CHAPTER 15 Integer Optimization

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Learning Objectives

After studying this chapter, you will be able to:

- Recognize when to use integer variables in optimization models.
- Incorporate integer variables into *Solver* models.
- Develop integer optimization models for practical applications such as workforce scheduling and location.
- Find alternative optimal solutions to integer optimization models.
- Formulate and solve optimization models with binary variables and logical constraints.
- Develop and solve mixed-integer optimization models such as facility location and fixed-cost models.

In the previous chapters, we saw that the variables in linear optimization models can assume any real value. For many practical applications, we need not be concerned with this assumption. For example, in deciding on the optimal number of cases of diapers to produce next month, we could use a linear model, since rounding a value like 5,621.63 would have little impact on the results. However, in a production-planning decision involving low-volume, high-cost items such as airplanes, an optimal value of 10.42 would make little sense, and a difference of one unit (rounded up or down) could have significant economic and production planning consequences.

In an **integer linear optimization model (integer program)**, some of or all the variables are restricted to being *whole numbers*. If only a subset of variables is restricted to being integer while others are continuous, we call this a **mixed-integer linear optimization model**. A special type of integer problem is one in which variables can be only 0 or 1; these are used to model logical yes-or-no decisions. Integer linear optimization models are generally more difficult to solve than pure linear optimization models but have many important applications in areas such as scheduling and supply chains.

Solving Models with General Integer Variables

Decision variables that we force to be integers are called **general integer variables**. We may specify any variable in an ordinary linear optimization model to be a general integer variable. Of course, if we solve the linear optimization model without the integer restrictions (called **linear program [LP] relaxation**) and the optimal solution happens to have all integer values, then it clearly would have solved the integer model. This is generally not the case, however. The algorithm used to solve integer optimization models begins by solving the LP relaxation and proceeds to enforce the integer restrictions using a systematic search process that involves solving a series of modified linear optimization problems. You need not worry about understanding how this is accomplished, because *Solver* takes care of the algorithmic details.

When using *Solver*, it is important to set a parameter called the *Integer Tolerance*. This value specifies when the *Solver* algorithm will terminate. By default, the *Integer Tolerance* is set to 0.05 within *Solver*. This means that *Solver* will stop if it finds an integer solution that is within 5% of the optimal solution. With this value, you may end up with a solution that is not the optimum, but is 95% of the way there. It does this for computational efficiency because many practical problems take a very long time to solve, even with today's technology (hours or even days!). A manager might be satisfied with a near-optimal solution that is guaranteed to be within a fixed percentage of the best if an answer is needed quickly. To find the guaranteed optimal integer solution, *Integer Tolerance* must be set to 0. To do this, click the *Options* button in the *Solver Parameters* dialog and ensure that the value of *Integer Optimality (%)* is 0.

Because integer models are discontinuous by their very nature, sensitivity information cannot be generated in the same manner as for linear models; therefore, no Sensitivity

EXAMPLE 15.1 Sklenka Skis Revisited

In Chapter 13, we developed a simple linear optimization model for finding the optimal product mix for a ski manufacturer. The model was

$$\begin{aligned} \textit{maximize Total Profit} &= 50 \text{ Jordanelle} + 65 \text{ Deercrest} \\ 3.5 \text{ Jordanelle} + 4 \text{ Deercrest} &\leq 84 \\ 1 \text{ Jordanelle} + 1.5 \text{ Deercrest} &\leq 21 \\ \text{Deercrest} - 2 \text{ Jordanelle} &\geq 0 \\ \text{Deercrest} &\geq 0 \\ \text{Jordanelle} &\geq 0 \end{aligned}$$

We saw that the optimal solution was to produce 5.25 pairs of Jordanelle skis and 10.5 pairs of Deercrest skis. Because the solution involves fractions, it would be beneficial to find the optimal solution for which the decision variables are integers. To do this, we simply add the constraints that Deercrest and Jordanelle must be integers to the model. Figure 15.1 shows the graphical illustration of the set of feasible values (dark blue dots) that satisfy all constraints as well as the integer restrictions.

To enforce integer restrictions on variables using *Solver*, click on *Integers* under the *Constraints* list and then click the *Add* button. In the *Add Constraint* dialog, enter the variable range in the *Cell Reference* field and choose *int* from the drop-down box as shown in Figure 15.2. We also need to ensure that we set the *Integer Tolerance* parameter to zero as discussed earlier. Figure 15.3 shows the resulting solution. Notice that the maximum value of the objective function for the model with integer restrictions is smaller than the linear optimization solution. This is expected because we have added an additional constraint (the integer restrictions). Whenever you add a constraint to a model, the value of the objective function can never improve and usually worsens. Figure 15.4 illustrates this graphically. As the profit line increases, the last feasible integer point through which it passes is (3, 12). Notice also that the optimal integer solution is not the same as the solution you would obtain from rounding the optimal solution to the LP relaxation.

Figure 15.1

Graphical Illustration of Feasible Integer Solutions for the Sklenka Ski Problem

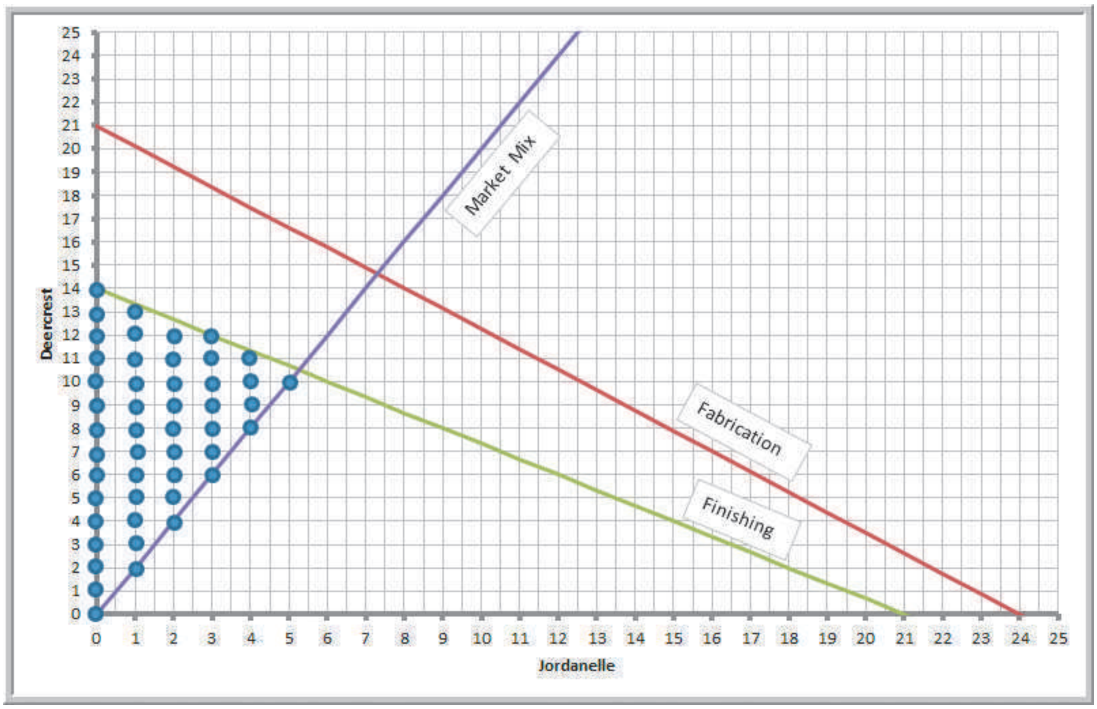


Figure 15.2

Defining General Integer Variables in *Solver*

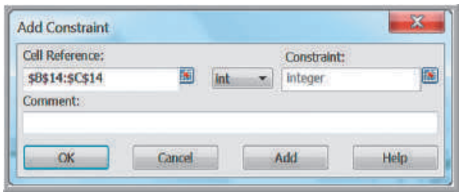


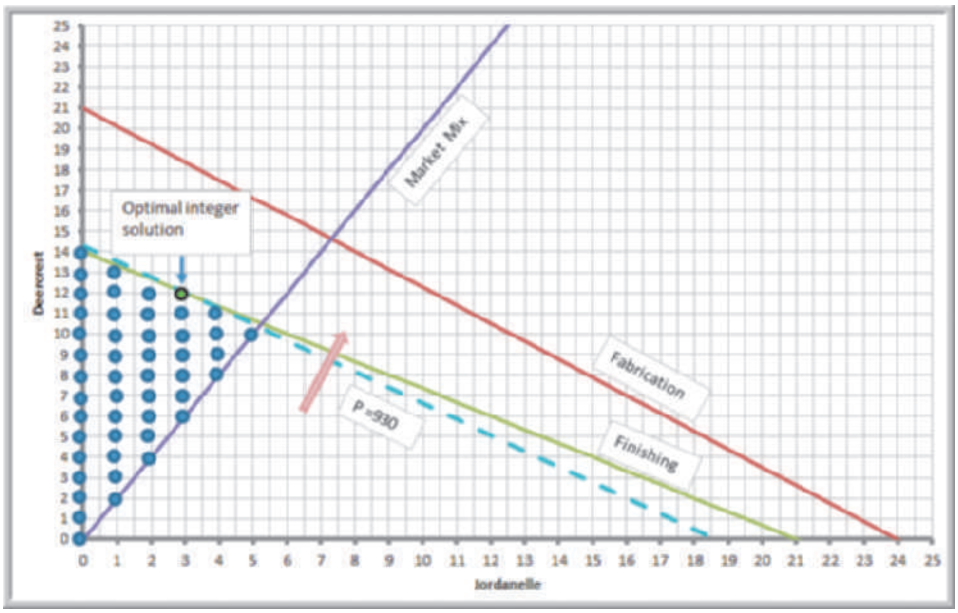
Figure 15.3

Optimal Integer Solution to Sklenka Skis Problem

	A	B	C	D
1	Sklenka Skis			
2				
3	Data			
4		Product		
5	Department	Jordanelle	Deercrest	Limitation (hours)
6	Fabrication	3.5	4	84
7	Finishing	1	1.5	21
8				
9	Profit/unit	$ 50.00	$ 65.00	
10				
11				
12	Model			
13		Jordanelle	Deercrest	
14	Quantity Produced	3	12	Hours Used
15	Fabrication	10.5	48	58.5
16	Finishing	3	18	21
17				
18				Excess Deercrest
19	Market mixture			6
20				
21				Total Profit
22	Profit Contribution	$ 150.00	$ 780.00	$ 930.00

Figure 15.4

Graphical Illustration of Optimal Integer Solution



report is provided by *Solver*—only the Answer report is available. To investigate changes in model parameters, it is necessary to re-solve the model.

If Sklenka Skis were a real situation, they would be producing thousands of pairs of skis for the world market. As we noted, it probably would not make much difference if they simply rounded the optimal linear optimization model. For other types of models, however, it is critical to enforce integer restrictions. For example, the paper industry needs to find the best mix of cutting patterns to meet demand for various sizes of paper rolls. In a similar fashion, sheet steel producers cut strips of different sizes from rolled coils of thin steel. For these types of problems, fractional values for the decision variables make no sense at all. Finding the best solution for such problems requires integer optimization.

EXAMPLE 15.2 A Cutting-Stock Problem

Suppose that a company makes standard 110-inch-wide rolls of thin sheet metal and slits them into smaller rolls to meet customer orders for widths of 12, 15, and 30 inches. The demands for these widths vary from week to week.

From a 110-inch roll, there are many different ways to slit 12-, 15-, and 30-inch pieces. A *cutting pattern* is a configuration of the number of smaller rolls of each type that are cut from the raw stock. Of course, we would want to use as much of the roll as possible to avoid costly scrap. For example, we could cut seven 15-inch rolls, leaving a 5-inch piece of scrap; or cut three 30-inch rolls and one 12-inch roll, leaving 8 inches of scrap. Finding good cutting patterns for a large set of end products is, in itself, a challenging problem. Suppose that the company has proposed the following cutting patterns:

	Size of End Item			
Pattern	**12 in.**	**15 in.**	**30 in.**	**Scrap**
1	0	7	0	5 in.
2	0	1	3	5 in.
3	1	0	3	8 in
4	9	0	0	2 in
5	2	1	2	11 in
6	7	1	0	11 in

Demands for the coming week are 500 12-inch rolls, 715 15-inch rolls, and 630 30-inch rolls. The problem is to develop a model that will determine how many 110-inch rolls to cut into each of the six patterns to meet demand and minimize scrap.

