一、图形学入门

常用公式：

- 已知三角形三点坐标，三角形面积为$S=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$
- 直线$Ax+By+C=0$，点$P(x_0,y_0)$到直线距离为$\frac{|Ax_0+By_0+C|}{\sqrt{A^2+B^2}}$
- $\vec{a}=(x_1,y_1,z_1),\vec{b}=(x_2,y_2,z_2)$，则
  $\vec{a}\times\vec{b}=i(y_1z_2-z_1y_2)+j(z_1x_2-x_1z_2)+k(x_1y_2-y_1x_2)$

焦距：眼到图像投影平面的距离

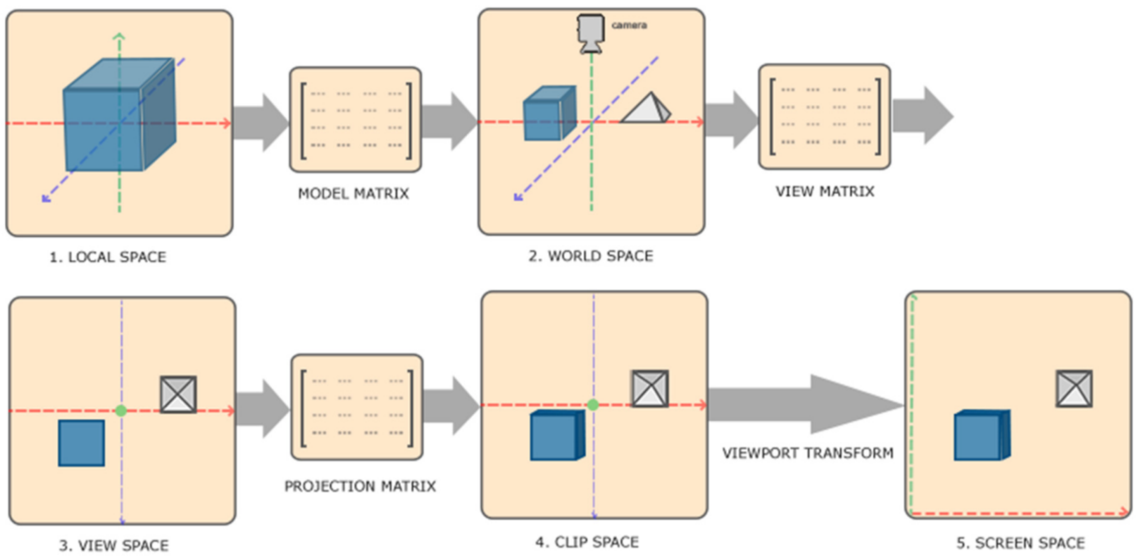
透视投影和**平行投影**的区别：透视投影焦距有限；平行投影焦距无限

光栅化（Rasterizaton）：将连续信号转变为离散像素的过程

双缓存图像：设置主要缓存和次要缓存，不断切换屏幕指向的缓存地址，以避免单屏幕显示的"显示，擦除"产生的闪烁

深度缓存（Z-buffer）：在光栅化每个像素点时，遍历几何体，同时记录该几何体在该像素点上的深度。如果下一个几何体深度更小，则该像素点替换成下个几何体的颜色。该方法用于解决几何体的先后问题。

二、坐标空间



在三维空间的计算中，**点的坐标**表示为：

$$(x,y,z,w)=(\frac{x}{w},\frac{y}{w},\frac{z}{w})$$

注：当$w=0$时，该坐标代表向量

变换矩阵

平移

$$T(\Delta x,\Delta y,\Delta z)=\begin{pmatrix}1&0&0&\Delta x\\0&1&0&\Delta y\\0&0&1&\Delta z\\0&0&0&1\end{pmatrix}$$

拉伸

$$T(x,y,z)=\begin{pmatrix}x&0&0&0\\0&y&0&0\\0&0&z&0\\0&0&0&1\end{pmatrix}$$

旋转

$$R_x(\theta)=\begin{pmatrix}1&0&0&0\\0&\cos\theta&-\sin\theta&0\\0&\sin\theta&\cos\theta&0\\0&0&0&1\end{pmatrix}$$

$$R_y(\theta)=\begin{pmatrix}\cos\theta&0&\sin\theta&0\\0&1&0&0\\-\sin\theta&0&\cos\theta&0\\0&0&0&1\end{pmatrix}$$

$$R_z(\theta)=\begin{pmatrix}\cos\theta&-\sin\theta&0&0\\\sin\theta&\cos\theta&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

$$R(\theta)=R_x(\theta)R_y(\theta)R_z(\theta)$$

注：法向量在不均匀缩放情况下直接乘以model矩阵会不垂直于交点，
$N'=mat3(transpose(model^{-1}))*N$

View矩阵

步骤为先将相机平移到原点，再旋转，所以将世界坐标系做逆变换就能得到View矩阵（相机位置为$(0,0,0)$，看向$-z$

$$M_v=R^{-1}T^{-1}$$

$$T^{-1}=\begin{pmatrix}1&0&0&-x_e\\0&1&0&-y_e\\0&0&1&-z_e\\0&0&0&1\end{pmatrix}，R^{-1}=\begin{pmatrix}x_{\hat{g}\times\hat{t}}&x_{\hat{t}}&x_{-\hat{g}}&0\\y_{\hat{g}\times\hat{t}}&y_{\hat{t}}&y_{-\hat{g}}&0\\z_{\hat{g}\times\hat{t}}&z_{\hat{t}}&z_{-\hat{g}}&0\\0&0&0&1\end{pmatrix}$$

