Final

1. Consider the pay-later call option. This has payoff $V_T = \max\{0, S_T - K\}$ at time T. The holder of the option does not pay a premium when the contract is set up,but must pay Q to the writer at expiry. The premium Q is only paid if the call ends up in-the-money, i.e. $S_T \geq K$.

   What is the value of Q?

   Answer: Reimagining the payoff of V_T at expiry time T with a long call and a short of a binary call, we have

   $$V_T = \max\{S - K, 0\} - Q\mathcal{H}(S - K)$$

   Now, using the Black-Scholes formulae as described in the textbook, we can say

   $$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) - Qe^{-r(T-t)}N(d_2)$$

   Of course at time $t = 0$ the value of the pay-later call option must be 0. This implies then that

   $$Q = \frac{Se^{r(T)}N(d_1)}{N(d_2)} - K$$

2. Use an arbitrage argument to prove that the price C of a European call option on a common stock with no dividends must satisfy

   $$C_2 \leq \lambda C_1 + (1 - \lambda)C_3$$

   where C_1, C_2 and C_3 are calls with the same expiry T, that have exercise prices $K_1 < K_2 < K_3$ respectively, and where λ is such that

   $$K_2 = \lambda K_1 + (1 - \lambda)K_3$$

   Answer: We first note that

   $$\begin{aligned} &K_2 = \lambda K_1 + (1 - \lambda)K_3 \\ &\Rightarrow K_2 - K_3 = \lambda K_1 - \lambda K_3 = \lambda(K_1 - K_3) \\ &\Rightarrow \lambda = \frac{K_2 - K_3}{K_1 - K_3} \text{ and } 1 - \lambda = \frac{K_1 - K_2}{K_1 - K_3} \end{aligned}$$

   Let's now construct a portfolio of selling C_2, buying λ units of C_1 and buying $1 - \lambda$ units of C_3. Thus, $\Pi = -C_2 + \lambda C_1 + (1 - \lambda)C_3$. At time of expiry T, we have

   $$\begin{aligned} \Pi(T) &= -max\{S_T - K_2, 0\} + \lambda max\{S_T - K_1, 0\} + (1 - \lambda)max\{S_T - K_3, 0\} \\ &= -max\{S_T - K_2, 0\} + \frac{K_2 - K_3}{K_1 - K_3}max\{S_T - K_1, 0\} + \frac{K_1 - K_2}{K_1 - K_3}\max\{S_T - K_3, 0\} \end{aligned}$$

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Written as a piecewise function we get:

$$\Pi(T) = \begin{cases} 0, & \text{if } S_T \leq K_1 \\ \frac{(S_T-K_1)(K_2-K_3)}{K_1-K_3}, & \text{if } K_1 < S_T \leq K_2 \\ -(S_T-K_2) + \frac{(S_T-K_1)(K_2-K_3)}{K_1-K_3}, & \text{if } K_2 < S_T \leq K_3 \\ \frac{(S_T-K_3)(K_1-K_2)}{K_1-K_3} - (S_T-K_2) + \frac{(S_T-K_1)(K_2-K_3)}{K_1-K_3}, & \text{if } K_3 < S_T \end{cases}$$

In the first interval,

$$0 \geq 0$$

In the second interval,

$$\frac{(S_T-K_1)(K_2-K_3)}{K_1-K_3} \geq 0$$

In the third interval,

$$\begin{aligned} -(S_T-K_2) + \frac{(S_T-K_1)(K_2-K_3)}{K_1-K_3} &= -\frac{(S_T-K_2)(K_1-K_3)}{K_1-K_3} + \frac{(S_T-K_1)(K_2-K_3)}{K_1-K_3} \\ &= \frac{(S_T-K_2)(K_1-K_3)-(S_T-K_1)(K_2-K_3)}{K_1-K_3} \\ &= \frac{S_TK_1+K_2K_3-S_TK_2-K_1K_3}{K_1-K_3} \\ &= \frac{S_T(K_1-K_2)-K_3(K_1-K_2)}{K_1-K_3} \\ &= \frac{(K_3-S_T)(K_1-K_2)}{K_1-K_3} \\ &\geq 0 \end{aligned}$$