Define X_i to be the number of 110-inch rolls to cut using cutting pattern i, for $i = 1, \ldots, 6$. Note that X_i needs to be a whole number because each roll that is cut generates a different number of end items. Thus, X_i will be modeled using general integer variables. Because the objective is to minimize scrap, the objective function is

$$\min 5X_1 + 5X_2 + 8X_3 + 2X_4 + 11X_5 + 11X_6$$

The only constraints are that end item demand must be met; that is, we must produce at least 500 12-inch rolls, 715 15-inch rolls, and 630 30-inch rolls. The number of end-item rolls produced is found by multiplying the number of end-item rolls produced by each cutting pattern by the number of 110-inch rolls cut using that pattern. Therefore, the constraints are

$$0X_1 + 0X_2 + 1X_3 + 9X_4 + 2X_5 + 7X_6 \geq 500 \quad \text{(12-inch rolls)}$$
$$7X_1 + 1X_2 + 0X_3 + 0X_4 + 1X_5 + 1X_6 \geq 715 \quad \text{(15-inch rolls)}$$
$$0X_1 + 3X_2 + 3X_3 + 0X_4 + 2X_5 + 0X_6 \geq 630 \quad \text{(30-inch rolls)}$$

Finally, we include nonnegativity and integer restrictions:

$$X_i \geq 0 \text{ and integer}$$

Figure 15.5 shows the cutting-stock model implementation on a spreadsheet (Excel file *Cutting Stock Model*) with the optimal solution. The constraint functions for the number produced in cells B23:D23 and the objective function in cell B26 are SUMPRODUCT functions of the decision variables in B15:B20 and the data in rows 5 through 10. The *Solver* model is shown in Figure 15.6.

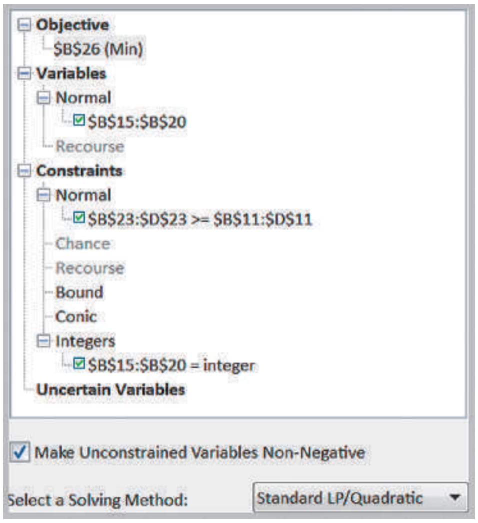
Figure 15.5

Spreadsheet Model and Optimal Solution for the *Cutting-Stock Model*

	A	B	C	D	E
1	Cutting Stock Model				
2					
3	Data				
4	Pattern	12-in rolls	15-in rolls	30-in rolls	Scrap
5	1	0	7	0	5
6	2	0	1	3	5
7	3	1	0	3	8
8	4	9	0	0	2
9	5	2	1	2	11
10	6	7	1	0	11
11	Demand	500	715	630	
12					
13	Model				
14		No. of rolls			
15	Pattern 1	73.00			
16	Pattern 2	210.00			
17	Pattern 3	0.00			
18	Pattern 4	56.00			
19	Pattern 5	0.00			
20	Pattern 6	0.00			
21					
22		12-in rolls	15-in rolls	30-in rolls	
23	Number produced	504	721	630	
24					
25		Total			
26	Scrap	1527			

Figure 15.6

Solver Model for Cutting-Stock Problem



Workforce-Scheduling Models

Workforce scheduling is a practical, yet highly complex, problem that many businesses face. Many fast-food operations hire students who can work in only small chunks of time during the week, resulting in a huge number of possible schedules. In such operations, customer demand varies by day of week and time of day, further complicating the problem of assigning workers to time slots. Similar problems exist in scheduling nurses in hospitals, flight crews in airlines, and many other service operations.

EXAMPLE 15.3 Brewer Services

Brewer Services contracts with outsourcing partners to handle various customer-service functions. The customer-service department is open Monday through Friday from 8 A.M. to 5 P.M. Calls vary over the course of a typical day. Based on a study of call volumes provided by one of the firm's partners, the minimum number of staff needed each hour of the day are as follows:

Hour	Minimum Staff Required
8–9	5
9–10	12
10–11	15
11–Noon	12
Noon–1	11
1–2	18
2–3	17
3–4	19
4–5	14

Mr. Brewer wants to hire some permanent employees and staff the remaining requirements using part-time employees who work 4-hour shifts (four consecutive hours starting as early as 8 A.M. or as late as 1 P.M.). Suppose that Mr. Brewer has five permanent employees. What is the minimum number of part-time employees he will need for each 4-hour shift to ensure meeting the staffing requirements?

Assuming that the five permanent employees work the full day, the part-time coverage requirements can be calculated by subtracting 5 from each of the time slots in the table. Define X_i to be the number of part-time employees that will work a 4-hour shift beginning at hour i, where $i = 1$ corresponds to an 8:00 A.M. start, $i = 2$ corresponds to a 9:00 A.M. start, and so on, with $i = 6$ corresponding to a 1:00 P.M. start as the last part-time shift. The objective is to minimize the total number of part-time employees:

$$\min X_1 + X_2 + X_3 + X_4 + X_5 + X_6$$

For each hour, we need to ensure that the total number of part-time employees who work that hour is at least as large as the minimum requirements. For example, only workers starting at 8:00 A.M. will cover the 8:00–9:00 time slot; thus,

$$X_1 \geq 0$$

Workers starting at either 8:00 A.M. or 9:00 A.M. will cover the second time slot; therefore,

$$X_1 + X_2 \geq 7$$

The remaining constraints are

$$X_1 + X_2 + X_3 \geq 10$$
$$X_1 + X_2 + X_3 + X_4 \geq 7$$
$$X_2 + X_3 + X_4 + X_5 \geq 6$$
$$X_3 + X_4 + X_5 + X_6 \geq 13$$
$$X_4 + X_5 + X_6 \geq 12$$
$$X_5 + X_6 \geq 14$$
$$X_6 \geq 9$$

All the variables must also be integers.

Figures 15.7 and 15.8 show the spreadsheet (Excel file *Brewer Services*) and *Solver* models for this example. The optimal solution is to hire 24 part-time workers.

Alternative Optimal Solutions

In looking at the solution, a manager might not be satisfied with the distribution of workers, particularly the fact that there are nine excess employees during the first hour. In most scheduling problems, many alternative optimal solutions usually exist. A little creativity in using the optimization model can help identify these.

Figure 15.7

Spreadsheet Model for *Brewer Services*

	A	B	C	D	E
1	**Brewer Services**				
2					
3	**Data**				
4		**Permanent Employees**			
5		5			
6			**Part Time Coverage**		
7	**Hour**	**Minimum Staff Required**	**Minimum Requirements**		
8	**8-9**	5	0		
9	**9-10**	12	7		
10	**10-11**	15	10		
11	**11-noon**	12	7		
12	**noon-1**	11	6		
13	**1-2**	18	13		
14	**2-3**	17	12		
15	**3-4**	19	14		
16	**4-5**	14	9		
17					
18	**Model**				
19				**Total part-time**	
20	**Shift**	**Number of PT employees**	**Hour**	**employees**	**Excess**
21	**1**	9	**8-9**	9	9
22	**2**	1	**9-10**	10	3
23	**3**	0	**10-11**	10	0
24	**4**	0	**11-noon**	10	3
25	**5**	5	**noon-1**	6	0
26	**6**	9	**1-2**	14	1
27	**Total**	24	**2-3**	14	2
28			**3-4**	14	0
29			**4-5**	9	0

	A	B	C	D	E
1	**Brewer Services**				
2					
3	**Data**				
4		**Permanent Employees**			
5		5			
6			**Part Time Coverage**		
7	**Hour**	**Minimum Staff Required**	**Minimum Requirements**		
8	**8-9**	5	=B8-$B$5		
9	**9-10**	12	=B9-$B$5		
10	**10-11**	15	=B10-$B$5		
11	**11-noon**	12	=B11-$B$5		
12	**noon-1**	11	=B12-$B$5		
13	**1-2**	18	=B13-$B$5		
14	**2-3**	17	=B14-$B$5		
15	**3-4**	19	=B15-$B$5		
16	**4-5**	14	=B16-$B$5		
17					
18	**Model**				
19				**Total part-time**	
20	**Shift**	**Number of PT employees**	**Hour**	**employees**	**Excess**
21	**1**	9	**8-9**	=B21	=D21-C8
22	**2**	1	**9-10**	=SUM(B21:B22)	=D22-C9
23	**3**	0	**10-11**	=SUM(B21:B23)	=D23-C10
24	**4**	0	**11-noon**	=SUM(B21:B24)	=D24-C11
25	**5**	5	**noon-1**	=SUM(B22:B25)	=D25-C12
26	**6**	9	**1-2**	=SUM(B23:B26)	=D26-C13
27	**Total**	=SUM(B21:B26)	**2-3**	=SUM(B24:B26)	=D27-C14
28			**3-4**	=SUM(B25:B26)	=D28-C15
29			**4-5**	=B26	=D29-C16

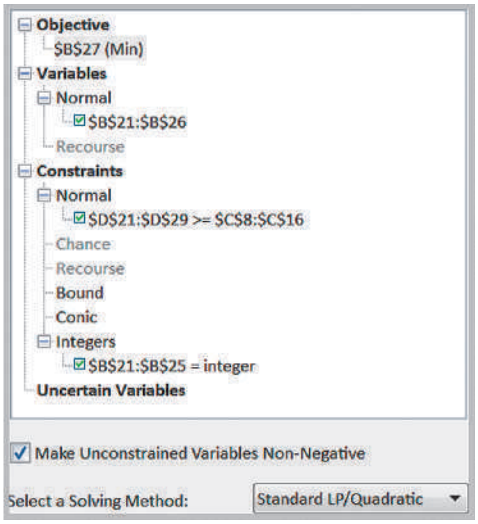


Figure 15.8

Solver Model for *Brewer Services*

EXAMPLE 15.4 Finding Alternative Optimal Solutions for *Brewer Services* Model

An easy way to find an alternative optimal solution that reduces the number of excess employees at 8:00 A.M. is to define a constraint setting the objective function equal to its optimal value and then changing the objective function to minimize the number of excess employees during the first hour. Figure 15.9 shows the modified *Solver* model with the constraint

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 24$$

and the new objective function to minimize the excess number of employees at 8:00 A.M., the value in cell E21. The solution is shown in Figure 15.10. In a "whack-a-mole" fashion, we now have 9 excess employees during the noon hour, a solution which isn't any better than the original one.

A better approach would be to define additional constraints to restrict the excess number of employees in the range E21:E29 to be less than or equal to some maximum number k and then attempt to minimize the original objective function. The *Solver* model is shown in Figure 15.11. If we do this, we find that the smallest value of k that results in a feasible solution is $k = 3$. The result is shown in Figure 15.12. We have achieved a better balance while still maintaining the minimum number of part-time employees.