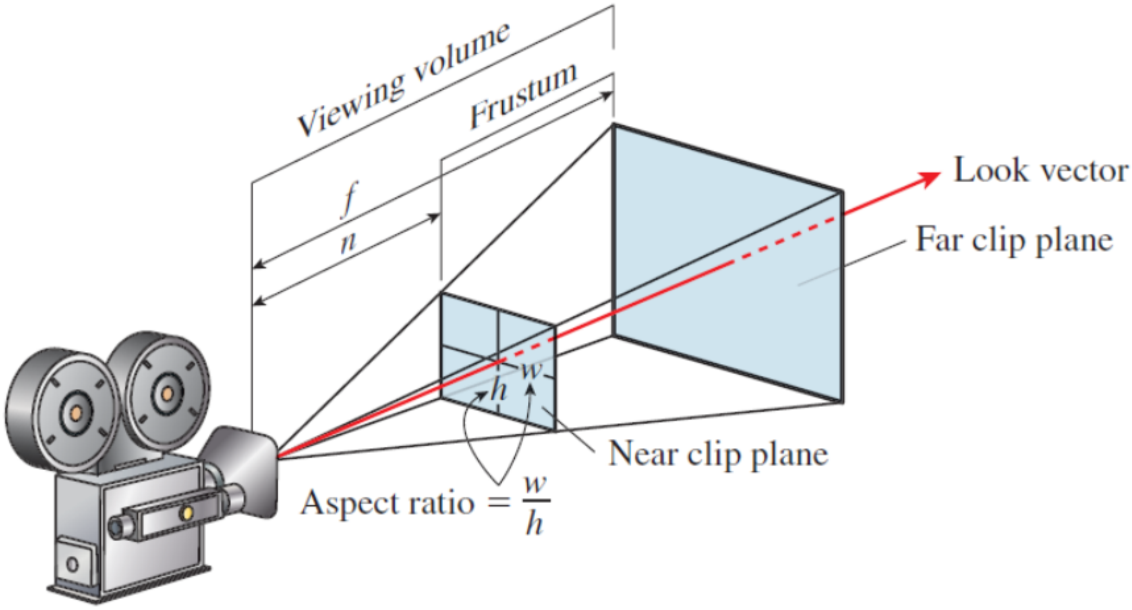
其中，$\hat{t}$向y转，$\hat{g}$向$-z$转，$\hat{g}\times\hat{t}$向x转

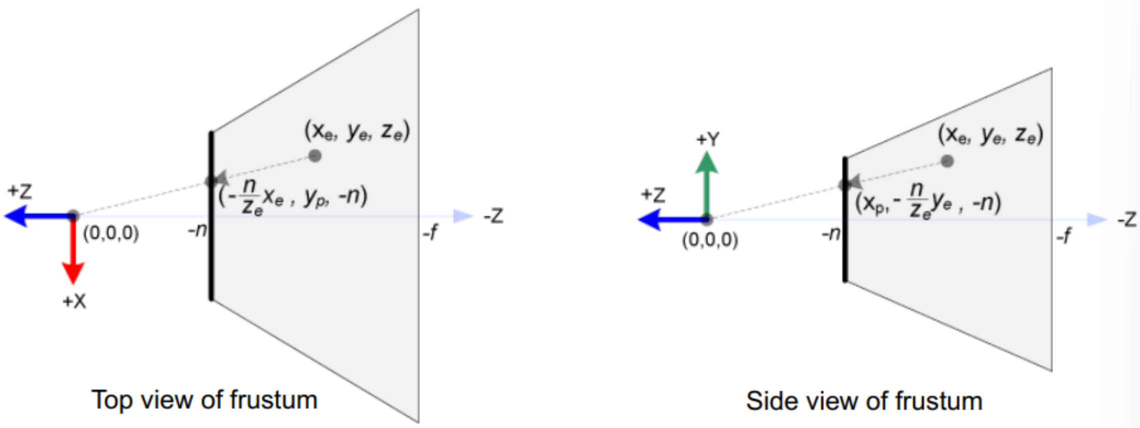
投影矩阵

透视投影

$$M_p = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}, where \begin{cases} t = n\tan\frac{fovY}{2} \\ r = aspect\_ratio \times n\tan\frac{fovY}{2} \\ l = -r \\ t = -b \end{cases}$$

证明:





Top view of frustum

Side view of frustum

$$x_p = \frac{-n \cdot x_e}{z_e} = \frac{n \cdot x_e}{-z_e}$$

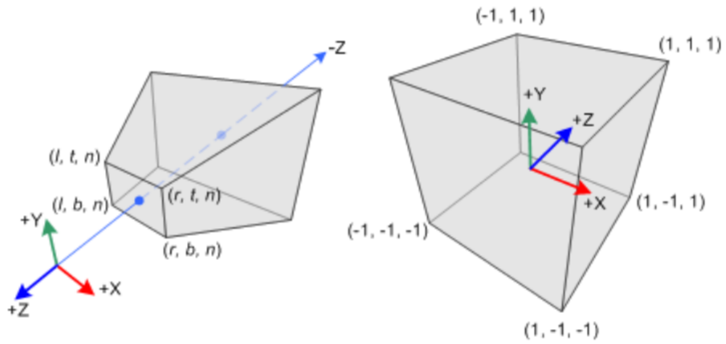
$$y_p = \frac{-n \cdot y_e}{z_e} = \frac{n \cdot y_e}{-z_e}$$