In the fourth interval,

$$\begin{aligned} &\frac{(S_T-K_3)(K_1-K_2)}{K_1-K_3} - (S_T-K_2) + \frac{(S_T-K_1)(K_2-K_3)}{K_1-K_3} \\ &\geq \frac{(S_T-K_3)(K_1-K_2)}{K_1-K_3} + 0 \\ &\geq 0 \end{aligned}$$

Thus we have shown that $\Pi(T) \geq 0$. To ensure the absence of arbitrage opportunities, it then follows that

$$\begin{aligned} &-C_2 + \lambda C_1 + (1-\lambda)C_3 \geq 0 \\ &\Rightarrow C_2 \leq \lambda C_1 + (1-\lambda)C_3 \end{aligned}$$

3. Use put-call parity to find the relationships between the deltas, gammas, vegas and thetas of European call and put options.

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Answer: By definition of the put call-parity, start with

$$C(S,t) - P(S,t) = S - Ee^{-r(T-t)}$$

The call and put options here have the same strike price E and expiration at time T. We let S represent the spot price at time t.

Now we take the derivative in terms of t to find the relationship between the deltas of the put and call.

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = \frac{\partial}{\partial S}S - \frac{\partial}{\partial S}Ee^{-r(T-t)}$$

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1$$

$$\boxed{\Delta_C = 1 + \Delta_P}$$

Next, in order to get the gammas, we simple continue from above

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1$$

$$\frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 P}{\partial S^2} = 0$$

$$\boxed{\Gamma_C = \Gamma_P}$$

Now we go back to put-call parity to get the vegas

$$C(S,t) - P(S,t) = S - Ee^{-r(T-t)}$$

$$\frac{\partial C}{\partial \sigma} - \frac{\partial P}{\partial \sigma} = \frac{\partial}{\partial \sigma}S - \frac{\partial}{\partial \sigma}Ee^{-r(T-t)} = 0$$

$$\boxed{\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma}}$$

Again going back to put-call parity to get the thetas,

$$C(S,t) - P(S,t) = S - Ee^{-r(T-t)}$$

$$\frac{\partial C}{\partial t} - \frac{\partial P}{\partial t} = \frac{\partial}{\partial t}S - \frac{\partial}{\partial t}Ee^{-r(T-t)} = -rEe^{-r(T-t)}$$

$$\boxed{\Theta_C = \Theta_P - rEe^{-r(T-t)}}$$

4. The price function of a European call option $C = C(S)$ as a function of spot price S (with a given expiration T and strike K) has an asymptote that is parallel to the line $C = S - K$. Find an equation for this asymptote. Use the Black-Scholes formula for the call price.

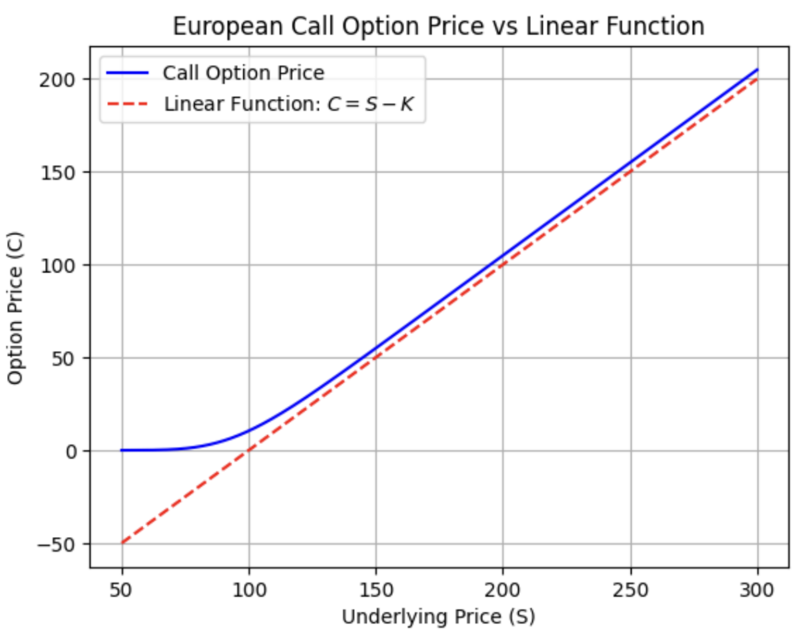
   Answer: Assuming no dividend, the Black-Scholes formula for the vanilla European call option price here is

$$C(S) = SN(d_1) - Ke^{-rT}N(d_2)$$

   where

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + T\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$$