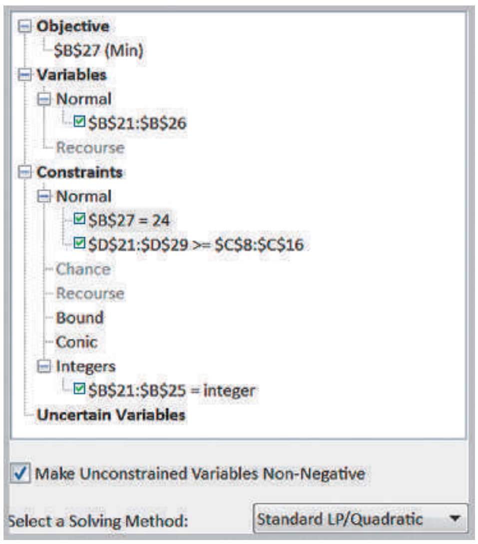


Figure 15.9

Modified *Solver* Model to Identify an Alternate Optimal Solution

Figure 15.10

Alternative Optimal Solution to Brewer Services Problem

	A	B	C	D	E
18	Model				
19				Total part-time	
20	Shift	Number of PT employees	Hour	employees	Excess
21	1	0	8-9	0	0
22	2	10	9-10	10	3
23	3	0	10-11	10	0
24	4	0	11-noon	10	3
25	5	5	noon-1	15	9
26	6	9	1-2	14	1
27	Total	24	2-3	14	2
28			3-4	14	0
29			4-5	9	0

Figure 15.11

Solver Model with Constraints on Excess Employees

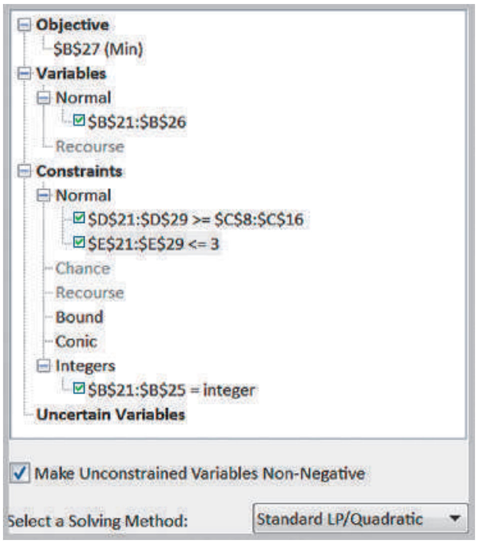


Figure 15.12

Improved Alternative Optimal Solution to Brewer Services Problem

	A	B	C	D	E
18	Model				
19				Total part-time	
20	Shift	Number of PT employees	Hour	employees	Excess
21	1	3	8-9	3	3
22	2	7	9-10	10	3
23	3	0	10-11	10	0
24	4	0	11-noon	10	3
25	5	2	noon-1	9	3
26	6	12	1-2	14	1
27	Total	24	2-3	14	2
28			3-4	14	0
29			4-5	12	3

Analytics in Practice: Sales Staffing at Qantas

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Service sector operations such as airlines, hotels, and restaurants must deal constantly with staffing problems in the face of fluctuating demand.[1] If the staff is too small, the firm cannot serve its customers well. This can result in lost sales and the loss of customer goodwill. A staff that is too large can meet customer demand, but labor costs might be excessive. Qantas Airways uses an integer programming model to determine the least-cost staff size in its telephone reservation system to meet projected demand.

The airline industry has been and continues to be an extremely competitive industry. Survival depends on maximizing efficiency in operations and capturing a sufficient share of the customer market. Qantas decided to analyze the size of its reservation staff because—as just discussed—an oversized staff is inefficient, but an undersized staff will result in lost market share. The fluctuation of demand over time makes the search for an optimal staff size a formidable task.

Qantas began its analysis by collecting demand data (number of calls) by month over a 2-year period. Then, for a 3-month period, data were collected on a half-hour basis. The data showed that demand varied by time of day and day of the week, but that for a given month, variation over weeks was insignificant. Therefore, a typical or average week could be used for a given month's planning purposes.

The integer programming model uses demand forecasts to optimize staff size over time. The following assumptions were made:

1. Shifts start only on the hour or half-hour.
2. Shifts start during the hours of 7:00 A.M. to 9:30 A.M., plus one shift that starts at 3:00 P.M. (7 possible shifts).
3. The length of shifts starting between 8:30 and 9:30 is 8.5 hours, with a 1-hour lunch; all other shifts are 8 hours with a 0.5-hour lunch.

Outputs from the model include the number of staff per shift, start and finish times of each shift, lunch schedule for each shift, and total staff needed for the day. Using the output of the daily integer optimization model, a manual approach was developed for devising a minimum workforce schedule permitting each employee two consecutive days off. This model and scheduling process saved more than $200,000 over 2 years in the Sydney office alone. Because of the success of this approach, similar approaches were later used in other offices and in other customer-contact areas, such as passenger sales and check-in facilities.

Integer Optimization Models with Binary Variables

Many optimization models require binary variables, which are variables that are restricted to being either 0 or 1. Mathematically, a **binary variable** x is simply a general integer variable that is restricted to being between 0 and 1:

$$0 \leq x \leq 1 \text{ and integer} \tag{15.1}$$

However, we usually just write this as $x = 0$ or 1. Binary variables enable us to model logical decisions in optimization models. For example, binary variables can be used to model decisions such as whether $(x = 1)$ or not $(x = 0)$ to place a facility at a certain location, whether or not to run a production line, or whether or not to invest in a certain stock.

[1]Based on A. Gaballa and W. Pearce, "Telephone Sales Manpower Planning at Qantas," *Interfaces*, 9, 3 (May, 1979): 1–9.

Project-Selection Models

One common example we present next is project selection, in which a subset of potential projects must be selected with limited resource constraints. Capital-budgeting problems in finance have a similar structure.

EXAMPLE 15.5 Hahn Engineering

Hahn Engineering's research and development group has identified five potential new engineering and development projects; however, the firm is constrained by its available budget and human resources. Each project is expected to generate a return (given by the net present value) but requires a fixed amount of cash and personnel. Because the resources are limited, all projects cannot be selected. Projects cannot be partially completed; thus, either the project must be undertaken completely or not at all. The data are given in Table 15.1. If a project is selected, it generates the full value of the expected return and requires the full amount of cash and personnel shown in Table 15.1. For example, if we select projects 1 and 3, the total return is \$180,000 + \$150,000 = \$330,000, and these projects require cash totaling \$55,000 + \$24,000 = \$79,000 and 5 + 2 = 7 personnel.

To model this situation, we define the decision variables to be binary, corresponding to either not selecting or selecting each project, respectively. Define $x_i = 1$ if project i is selected and 0 if it is not selected. By multiplying these binary variables by the expected returns, the objective function is

$$\text{maximize } \$180{,}000x_1 + \$220{,}000x_2 + \$150{,}000x_3 + \$140{,}000x_4 + \$200{,}000x_5$$

Because cash and personnel are limited, we have the following constraints:

$$\$55{,}000x_1 + \$83{,}000x_2 + \$24{,}000x_3 + \$49{,}000x_4 + \$61{,}000x_5 \le \$150{,}000 \quad \text{(cash limitation)}$$

$$5x_1 + 3x_2 + 2x_3 + 5x_4 + 3x_5 \le 12 \quad \text{(personnel limitation)}$$

Note that if projects 1 and 3 are selected, then $x_1 = 1$ and $x_3 = 1$, and the objective and constraint functions, equal

$$\text{return} = \$180{,}000(1) + \$220{,}000(0) + \$150{,}000(1) + \$140{,}000(0) + \$200{,}000(0) = \$330{,}000$$

$$\text{cash required} = \$55{,}000(1) + \$83{,}000(0) + \$24{,}000(1) + \$49{,}000(0) + \$61{,}000(0) = \$79{,}000$$

$$\text{personnel required} = 5(1) + 3(0) + 2(1) + 5(0) + 3(0) = 7$$

This model is easy to implement on a spreadsheet, as shown in Figure 15.13 (Excel file *Hahn Engineering Project Selection*). The decision variables are defined in cells B11:F11. By multiplying these values by the data for each project in rows 5–7, we can easily compute the total return, cash used, and personnel used for the projects that are selected in rows 12–14. The objective function is computed in cell G12 as the sum of the returns for the selected projects. Similarly, the amounts of cash and personnel used are also summed for the projects selected, representing the constraint functions in cells G13 and G14. The optimal solution is to select projects 1, 3, and 5 for a total return of \$530,000.

Table 15.1

Project-Selection Data

	Project 1	Project 2	Project 3	Project 4	Project 5	Available Resources
Expected return (NPV)	\$180,000	\$220,000	\$150,000	\$140,000	\$200,000	
Cash requirements	\$55,000	\$83,000	\$24,000	\$49,000	\$61,000	\$150,000
Personnel requirements	5	3	2	5	3	12

Figure 15.13

Spreadsheet Model for Project-Selection Problem

	A	B	C	D	E	F	G
1	Hahn Engineering						
2							
3	Data						
4		Project 1	Project 2	Project 3	Project 4	Project 5	Available
5	Expected Return (NPV)	$180,000	$220,000	$150,000	$140,000	$200,000	Resources
6	Cash requirements	$ 55,000	$ 83,000	$ 24,000	$ 49,000	$ 61,000	$ 150,000
7	Personnel requirements	5	3	2	5	3	12
8							
9	Model						
10							
11	Project selection decisions	1	0	1	0	1	Total
12	Return	$180,000	$ -	$150,000	$ -	$200,000	$ 530,000
13	Cash Used	$ 55,000	$ -	$ 24,000	$ -	$ 61,000	$ 140,000
14	Personnel Used	5	0	2	0	3	10

The *Solver* model is shown in Figure 15.14. To invoke the binary constraints on the variables, use the same process as defining integer variables, but choose *bin* from the drop-down box in the *Add Constraint* dialog. The resulting constraint is $B11:$F11 = binary, as shown in the *Solver* model.

As we noted, sensitivity analysis for integer optimization can be conducted only by re-solving the model for changes in the data. In the project-selection problem, it would probably benefit the manager to determine the impact of additional resources on the total expected return. First, note that if all projects are selected, they would require $272,000 in cash and 18 personnel. By setting all the decision variables to 1, we obtain a return of $890,000. As the amount of cash and personnel vary from the base case to this extreme, we may find the optimal returns, as shown in Figure 15.15. The color-coded

Figure 15.14

Solver Model for *Hahn Engineering Project Selection* Problem

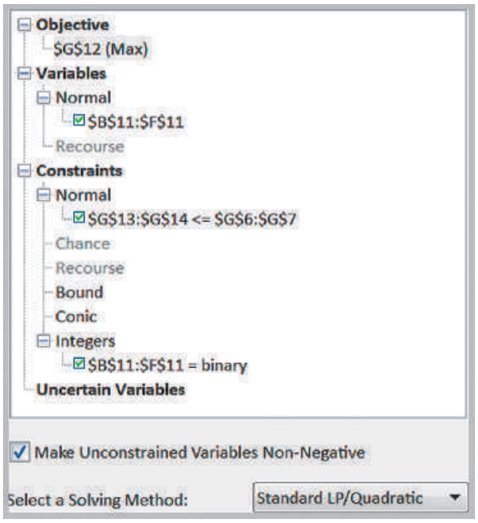


Figure 15.15

Sensitivity Analysis of Optimal Returns for Project-Selection Model

	I	J	K	L	M	N	O	P	Q
2		Cash							
3	Personnel	$150,000	$170,000	$190,000	$210,000	$230,000	$250,000	$270,000	$272,000
4	12	$ 530,000	$ 570,000	$ 570,000	$ 600,000	$ 600,000	$ 600,000	$ 600,000	$ 600,000
5	13	$ 530,000	$ 570,000	$ 570,000	$ 600,000	$ 750,000	$ 750,000	$ 750,000	$ 750,000
6	14	$ 530,000	$ 570,000	$ 570,000	$ 600,000	$ 750,000	$ 750,000	$ 750,000	$ 750,000
7	15	$ 530,000	$ 570,000	$ 670,000	$ 670,000	$ 750,000	$ 750,000	$ 750,000	$ 750,000
8	16	$ 530,000	$ 570,000	$ 670,000	$ 670,000	$ 750,000	$ 750,000	$ 750,000	$ 750,000
9	17	$ 530,000	$ 570,000	$ 670,000	$ 670,000	$ 750,000	$ 750,000	$ 750,000	$ 750,000
10	18	$ 530,000	$ 570,000	$ 670,000	$ 670,000	$ 750,000	$ 750,000	$ 750,000	$ 890,000

regions in the matrix show combinations of personnel and cash with the same minimal values of the return; such a visual display is often called a **heat map**, and it allows you to easily identify different solutions. This information can help the manager evaluate the trade-offs between increasing the expected return and acquiring additional resources. The upper-left-hand corner of each colored region (shown boxed in the figure) represents the lowest amount of resources required to achieve that return. For example, the company can improve the return by $40,000 by increasing its cash availability by $20,000 with no additional personnel or improve the return by $140,000 by increasing the cash availability by $40,000 with three additional personnel. Although the best decision may not be clear, analysis provides the decision maker with better information to make an informed choice.

Using Binary Variables to Model Logical Constraints

Binary variables allow us to model a wide variety of logical constraints. For example, suppose that if project 1 is selected, then project 4 must also be selected. Your first thought might be to incorporate an IF function in the Excel model; however, recall that we noted in Chapter 13 that such functions destroy the linearity property of the Excel model; therefore, we need to express such constraints differently. (We do, however, address these issues further in the next chapter.) If project 1 is selected, then $x_1 = 1$, and we want to force x_4 to be 1 also. This can be done using the following constraint:

$$x_4 \geq x_1$$

Mathematically, if $x_1 = 1$ then this constraint implies that $x_4 \geq 1$ and, consequently, x_4 must equal 1. If $x_1 = 0$, then $x_4 \geq 0$ and x_4 can be either 0 or 1. Table 15.2 summarizes how to model a variety of logical conditions using binary variables.

