Math Type-up

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Abstract

Given a Hamiltonian system in R^2 with nonzero degenerate quadratic part, our goal is to classify the critical point at (0,0) of said system in both stable and weakly unstable cases. For the stable case, the ideal result is to show there exists some symplectic transformation τ such that $H \circ \tau = \xi^2 + a(x)$. If not possible, we'd like to show the function is similar up to an error dependent on x only. $H \circ \tau = (\xi^2 + x^{2m})(1 + O(|x|)$ or if not possible, we are left with dependence on ξ too $H \circ \tau = (\xi^2 + x^{2m})(1 + O(|x| + |\xi|)$.

1 Introduction

We will begin by establishing notation and some assumptions on the Hamiltonian.

1.1 Assumptions

- H analytic function with degenerate quadratic part, possibly polynomial, of x and ξ in R^2
- For the stable case, $H > 0$ at all points near the origin. Because of this assumption, there must exist some even dimensional term of x only: If every term has dependence on ξ, then $x = \epsilon, \xi = 0 \Rightarrow H = 0$ breaking assumption. Take the lowest degree x-only term, as we shown one exists. This term must be of even degree because if not, $x = -\epsilon, \xi = 0 \Rightarrow H < 0$ for small enough ϵ.
- For the unstable case, H can equal 0 near the origin.
- Use canonical symplectic form $\omega = dx \wedge d\xi$, or in matrix form, $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- hello

1.2 Notation

- Let $H(x, \xi) = H_2 + H_3 + H_4 + ... = \xi^2 + H_3 + H_4 + ...$ where H_i are homogeneous functions of degree i. $H_2 = \xi^2$ makes the quadratic part degenerate, with matrix form $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Observe, the determinant of the matrix is 0.
- Using the same notation from [Hofer], $\tau = exp(X_p)$ with X_p being the time 1 map of a Hamiltonian vector field, which is therefore a symplectic transformation.
- Define repeated applications of the Hamiltonian field as X_p^n, where for example, $X_p^3 H = X_p(X_p(X_p H))$. $n = 0$ is equivalent to the identity. When taylor expanded,

  $$H \circ \tau = H(e^{X_p}) = H + X_p H + \frac{X_p^2 H}{2} + ... = \sum_{n=0}^{\infty} \frac{X_p^n H}{n!}$$

  Equivalently, this can be written as a closed form recursively with poisson brackets,

  $$K_n = \frac{1}{n}\{P, K_{n-1}\}, K_0 = H, \, H \circ \tau = \sum K_n$$

- Denote $a^n(x)$ as a polynomial function of x up to degree n. For example $a^3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ where a_i are real numbers.
- hello

2 Examples, Theorems, Stuff

We will show the general procedure for pushing all terms of a degree to a higher order.

2.1 General procedure assuming even degree x only term is x^4

Let $H = \xi^2 + x^4 + H_3 + H_4 + ...$, H can be a polynomial in which case "..." ends Recall $X_H = H_\xi \partial x - H_x \partial \xi$ and $X_H p = -X_p H$

Construct symplectic transformation, τ_3, such that $H \circ \tau_3 = \xi^2 + ax^3 + H_4 + ...$
i.e, we want to push back every term in H_3 that has dependence on ξ. In this example, $ax^3 = 0$ because of the assumption on smallest x only term assumption. By linearity of the map and structure of H, $H \circ \tau = \xi^2 + H_3 + X_p(H_2) + h.o.t.$ Therefore, we want $H_3 = -X_p(H_2) = X_{H_2}p$
$H_2 = \xi^2, p = C_{3,0}x^3 + C_{2,1}x^2\xi + C_{1,2}x\xi^2 + C_{0,3}\xi^3$
$X_{H_2}p = (2\xi\partial x)(p) = 6C_{3,0}x^2\xi + 4C_{2,1}x\xi^2 + 2C_{1,2}\xi^3$

Therefore there exists constants $C_{3,0}, C2, 1, C1, 2$ s.t $X_{H_2}p = H_3$. $H \circ \tau_3 = H + X_pH + h.o.t = \xi^2 + x^4 + H4 + H5 + h.o.t.$ Note H_4 and so on may have changed from the original after.