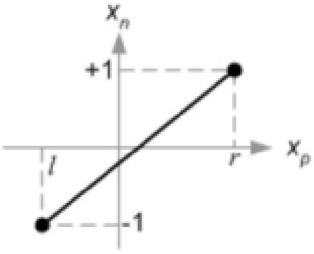
$$\begin{pmatrix} x_{clip} \\ y_{clip} \\ z_{clip} \\ w_{clip} \end{pmatrix} = M_{projection} \cdot \begin{pmatrix} x_{eye} \\ y_{eye} \\ z_{eye} \\ w_{eye} \end{pmatrix} \qquad \begin{pmatrix} x_{ndc} \\ y_{ndc} \\ z_{ndc} \end{pmatrix} = \begin{pmatrix} x_{clip}/w_{clip} \\ y_{clip}/w_{clip} \\ z_{clip}/w_{clip} \end{pmatrix}$$

$$\begin{pmatrix} x_c \\ y_c \\ z_c \\ w_c \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_e \\ y_e \\ z_e \\ w_e \end{pmatrix}, \qquad \therefore w_c = -z_e$$

Normalized device coordinate (NDC)

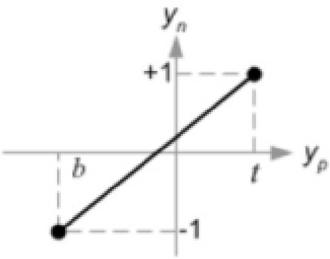
- Range normalization
  - x-coordinate: [l, r] to [-1, 1]
  - y-coordinate: [b, t] to [-1, 1]
  - z-coordinate: [n, f] to [-1, 1]





Mapping from x_p to x_n

$$
\begin{aligned}
x_n &= \frac{1-(-1)}{r-l}\cdot x_p + \beta \\
1 &= \frac{2r}{r-l} + \beta \qquad (\text{substitute } (r,1) \text{ for } (x_p, x_n)) \\
\beta &= 1 - \frac{2r}{r-l} = \frac{r-l}{r-l} - \frac{2r}{r-l} \\
&= \frac{r-l-2r}{r-l} = \frac{-r-l}{r-l} = -\frac{r+l}{r-l} \\
\therefore x_n &= \frac{2x_p}{r-l} - \frac{r+l}{r-l}
\end{aligned}
$$



Mapping from y_p to y_n

$$
\begin{aligned}
y_n &= \frac{1-(-1)}{t-b}\cdot y_p + \beta \\
1 &= \frac{2t}{t-b} + \beta \qquad (\text{substitute } (t,1) \text{ for } (y_p, y_n)) \\
\beta &= 1 - \frac{2t}{t-b} = \frac{t-b}{t-b} - \frac{2t}{t-b} \\
&= \frac{t-b-2t}{t-b} = \frac{-t-b}{t-b} = -\frac{t+b}{t-b} \\
\therefore y_n &= \frac{2y_p}{t-b} - \frac{t+b}{t-b}
\end{aligned}
$$

$$
\begin{aligned}
x_n &= \frac{2x_p}{r-l} - \frac{r+l}{r-l} \qquad \left(x_p = \frac{nx_e}{-z_e}\right) \\
&= \frac{2\cdot\frac{n\cdot x_e}{-z_e}}{r-l} - \frac{r+l}{r-l} \\
&= \frac{2n\cdot x_e}{(r-l)(-z_e)} - \frac{r+l}{r-l} \\
&= \frac{\frac{2n}{r-l}\cdot x_e}{-z_e} - \frac{r+l}{r-l} \\
&= \frac{\frac{2n}{r-l}\cdot x_e}{-z_e} + \frac{\frac{r+l}{r-l}\cdot z_e}{-z_e} \\
&= \underbrace{\left(\frac{2n}{r-l}\cdot x_e + \frac{r+l}{r-l}\cdot z_e\right)}_{x_c} \Big/ -z_e
\end{aligned}
$$

$$
\begin{aligned}
y_n &= \frac{2y_p}{t-b} - \frac{t+b}{t-b} \qquad \left(y_p = \frac{ny_e}{-z_e}\right) \\
&= \frac{2\cdot\frac{n\cdot y_e}{-z_e}}{t-b} - \frac{t+b}{t-b} \\
&= \frac{2n\cdot y_e}{(t-b)(-z_e)} - \frac{t+b}{t-b} \\
&= \frac{\frac{2n}{t-b}\cdot y_e}{-z_e} - \frac{t+b}{t-b} \\
&= \frac{\frac{2n}{t-b}\cdot y_e}{-z_e} + \frac{\frac{t+b}{t-b}\cdot z_e}{-z_e} \\
&= \underbrace{\left(\frac{2n}{t-b}\cdot y_e + \frac{t+b}{t-b}\cdot z_e\right)}_{y_c} \Big/ -z_e
\end{aligned}
$$

$$\begin{pmatrix} x_c \\ y_c \\ z_c \\ w_c \end{pmatrix} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & A & B \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_e \\ y_e \\ z_e \\ w_e \end{pmatrix}, \qquad z_n = z_c/w_c = \frac{Az_e + Bw_e}{-z_e}$$

$$\begin{cases} \dfrac{-An+B}{n} = -1 \\ \dfrac{-Af+B}{f} = 1 \end{cases} \rightarrow \begin{cases} -An+B = -n \\ -Af+B = f \end{cases}$$

平行矩阵

$$M_o = \begin{pmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & \frac{t+b}{t-b} \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

观察矩阵

$[-1,1]^2 \rightarrow [0, width] \times [0, height]$

$$M_viewport = \begin{pmatrix} \frac{width}{2} & 0 & 0 & \frac{width}{2} \\ 0 & \frac{height}{2} & 0 & \frac{height}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

光栅化

Bresenham算法

在平面直角坐标系上，假设有一条直线，方程为 $f(x,y) = Ax + By + C, A = \Delta y, B = -\Delta x, C = (\Delta x)b$，注意这里斜率不能过大。从$(x_0, y_0)$开始，决定需要填色的像素是$(x_0+1, y_0)$还是$(x_0+1, y_0+1)$，目标是寻找其中一个距离直线更近的点

