for each month over the next year. In addition, he has estimated his monthly expenses, which vary because of scheduled payments for insurance, utilities, tuition and books, and so on. The following table summarizes his estimates:

| Month | Income | Expenses |
|---|---|---|
| 1. January | $3,400 | $3,360 |
| 2. February | $3,400 | $2,900 |
| 3. March | $3,400 | $6,600 |
| 4. April | $9,500 | $2,750 |
| 5. May | $3,400 | $2,800 |
| 6. June | $5,000 | $6,800 |
| 7. July | $4,600 | $3,200 |
| 8. August | $3,400 | $3,600 |
| 9. September | $3,400 | $6,550 |
| 10. October | $3,400 | $2,800 |
| 11. November | $3,400 | $2,900 |
| 12. December | $5,000 | $6,650 |

Jason has identified several short-term investment opportunities:

- a 3-month CD yielding 0.60% at maturity
- a 6-month CD yielding 1.42% at maturity
- an 11-month CD yielding 3.08% at maturity
- a savings account yielding 0.0375% per month

To ensure enough cash for emergencies, he would like to maintain at least $2,000 in the savings account. Jason's objective is to maximize his cash balance at the end of the year. Develop a linear optimization model to find the best investment strategy.

**24.** Pavlick Products supplies a key component for automobile interiors to U.S. assembly plants. The components can be manufactured in China or Mexico. Unit cost in China is $333, and the unit cost in Mexico is $350. However, shipping costs per 500 units are $10,000 from China, and only $2,000 from Mexico and are expected to increase 4% each month from China and 1% each month from Mexico. Each unit is sold to the automotive customer for $400. Contracts with the Chinese vendor require that a minimum of 2,500 units be produced each month. Demand for the next 12 months is estimated to be:

| | Demand |
|---|---|
| January | 14,000 |
| February | 16,000 |
| March | 14,000 |
| April | 14,000 |
| May | 16,000 |
| June | 10,500 |
| July | 14,000 |
| August | 20,000 |
| September | 20,000 |
| October | 16,000 |
| November | 14,000 |
| December | 10,500 |

The Mexican plant is new and is gearing up production; its capacity will increase over the next year as follows:

| | Mexican Plant Capacity |
|---|---|
| January | 0 |
| February | 2,500 |
| March | 5,000 |
| April | 7,500 |
| May | 10,000 |
| June | 12,500 |
| July | 15,000 |
| August | 15,000 |
| September | 15,000 |
| October | 15,000 |
| November | 15,000 |
| December | 15,000 |

How should the company source production to maximize total profit?

**25.** Michelle is a business student who plans to attend medical school. The average state university medical school education expense can cost around $35,000 per year and is escalating rapidly. Michelle created a spreadsheet model to calculate the total expenses for each year of medical school, including both education and living expenses. Her estimates are Year 1: $57,067, Year 2: $56,572, Year 3: $67,846, and Year 4: $55,662. She is considering three loan options: the Stafford loan, a 6.8% loan with a cap of $47,167 that does not accrue interest during medical school; the Graduate Plus loan, a 7.9% loan with no cap that does accrue interest during medical school; and a private bank loan, a 5.9% loan with a cap of $30,000, also with accruing interest during medical school. Assume that each loan will be paid over 25 years after graduation. Michelle currently has $39,500 saved from investments, family gifts, and work, and will receive an additional $4,500 in gifts from her grandparents in years 2 through 4. Develop and solve an optimization model to determine how much money to fund from each type of loan to minimize the amount of interest that will have to be paid on the loans. (Hint: use the Excel function CUMIPMT to find the total interest that will be paid over the life of a loan. For example, if a 30-year loan for $100,000 has an interest rate of 9%, then the formula = − CUMIPMT(9%, 30, 100,000, 1, 30, 0) will yield $192,009 cumulative interest paid between years 1 and 30. (Note that this function yields a negative value so include the minus sign.)