For a general degree H_k to be pushed forward. The hamiltonian would be generated from a polynomial of degree k, and there exists coefficients such that $X_p(H_2) = H_k$ because this is equivalent to $2\xi\partial_x(p_k) = H_k$, and by existence of ODEs there exists p_k that satisfies this. The degrees have to pushed back sequentially however as each τ_k can generate new terms of degree k+1 and higher. Therefore for a given k, we can push the Hamiltonian to $H \circ \tau_3 \circ .. \circ \tau_k = \xi^2 + a^k(x) + H_{k+1} + H_{k+2} + ...$

3 Henry's stuff

3.1 Claim: $X_p^r H, r \geq 0$ creates new polynomials with the same number of terms or less

Let H be a polynomial with $v+3, v \geq 0$ terms and following the same assumptions from before.

$$\text{H} = \xi^2 + x^{2n} + c_i x^j \xi^k + \textstyle\sum_{q=1}^v c_{i_q} x^{j_q} \xi^{k_q} = B_0\xi^2 + A_0 x^{2n} + C_0 x^j \xi^k + \sum_{q=1}^v D_{0,q} x^{j_q} \xi^{k_q}$$

WLOG, let us crank out $c_i x^j \xi^k$. Therefore,

$$p = \tfrac{ci}{2(j+1)} x^{j+1}\xi^{k-1}, \quad p_x = \tfrac{c_i}{2} x^j \xi^{k-1}, \quad p_\xi = \tfrac{c_i(k-1)}{2(j+1)} x^{j+1}\xi^{k-2}.$$

$$\begin{aligned} X_pH = &\tfrac{c_i(k-1)}{2(j+1)} x^{j+1}\xi^{k-2}(2nx^{2n-1} + kc_i x^{j-1}\xi^k + \textstyle\sum_{q=1}^v j_q c_{i_q} x^{j_q-1}\xi^{k_q}) \\ &- \tfrac{c_i}{2} x^j \xi^{k-1}(2\xi + jc_i x^j \xi^{k-1} + \textstyle\sum_{q=1}^v j_q c_{i_q} x^{j_q}\xi^{k_q-1}) \end{aligned}$$

Rename coefficients for simplicity

$$= A_1 x^{2n+j}\xi^{k-2} + B_1 x^k \xi^j + C_1 x^{2j}\xi^{2k-2} + \textstyle\sum_{q=1}^v D_{1,q} x^{j_q+j}\xi^{k_q+(k-2)}$$

Notice this creates the same number of terms as the original H. Note also that coefficients can zero out leading to less terms than the original.
Suppose the claim is true for some int r. Therefore

$$X_p^r H = A_r x^a \xi^b + B_r x^c \xi^d + C_r x^e \xi^f + \textstyle\sum_{q=1}^n D_{r,q} x^g \xi^h$$

$$\begin{aligned} X_p^{r+1}H = &\tfrac{c_i(k-1)}{2(j+1)} x^{j+1}\xi^{k-2}((aA_r x^{a-1}\xi^b + cB_r x^{c-1}\xi^d + eC_r x^{e-1}\xi^f + + \textstyle\sum_{q=1}^v gD_{r,q} x^{g-1}\xi^h) - \\ &\tfrac{c_i}{2} x^j \xi^{k-1}(bA_r x^a \xi^{b-1} + dB_r x^c \xi^{d-1} + fC_r x^e \xi^{f-1} + \textstyle\sum_{q=1}^v hD_{r,q} x^g \xi^{h-1}) \end{aligned}$$

$$= A_{r+1} x^{a+j}\xi^{b+j-2} + B_{r+1} x^{c+j}\xi^{d+k-2} + C_{r+1} x^{e+j}\xi^{e+k-2} + \textstyle\sum_{q=1}^v D_{r+1,q} x^{g+j}\xi^{h+k-2}$$

Therefore by induction, $X_p^r H$ always has the same number of terms or less. Moving forward, since each new term comes from a part of the old term, we look at the growth of coefficients of each term separately. In future proofs, we will name the terms "A term" or "C term" referencing the terms with coefficients A_i or C_i respectively, with C term being a fully generic term, A and B terms more specific to show connection to our Hamiltonian. And WLOG, we will also only look at 4 terms or less as the summation of all the terms behave the same, and linearity of applying each X_p allows us to look at each term in isolation.