Table 15.2

Modeling Logical Conditions Using Binary Variables

Logical Condition	Constraint Model Form
If *A*, then *B*	$B \geq A$ or $B - A \geq 0$
If not *A*, then *B*	$B \geq 1 - A$ or $A + B \geq 1$
If *A*, then not *B*	$B \leq 1 - A$ or $A + B \leq 1$
At most one of *A* and *B*	$A + B \leq 1$
If *A*, then *B* and *C*	($B \geq A$ and $C \geq A$) or $B + C \geq 2A$
If *A* and *B*, then *C*	$C \geq A + B - 1$ or $A + B - C \leq 1$

EXAMPLE 15.6 Adding Logical Constraints into the Project-Selection Model

Suppose that we want to ensure that if project 1 is selected, then project 4 is selected, and that at most one of projects 1 and 3 can be selected in the Hahn Engineering model. To incorporate the constraint $x_4 \geq x_1$, write it as $x_4 - x_1 \geq 0$ by defining a cell for the constraint function $x_4 - x_1$ (cell B17 in Figure 15.16). Similarly, for the constraint $x_1 + x_3 \leq 1$, define a cell for $x_1 + x_3$ (cell B18 in Figure 15.16). Then add these constraints to the *Solver* model, as shown in Figure 15.17 (Excel file *Hahn Engineering Project Selection with Logical Conditions*). In the optimal solution, we do not select project 1, although project 4 is selected anyway. With the additional constraints, the expected return is smaller than the original solution.

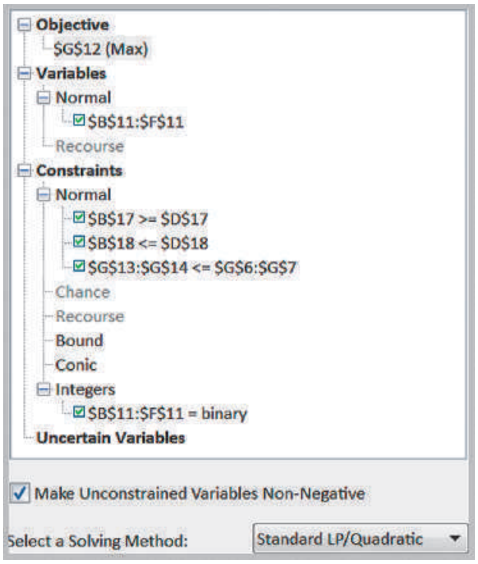
Figure 15.16

Modified Project-Selection Model with Logical Conditions

	A	B	C	D	E	F	G
1	Hahn Engineering Model with Logical Constraints						
2							
3	Data						
4		Project 1	Project 2	Project 3	Project 4	Project 5	Available
5	Expected Return (NPV)	$180,000	$220,000	$150,000	$140,000	$200,000	Resources
6	Cash requirements	$ 55,000	$ 83,000	$ 24,000	$ 49,000	$ 61,000	$ 150,000
7	Personnel requirements	5	3	2	5	3	12
8							
9	Model						
10							
11	Project selection decisions	0	0	1	1	1	Total
12	Return	$ -	$ -	$150,000	$140,000	$200,000	$ 490,000
13	Cash Used	$ -	$ -	$ 24,000	$ 49,000	$ 61,000	$ 134,000
14	Personnel Used	0	0	2	5	3	10
15			=E11-B11				
16	Logical conditions						
17	If project 1 then project 4	1	≥	0			
18	At most one of projects 1 and 3	1	≤	1			
19			=B11+D11				
20							

Figure 15.17

Modified *Solver* Model with Logical Conditions



Location Models

Integer optimization models have wide applications in locating facilities. The following is an example of a "covering" problem, one in which we seek to choose a subset of locations that serve, or cover, all locations in a service area.

EXAMPLE 15.7 Anderson Village Fire Department

Suppose that an unincorporated village wishes to find the best locations for fire stations. Assume that the village is divided into smaller districts, or neighborhoods, and that transportation studies have estimated the response time for emergency vehicles to travel between each pair of districts. The village wants to locate the fire stations so that all districts can be reached within an 8-minute response time. The following table shows the

(continued)

estimated response time in minutes between each pair of districts:

From/To	1	2	3	4	5	6	7
1	0	2	10	6	12	5	8
2	2	0	6	9	11	7	10
3	10	6	0	5	5	12	6
4	6	9	5	0	9	4	3
5	12	11	5	9	0	10	8
6	5	7	12	4	10	0	6
7	8	10	6	3	8	6	0

Define $X_j = 1$ if a fire station is located in district j and 0 if not. The objective is to minimize the number of fire stations that need to be built:

$$\min X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7$$

Each district must be reachable within 8 minutes by some fire station. Thus, from the table, for example, we see that to be able to respond to district 1 in 8 minutes or less, a station must be located in either district 1, 2, 4, 6, or 7. Therefore, we must have the constraint:

$$X_1 + X_2 + X_4 + X_6 + X_7 \geq 1$$

Similar constraints may be formulated for each of the other districts:

$$X_1 + X_2 + X_3 + X_6 \geq 1$$
$$X_2 + X_3 + X_4 + X_5 + X_7 \geq 1$$
$$X_1 + X_3 + X_4 + X_6 + X_7 \geq 1$$
$$X_3 + X_5 + X_7 \geq 1$$
$$X_1 + X_2 + X_4 + X_6 + X_7 \geq 1$$
$$X_1 + X_3 + X_4 + X_5 + X_6 + X_7 \geq 1$$

Figure 15.18 shows a spreadsheet model for this problem (Excel file *Anderson Village Fire Station Location Model*). To develop the constraints in the model, we construct a matrix by converting all response times that are within 8 minutes to 1s and those that exceed 8 minutes to 0s. Then the constraint functions for each district are simply the SUMPRODUCT of the decision variables and the rows of this matrix, making the *Solver* model, shown in Figure 15.19, easy to define. For instance, the formula in cell I20 is = SUMPRODUCT($B$28:$H$28,B20:H20). For this example, the solution is to site fire stations in districts 3 and 7.

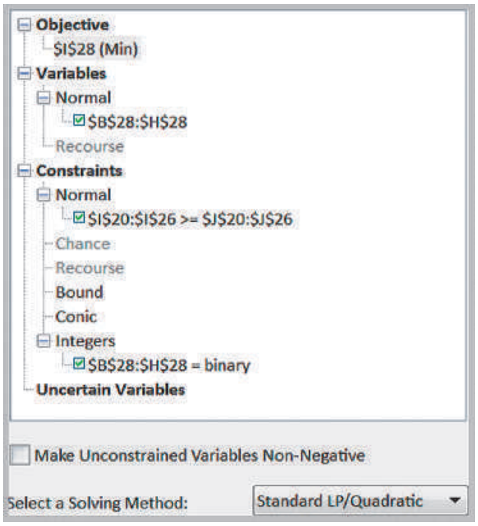
	A	B	C	D	E	F	G	H	I	J
1	Anderson Village Fire Station Model									
2										
3	Data									
4										
5	Response time	8 minutes								
6										
7	Response Times									
8	From/To	1	2	3	4	5	6	7		
9	1	0	2	10	6	12	5	8		
10	2	2	0	6	9	11	7	10		
11	3	10	6	0	5	5	12	6		
12	4	6	9	5	0	9	4	3		
13	5	12	11	5	9	0	10	8		
14	6	5	7	12	4	10	0	6		
15	7	8	10	6	3	8	6	0		
16										
17	Model									
18										
19	From/To	1	2	3	4	5	6	7	Covered?	Requirement
20	1	1	1	0	1	0	1	1	1	1
21	2	1	1	1	0	0	1	0	1	1
22	3	0	1	1	1	1	0	1	2	1
23	4	1	0	1	1	0	1	1	2	1
24	5	0	0	1	0	1	0	1	2	1
25	6	1	1	0	1	0	1	1	1	1
26	7	1	0	1	1	1	1	1	2	1
27									Total	
28	Location	0	0	1	0	0	0	1	2	

Figure 15.18

Spreadsheet Model for *Anderson Village Fire Station Location Model*

Figure 15.19

Solver Model for Anderson Village Fire Station Location



Parameter Analysis

Suppose that the Anderson Village township's board of trustees wants to better understand the trade-offs between the response time and minimum number of fire stations needed. We could change the value of the response time in cell B5 and resolve the model or use the *Analytic Solver Platform Parameter Analysis* feature that we described in Chapter 13.

EXAMPLE 15.8 Parameter Analysis for Response Time

As described in Chapter 13, first choose an empty cell and define the parameter range (in this case, choose cell D5 with a lower value of 5 and upper value of 10) and then reference the defined parameter cell in place of the response time (cell B5) in the model. Then, choose *Parameter Analysis* from the *Optimization* list within the *Reports* menu in the *Analytic Solver Platform* ribbon. Select the variable cells, objective function cell, and parameter cell in the *Multiple Optimizations Report* dialog and change *Major Axis Points* to 6, and the model will be run for each response time from 5 through 10.

Figure 15.20 shows the report summary using *Analytic Solver Platform Parameter Analysis* (we added the descriptive titles in rows 1 and 2). In column A are the parameterized values of the response time. The 1s in columns B through H show where the fire stations should be located. Column I shows the minimum number of fire stations required. These results show the maximum response time can be reduced to 6 minutes while still using only two fire stations (the model solution yields districts 1 and 3). This would clearly be a better alternative. Also, if the response time is increased by only 1 minute from its original target, the township could save the cost of building a second facility. Of course, such decisions need to be evaluated carefully.

Figure 15.20

Parameter Analysis Report

	A	B	C	D	E	F	G	H	I
1	Location	1	2	3	4	5	6	7	
2	Response Time								Min. Number of Sites
3	$D$5	$B$28	$C$28	$D$28	$E$28	$F$28	$G$28	$H$28	$I$28
4	5	1	0	1	1	0	0	0	3
5	6	1	0	1	0	0	0	0	2
6	7	1	0	1	0	0	0	0	2
7	8	0	0	1	0	0	0	1	2
8	9	0	0	0	1	0	0	0	1
9	10	0	0	0	1	0	0	0	1

A Customer-Assignment Model for Supply Chain Optimization

Supply chain optimization is one of the broadest applications of integer optimization and is used extensively today as companies seek to reduce logistics costs and improve customer service in tough economic environments. Although many applications involve optimization models with both normal and binary variables, which we describe in the next section, some applications require only binary variables.

Suppose that a company has numerous potential locations for distribution centers that will ship products to many customers and wants to redesign its supply chain by selecting a fixed number of distribution centers. In an effort to provide exceptional customer service, some companies have a single-sourcing policy—that is, every customer can be supplied from only one distribution center. The problem is to determine how to assign customers to the distribution centers so as to minimize the total cost of shipping to the customers.

Define $X_{ij} = 1$ if customer j is assigned to distribution center i and 0 if not; $Y_i = 1$ if distribution center i is chosen from among a set of potential locations; and $C_{ij} =$ the total cost of satisfying the demand of customer j from distribution center i. We wish to minimize the total cost, ensure that every customer is assigned to one and only one distribution center, and select k distribution centers from the set of potential locations. This can be accomplished by the following model:

$$\min \sum_i \sum_j C_{ij} X_{ij}$$

$$\sum_i X_{ij} = 1, \text{ for every } j$$

$$\sum_i Y_i = k$$

$$X_{ij} \le Y_i, \text{ for every } i \text{ and } j$$

$$X_{ij} \text{ and } Y_i \text{ are binary}$$

The first constraint ensures that each customer is assigned to exactly one distribution center. The next constraint limits the number of distribution centers selected. The final constraint ensures that customer j cannot be assigned to distribution center i unless that distribution center is selected in the supply chain. This is similar to the logical constraints we described in Table 15.2. If $Y_i = 1$, then any customer may be assigned to distribution center i; if $Y_i = 0$, then X_{ij} is forced to be 0 for all customers j because distribution center i is not selected.

EXAMPLE 15.9 Paul & Giovanni Foods

Paul & Giovanni Foods distributes supplies to restaurants in five major cities: Houston, Las Vegas, New Orleans, Chicago, and San Francisco. In a study to reconfigure their supply chain, they have identified four possible locations for distribution centers: Los Angeles, Denver, Pensacola, and Cincinnati. The costs of supplying each customer city from each possible distribution center are shown next:

Sourcing Costs	Houston	Las Vegas	New Orleans	Chicago	San Francisco
Los Angeles	$40,000	$11,000	$75,000	$70,000	$60,000
Denver	$72,000	$77,000	$120,000	$30,000	$75,000
Pensacola	$24,000	$44,000	$45,000	$80,000	$90,000
Cincinnati	$32,000	$55,000	$90,000	$20,000	$105,000

P&G Foods wishes to determine the best supply chain configuration to minimize cost.