   To visualize what is going on, I made the graph below where I fixed all variables besides S:



   From the graph, it is obvious that as $S \to \infty$, we become parallel to the line $C = S - K$. As $S \to \infty$, $d_1 \to \infty$ so $N(d_1) \to 1$. Similarly, As $S \to \infty$, $d_2 \to \infty$ so $N(d_2) \to 1$. Thus, the function of the asymptote is $C = S - Ke^{-rT}$

5. Let S_t be an Ito process that satisfies the SDE

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

   Use Ito's Lemma to find SDEs satisfied by

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(a) $f(S) = aS + b$
Answer:

$$
\begin{aligned}
df(S_t) &= \frac{\partial f}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS_t)^2 \\
&= adS_t \\
&= a\alpha S_t dt + a\sigma S_t dW_t
\end{aligned}
$$

(b) $g(S) = S^a$
Answer:

$$
\begin{aligned}
dg(S_t) &= \frac{\partial g}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial S^2}(dS_t)^2 \\
&= aS_t^{a-1}dS_t + \frac{a(a-1)}{2}S_t^{a-2}(dS_t)^2 \\
&= aS_t^{a-1}(\alpha S_t dt + \sigma S_t dW_t) + \frac{1}{2}a(a-1)S_t^{a-2}(\alpha S_t dt + \sigma S_t dW_t)^2 \\
&= a\alpha S_t^a dt + a\sigma S_t^a dW_t + \frac{1}{2}a(a-1)S_t^{a-2}(\alpha^2 S_t^2 dt^2 + 2\alpha\sigma S_t^2 dt dW_t + \sigma^2 S_t^2 dW_t^2) \\
&= a\alpha S_t^a dt + a\sigma S_t^a dW_t + \frac{1}{2}a(a-1)S_t^{a-2}(0 + 0 + \sigma^2 S_t^2 dt) \\
&= a\alpha S_t^a dt + a\sigma S_t^a dW_t + \frac{1}{2}a(a-1)\sigma^2 S_t^a dt \\
&= aS_t^{\alpha}(\alpha + \frac{1}{2}(a-1)\sigma^2)dt + a\sigma S_t^a dW_t
\end{aligned}
$$

(c) $h(S,t) = S^a e^{bt}$
Answer:

$$
\begin{aligned}
dh(S,t) &= \frac{\partial h}{\partial S}dS_t + \frac{\partial h}{\partial t}dt + \frac{1}{2}\frac{\partial^2 h}{\partial S^2}(dS_t)^2 \\
&= aS_t^{a-1}e^{bt}dS_t + bS_t^a e^{bt}dt + \frac{1}{2}a(a-1)S_t^{a-2}e^{bt}(dS_t)^2 \\
&= aS_t^{a-1}e^{bt}(\alpha S_t dt + \sigma S_t dW_t) + bS_t^a e^{bt}dt + \frac{1}{2}a(a-1)S_t^{a-2}e^{bt}(\sigma^2 S_t^2 dt) \\
&= a\alpha S_t^a e^{bt}dt + a\sigma S_t^a e^{bt}dW_t + bS_t^a e^{bt}dt + \frac{1}{2}a(a-1)\sigma^2 S_t^a e^{bt}dt \\
&= S_t^a e^{bt}(a\alpha + b + \frac{1}{2}a(a-1)\sigma^2)dt + a\sigma S_t^a e^{bt}dW_t
\end{aligned}
$$

6. Assume, as usual, that our market offers a cash bond with price $B_t = e^{rt}$, and a risky asset whose price S_t is an Ito process with