**26.** Marketing managers have various media alternatives, such as radio, TV, magazines, and so on, in which to advertise and must determine which to use, the number of insertions in each, and the timing of insertions to maximize advertising effectiveness within a limited budget. Suppose that three media options are available to Kernan Services Corporation: radio, TV, and magazine. The following table provides some information about costs, exposure values, and bounds on the permissible number of ads in each medium desired by the firm. The exposure value is a measure of the number of people exposed to the advertisement and is derived from market research studies, and the client's objective is to maximize the total exposure value. The company would like to achieve a total exposure value of at least 90,000.

| | Medium Cost/Ad | Exposure Value/Ad | Min Units | Max Units |
|---|---|---|---|---|
| Radio | $500 | 2,000 | 0 | 15 |
| TV | $2,000 | 4,000 | 10 | |
| Magazine | $200 | 2,700 | 6 | 12 |

How many of each type of ad should be placed to minimize the cost of achieving the minimum required total exposure? Use the auxiliary variable approach to model this problem, and write a short memo to the marketing manager explaining the solution and sensitivity information.

**27.** Klein Industries manufactures three types of portable air compressors: small, medium, and large, which have unit profits of $20.50, $34.00, and $42.00, respectively. The projected monthly sales are as follows:

| | Small | Medium | Large |
|---|---|---|---|
| Minimum | 14,000 | 6,200 | 2,600 |
| Maximum | 21,000 | 12,500 | 4,200 |

The production process consists of three primary activities: bending and forming, welding, and painting. The amount of time in minutes needed to process each product in each department is as follows:

| | Small | Medium | Large | Available Time |
|---|---|---|---|---|
| Bending/forming | 0.4 | 0.7 | 0.8 | 23,400 |
| Welding | 0.6 | 1.0 | 1.2 | 23,400 |
| Painting | 1.4 | 2.6 | 3.1 | 46,800 |

How many of each type of air compressor should the company produce to maximize profit?

**a.** Formulate and solve a linear optimization model using the auxiliary variable cells method and write a short memo to the production manager explaining the sensitivity information.

**b.** Solve the model without the auxiliary variables and explain the relationship between the reduced costs and the shadow prices found in part a.

**28.** Fruity Juices, Inc., produces five different flavors of fruit juice: apple, cherry, pomegranate, orange, and pineapple. Each batch of product requires processing in three departments (blending, straining, and bottling). The relevant data (per 1,000-gallon batches) are shown next.

| | Time Required in Minutes/Batch | | | | | |
|---|---|---|---|---|---|---|
| | Apple | Cherry | Pomegranate | Orange | Pineapple | Minutes Avail. |
| Blend | 23 | 22 | 18 | 19 | 19 | 5,000 |
| Strain | 22 | 40 | 20 | 31 | 28 | 3,000 |
| Bottle | 10 | 10 | 10 | 10 | 10 | 5,000 |

| | Profit and Sales Potential | | | | |
|---|---|---|---|---|---|
| | Apple | Cherry | Pomegranate | Orange | Pineapple |
| Profit ($/1,000 gal) | $800 | $320 | $1,120 | $1,440 | $800 |
| Max Sales (000) | 20 | 30 | 50 | 50 | 20 |
| Min Sales (000) | 10 | 15 | 20 | 40 | 10 |

**a.** Formulate a linear program to find the amount of each product to produce.

**b.** Implement your model on a spreadsheet and find an optimal solution with *Solver*.

**c.** What effect would an increase of capacity in the straining department have on profit?

**29.** Worley Fluid Supplies produces three types of fluid-handling equipment: control valves, metering pumps, and hydraulic cylinders. All three products require assembly and testing before they can be shipped to customers.

| | Control Valve | Metering Pump | Hydraulic Cylinder |
|---|---|---|---|
| Assembly time (min) | 45 | 20 | 30 |
| Testing time (min) | 20 | 15 | 25 |
| Profit/unit | $372 | $174 | $288 |
| Maximum sales | 20 | 50 | 45 |
| Minimum sales | 5 | 12 | 22 |

A total of 3,150 minutes of assembly time and 2,100 minutes of testing time are available next week.

**a.** Develop a linear optimization model to determine how many pieces of equipment the company should make next week to maximize profit contribution.