Define $X_{ij} = 1$ if customer city j is assigned to distribution center i and 0 if not and $Y_i = 1$ if distribution center i is chosen from among a set of potential locations. The integer optimization model is

$$\text{minimize } \$40{,}000X_{11} + \$11{,}000X_{12} + \$75{,}000X_{13} + \$70{,}000X_{14} + \$60{,}000X_{15} + \$72{,}000X_{21} + \$77{,}000X_{22} + \$120{,}000X_{23} + \$30{,}000X_{24} + \$75{,}000X_{25} + \$24{,}000X_{31} + \$44{,}000X_{32} + \$45{,}000X_{33} + \$80{,}000X_{34} + \$90{,}000X_{35} + \$32{,}000X_{41} + \$55{,}000X_{42} + \$90{,}000X_{43} + \$20{,}000X_{44} + \$105{,}000X_{45}$$

$$X_{11} + X_{21} + X_{31} + X_{41} = 1$$

$$X_{12} + X_{22} + X_{32} + X_{42} = 1$$

$$X_{13} + X_{23} + X_{33} + X_{43} = 1$$

$$X_{14} + X_{24} + X_{34} + X_{44} = 1$$

$$X_{15} + X_{25} + X_{35} + X_{45} = 1$$

$$Y_1 + Y_2 + Y_3 + Y_4 = k$$

$X_{ij} \le Y_i$, for every i and j (e.g., $X_{11} \le Y_1$, $X_{21} \le Y_1$, and so on)

X_{ij} and Y_i are binary

Figure 15.21 shows a spreadsheet model and the optimal solution for $k = 2$ (Excel file *Paul & Giovanni Foods*); Figure 15.22 shows the *Solver* model. We see the distribution centers in Los Angeles and Cincinnati should be chosen, with Los Angeles serving Las Vegas, New Orleans, and San Francisco and Cincinnati serving Houston and Chicago.

This model can easily be used to evaluate alternatives for different values of k using parameter analysis techniques. For example, when $k = 1$, the model selects Los Angeles with a total cost of \$256,000; when $k = 3$, Los Angeles, Cincinnati, and Pensacola are chosen with a minimum cost of \$160,000; and if all four distribution centers are chosen, the same solution results. The supply chain manager can use this information to determine the trade-offs associated with opening different numbers of distribution centers.

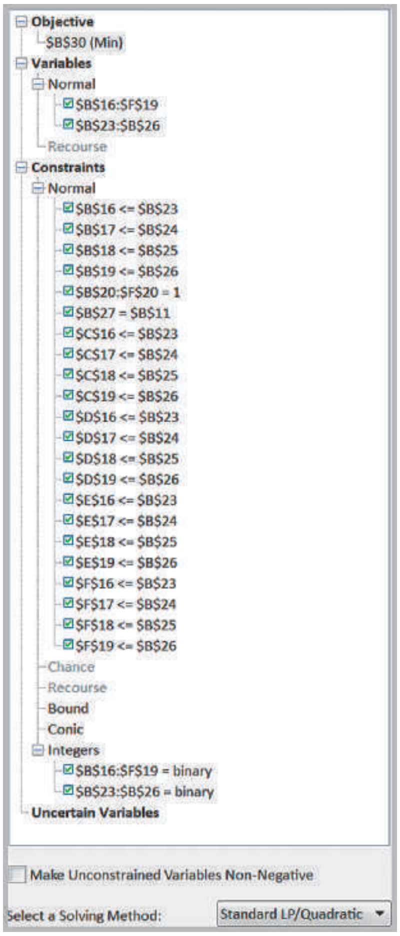
Figure 15.21

Spreadsheet Model and Optimal Solution for *Paul & Giovanni Foods* for $k = 2$

	A	B	C	D	E	F
1	Paul & Giovanni Foods					
2						
3	Data					
4						
5	Sourcing Costs	Houston	Las Vegas	New Orleans	Chicago	San Francisco
6	Los Angeles	$40,000	$11,000	$75,000	$70,000	$60,000
7	Denver	$72,000	$77,000	$120,000	$30,000	$75,000
8	Pensacola	$24,000	$44,000	$45,000	$80,000	$90,000
9	Cincinnati	$32,000	$55,000	$90,000	$20,000	$105,000
10						
11	Number of DCs	2				
12						
13	Model					
14						
15	Customer Assignments	Houston	Las Vegas	New Orleans	Chicago	San Francisco
16	Los Angeles	0	1	1	0	1
17	Denver	0	0	0	0	0
18	Pensacola	0	0	0	0	0
19	Cincinnati	1	0	0	1	0
20	Sum	1	1	1	1	1
21						
22	DCs Chosen					
23	Los Angeles	1				
24	Denver	0				
25	Pensacola	0				
26	Cincinnati	1				
27	Sum	2				
28						
29		Total				
30	Cost	$ 198,000				

Figure 15.22

Solver Model for *Paul & Giovanni Foods*



Analytics in Practice: Supply Chain Optimization at Procter & Gamble

In 1993, Procter & Gamble began an effort entitled Strengthening Global Effectiveness (SGE) to streamline work processes, drive out non-value-added costs, and eliminate duplication.[2]

A principal component of SGE was the North American Product Supply Study, designed to reexamine and reengineer P&G's product-sourcing and distribution system for its North American operations, with an emphasis on plant consolidation. Prior to the study, the North American supply chain consisted of hundreds of suppliers, more than 50 product categories, more than 60 plants, 15 distribution centers, and more than 1,000 customers. The need to consolidate plants was driven by the move to global brands and common packaging, and the need to reduce manufacturing expense, improve speed to market, avoid major capital investments, and deliver better consumer value.

P&G had a policy of single sourcing; therefore, one of the key submodels in the overall optimization effort was the customer assignment optimization model

[2]Based on Jeffrey D. Camm, Thomas E. Chorman, Franz A. Dill, James R. Evans, Dennis J. Sweeney, and Glenn W. Wegryn, "Blending OR/MS, Judgment, and GIS: Restructuring P&G's Supply Chain," *Interfaces*, 27, 1 (January–February, 1997): 128–142.

Kheng Guan Toh/Shutterstock.com

described in this section to identify optimal distribution center locations in the supply chain and to assign customers to the distribution centers. Customers were aggregated into 150 zones. The parameter *k* was varied by the analysis team to examine the effects of choosing different numbers of locations. This model was used in conjunction with a simple transportation model for each of 30 product categories. Product-strategy teams used these models to specify plant locations and capacity options and optimize the flow of product from plants to distribution centers and customers. In reconfiguring the supply chain, P&G realized annual cost savings of more than $250 million.

Mixed-Integer Optimization Models

Many practical applications of optimization involve a combination of continuous variables and binary variables. This provides the flexibility to model many different types of complex decision problems.

Plant Location and Distribution Models

Suppose that in the GAC transportation model example discussed in Chapter 14, demand forecasts exceed the existing capacity and the company is considering adding a new plant from among two choices: Fayetteville, Arkansas, or Chico, California. Both plants would have a capacity of 1,500 units, but only one can be built. Table 15.3 shows the revised data.

The company now faces two decisions. It must decide which plant to build and then how to best ship the product from the plant to the distribution centers. Of course, one approach would be to solve two separate transportation models, one that includes the Fayetteville plant and the other that includes the Chico plant. However, we demonstrate how to answer both questions simultaneously, because this provides the most efficient approach, especially if the number of alternative locations is large. The difference between this situation and the customer-assignment model in the previous section is that single sourcing is not required; therefore, any distribution center may receive some of its demand from more than one plant.

Table 15.3 Plant Location Data

	Distribution Center				
Plant	Cleveland	Baltimore	Chicago	Phoenix	Capacity
Marietta	$12.60	$14.35	$11.52	$17.58	1,200
Minneapolis	$9.75	$16.26	$8.11	$17.92	800
Fayetteville	$10.41	$11.54	$9.87	$11.64	1,500
Chico	$13.88	$16.95	$12.51	$8.32	1,500
Demand	300	500	700	1,800	

EXAMPLE 15.10 A Mixed-Integer Plant Location Model

To build an optimization model to simultaneously choose which location to build the plant and how to ship the product from the plants to the distribution centers, define a binary variable for the decision of which plant to build: $Y_1 = 1$ if the Fayetteville plant is built and $Y_2 = 1$ if the Chico plant is built; and define normal variables X_{ij}, representing the amount shipped from plant i to distribution center j. The objective function now includes terms for the proposed plant locations as well as the existing ones:

$$\begin{aligned}\text{minimize } & 12.60X_{11} + 14.35X_{12} + 11.52X_{13} + 17.58X_{14}\\ & + 9.75X_{21} + 16.26X_{22} + 8.11X_{23} + 17.92X_{24}\\ & + 10.41X_{31} + 11.54X_{32} + 9.87X_{33} + 11.64X_{34}\\ & + 13.88X_{41} + 16.95X_{42} + 12.51X_{43} + 8.32X_{44}\end{aligned}$$

Capacity constraints for the Marietta and Minneapolis plants remain as before. However, for Fayetteville and Chico, we can allow shipping from those locations only if a plant is built there. In other words, if we do not build a plant in Fayetteville (if $Y_1 = 0$), for example, then we must ensure that the amount shipped from Fayetteville to any distribution center must be zero, or $X_{3j} = 0$ for $j = 1$ to 4. To do this, we multiply the capacity by the binary variable corresponding to the location:

$$X_{11} + X_{12} + X_{13} + X_{14} \le 1{,}200$$
$$X_{21} + X_{22} + X_{23} + X_{24} \le 800$$
$$X_{31} + X_{32} + X_{33} + X_{34} \le 1{,}500Y_1$$
$$X_{41} + X_{42} + X_{43} + X_{44} \le 1{,}500Y_2$$

Note that if the binary variable is zero, then the right-hand side of the constraint is zero, forcing all shipment variables to be zero also. If, however, a particular Y-variable is 1, then shipping up to the plant capacity is allowed. The demand constraints are the same as before, except that additional variables corresponding to the possible plant locations are added and new demand values are used:

$$X_{11} + X_{21} + X_{31} + X_{41} = 300$$
$$X_{12} + X_{22} + X_{32} + X_{42} = 500$$
$$X_{13} + X_{23} + X_{33} + X_{43} = 700$$
$$X_{14} + X_{24} + X_{34} + X_{44} = 1{,}800$$

To guarantee that only one new plant is built, we must have

$$Y_1 + Y_2 = 1$$

Finally, we have nonnegativity for the continuous variables: $X_{ij} \ge 0$, for all i and j.

Figure 15.23 shows the spreadsheet model (Excel file *Plant Location Model*) and optimal solution. Note that in addition to the continuous variables X_{ij}, in the range B16:E19, we defined binary variables Y_i in cells I16 and I17. Cells J16 and J17 represent the constraint functions $1{,}500Y_1 - X_{31} - X_{32} - X_{33} - X_{34}$ and $1500Y_2 - X_{41} - X_{42} - X_{43} - X_{44}$, respectively. These are restricted to be greater than or equal to zero to enforce the capacity constraints at the potential locations in the *Solver* model (Figure 15.24). You should closely examine the other constraints in the *Solver* model to verify that they are correct. The solution specifies selecting the Chico location. Models of this type are commonly used in supply chain design and other facility location applications.