解决：将$x = x_0 + 1, y = y_0 + \frac{1}{2}$代入直线方程，若$f(x_0+1, y_0+\frac{1}{2}) <= 0$，则选择$(x_0+1, y_0)$涂色，若$f(x_0+1, y_0+\frac{1}{2}) > 0$，则选择$(x_0+1, y_0+1)$涂色

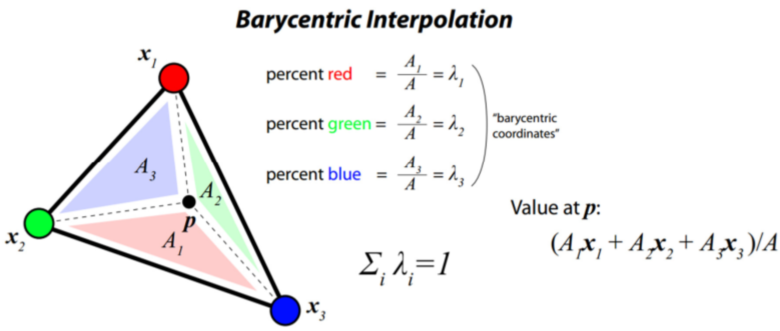
三角形取点

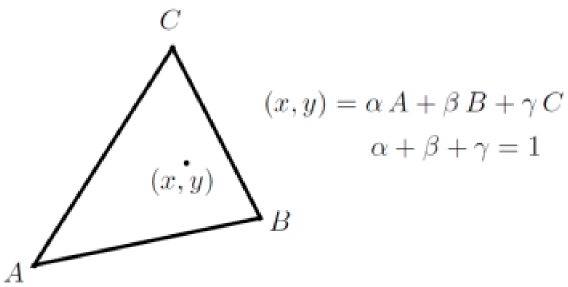
对于三角形每一条直线，做Bresenham算法：$P_i=(X_i,Y_i)$，$dX_i=X_{i+1}-X_i$，$dY_i=Y_{i+1}-Y_i$

$$E_i(x,y)=(x-X_i)dY_i-(y-Y_i)dX_i\begin{cases}=0: point\ on\ edge\\>0: outside\ edge\\<0: inside\ edge\end{cases}$$

取所有点，当三条直线$E_i(x,y)$同时满足<0时，该点为三角形内的点

上色





$$\alpha=\frac{-(x-x_B)(y_C-y_B)+(y-y_B)(x_C-x_B)}{-(x_A-x_B)(y_C-y_B)+(y_A-y_B)(x_C-x_B)}$$

$$\beta=\frac{-(x-x_C)(y_A-y_C)+(y-y_C)(x_A-x_C)}{-(x_B-x_C)(y_A-y_C)+(y_B-y_C)(x_A-x_C)}$$

$$\gamma=1-\alpha-\beta$$

三、曲线和三角化

德劳内三角剖分法（构造Mesh）：任意三角形的外接圆内没有第四个点

翻转 Flipping：如果两个三角形未满足德劳内，则换中间连线

泰森多边形 Voronoi Diagram：将平面划分为不同区域（每个区域只有一个点），使该区域到区域中的点距离最近，这些区域叫做**Voronoi Cells**，可由中垂线法生成。

泰森多边形和德劳内三角是对偶关系，连接泰森多边形相邻的点即可获得德劳内三角剖分

四、几何模型

椭球模型：$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

椭球坐标变换：$x = a\cos(u)\cos(v), y = b\cos(u)\sin(v), z = c\sin(u)$，$-\frac{\pi}{2} \le u \le \frac{\pi}{2}, -\pi \le v \le \pi$

对于曲线某点的切线，直接求偏导；对于曲面某点的法向量，先按照两个不同方向求偏导，得两个切线，再对两个切线的方向向量做叉乘

插值问题：给定值的映射序列，找到满足该映射的函数

多项式插值：对于$n+1$个数据，构造一个最大项为n次的多项式，将数据带入求解

多项式插值的震荡现象：当多项式阶数很高时，会在边缘点发生强烈的震荡

样条插值：样条插值是一种**分段定义**的低阶多项式函数，但阶数也不能过低，需要保证一定的曲线连续性，通常考虑三阶的样条插值

三阶样条插值：定义$p(x) = a + bx + cx^2 + dx^3$，其具有二阶导数连续性，在结点处一致

给定定义在$[a,b]$上的函数$f(x)$和结点集合$a = x_0 < x_1 < \ldots < x_n = b$，样条曲线为：

$$S(x) = \begin{cases} a_0 + b_0(x-x_0) + c_0(x-x_0)^2 + d_0(x-x_0)^3, & if\ x_0 \le x \le x_1 \\ a_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_1)^3, & if\ x_1 \le x \le x_2 \\ \ldots \\ a_{n-1} + b_{n-1}(x-x_{n-1}) + c_{n-1}(x-x_{n-1})^2 + d_{n-1}(x-x_{n-1})^3, & if\ x_{n-1} \le x \le x_n \end{cases}$$

同时要满足：对于每个结点，S值与f值一致；连续两条样条曲线共有点的值一致；连续两条样条曲线在共有点的一阶导数一致；连续两条样条曲线在共有点的二阶导数一致

还要满足下述其一：$S''(x_0) = S''(x_n) = 0$（自然样条）或$S'(x_0) = f'(x_0), S'(x_n) = f'(x_n)$（紧凑样条）

Bernstein Basis：$b_{v,n}(x) = \binom{n}{v} x^v (1-x)^{n-v}$

贝塞尔曲线：$B(t) = \sum_{i=0}^{n} b_{i,n}(t) P_i = \sum_{i=0}^{n} \binom{n}{v} t^i (1-t)^{n-i} P_i$，其中$P_i$是控制点

de Casteljau：$\beta_i^{(0)} = \beta_i, i = 0, \ldots, n$ and $\beta_i^{(j)} = \beta_i^{(j-1)}(1-t_0) + \beta_{i+1}^{(j-1)} t_0, i = 0, \ldots, n-j, j = 1, \ldots, n$