3.2 Claim: all mixed terms with just ξ^1 can be cranked out in a symplectic transformation that ends.

$$\text{Given } H = \xi^2 + x^{2n} + ax^m\xi \Rightarrow$$

$$p = \tfrac{a}{2(m+1)}x^{m+1}$$
$$\dot{x} = 0, \dot{\xi} = -\tfrac{a}{2}x^m,$$
$$x(0) = y, \xi(0) = \eta \Rightarrow$$

$$x(t) = y, \xi(t) = -\tfrac{a}{2}x^m t + \eta \Rightarrow$$

$$H \circ \tau = (-\tfrac{a}{2}x^m t + \eta)^2 + y^{2n} + ay^m(-\tfrac{a}{2}x^m t + \eta) =$$

$$\tfrac{a^2}{4}y^{2m} - a\eta y^m + \eta^2 + y^{2n} - \tfrac{a^2}{2}y^{2m} + a\eta y^m =$$

$$\tfrac{3a^2}{4}x^{2m} + \xi^2 + x^{2n}$$
(after renaming variables back to x and ξ).

3.3 Claim: Can apply transformations one by one to crank out terms

Trivial from linearity of the transformations we are applying and how they only affect the same degree H one time, to crank out the one we want. *maybe would be good to show*

3.4 Claim: The coefficients of each term can be defined recursively, and these terms can fully be found recursively as well.

For term $C_0x^j\xi^k$, when cranking out $ax^q\xi^r$, with r ≥ 2, given r = 0 cannot be cranked out, and r = 1 can be cranked out in an ending transformation as seen above. we will show the corresponding term in $X_p^{m+1}H$ is

$$C_{m+1}x^?\xi^? = C_m(\tfrac{a(r-1)(degree(x) - a(q+1)(degree(\xi))}{2(q+1)})x^{degree(x)+q}\xi^{degree(\xi)+r-2},$$

where degree(x) and degree(xi) refers to the degree of x and ξ from the m-th term. Because degrees of x and ξ grow linearly this is equivalent to

$$C_m(\tfrac{a(r-1)(j+(m)q) - a(q+1)(k+m(r-2))}{2(q+1)})x^{j+(m+1)q}\xi^{k+(m+1)(r-2)} = C_{m+1}x^{j+(m+1)q}\xi^{k+(m+1)(r-2)}$$

Given $H = B_0\xi^2 + A_0x^{2n} + C_0x^j + \xi^k + ax^q\xi^r$, and WLOG, cranking out $ax^q\xi^r$ Note, normally $B_0 = 1$, and H can be re-scaled to ensure this, but use A_0 for clarity.

$$p = \tfrac{a}{2(q+1)}x^{q+1}\xi^{r-1}, \qquad p_\xi = \tfrac{a(r-1)}{2(q+1)}x^{q+1}\xi^{r-2}, \qquad p_x = \tfrac{a}{2}x^q\xi^{r-1}$$

$$X_pH = \tfrac{a(r-1)}{2(q+1)}x^{q+1}\xi^{r-2}(2nA_0x^{2n-1} + jC_0x^{j-1}\xi^k + qax^{q-1}\xi^r)$$
$$-\tfrac{a}{2}x^q\xi^{r-1}(2B_0\xi + kC_ox^j\xi^{k-1} + rax^q\xi^{r-1})$$

$$= A_0(\tfrac{a(r-1)(2n+0q) - a(q+1)(0+0(r-2))}{2(q+1)})x^{2n+1q}\xi^{0+1(r-2)}$$

$$+B_0(\tfrac{a(r-1)(0+0q)-a(q+1)(2+0(r-2))}{2(q+1)})x^{0+1q}\xi^{2+1(r-2)}$$
$$+C_0(\tfrac{a(r-1)(j+0q)-a(q+1)(k+0(r-2))}{2(q+1)})x^{j+1q}\xi^{k+1(r-2)}$$
$$+D_0(\tfrac{a(r-1)(q+0q)-a(q+1)(r+0(r-2))}{2(q+1)})x^{q+1q}\xi^{r+1(r-2)},$$
$$D_0 = a$$

$$= A_1 x^{2n+1q}\xi^{0+1(r-2)} + B_1 x^{0+1q}\xi^{2+1(r-2)}$$
$$+C_1 x^{j+1q}\xi^{k+1(r-2)} + D_1 x^{q+1q}\xi^{r+1(r-2)}$$