Binary Variables, IF Functions, and Nonlinearities in Model Formulation

You may be wondering about why we need to express the constraints in the following fashion to ensure that if we don't build a plant, then we must ensure that no product is shipped from that plant:

$$X_{31} + X_{32} + X_{33} + X_{34} \le 1{,}500Y_1$$
$$X_{41} + X_{42} + X_{43} + X_{44} \le 1{,}500Y_2$$

	A	B	C	D	E	F	G	H	I	J
1	Plant Location Model									
2										
3	Data									
4			Distribution Center							
5	Plant	Cleveland	Baltimore	Chicago	Phoenix	Capacity				
6	Marietta	$ 12.60	$ 14.35	$ 11.52	$ 17.58	1200				
7	Minneapolis	$ 9.75	$ 16.26	$ 8.11	$ 17.92	800				
8	Fayetteville	$ 10.41	$ 11.54	$ 9.87	$ 11.64	1500				
9	Chico	$ 13.88	$ 16.95	$ 12.51	$ 8.32	1500				
10	Demand	300	500	700	1800					
11										
12	Model									
13										
14	Amount Shipped		Distribution Center							
15	Plant	Cleveland	Baltimore	Chicago	Phoenix	Total shipped			New Plant Chosen	Surplus Capacity
16	Marietta	200	500	0	300	1000		Fayetteville	0	0
17	Minneapolis	100	0	700	0	800		Chico	1	0
18	Fayetteville	0	0	0	0	0		Total	1	
19	Chico	0	0	0	1500	1500				
20	Demand met	300	500	700	1800					
21										
22	Total cost									
23	$34,101.00									

	A	B	C	D	E	F	G	H	I	J
1	Plant Location Model									
2										
3	Data									
4			Distribution Center							
5	Plant	Cleveland	Baltimore	Chicago	Phoenix	Capacity				
6	Marietta	12.6	14.35	11.52	17.58	1200				
7	Minneapolis	9.75	16.26	8.11	17.92	800				
8	Fayetteville	10.41	11.54	9.87	11.64	1500				
9	Chico	13.88	16.95	12.51	8.32	1500				
10	Demand	300	500	700	1800					
11										
12	Model									
13										
14	Amount Shipped		Distribution Center							
15	Plant	Cleveland	Baltimore	Chicago	Phoenix	Total shipped			New Plant Chosen	Surplus Capacity
16	Marietta	200	500	0	300	=SUM(B16:E16)		Fayetteville	0	=F8*I16-F18
17	Minneapolis	100	0	700	0	=SUM(B17:E17)		Chico	1	=F9*I17-F19
18	Fayetteville	0	0	0	0	=SUM(B18:E18)		Total	=SUM(I16:I17)	
19	Chico	0	0	0	1500	=SUM(B19:E19)				
20	Demand met	=SUM(B16:B19)	=SUM(C16:C19)	=SUM(D16:D19)	=SUM(E16:E19)					
21										
22	Total cost									
23	=SUMPRODUCT(B6:E9,B16:E19)									

Figure 15.23

Spreadsheet Model for *Plant Location Model*

Frequent users of Excel might immediately focus on the "if" condition and want to model the problem on the spreadsheet using a logical IF function to define the capacity in cells F8 and F9. For example, we might enter the formula =IF(I16=1, 1500, 0) into cell F8. This says that if the Fayetteville plant is chosen, then the available capacity is 1,500; otherwise it is zero. Simple, right? From a spreadsheet perspective, there is nothing wrong with this. However, from a linear optimization perspective, the use of an IF function no longer preserves the linearity of the model (technically, the model would be called *nonsmooth*) and we would get an error message in trying to solve the model using a linear-based *Solver* algorithm. Similarly, you might think to model the constraint as $X_{31}Y_1 + X_{32}Y_1 + X_{33}Y_1 + X_{34}Y_1 \leq 1{,}500$. Although this is logically correct, multiplying the two variables together results in a nonlinear function. Both nonsmooth and nonlinear models are much more difficult to solve than linear models. You can learn about these in the online Supplementary Chapter A. So for now, it is important that the models we develop retain linear characteristics.

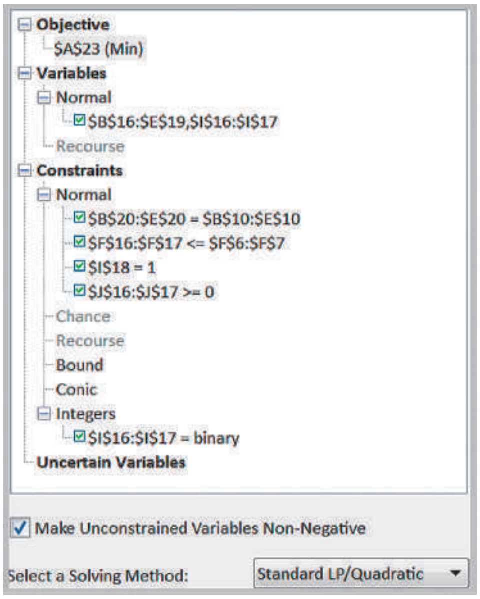


Figure 15.24

Solver Model for Plant Location Problem

Fixed-Cost Models

Many business problems involve fixed costs; they are either incurred in full or not at all. Binary variables can be used to model such problems in a similar fashion as we did for the plant location model.

EXAMPLE 15.11 Incorporating Fixed Costs into the *K&L Designs* Model

Consider the multiperiod production-inventory-planning model for K&L Designs that we developed in Chapter 14. Suppose that the company must rent some equipment, which costs $65 for 3 months. The equipment can be rented or returned each quarter, so if nothing is produced in a quarter, it makes no sense to incur the rental cost.

The fixed costs can be incorporated into the model by defining an additional set of variables:

Y_A = 1 if production occurs during the autumn and 0 if not

Y_W = 1 if production occurs during the winter and 0 if not

Y_S = 1 if production occurs during the spring and 0 if not

Then, the objective function becomes

$$\text{minimize } 11P_A + 14P_W + 12.50P_S + 1.20I_A + 1.20I_W + 1.20I_S + 65(Y_A + Y_W + Y_S)$$

The basic material balance equations are the same:

$$P_A - I_A = 150$$
$$P_W + I_A - I_W = 400$$
$$P_S + I_W - I_S = 50$$

However, we must ensure that whenever a production variable, P, is positive, the corresponding Y variable is equal to 1; conversely, if the Y variable is 0 (you don't rent the equipment), then the corresponding production variable must also be 0. This can be accomplished with the following constraints:

$$P_A \le 600Y_A$$
$$P_W \le 600Y_W$$
$$P_S \le 600Y_S$$

Note that if any Y is 0 in a solution, then P is forced to be zero, and if P is positive, then Y must be 1. Because we don't know how much the value of any production variable will be, we use 600, which is the sum of the demands over the time horizon, to multiply by Y. So when Y is 1, any amount up to 600 units can be produced. Actually any large number can be used, so long as it doesn't restrict the possible values of P. Generally, the smallest value should be used for efficiency. Finally, P_A, P_W, and P_S must be nonnegative, and Y_A, Y_W, and Y_S are binary.

Figure 15.25

Spreadsheet Model for *K&L Designs Fixed-Cost Model*

	A	B	C	D
1	K&L Designs Fixed Cost Model			
2				
3	Data			
4				
5	Cost	Quarter 1	Quarter 2	Quarter 3
6	Production	$ 11.00	$ 14.00	$ 12.50
7	Inventory	$ 1.20	$ 1.20	$ 1.20
8	Demand	150	400	50
9	Fixed cost	$ 65.00	$ 65.00	$ 65.00
10				
11	Model			
12				
13		Quarter 1	Quarter 2	Quarter 3
14	Production	600	0	0
15	Inventory	450	50	0
16	Binary	1	0	0
17				
18	Binary constraints	600	0	0
19	Net production	150	400	50
20				
21		Cost		
22	Total	$7,265.00		

	A	B	C	D
1	K&L Designs Fixed Cost Model			
2				
3	Data			
4				
5	Cost	Quarter 1	Quarter 2	Quarter 3
6	Production	11	14	12.5
7	Inventory	1.2	1.2	1.2
8	Demand	150	400	50
9	Fixed cost	65	65	65
10				
11	Model			
12				
13		Quarter 1	Quarter 2	Quarter 3
14	Production	600	0	0
15	Inventory	450	50	0
16	Binary	1	0	0
17				
18	Binary constraints	=600*B16	=600*C16	=600*D16
19	Net production	=B14-B15	=C14-C15+B15	=D14-D15+C15
20				
21		Cost		
22	Total	=SUMPRODUCT(B6:D7,B14:D15) + 65*(B16+C16+D16)		

Figure 15.26

Solver Model for K&L Designs Fixed-Cost Problem

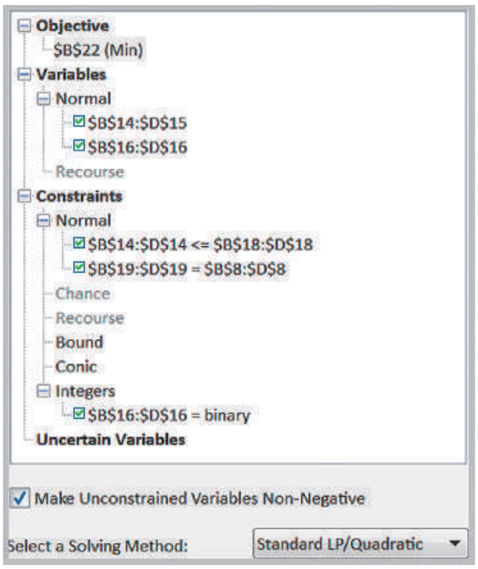


Figure 15.25 shows a spreadsheet implementation for this model with the optimal solution (Excel file *K&L Designs Fixed Cost Model*). Figure 15.26 shows the *Solver* model. With the fixed costs of equipment, it is better to produce everything in the first quarter and carry the inventory, in contrast to the solution we found in Chapter 14.

You might observe that this model does not preclude feasible solutions in which a production variable is 0 while its corresponding Y-variable is 1. This implies that we incur the fixed cost even though no production is incurred during that time period. Although such a solution is feasible, it can never be optimal, because a lower cost could be obtained by setting the Y-variable to 0 without affecting the value of the production variable, and the solution algorithm will always ensure this. Therefore, it is not necessary to explicitly try to incorporate this in the model.

Key Terms

Binary variable
General integer variables
Heat map
Integer linear optimization model (integer program)
Linear program (LP) relaxation
Mixed-integer linear optimization model

Problems and Exercises

Note: *Data for most of these problems are provided in the Excel files* Chapter 14 Problem Data *(for Problems 1–4) or* Chapter 15 Problem Data *to facilitate model building. Worksheet tabs correspond to problem scenario names.*

1. Solve Problem 6 in Chapter 13 to ensure that the number of minutes of each type of ad are integer valued. How much difference is there between the optimal integer solution and the linear optimization solution? Would rounding the continuous solution have provided the optimal integer solution?

2. Solve the *J&M Manufacturing* model in Chapter 14 to ensure that the number of units produced is integer valued. How much difference is there between the optimal integer solution and the linear optimization solution?

3. Solve the Toy Manufacturing model in Problem 5 of Chapter 14 with the restriction that the number of units manufactured must be an integer. Compare your solution with the linear optimization solution.

4. Solve the media selection model in Problem 21 of Chapter 14 with the restriction that the number of ads placed must be integer. Compare your solution with the linear optimization solution.

5. Solve the following as integer optimization model:

   Maximize$Z = a + 4b$;subjectto$2a + 4b <= 7$;$5a + 3b <= 15$;and a, b are integers satisfying non-negativity constraints.

6. The Gardner Theater, a community playhouse, needs to determine the lowest-cost production budget for an upcoming show. Specifically, they have to determine which set pieces to construct and which, if any, set pieces to rent from another local theater at a predetermined fee. However, the organization has only two weeks to fully construct the set before the play goes into technical rehearsals. The theater has two part-time carpenters who work up to 12 hours a week each at \$10 an hour. Additionally, the theater has a part-time scenic artist who can work 15 hours per week to paint the set and props as needed at a rate of \$15 per hour.

   The set design requires 20 flats (walls), 2 hanging drops with painted scenery, and 3 large wooden tables (props). The number of hours required for each piece for carpentry and painting is shown below:

Carpentry	Painting	
Flats	0.5	2.0
Hanging Drops	2.0	12.0
Props	3.0	4.0

   Flats, hanging drops, and props can also be rented at a cost of \$75, \$500, and \$350 each, respectively. How many of each units should be built by the theater and how many should be rented to minimize total costs?

7. Van Nostrand Hospital must schedule nurses so that the hospital's patients are provided with adequate care. At the same time, in the face of tighter competition in the health-care industry, careful attention must be paid to keeping costs down. From historical records, administrators can project the *minimum* number of nurses to have on hand for the various times of day and days of the week. The nurse-scheduling problem seeks to find the minimum total number of nurses required to provide adequate care. Nurses start work at the beginning of one of the 4-hour shifts given next and work for 8 hours. Formulate and solve the nurse-scheduling problem as an integer program for one day for the data below.