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

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Assume that α, σ, r are constants with $\sigma > 0, r > 0$
Let $\mathbb{Q}$ be the risk-neutral measure for which $e^{-rt}S_t$ is $\mathbb{Q}$-martingale, and W_t is Brownian motion for $\mathbb{Q}$,

$$d\tilde{W}_t = dW_t + \theta dt \quad \theta = \frac{\alpha - r}{\sigma}$$

(a) Prove that the price process S_t^2 is not tradable in our market, i.e., the discounted process $e^{-rt}S_t^2$ is not a $\mathbb{Q}$-martingale
Hint. Calculate $d(e^{-rt}S_t^2)$

Answer: Let $f(S,t) = e^{-rt}S_t^2$. Then,

$$\begin{aligned}
df(S,t) &= \frac{\partial f}{\partial S}dS_t + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 h}{\partial S^2}(dS_t)^2 \\
&= 2S_te^{-rt}dS_t - rS_t^2e^{-rt}dt + e^{-rt}(dS_t)^2 \\
&= 2S_te^{-rt}(\alpha S_tdt + \sigma S_tdW_t) - rS_t^2e^{-rt}dt + e^{-rt}(\sigma^2S_t^2dt) \\
&= 2\alpha S_t^2e^{-rt}dt + 2\sigma S_t^2e^{-rt}dW_t - rS_t^2e^{-rt}dt + \sigma^2S_t^2e^{-rt}dt \\
&= S_t^2e^{-rt}(2\alpha - r + \sigma^2)dt + 2\sigma S_t^2e^{-rt}dW_t \\
&= S_t^2e^{-rt}(2\alpha - r + \sigma^2)dt + 2\sigma S_t^2e^{-rt}d\tilde{W}_t - 2(\alpha - r)S_t^2e^{-rt} \\
&= S_t^2e^{-rt}(r + \sigma^2)dt + 2\sigma S_t^2e^{-rt}d\tilde{W}_t
\end{aligned}$$

From the above, it is obvious the drift term is not zero which proves the claim that the discounted process $e^{-rt}S_t^2$ is not a $\mathbb{Q}$-martingale.

(b) Show that S_t^a is the price process of a tradable asset if and only if $a = 1$ or $a = \frac{-2r}{\sigma^2}$

Answer: Looking at the answer in $(5c)$ and setting $b = -r$ and $dW_t = d\tilde{W}_t - \frac{\alpha - r}{\sigma}dt$,we know

$$\begin{aligned}
d(e^{-rt}S_t^\alpha) &= S_t^ae^{-rt}(a\alpha - r + \frac{1}{2}a(a-1)\sigma^2)dt + a\sigma S_t^ae^{-rt}(d\tilde{W}_t - \frac{\alpha - r}{\sigma}dt) \\
&= S_t^ae^{-rt}(a\alpha - r + \frac{1}{2}a(a-1)\sigma^2)dt + a\sigma S_t^ae^{-rt}d\tilde{W}_t - a(\alpha - r)S_t^ae^{-rt}dt \\
&= S_t^ae^{-rt}(a\alpha - r + \frac{1}{2}a(a-1)\sigma^2 - a(\alpha - r))dt + a\sigma S_t^ae^{-rt}d\tilde{W}_t
\end{aligned}$$

To find when this is a tradable asset, we need to make the drft term equal to 0.

So,

$$
\begin{aligned}
0 &= a\alpha - r + \frac{1}{2}a(a-1)\sigma^2 - a(\alpha - r)\\
&= \frac{\sigma^2}{2}a^2 + (\alpha + \frac{\sigma^2}{2} - \alpha + r)a - r\\
&= \frac{\sigma^2}{2}a^2 + (\frac{\sigma^2 + 2r}{2})a - r
\end{aligned}
$$