贝塞尔曲面：$P(u,v) = \sum_{i=1}^{n} \sum_{j=0}^{m} B_i^n(u) B_j^m(v) k_{i,j}$，其有$(n+1)(m+1)$个控制点定义

B样条曲线：$S_{n,t}(x) = \sum_{i=0}^{degree} \alpha_i B_{i,n}(x)$

Coxde Boor法解B样条曲线：

$$B_{i,1}(x) = \begin{cases} 1, & t_i \le x \le t_{i+1} \\ 0, & otherwise \end{cases}$$

$$B_{i,k}(x) = \frac{x - t_i}{t_{i+k-1} - t_i} B_{i,k-1}(x) + \frac{t_{i+k} - x}{t_{i+k} - t_{i+1}} B_{i+1,k-1}(x)$$

Catmull细分曲面：

- 增加面中心点：$v_F = \sum_{i=1}^{n} \frac{1}{n} v_i$
- 增加边点：$v_E = \frac{v+w+v_{F_1}+v_{F_2}}{4}$，其中$v, w$是边端点，$v_{F_1}, v_{F_2}$是生成的相邻面中心点
- 改变原来点的值：$v' = \frac{1}{n}Q + \frac{2}{n}R + \frac{n-3}{n}v$，其中$n$为面的顶点数，如正方形为$n=3$，$Q$为所有相邻生成面中心点的均值，$R$为所有和该点相邻的生成的边点的均值

五、几何渲染

冯模型 Phong reflection model

光的组成：

- 环境光Ambient：**常量**光源（与观看角度无关）
- 漫射光Diffuse：均匀发射向任意方向的光源（与观看角度无关）
- 镜面光Specular：向特定方向反射的光源（和观看角度有关）

注：对于物体也定义了如上三种属性

$$I_p = k_a i_a + \sum_{m\in lights} (k_d(\hat{L_m}\cdot\hat{N})i_{m,d} + k_s(\hat{R_m}\cdot\hat{V})^{\alpha} i_{m,s})$$

其中k_a, k_d, k_s分别是环境光、漫反射、镜面反射常量，和光属性相关；$i_a, i_{m,d}, i_{m,s}$和物体属性相关；$\hat{L}_m$代表归一后光射入的反向向量，$\hat{N}$代表归一后光和物体交点物体的法向量，二者点乘得夹角θ；$\hat{R}_m$是出射光线，且$\hat{R}_m = 2(\hat{L}_m\cdot\hat{N})\hat{N} - \hat{L}_m$，$\hat{V}$代表从交点指向相机的向量，$\alpha$控制shininess，值越低，图形越善亮

布林冯模型 Blinn-Phone

定义$H = \frac{\hat{L}_m+\hat{V}}{||\hat{L}_m+\hat{V}||}$，用$(\hat{N}\cdot H)^{\alpha'}$代替冯模型中$(\hat{R_m}\cdot\hat{V})^{\alpha}$

XYZ颜色和RGB颜色

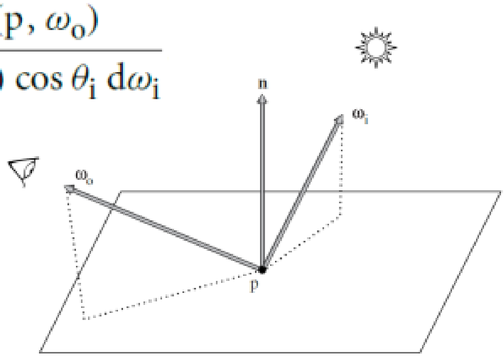
根据能量，定义$X = k\int_\lambda \Phi(\lambda)\bar{x}(\lambda)d\lambda, Y = k\int_\lambda \Phi(\lambda)\bar{y}(\lambda)d\lambda, Z = k\int_\lambda \Phi(\lambda)\bar{z}(\lambda)d\lambda$

进行归一化得$x = \frac{X}{X+Y+Z}, y = \frac{Y}{X+Y+Z}, z = \frac{Z}{X+Y+Z}$

将红绿蓝单独提取，定义$R = \int_\lambda S(\lambda)\bar{r}(\lambda)d\lambda, G = \int_\lambda S(\lambda)\bar{g}(\lambda)d\lambda, B = k\int_\lambda S(\lambda)\bar{b}(\lambda)d\lambda$

双向反射分布函数 BRDF

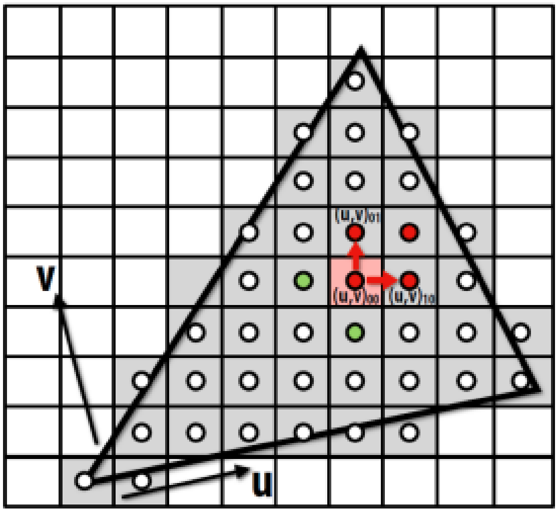
$$f_r(p,\omega_o,\omega_i) = \frac{dL_o(p,\omega_o)}{dE(p,\omega_i)} = \frac{dL_o(p,\omega_o)}{L_i(p,\omega_i)\cos\theta_i\, d\omega_i}$$

