Sets 2

Cantor

N IS UNCOUNTABLY INFINITE

Cantor came up with the diagonal argument (Snake trick with matrices), used to prove that a countably infinite union of countably infinites set is countably infinite, the idea that sizes of sets should be understood via bijections (A ~ B for A, B sets) as well as the notions of countably infinite and uncountably infinite.

PROOF: Assume A is countably infinite $\iff A = \{s_1, s_2, s_3\} where s_j = \{x_m{}^j\}_{m=1,2,...}$ for $x_m{}^j = 0 or x_m{}^j = 1$ We construct a sequence of 0's and 1's that cannot be in the enumeration $\{s_1, s_2, s_3....\}$.

s_0 differs from each s_j by the jth element => $s_0 \notin \{s_0, s_1, ...\}$, but s_0 is a sequence of 0's and 1's => $s_0 \in A => <=$

Corollary The power set P(N) of N is uncountably infinite

Remark Recall the proof that if B is a set with m elements #(B) = m, then its power set P(B) has 2^m elements based on the on/off idea. For each element of B we have the choice to include it in our subset ("on") or not to include it ("off"). Therefore we have 2 choices of each element and #(B) = m, #P(B) = 2^m.

PROOF N ~ J, we can write N = $\{x_1, x_2, x_3, ...\} = \{0, 1, 2, 3, ...\}$. When we form a subset of N, for each i we can include x_i, or leave it out. Say we represent including x_i by 1 and leaving x_i out by 0, then each subset of N can be represented as a sequence of 0's and 1's. In fact there is a one-to-one correspondence between the subset of N and the sequences of 0's and 1's. Therefore P(N) ~ A where A is the set of all sequences of 0's and 1's, but we showed in the previous theorem that A is uncountably infinite => P(N) is likewise uncountably infinite. Q.E.D

R IS UNCOUNTABLY INFINITE

We'll use the one to one correspondence with the set of sequences of 0's and 1's in order to prove R is uncountably infinite.

1. We show R ~ (0,1) via a clever bijection

2. Set up a correspondence between the interval (0,1) and the set A of all sequences of 0's and 1's via a binary expansion.

    - **Proposition**: R is in bijective correspondence with the interval (0,1)
    - **Remark**: $(0, 1) \subsetneq R$, but we saw infinite sets can be one to one correspondence with are of their proper subsets.
    - **PROOF**: Recall from trigonometry that $\tan(-\frac{\pi}{2}, \frac{\pi}{2}) \to R$ is a bijection. Since tan x is a bijection, R ~ $(-\frac{pi}{2}, \frac{\pi}{2})$. We now use a linear function, a bijection to show (0,1) ~ $(-\frac{\pi}{2}, \frac{\pi}{2})$ $g(x) = \pi x - \frac{\pi}{2}$ The composition of 2 bijection is itself a bijection. $\to \tan(g(x)) = \tan(\pi x - \frac{\pi}{2})$ is a bijection from (0,1) to R. The map we want $f : R \to (0, 1)$ is its inverse $f(x) = (\tan(\pi x) - \frac{\pi}{2})^{-1}$ as the inverse of a bijection is itself a bijection. Q.E.D

3. To each $x \in (0, 1)$ we want to associate 0. $\{x_1, x_2, x_3....\}$ where the decimal $\{x_1, x_2...\}$ is a sequence of 0's

4. and 1's. In other words we are giving a binary expansion of $x \in (0, 1)$ as $x_1, x_2, x_3.... = \sum_{m=1}^{\inf} \frac{1}{2^m} x_m$

Countability of sets 4

Argument that R is uncounably infinite in 2 steps

1. R ~ (0,1)
2. Set up a correspondance between (0,1) and the set A of all sequences of 0's and 1's via a binary representation. To each $x \in (0, 1)$, we want to associate $0, x_1, x_2, x_3, ...$ where often decimal $\{x_1, x_2, ...\}$ is a squence of 0's and 1's. In other words we are gicing a binary expansion of every $x \in (0, 1)$ as $0, x_1, X_2, x_3 \cdots = 0 + x_1 \frac{1}{2} + x_2 + \frac{1}{4} \cdots = \sum_{m=1}^{\infty} x_m \frac{1}{2^m}$

Similarly any $x \in (1,0) that is a sum of the form $\frac{1}{2^{p_1}} + \frac{1}{2^{p_2}}$ for, $

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How can we represent $x = \frac{1}{1^{p_1}}......$ in an easier to understand form.

A

Do some algebra:

$x = \frac{1}{2^{p1}} .... = \frac{2^{pk-1}}{2^{pk-p1} - 2^{p2}} .... + \frac{1}{2^{pk}} = frac2^{pk-p1} + 2^{pk-p2} .... 2^{p}k = \frac{\text{odd natual numbers}}{\text{powers of two}} = \frac{m}{2^{m}} form \in N \text{ and } m \in N^{\star} \text{ as } p1 <$ $p2 < ...pk$ So the difference between $pk - p...$, as all positive integers.

So the sequence in (0,1) that has this discussed binary expansion is B = $\{frac12 + frac14 + \frac{3}{4} + \frac{1}{8}...\}$ NOTE that B is countably infinite as each set $Bm = \{0 < \frac{odd}{2^{m}} <\}$ is finite, $B = U_{m=1}^{\infty} Bm$ is countable by one corolloary , and the countably infinite sequence {1/2,1/4,1/8....} subset of B which means the countably set B must be countably infinite.