Shift	Time	Minimum Number of Nurses Needed
1	12:00 A.M.–4:00 A.M.	5
2	4:00 A.M.–8:00 A.M.	12
3	8:00 A.M.–12:00 P.M.	14
4	12:00 P.M.–4:00 P.M.	8
5	4:00 P.M.–8:00 P.M.	14
6	8:00 P.M.–12:00 A.M.	10

8. Joe is an active 26-year-old male who lifts weights 6 days a week. His rigorous training program requires a diet that will help his body recover efficiently. He is also a graduate student who is looking to minimize the cost of consuming his favorite foods. Joe is trying to gain weight, or at least maintain his current body weight so he is not concerned about calories. His personal trainer suggests at least 300 grams of protein, 95 grams of fat, 225 grams of carbohydrates, and no more than 110 grams of sodium per day. His favorite foods are all items that he is familiar with preparing as shown in the table below. He is willing to consume multiple servings of each food per day to meet his requirements, although he cannot eat more than one steak per day and does not want to eat more than three pulled pork sandwiches a day. He needs to consume at least two servings of broccoli per day, and one serving of carrots but is willing to eat two servings of carrots if necessary. Joe likes a certain brand of nutrition bars, but he would not eat more than one. Unless previously noted, he does not want more than five servings of any one food. How many servings of each food should he have in an optimal daily diet?

Food	Protein (grams)	Fat (grams)	Carbohydrates (grams)	Sodium (grams)	Cost/Serving	Max Servings
Chicken Breast	40	10	2	6	$4.99	5
Steak	49	16	3	11	$8.99	1
Pulled Pork Sandwich	27	16	27	19	$3.99	3
Salmon Filet	39	15.5	1	5	$5.15	5
Rolled Oats	9	1	27	9	$0.80	5
Baked Potato	4	0	34	18	$1.50	5
Nutrition Bar	19	18	17	3	$3.00	1
Serving of Broccoli	2	0	6	2	$0.50	5
Serving of Carrots	1	1	7	2	$0.50	2

9. Jubilee Works has three types of jobs in one of its plant and needs to assign it to three men. Each assignment of a man to a job fetches different cost of performing that job. The data below shows the cost matrix ($) for assignment of the three men to three jobs.

	J1	J2	J3
M1	5	8	9
M2	6	7	11
M3	8	9	10

Use this data to construct and solve an integer optimization model for finding the assignments to minimize costs.

10. A building contractor has just won a contract to build a municipal library building. His present labor work force is inadequate to take this work immediately as he has already got other jobs on hand. Therefore he can either hire new labor on full-time basis (for 8-hours day each) at $80 per day or allow over time to existing labor (for 5-hour day each) which will cost $ 110 per day. The contractor wants to limit his extra payment to $ 1000 per day and utilize no more than 20 laborers (either full-time or part-time) because of limited supervision. He estimates that new labor employed will generate $30 per day as profit while an overtime worker will generate $50 per day. Develop an integer optimization model and solve it to aid the building contractor in deciding optimal labor mix.

11. Fuller Legal Services wants to determine how much time to allocate to four different services: business consulting, criminal work, nonprofit consulting, and wills/trusts. Mr. Fuller has determined the average hourly fees and the minimum and maximum hours (for consulting and criminal work) and cases (for wills/trusts) that he would like to spend on each. He has no shortage of demand for his services. The relevant data are shown below:

	Billables/hr	Minimum Hours	Maximum Hours
Business Consulting	$200.00	30.00	45.00
Criminal Work	$150.00	20.00	100.00
Nonprofit Consulting	$100.00	35.00	70.00

	Billables/Client	Minimum Cases	Maximum Cases	Hours/Case	Hours Worked per Month
Wills/Trusts	$3,000.00	2.00	6.00	17	200.00

Develop an optimization model to maximize monthly revenue.

12. Four items are considered for loading on an airplane, which has a capacity to load up to 25 metric ton. The weights and values of the items are provided in the table. Which items and what quantities should be loaded onto the plane so as to maximize the value of the cargo transported? Formulate this as integer optimization model.

Item	a	b	c	d
Weight (tons)	2	7	5	3
Value (per unit)	10	36	25	14

13. A software-support division of Blain Information Services has eight projects that can be performed. Each project requires different amounts of development time and testing time. In the coming planning period, 1,150 hours of development time and 900 hours of testing time are available, based on the skill mix of the staff. The internal transfer price (revenue to the support division) and the times required for each project are shown in the table. Which projects should be selected to maximize revenue?

Project	Development Time	Testing Time	Transfer Price
1	80	67	$23,520
2	248	208	$72,912
3	41	34	$12,054
4	10	92	$32,340
5	240	202	$70,560
6	195	164	$57,232
7	269	226	$79,184
8	110	92	$32,340

14. The Kelmer Performing Arts Center offers a series of four programs that includes jazz, bluegrass, folk, classical, and comedy. The Program Coordinator needs to determine which acts to choose for next year's series. She assigned an "impact" rating to each artist that reflects how well the act meets the Center's mission and provides community value. This rating is on a scale from 1 to 4, with 4 being the greatest impact and 1 being the least impact. The theater has 500 seats with an average ticket price of $12. Based on an estimate of the potential sales, the revenue from each artist is calculated. The center has a budget of $20,000 and would like the total impact factor to be at least 12, reflecting an average impact per artist of at least 3. To avoid duplication of genres, at most one of artists 2, 7, and 9 may be chosen, and at most one of artists 3 and 6 may be chosen. Finally, the center wishes to maximize its revenue. Data are shown below.

Develop and solve an optimization model to find the best program schedule to maximize the total profit.

Artist	Cost	Impact	Ticket Estimate
1	$7,000.00	3	350
2	$975.00	4	500
3	$1,500.00	3	350
4	$5,000.00	3	400
5	$8,000.00	2	400
6	$1,500.00	3	300
7	$6,500.00	4	500
8	$3,000.00	2	350
9	$2,500.00	4	400

15. Dannenfelser Design works with clients in three major project categories: architecture, interior design, and combined. Each type of project requires an estimated number of hours for different categories of employees, as shown in the following table.

	Architecture	Interior Design	Combined	Hourly Rate
Principal	15	5	18	$150
Sr. designer	25	35	40	$110
Drafter	40	30	60	$75
Administrator	5	5	8	$50

In the coming planning period, 184 hours of principal time, 414 hours of senior designer time, 588 hours of drafter time, and 72 hours of administrator time are available. Revenue per project averages $12,900 for architecture, $11,110 for interior design, and $18,780 for combined projects. The firm would like to work on at least one of each type of project for exposure among clients. Assuming that the firm has more demand than they can possibly handle, find the best mix of projects to maximize profit.

16. Anya is a part-time business student who works full time and is constantly on the run. She recognized the challenge of eating a balanced diet and wants to minimize cost while meeting some basic nutritional requirements. Based on some research, she found that a very active woman should consume 2,250 calories per day. According to one author's guidelines, the following daily nutritional requirements are recommended in the table at the right.

Source	Recommended Intake (Grams)
Fat	Maximum 75
Carbohydrates	Maximum 225
Fiber	Maximum 30
Protein	At least 168.75

Food	Cost/Serving	Calories	Fat	Carbs	Fiber	Protein
Turkey sandwich	$4.69	530	14	73	4	28
Baked-potato soup	$3.39	260	16	23	1	6
Whole-grain chicken sandwich	$6.39	750	28	83	10	44
Bacon turkey sandwich	$5.99	770	28	84	5	47
Southwestern refrigerated chicken wrap	$3.69	220	8	29	15	21
Sesame chicken refrigerated chicken wrap	$3.69	250	10	26	15	26
Yogurt	$0.75	110	2	19	0	5
Raisin bran with skim milk	$0.40	270	1	58	8	12
Cereal bar	$0.43	110	2	22	0	1
1 cup broccoli	$0.50	25	0.3	4.6	2.6	2.6
1 cup carrots	$0.50	55	0.25	13	3.8	1.3
1 scoop protein powder	$1.29	120	4	5	0	17

She chose a sample of meals in the table above that could be obtained from healthier quick-service restaurants around town as well as some items that could be purchased at the grocery store.

Anya does not want to eat the same entrée (first six foods) more than once each day but does not mind eating breakfast or side items (last five foods) twice a day and protein powder-based drinks up to four times a day, for convenience. Develop an integer linear optimization model to find the number of servings of each food choice in a daily diet to minimize cost and meet the nutritional targets.

17. Josh Steele manages a professional choir in a major city. His marketing plan is focused on generating additional local demand for concerts and increasing ticket revenue and also gaining attention at the national level to build awareness of the ensemble across the country. He has $20,000 to spend on media advertising. The goal of the advertising campaign is to generate as much local recognition as possible while reaching at least 4,000 units of national exposure. He has set a limit of 100 total ads. Additional information is shown next.

Media	Price	Local Exposure	National Exposure	Limit
FM radio spot	$80.00	110	40	30
AM radio spot	$65.00	55	20	30
Cityscape ad	$250.00	80	5	24
MetroWeekly ad	$225.00	65	8	24
Hometown paper ad	$500.00	400	70	10
Neighborhood paper ad	$300.00	220	40	10
Downtown magazine ad	$55.00	35	0	15
Choir journal ad	$350.00	10	75	12
Professional organization magazine ad	$300.00	20	65	12

The last column sets limits on the number of ads to ensure that the advertising markets do not become saturated.

a. Find the optimal number of ads of each type to run to meet the choir's goals by developing and solving an integer optimization model.

b. What if he decides to use no more than six different types of ads? Modify the model in part (a) to answer this question.

18. Timberland Inc. produces cars and has 4 plants and 6 sales depots. The data below depicts the transportation cost (of moving the car from plant to sales depot), fixed cost, and demand schedule:

Plant	1	2	3	4	5	6	Production units	Fixed costs
1	80	15	30	70	40	120	40	430
2	60	85	35	10	20	60	30	300
3	20	70	20	15	30	40	50	370
4	40	30	22	30	26	100	45	180
Demand units	20	10	15	7	9	25	86/165	1280

The total production is 165 cars and the demand is 86 cars. Since production is more than demand, management wishes to shut down some plants if required. Develop an integer optimization model to determine where to setup plants and sales depots (production and distribution system) such that cost is minimized.

19. Cady Industries produces custom induction motors for specific customer applications. Each motor can be configured from different options for horsepower, the driveshaft forming process, spider bar component material, rotor plate process, type of bearings, tophat (a system of channels encased in a box that is placed on top of the motor to reduce airflow velocity both entering and exiting the motor) design, torque direction, and an optional mounting base.

	Cost	Time Requirement (Days)
Horsepower		
1000 HP	$155,000	32
5000 HP	$165,000	36
10000 HP	$180,000	42
15000 HP	$205,000	50
Shaft		
Heat-Rolled	$10,000	10
Oil-Quenched	$5,000	16
Forged	$15,000	8
Spider Bar Material		
Copper	$10,000	4
Aluminum	$2,500	8
Rotor Plates		
Laser-Cut	$12,500	5
Machine-Punched	$7,500	12
Bearings		
Sleeve	$5,000	4
Anti-Friction	$5,000	4
Oil Well	$3,000	2
Oil guard	$5,000	4
Tophat Design		
Box	$5,000	15
V-Box	$20,000	15
Torque Direction		
Vertical	$35,000	10
Horizontal	$40,000	6
Optional Base	$75,000	10

Copper spider bars are required on 10,000 and 15,000 horsepower motors. If a V-box tophat is required, a horizontal torque direction must be used. Finally, if the optional base is required, a horizontal torque direction must be chosen.

a. Develop and solve an optimization model to find the minimum cost configuration of a motor.

b. Develop and solve an optimization model to find the configuration that can be completed in the shortest amount of time.

c. Customer A has a new plant opening in 90 days and needs a motor with at least 5,000 horsepower. The customer has specified that sleeve bearings be installed for easy maintenance and a V-box tophat is required to meet airflow velocity limitations. Find the optimal configuration that can be built within the 90-day requirement.

d. Customer B has a budget of $365,000 and requires a motor with 15,000 horsepower, a heat-rolled shaft, and the optional base. They want the highest-quality product, which implies that they are willing to maximize the cost up to the budget limitation. Find the optimal configuration that will meet these requirements.

20. For the *General Appliance Corporation* transportation model discussed in Chapter 14, suppose that the company wants to enforce a single sourcing constraint that each distribution center be served from only one plant. Assume that the capacity at the Marietta plant is 1,500. Set up and solve a model to find the minimum cost solution.

21. For the Shafer Office Supplies problem (Problem 15 in Chapter 14), suppose that the company wants to enforce a single sourcing constraint that each retail store be served only from one distribution center. Set up and solve a model to find the minimum cost solution.