Suppose the claim is true for some $m \geq 0$. Now that there are clear parallels for each term, WLOG, only look at C term, the most general term. The C term for $X_p^m H = C_m x^{j+mq}\xi^{k+m(r-2)}$. Now the first lemma above also shows that the C term in X_p^{m+1} depends only on the previous C term and is:

$$\tfrac{a(r-1)}{2(q+1)}x^{q+1}\xi^{r-2}(C_m(j+mq)x^{j+mq-1}\xi^{k+m(r-2)}) - \tfrac{a}{2}x^q\xi^{r-1}(C_m(k+m(r-2))x^{j+mq}\xi^{k+m(r-2)-1})$$

$$= C_m(\tfrac{a(r-1)(j+mq)-a(q+1)(k+m(r-2))}{2(q+1)})x^{j+(m+1)q}\xi^{k+(m+1)(r-2))}$$

$$= C_{m+1}x^{j+(m+1)q}\xi^{k+(m+1)(r-2))}$$

Therefore true for all integers m by induction.

3.5 Claim: Bounds for coefficients to ensure coefficients of higher degree Taylor polynomial terms converge to 0

We will show for an arbitrary term, $C_0 x^j \xi^k$, when undergoing the transformation for another arbitrary term (could be the same term), $ax^q\xi^r$,
if $|a| < |\frac{2(q+1)}{(r-1)q-(q+1)(r-2)}|$ coefficients will converge to 0, and if $(r-1)q-(q+1)(r-2) = 0$, then convergence occurs for all a.

$$\text{Let } r_m = \tfrac{a(r-1)(j+mq)-a(q+1)(k+m(r-2))}{2(q+1)} \Rightarrow C_{m+1} = C_m \cdot r_m.$$

Recall the Taylor expansion of $H \circ \tau = H + X_p H + \frac{X_p^2 H}{2!} + ... = \sum_{n=1}^{\infty} \frac{X_p^n H}{n!}$ includes n! in the denominator, so if coefficients in $X_p^n H$ grow slower than factorial rates, then coefficients will converge to 0. In other words,

$$|r_{m+1}| < |r_m| + 1$$

$$r_{m+1} = \tfrac{a(r-1)(j+(m+1)q)-a(q+1)(k+(m+1)(r-2))}{2(q+1)}$$

$$= \tfrac{a(r-1)(j+mq)-a(q+1)(k+m(r-2))+a(r-1)q+a(q+1)(r-2)}{2(q+1)}$$

$$= r_m + \tfrac{a(r-1)q+a(q+1)(r-2)}{2(q+1)} \Rightarrow$$

$$|r_{m+1}| = |r_m + \tfrac{a(r-1)q+a(q+1)(r-2)}{2(q+1)}| \leq |r_m| + |\tfrac{a(r-1)q+a(q+1)(r-2)}{2(q+1)}| \Rightarrow$$

$$|r_{m+1}| < |r_m| + 1 \Leftrightarrow |\tfrac{a(r-1)q+a(q+1)(r-2)}{2(q+1)}| < 1$$

$$\Leftrightarrow |a\tfrac{(r-1)q+(q+1)(r-2)}{2(q+1)}| < 1$$

$$\Leftrightarrow |a||\tfrac{(r-1)q+(q+1)(r-2)}{2(q+1)}| = 0 < 1,$$
if $[(r-1)q+(q+1)(r-2)] = 0$, or $q = r-2$

or

$$|a| < |\tfrac{2(q+1)}{(r-1)q+(q+1)(r-2)}|, \text{ when } q \neq r-2$$

Therefore when $q = r-2$, higher order coefficients always converge to 0. Otherwise, $|a| < |\frac{2(q+1)}{(r-1)q+(q+1)(r-2)}|$ is the necessary and sufficient condition.