Using the quadratic formula to solve the equation, we get

$$
\begin{aligned}
a &= \frac{-\frac{\sigma^2+2r}{2} \pm \sqrt{(\frac{\sigma^2+2r}{2})^2 - 2\sigma^2 r}}{\sigma^2}\\
&= \frac{-\frac{\sigma^2+2r}{2} \pm \sqrt{\frac{(\sigma^2+2r)^2-8\sigma^2 r}{4}}}{\sigma^2}\\
&= \frac{-\frac{\sigma^2+2r}{2} \pm \sqrt{\frac{\sigma^4+4r\sigma^2+4r^2-8\sigma^2 r}{4}}}{\sigma^2}\\
&= \frac{-\frac{\sigma^2+2r}{2} \pm \sqrt{\frac{(\sigma^2-2r)^2}{4}}}{\sigma^2}\\
&= \frac{-\frac{\sigma^2+2r}{2} \pm \frac{\sigma^2-2r}{2}}{\sigma^2}\\
&= \frac{-\sigma^2 - 2r \pm (\sigma^2 - 2r)}{2\sigma^2}\\
&= \frac{-2r}{\sigma^2}, 1
\end{aligned}
$$

Thus, S_t^a is the price process of a tradable asset if and only if $a = 1$ or $a = \frac{-2r}{\sigma^2}$

(c) Find all values of a for which $V = S^a$ is a solution of the Black-Scholes PDE,

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0
$$

Remark. Solutions of the PDE that have no expiration date are "perpetual options".

Answer: Substituting $V = S^a$ into the Black-Scholes PDE we get

$$
\begin{aligned}
0 &= 0 + \frac{1}{2}\sigma^2 S^2 a(a-1)S_t^{a-2} + rSaS_t^{a-1} - rS^a\\
&= \frac{1}{2}\sigma^2 S^a a^2 - \frac{1}{2}\sigma^2 S^a a + raS^a - rS^a\\
&= S^a(\frac{\sigma^2}{2}a^2 + \frac{\sigma^2 + 2r}{2}a - r)
\end{aligned}
$$

Notice we are now solving the same equation as we did in the above question. Thus, we know $V = S^a$ is a solution of the Black-Scholes PDE for $a = 1$ or $a = \frac{-2r}{\sigma^2}$

7. A certain stock pays dividends twice each year. Each dividend payment is 3% of the share price.
The current share price is $S_0 = \$100$. Volatility is $\sigma = 40\%$. The risk-free interest rate is $r = 5\%$.
Note: For the problems here I assume the dividend payouts are in 6 months and a year

   (a) Calculate the price of a forward contract on the stock with a delivery date of $T = 1$ year.

   Answer:

$$\begin{aligned} F &= (S_0 - PV(\text{dividend})e^{rt} \\ &= \left(100 - \left(\frac{.03(100)}{1+0.5(0.05)} + \frac{.03(100)}{1+1(0.05)}\right)\right) e^{0.05(1)} \\ &= \left(100 - \frac{3}{1.025} - \frac{3}{1.05}\right) e^{.05} \\ &\approx \$99.05 \end{aligned}$$

   (b) Calculate the price of an at-the-money European call option on this stock with strike price $K = \$100$ and time to expiration $T = 1$ (one year).

   Answer:

$$\begin{aligned} C &= e^{-rt} F N\left(\frac{\log\frac{F}{K}}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) - Ke^{-rT} N\left(\frac{\log\frac{F}{K}}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right) \\ &= 99.05e^{-0.05} N\left(\frac{\log\frac{99.05}{100}}{0.4} + 0.2\right) - 100e^{-0.05} N\left(\frac{\log\frac{99.05}{100}}{0.4} - 0.2\right) \\ &\approx \$14.56 \end{aligned}$$

   (c) Use put-call parity to calculate the price of a European put option with the same strike and expiration ($K = 100, T = 1$).

   Answer: From the put-call parity we know

$$\begin{aligned} &C - P = e^{-rT}(F - K)S \\ &\Rightarrow P = C - e^{-rT}(K - F) = 14.56 + e^{-0.05}(100 - 99.05) \approx \$15.46 \end{aligned}$$

Final

State all prices (forward, call, put) in dollars with 2 decimals accuracy.
Show all the steps in your calculations. You may use a calculator, and you may use software to find the values of $N(d_1), N(d_2)$.