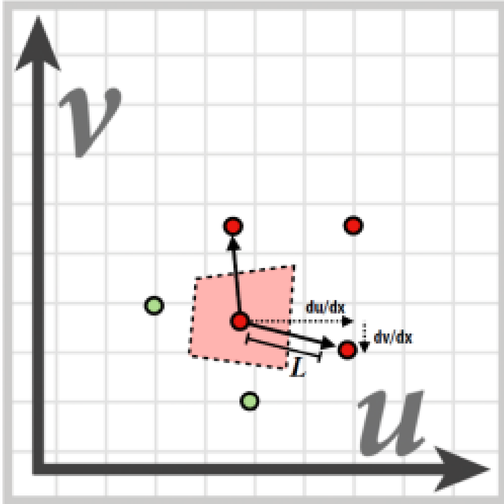


$$L_0(p,\omega_0) = \int_{s^2} f(p,\omega_0,\omega_i)L_i(p,\omega_i)|\cos\theta_i|d\omega_i$$

折射：$\eta_i \sin\theta_i = \eta_t \sin\theta_t$

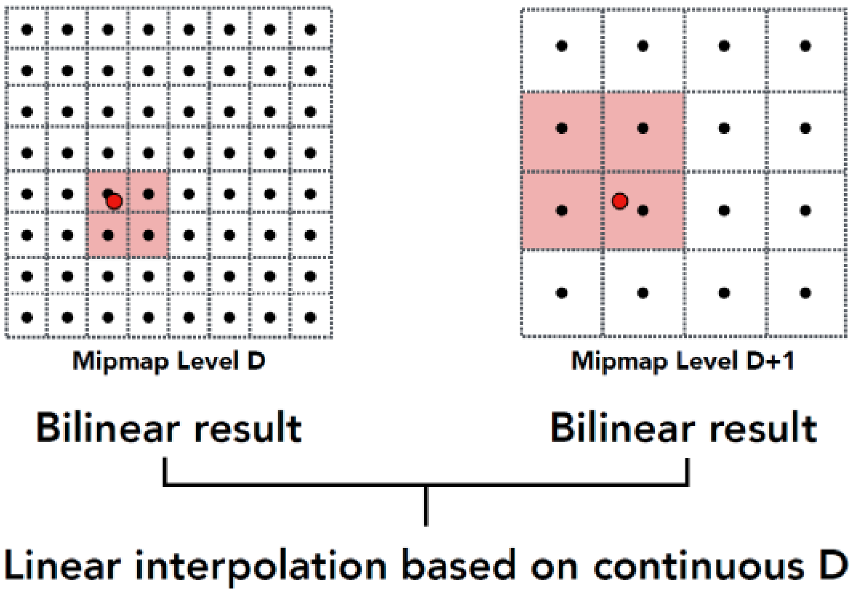
六、纹理





$$D = \log_2 L$$

$$L = \max\left(\sqrt{\left(\frac{du}{dx}\right)^2+\left(\frac{dv}{dx}\right)^2}, \sqrt{\left(\frac{du}{dy}\right)^2+\left(\frac{dv}{dy}\right)^2}\right)$$



七、光线追踪

1. 计算光线$r(t)=o+td$和圆$f(x)=|x|^2-1$交点:

   将光线方程带入圆方程得$f(r(t))=|o+td|^2-1\rightarrow|d|^2t^2+2(o\cdot d)t+|o|^2-1=0$

$$t=-o\cdot d\pm\sqrt{(o\cdot d)^2-|o|^2+1}$$

2. 计算光线$r(t)=o+td$和平面$N^Tx=c$交点:

   $N^Tr(t)=c\rightarrow N^T(o+td)=c\rightarrow t=\frac{c-N^To}{N^Td}\rightarrow r(t)=o+\frac{c-N^To}{N^Td}d$

3. 判断一个点是否在三角形内部？需要满足

$$p(b_1,b_2)=(1-b_1-b_2)p_0+b_1p_1+b_2p_2, b_1\geq0, b_2\geq0, b_1+b_2\leq1$$

   将直线方程代入得

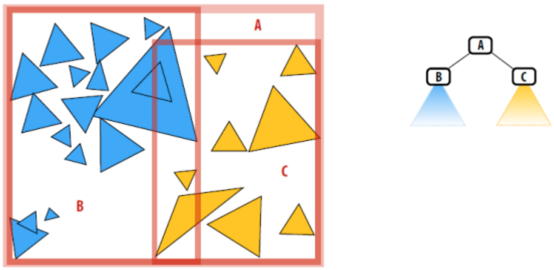
$$o + td = (1 - b_1 - b_2)p_0 + b_1 p_1 + b_2 p_2, b_1 \geq 0, b_2 \geq 0, b_1 + b_2 \leq 1$$

$$\begin{pmatrix} -\mathbf{d} & \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \mathbf{s}$$

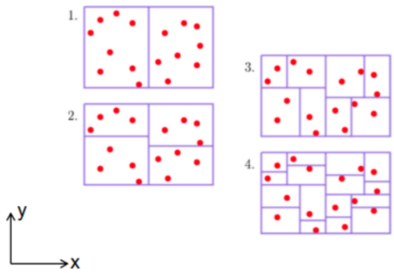
$$\begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \frac{1}{(\mathbf{d} \times \mathbf{e}_2) \cdot \mathbf{e}_1} \begin{bmatrix} (\mathbf{s} \times \mathbf{e}_1) \cdot \mathbf{e}_2 \\ (\mathbf{d} \times \mathbf{e}_2) \cdot \mathbf{s} \\ (\mathbf{s} \times \mathbf{e}_1) \cdot \mathbf{d} \end{bmatrix} \Rightarrow \begin{bmatrix} t \\ b_1 \\ b_2 \end{bmatrix} = \frac{1}{\mathbf{s}_1 \cdot \mathbf{e}_1} \begin{bmatrix} \mathbf{s}_2 \cdot \mathbf{e}_2 \\ \mathbf{s}_1 \cdot \mathbf{s} \\ \mathbf{s}_2 \cdot \mathbf{d} \end{bmatrix}$$