22. Premier Paints supplies to major contractors. One of their contracts for a specialty paint requires them to supply 750, 500, 400, and 950 gallons over the next 4 months. To produce this paint requires a shutdown and cleaning of one of their manufacturing departments at a cost of $1,000. The entire contract requirement can be produced during the first month in one production run; however, the inventory that must be held until delivery costs $0.75 per gallon per month. If the paint is produced in other months, then the cleaning costs are incurred during each month of production. Formulate and solve an integer optimization model to determine the best monthly production schedule to meet delivery contracts and minimize total costs.

23. Chris Corry has a company-sponsored retirement plan at a major brokerage firm. He has the following funds available:

Fund	Risk	Type	Return
1	High	Stock	11.98%
2	High	Stock	13.18%
3	High	Stock	9.40%
4	High	Stock	7.71%
5	High	Stock	8.35%
6	High	Stock	16.38%
7	Medium	Blend	4.10%
8	Medium	Blend	12.52%
9	Medium	Blend	8.62%
10	Medium	Blend	11.14%
11	Medium	Blend	8.78%
12	Low	Blend	9.44%
13	Low	Blend	8.38%
14	Low	Bond	7.65%
15	Low	Bond	6.90%
16	Low	Bond	5.53%
17	Low	Bond	6.30%

His financial advisor has suggested that at most 40% of the portfolio should be composed of high-risk funds. At least 25% should be invested in bond funds, and at most 40% can be invested in any single fund. At least six funds should be selected, and if a fund is selected, it should be funded with at least 5% of the total contribution.

Develop and solve an integer optimization model to determine which funds should be selected and what percentage of his total investment should be allocated to each fund.

24. The Spurling Group is considering using magazine outlets to advertise their online Web site. The company has identified seven publishers. Each publisher breaks down its subscriber base into a number of groups based on demographics and location. These data are shown in the table.

Publisher	Groups	Subscribers/Group	Cost/Group
A	5	460,000	$1,560
B	10	50,000	$290
C	4	225,000	$1,200
D	20	24,000	$130
E	5	1,120,000	$2,500
F	1	1,700,000	$7,000
G	2	406,000	$1,700

The company has set a budget of $25,000 for advertising and wants to maximize the number of subscribers exposed to their ads. However, publishers B and D are competitors and only one of these may be chosen. A similar situation exists with publishers C and G. Formulate and solve an integer optimization model to determine which publishers to select and how many groups to purchase for each publisher.

25. Tunningley Services is establishing a new business to serve customers in the Ohio, Kentucky, and Indiana region around the Cincinnati Ohio area. The company has identified 15 key market areas and wants to establish regional offices to meet the goal of being able to travel to all key markets within 60 minutes. The data file *Tunningley.xlsx* provides travel times in minutes between each pair of cities.

a. Develop and solve an optimization model to find the minimum number of locations required to meet their goal.

b. Suppose they change the goal to 90 minutes. What would be the best solution?

26. Tindall Bookstores is a major national retail chain with stores located principally in shopping malls. For many years, the company has published a Christmas catalog that was sent to current customers on file. This strategy generated additional mail-order business, while also attracting customers to the stores. However, the cost-effectiveness of this strategy was never determined. In 2008, John Harris, vice president of marketing, conducted a major study on the effectiveness of direct-mail delivery of Tindall's Christmas catalog. The results were favorable: Patrons who were catalog recipients spent more, on average, than did comparable nonrecipients. These revenue gains more than compensated for the costs of production, handling, and mailing, which had been substantially reduced by cooperative allowances from suppliers.

With the continuing interest in direct mail as a vehicle for delivering holiday catalogs, Harris continued to investigate how new customers could most effectively be reached. One of these ideas involved purchasing mailing lists of magazine subscribers through a list broker. To determine which magazines might be more appropriate, a mail questionnaire was administered to a sample of current customers to ascertain which magazines they regularly read. Ten magazines were selected for the survey. The assumption behind this strategy is that subscribers of magazines that a high proportion of current customers read would be viable targets for future purchases at Tindall stores. The question is which magazine lists should be purchased to maximize reaching of potential customers in the presence of a limited budget for purchasing lists.

Data from the customer survey have begun to trickle in. The information about the 10 magazines to which a customer subscribes is provided on the returned questionnaire. Harris has asked you to develop a prototype model, which later can be used to decide which lists to purchase. So far only 53 surveys have been returned. To keep the prototype model manageable, Harris has instructed you to go ahead with the model development using the data from the 53 returned surveys. These data are shown in Table 15.4. The costs of the first 10 lists are given next, and your budget is $3,000.

Data for Tindall Bookstores Survey

List	1	2	3	4	5	6	7	8	9	10
Cost (000)	$1	$1	$1	$1.5	$1.5	$1.5	$1	$1.2	$0.5	$1.1

a. What magazines should be chosen to maximize overall exposure?

b. Conduct a budget sensitivity analysis on the Tindall magazine list–selection problem. Solve the problem for a variety of budgets and graph percentage of total reach (number reached/53) versus budget amount. As an analyst, make a recommendation as to when an increment in budget is no longer warranted.

Table 15.4
Survey Results

Customer	Magazines	Customer	Magazines
1	10	28	4, 7
2	1, 4	29	6
3	1	30	3, 4, 5, 10
4	5, 6	31	4
5	5	32	8
6	10	33	1, 3, 10
7	2, 9	34	4, 5
8	5, 8	35	1, 5, 6
9	1, 5, 10	36	1, 3
10	4, 6, 8, 10	37	3, 5, 8
11	6	38	3
12	3	39	2, 7
13	5	40	2, 7
14	2, 6	41	7
15	8	42	4, 5, 6
16	6	43	None
17	4, 5	44	5, 10
18	7	45	1, 2
19	5, 6	46	7
50	2, 8	47	1, 5, 10
21	7, 9	48	3
22	6	49	1, 3, 4
23	3, 6, 10	50	None
24	None	51	2, 6
25	5, 8	52	None
26	3, 10	53	2, 5, 8, 9, 10
27	2, 8		

Case: Performance Lawn Equipment

PLE produces its most popular model of lawn tractor in its Kansas City and Santiago plants and ships these units to major distribution centers in Atlanta, Caracas, Melbourne, Mexico City, London, Shanghai, and Toronto. Unit shipping costs can be found in the PLE database. Both the Kansas City and the Santiago plants have a maximum annual capacity of 60,000 units. Long-term forecasts of annual demands at the distribution centers within 5 years that PLE wants to plan for are

Atlanta—60,000

Caracas—10,000

Melbourne—6,000

Mexico City—4,000

London—40,000

Toronto—5,000

Shanghai—50,000

To support its growing sales, PLE is considering adding additional plants. The capacities of the proposed plants and the fixed costs of construction can be found in the PLE database. If a new plant is constructed, only one of the two potential capacities can be considered. Locations being considered are Birmingham, Alabama; Singapore; Frankfurt, Germany; Mumbai, India; and Auckland, New Zealand. Other options are to increase the capacities of the existing plants in Kansas City and Santiago. Fixed costs of constructing new facilities or expanding the existing plants can be found in the PLE database. Develop and solve an optimization model to identify the best location for the new plants and transportation allocations to meet demand. Some members of the executive committee are concerned that the estimate for the China market (demand at Shanghai) is too uncertain and may range from 20,000 to 60,000 units. In addition, it was suggested that the capacity of the Kansas City plant be reduced to save distribution costs. Write a report explaining your solution and any recommendations you may develop after conducting appropriate sensitivity analyses with the model to address the concerns of the executive committee.

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SUPPLEMENTARY CHAPTER A

Nonlinear and Non-Smooth Optimization

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This supplementary chapter is available online at www.pearsonhighered.com/evans

Learning Objectives

After studying this chapter, you will be able to:

- Recognize when to use nonlinear optimization models.
- Develop and solve nonlinear optimization models for different applications.
- Interpret *Solver* reports for nonlinear optimization.
- Use empirical data and line-fitting techniques in nonlinear optimization.
- Recognize a quadratic optimization model.
- Identify non-smooth optimization models and when to use *Evolutionary Solver*.
- Formulate and solve sequencing and scheduling models using *Solver*'s *alldifferent* constraint.

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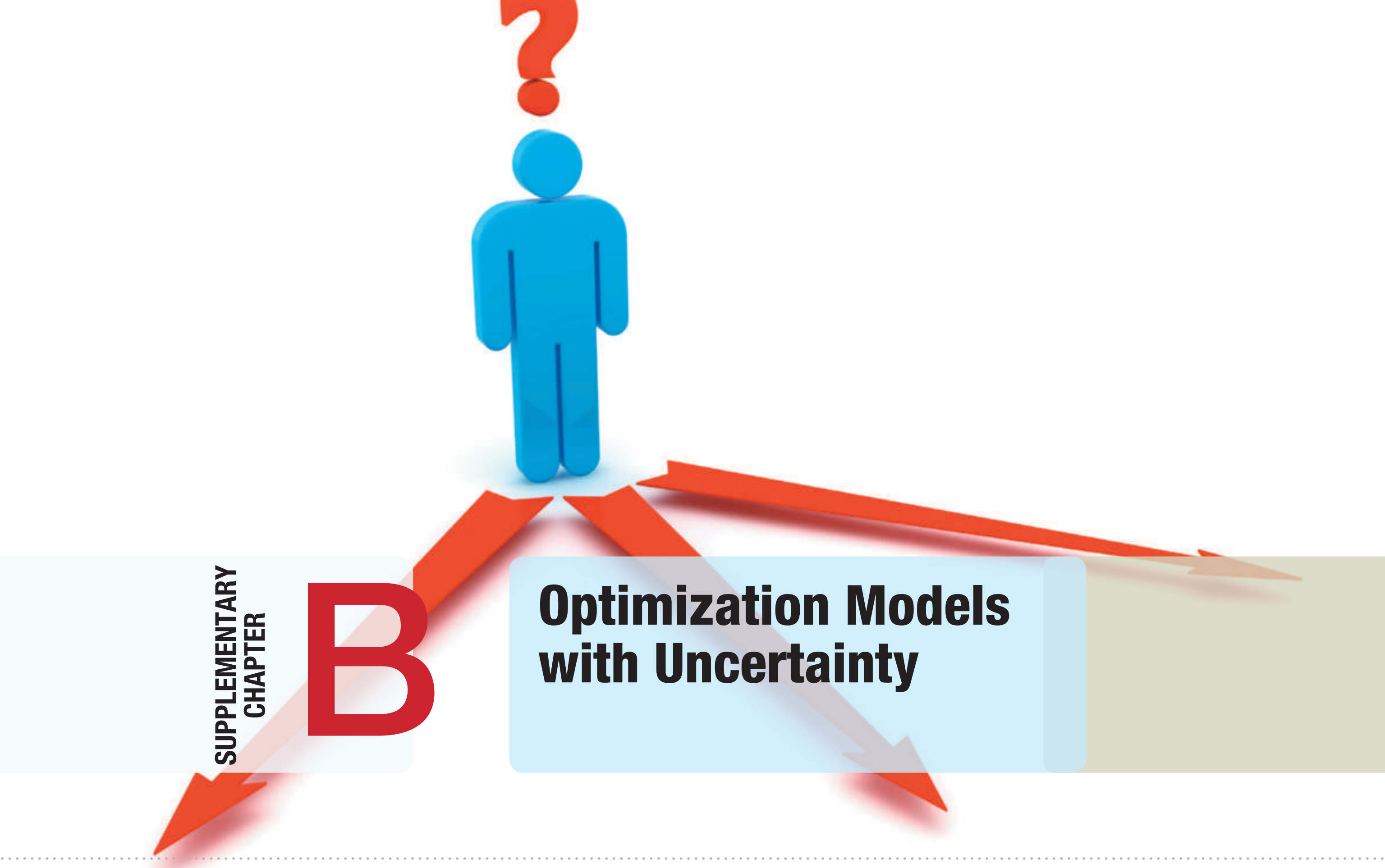

SUPPLEMENTARY CHAPTER

B Optimization Models with Uncertainty

Tim Arbaev/Shutterstock.com

This supplementary chapter is available online at www.pearsonhighered.com/evans

Learning Objectives

After studying this chapter, you will be able to:

- Evaluate risk in solutions to optimization models using Monte Carlo simulation.
- Solve optimization models with chance constraints.
- Use multiple parameterized simulations in *Analytic Solver Platform* to find optimal solutions in simulation models with decision variables.
- Use *Analytic Solver Platform* to combine simulation modeling and optimization to maximize or minimize the expected value of a model output.
- Incorporate uncertainty into optimization models such as project selection.