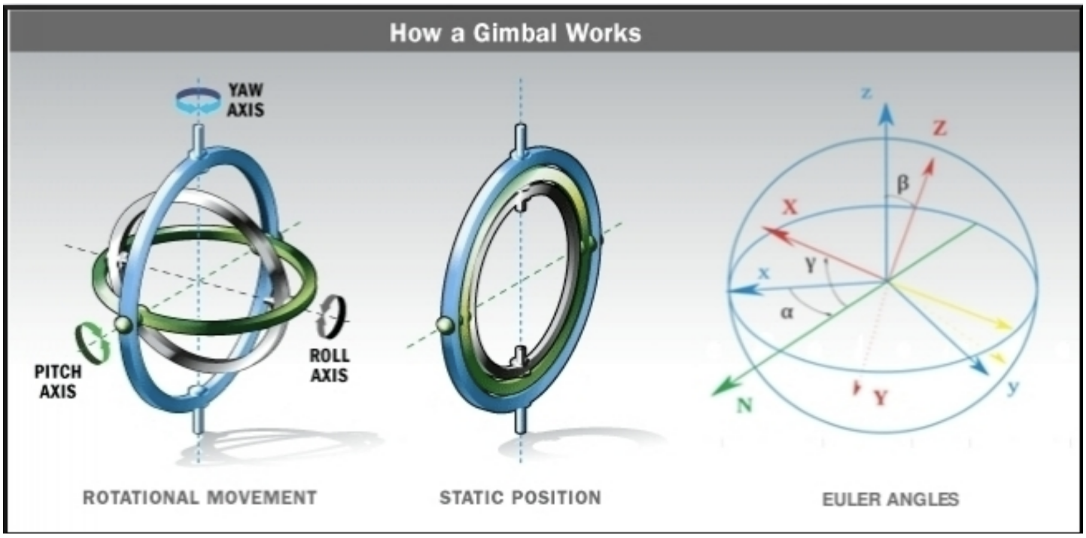
4160 Lecture 5

date: 2/6/2018

Recap

- Representing Rotations
  - $R = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \sin\theta + (I - uu^T)\cos\theta + uu^T$
  - $u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \|u\| = 1$
- Rodrigues Formula
  - $v_{rot} = v\cos\theta + (u \times v)\sin\theta + u(u \cdot v)(1 - \cos\theta)$
- Alternative 3
  - Euler Angle
  - 

  - $R_z(\theta_z)R_y(\theta_y)R_x(\theta_x)$

Rotation using Quaternions

- A quaternion for rotation
  - $q = [\cos\frac{\theta}{2}, \sin\frac{\theta}{2} n] = [s, (x, y, z)]$
  - Here n is the rotation axis(a unit vector)
  - A generalized complex number
  - $q = [s, v] = [s, (x, y, z)] = s + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$

- $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$
- $\mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}$
- $q_1 + q_2 = [s_1 + s_2, v_1 + v_2]$
- $q_1 q_2 = [s_1 s_2 - v_1 \cdot v_2, v_1 \times v_2 + s_1 v_2 + s_2 v_1]$
- Conjugate: $q = [s, v], q^* = [s, -v]$

- Algebraic Properties
  - identity: $I = [1, 0]$
  - 2-norm
  - inverse: $qq^{-1} = q^{-1}q = I$, $q^{-1} = q^*$
  - $a(bc) = (ab)c$
  - dot product

- Quaternion Exponential & Logarithm
  - $\log[\cos\theta, \sin\theta v] = [0, \theta v]$
  - $\exp([0, \theta v]) = [\cos\theta, \sin\theta v]$

- Arbitrary exponential
  - $q^t = \exp(t \log q)$
  - $q^a q^b = q^{a+b}$
  - $(q^a)^b = q^{ab}$
  - $qp^t q^* = (qpq^*)^t$

- Rotate with Quaternions
  - let $p = [0, r]$
  - $q = [\cos\theta, \sin\theta\mathbf{n}]$
  - rotate a vector r by 2θ around axis $\mathbf{n}$

$$p' = qpq^{-1}$$