3.6 Claim: All polynomials can be cranked out as desired

We have shown there exists a bound on coefficient for each term to be cranked out. By the claim above, we can partition each term with coefficients greater than the bound. Given term, $Cx^q\xi^q$. Let b = bound from the above derivation. if $|C| > b$ there exists some natural number k, s.t $|\frac{C}{k}| < b$, therefore the term can be partitioned into $\sum_{i=1}^{k} \frac{C}{k} x^q \xi^r$ and each term can be then cranked out. Note k is a natural number so by ALT, the sum of the coefficients of the higher order terms will also converge to 0. This will generate more higher order terms than before, but they will all still converge to 0, as the new terms don't matter, since convergence only depends on the cranked out term.

3.7 Claim: We can push H to $(\xi^2 + a^n(x))(1 + O(|x| + |\xi|)$ without shrinking domain

Given polynomial $H = \xi^2 + H_3 + ... + x^{2m} + H_{2m} + ... + H_k$ We know we can push past an arbitrary degree n with $\tau = \tau_3 \circ \tau_4 \circ ... \circ \tau_n$ constructed as described in this paper. $H \circ \tau = \xi^2 + a^n(x) + h.o.t.$ Let us look on some domain around (0,0) s.t $|x|, |\xi| < 1$.

Our purpose here is to examine the higher order terms in an attempt to factor it to a higher order. These higher order terms are all of degree n+1 or greater. Approach assuming H_{n+1} does not include $ax^n\xi$ because if it does, we have shown that that term can be pushed back to a x only term in an ending transformation. Therefore every term has either ξ^2, x^{n+1}, or both, simply from the degree of the terms. Every term with x^{n+1} factors into $a^n(x)$ and all remaining terms factor into ξ^2.

$$H = \xi^2(1 + a\xi + bx + c\xi^2 + dx^2 + ex\xi + ...) + a^n(x)(1 + f\xi + gx + ...)$$

[1]

By the restriction on the domain, each term $a|x|^i|\xi|^j \leq a|x||\xi| \leq |x| + |\xi|$ Because we know coefficients of higher order terms converge to 0 at a factorial speed, the series of coefficients converge to a number. [2] . Call that number m_1 and m_2 for the two series. Therefore each side can be factored into

$$H = \xi^2(1 + O(|x| + |\xi|)) + a^n(x)(1 + O(|x| + |\xi|))$$

$$= (\xi^2 + a^n(x)(1 + O(|x| + |\xi|)))$$

We would like to remove dependence on ξ in the error term.

4 Fun discoveries

- $ax^2\xi^2$ doesn't end, but creates 2 terms with ξ^2 after which all future terms only have x dependence.
- $ax^n\xi^2$ creates only terms with ξ^2 or x only terms, i.e dependence on ξ^2 is controlled and can be pushed back as much as desired.
- Hello

[1] My only real problem here is $a^n(x)$ doesn't factor as nicely as it looks. We could just do a^{2m} which will factor like that, but it should work once we get into big O notation, I'm just not sure how to get there.

[2] I'm pretty sure this is true by the ratio test. Even if the first k terms are growing, that's always finite and then the future terms converge according to ratio test.

5 References

Cite this better but just list for now I guess:

- Hofer, H., Zehnder, E. (1995). Symplectic invariants and Hamiltonian dynamics. In: Hofer, H., Taubes, C.H., Weinstein, A., Zehnder, E. (eds) The Floer Memorial Volume. Progress in Mathematics, vol 133. Birkhäuser Basel. https://doi.org/10.1007/978-3-0348-9217-9-21
- Ito, Hidekazu. "Convergence of Birkhoff normal forms for integrable systems.." Commentarii mathematici Helvetici 64.3 (1989): 412-461. ¡http://eudml.org/doc/140164¿. [3]

[3] Haven't actually used this yet so maybe don't need it