图形空间分割

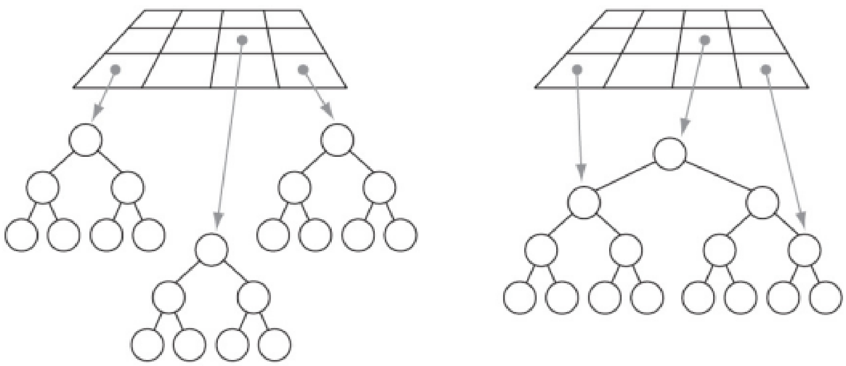
- Octree：正方形不断分割成四个正方形，直至分割后正方形中图形数少于一定值，树状存储
- BVH：以图形将空间分割，将结点的基元分成不相连集合



- K-D树：基元表示法，每个基元都由其中心点表示



- Hybrid：每一个划分空间包含独立或是混合的树



差异定义

给定N维序列P和一个采样序列B，P关于B的差异定义为

$$D_N(B,P) = \sup_{b \in B} \left| \frac{\sharp\{x_i \in b\}}{N} - \lambda(b) \right|$$

$$B = \{[0, v_1] \times [0, v_2] \times \cdots \times [0, v_s]\}$$

$$P = x_1, \ldots, x_N$$

$\sharp\{x_i \in b\}$ is the number of points in b

$\lambda(b)$ is the volume of b

八、采样

球分布变换：$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$，假设我们想根据$p(r,\theta)$采样，先计算Jacobian矩阵的模，$|J_T| = r^2\sin\theta$，则有$p(r,\theta,\phi) = r^2\sin\theta p(x,y,z)$。根据球坐标定义的固定角$d\omega = \sin\theta d\theta d\phi$，则$p(\theta,\phi)d\theta d\phi = p(\omega)d\omega \to p(\theta,\phi) = \sin\theta p(\omega)$

- Sampling a unit disk uniformly
  - Wrong approach: $r = \xi_1, \theta = 2\pi\xi_2$
  - PDF p(x,y) by normalization is: $p(x,y) = 1/\pi$
  - Transform into polar coordinate: $p(r,\theta) = r/\pi$ $\quad p(r,\theta) = r\ p(x,y)$
  - Compute the marginal and conditional densities

$$p(r) = \int_0^{2\pi} p(r,\theta)\,d\theta = 2r$$

$$p(\theta|r) = \frac{p(r,\theta)}{p(r)} = \frac{1}{2\pi}$$

  - Integrating and inverting to find P(r), P-1(r), P(θ), and P-1(θ)

$$r = \sqrt{\xi_1}$$
$$\theta = 2\pi\xi_2$$

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- Uniformly sampling a hemisphere
  - Uniform sampling means $p(\omega) = c$
  - Normalization:

$$p(\theta,\phi) = \sin\theta\ p(\omega)$$

$$\int_{\mathcal{H}^2} p(\omega)\,d\omega = 1 \Rightarrow c\int_{\mathcal{H}^2} d\omega = 1 \Rightarrow c = \frac{1}{2\pi} \longrightarrow p(\omega) = 1/(2\pi) \longrightarrow p(\theta,\phi) = \sin\theta/(2\pi)$$

  - Consider sampling θ:

$$p(\theta) = \int_0^{2\pi} p(\theta,\phi)\,d\phi = \int_0^{2\pi} \frac{\sin\theta}{2\pi}\,d\phi = \sin\theta$$

  - Compute the conditional density for φ:

$$p(\phi|\theta) = \frac{p(\theta,\phi)}{p(\theta)} = \frac{1}{2\pi}$$

- Use 1D inversion technique to sample:

$$P(\theta) = \int_0^{\theta} \sin\theta'\, d\theta' = 1 - \cos\theta$$

$$P(\phi|\theta) = \int_0^{\phi} \frac{1}{2\pi}\, d\phi' = \frac{\phi}{2\pi}$$

- Inversion is straightforward

$$\theta = \cos^{-1}\xi_1$$
$$\phi = 2\pi\xi_2.$$

$$\rightarrow$$

$$x = \sin\theta\cos\phi = \cos(2\pi\xi_2)\sqrt{1-\xi_1^2}$$
$$y = \sin\theta\sin\phi = \sin(2\pi\xi_2)\sqrt{1-\xi_1^2}$$
$$z = \cos\theta = \xi_1$$