**b.** Implement your model on a spreadsheet and find an optimal solution.

**c.** Explain the sensitivity information for the objective coefficients. What happens if the profit on hydraulic cylinders is decreased by $10?

**d.** Due to scheduled maintenance, the assembly time is expected to be only 3,000 minutes. How will this affect the solution?

**e.** A worker in the testing department has to take a personal leave because of a death in the family and will miss 3 days (24 hours). How will this affect the optimal solution?

**f.** Use the auxiliary variable technique to handle the bound constraints and generate all shadow prices.

**30.** Beverly Ann Cosmetics has created two new perfumes: Summer Passion and Ocean Breeze. It costs $5.25 to purchase the fragrance needed for each bottle of Summer Passion and $4.70 for each bottle of Ocean Breeze. The marketing department has stated that at least 30% but no more than 70% of the product mix be Summer Passion; the forecasted monthly demand is 7,000 bottles and is estimated to increase by 8 bottles for each $1 spent on advertising. For Ocean Breeze, the demand is forecast to be 12,000 bottles and is expected to increase by 15 bottles for each $1 spent on advertising. Summer Passion sells for $42.00 per bottle and Ocean Breeze, for $30.00 per bottle. A monthly budget of $100,000 is available for both advertising and purchase of the fragrances. Develop and solve a linear optimization model to determine how much of each type of perfume should be produced to maximize the net profit.

## Case: Performance Lawn Equipment

Elizabeth Burke wants to develop a model to more effectively plan production for the next year. Currently, PLE has a planned capacity of producing 9,100 mowers each month, which is approximately the average monthly demand over the previous year. However, looking at the unit sales figures for the previous year, she observed that the demand for mowers has a seasonal fluctuation, so with this "level" production strategy, there is overproduction in some months, resulting in excess inventory buildup and underproduction in others, which may result in lost sales during peak demand periods.

In discussing this with her, she explained that she could change the production rate by using planned overtime or undertime (producing more or less than the average monthly demand), but this incurs additional costs, although it may offset the cost of lost sales or of maintaining excess inventory. Consequently, she believes that the company can save a significant amount of money by optimizing the production plan.

Ms. Burke saw a presentation at a conference about a similar model that another company used but didn't fully understand the approach. The PowerPoint notes didn't have all the details, but they did explain the variables and the types of constraints used in the model. She thought they would be helpful to you in implementing an optimization model. Here are the highlights from the presentation:

Variables:

$X_t$ = planned production in period $t$

$I_t$ = inventory held at the end of period $t$

$L_t$ = number of lost sales incurred in period $t$

$O_t$ = amount of overtime scheduled in period $t$

$U_t$ = amount of undertime scheduled in period $t$

$R_t$ = increase in production rate from period $t - 1$ to period $t$

$D_t$ = decrease in production rate from period $t - 1$ to period $t$

Material balance constraint:

$$X_t + I_{t-1} - I_t + L_t = \text{demand in month } t$$

Overtime/undertime constraint:

$$O_t - U_t = X_t - \text{normal production capacity}$$

Production rate-change constraint:

$$X_t - X_{t-1} = R_t - D_t$$

Ms. Burke also provided the following data and estimates for the next year: unit production cost = \$70.00; inventory-holding cost = \$1.40 per unit per month; lost sales cost = \$200 per unit; overtime cost = \$6.50 per unit; undertime cost = \$3.00 per unit; and production-rate-change cost = \$5.00 per unit, which applies to any increase or decrease in the production rate from the previous month. Initially, 900 units are expected to be in inventory at the beginning of January, and the production rate for December 2012 was 9,100 units. She believes that monthly demand will not change substantially from last year, so the sales figures for last year in the PLE database should be used for the monthly demand forecasts.

Your task is to design a spreadsheet that provides detailed information on monthly production, inventory, lost sales, and the different cost categories and solve a linear optimization model for minimizing the total cost of meeting demand over the next year. Compare your solution with the level production strategy of producing 9,100 units each month. Interpret the Sensitivity report, and conduct an appropriate study of how the solution will be affected by changing the assumption of the lost sales costs. Summarize all your results in a report to Ms. Burke.