- **Cosine-weighted hemisphere sampling**
  - It is useful to have a cosine distribution over the hemisphere (the incident cosine term)
  - We require: $p(\omega) \propto \cos\theta$
  - Derive as before:

$$\int_{\mathcal{H}^2} c\, p(\omega)\, d\omega = 1 \qquad d\omega = \sin\theta\, d\theta\, d\phi \qquad p(\theta,\phi) = \sin\theta\, p(\omega)$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} c\cos\theta\sin\theta\, d\theta\, d\phi = 1$$

$$c\, 2\pi \int_0^{\pi/2} \cos\theta\sin\theta\, d\theta = 1$$

$$c = \frac{1}{\pi}$$

$$\rightarrow \boxed{p(\theta,\phi) = \frac{1}{\pi}\cos\theta\sin\theta}$$

重要性采样：

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$$

如果$p(x)$和$f(x)$近似，则从$p(x)$中采样收敛很快

九、重建

在一个像素附近插值采样以进行重建，计算一个加权平均数：

$$I(x,y) = \frac{\sum_i f(x-x_i, y-y_i)L(x_i,y_i)}{\sum_i f(x-x_i, y-y_i)}$$

- $L(x_i, y_i)$: the radiance value of the i-th sample at (x_i,y_i)
- f is a filter function

高斯滤波：轻微模糊

米切尔滤波：接收负值，提高边缘锐度

十、全局光照

双向路径追踪：

- **Fixed-length bounce rendering**
  - Enumerating all possibilities
    - s : number of light source ray path
    - t : number of camera ray path
    - Rendering is based on a fixed length L: s+t=L
  - s+t=1: direct emission

光子估计：光源发射，环境存储光子信息，当作二次光源

- **Radiance estimate**
  - Incoming flux is approximated using the photon map
  - Searching the nearest n photons
  - Each photon p has equal power (energy)

$$L_r(x,\vec{\omega}) = \int_{\Omega_x} f_r(x,\vec{\omega}',\vec{\omega})\frac{d\Phi_i(x,\vec{\omega}')}{dA_i} \approx \sum_{p=1}^{n} f_r(x,\vec{\omega}_p,\vec{\omega})\frac{\Delta\Phi_p(x,\vec{\omega}_p)}{\Delta A}$$

  - Assuming that the surface is locally flat
    - Projecting the sphere onto the tangent surface

$$\Delta A = \pi r^2$$

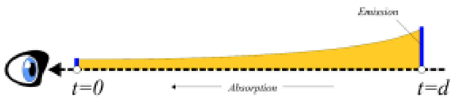
十一、体积渲染

体积渲染方程：

- **Volume rendering equation**
  - Integrating along the direction of light

$$I(D) = I_0\, e^{-\int_{s_0}^{D} \kappa(t)\,\mathrm{d}t} + \int_{s_0}^{D} q(s)\, e^{-\int_{s}^{D} \kappa(t)\,\mathrm{d}t}\,\mathrm{d}s$$

  - Integral in terms of transparency



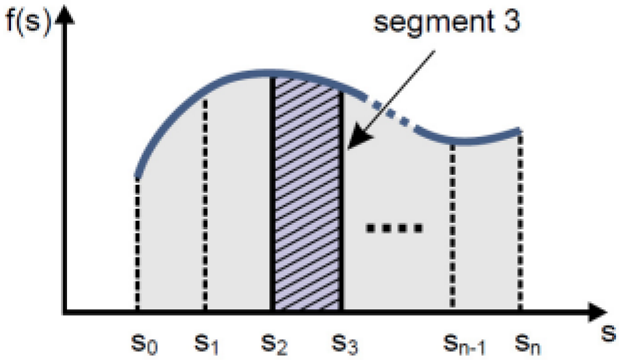
$$I(D) = I_0\, T(s_0, D) + \int_{s_0}^{D} q(s) T(s, D)\,\mathrm{d}s$$

Reduction of energy

Production of energy

离散化

$$I(s_i) = I(s_{i-1}) T(s_{i-1}, s_i) + \int_{s_{i-1}}^{s_i} q(s) T(s, s_i)\,\mathrm{d}s$$



- **Ray integral**
  - Quantity definition $I(s_i) = I(s_{i-1}) T(s_{i-1}, s_i) + \int_{s_{i-1}}^{s_i} q(s) T(s, s_i)\,\mathrm{d}s$

$$T_i = T(s_{i-1}, s_i)\,, \quad c_i = \int_{s_{i-1}}^{s_i} q(s) T(s, s_i)\,\mathrm{d}s$$

$$\Downarrow$$

$$I(D) = I(s_n) = I(s_{n-1}) T_n + c_n = \left(I(s_{n-2}) T_{n-1} + c_{n-1}\right) T_n + c_n = \ldots$$

$$\Downarrow$$

$$I(D) = \sum_{i=0}^{n} c_i \prod_{j=i+1}^{n} T_j\,, \quad \text{with } c_0 = I(s_0)$$

- **Compositing schemes**
  - **Front-to-back composition**
    - Viewing rays are traversed from eye into the volume

$$\hat{C}_i = \hat{C}_{i+1} + \hat{T}_{i+1} C_i \qquad C_{\mathrm{dst}} \leftarrow C_{\mathrm{dst}} + (1 - \alpha_{\mathrm{dst}}) C_{\mathrm{src}}$$
$$\hat{T}_i = \hat{T}_{i+1}(1 - \alpha_i) \xrightarrow{\text{Alpha blending}} \alpha_{\mathrm{dst}} \leftarrow \alpha_{\mathrm{dst}} + (1 - \alpha_{\mathrm{dst}}) \alpha_{\mathrm{src}}$$

  - **Back-to-front composition**
    - Viewing rays are traversed from back of the volume into the eye

$$\hat{C}_i = \hat{C}_{i-1}(1 - \alpha_i) + C_i$$
$$\hat{T}_i = \hat{T}_{i-1}(1 - \alpha_i) \xrightarrow{\text{Alpha blending}} C_{\mathrm{dst}} \leftarrow (1 - \alpha_{\mathrm{src}}) C_{\mathrm{dst}} + C_{\mathrm{src}}$$