Probability and Statistics Homework 7

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1. Suppose that a random variable X has the density given by

$$f(x) = \begin{cases} 0, & x \leq 0 \\ cx, & x \in (0,2) \\ 0, & x \geq 2, \end{cases}$$

where c is a certain constant.

(a) Find c.

(b) Find the cumulative distribution function of X.

(c) Find $P(X > 1)$.

(d) Find EX

(e) Find $Var(X)$.

(f) Find a formula for the density of $Y = e^X$.

(g) Find Ee^X.

Answer:
(a) For that $\int_{-\infty}^{\infty} f(x)\ dx = 1$, we have

$$\begin{aligned} \int_0^2 cx\ dx &= \frac{1}{2}cx^2\Big|_0^2 \\ &= 2c = 1 \\ \Rightarrow c &= \frac{1}{2} \end{aligned}$$

(b) According to defintion, $F_X(y) = P(-\infty < x \leq y)$.

1. For $y \leq 0$,

$$F_X(y) = 0$$

2. For $y \in (0,2)$,

$$F_X(y) = \int_0^y \frac{1}{2}x\ dx = \frac{1}{4}y^2$$

3. For $y \geq 2$,

$$F_X(y) = 1$$

Therefore, the cdf of X is

$$F_X(y) = \begin{cases} 0, & y \leq 0 \\ \frac{1}{4}y^2, & y \in (0,2) \\ 1, & y \geq 2 \end{cases}$$

(c) According to the property of cdf

$$\begin{aligned} P(X > 1) &= 1 - P(X \leq 1) \\ &= 1 - F_X(y = 1) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

(d) According to the definition of expectation

$$\begin{aligned} EX &= \int_{-\infty}^{\infty} xf(x)\ dx \\ &= \int_0^2 \frac{1}{2}x^2\ dx \\ &= \frac{4}{3} \end{aligned}$$

(e) Since $Var(X) = E(X^2) - (E[X])^2$, and we have EX in (d), we are now calculate $E[X^2]$.

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)\ dx \\ &= \int_0^2 \frac{1}{2}x^3\ dx \\ &= 2 \end{aligned}$$

Therefore, the Variance of X is

$$\begin{aligned} Var(x) &= E[X^2] - (E[X])^2 \\ &= 2 - (\frac{4}{3})^2 \\ &= \frac{2}{9} \end{aligned}$$

(f) Considering the cdf of Y, for $x \in (0,2)$, $y \in (1, e^2)$

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(e^X \leq y) = P(x \leq \ln y) \\
&= F_X(\ln y)
\end{aligned}
$$

To find the pdf of Y, we do derivate to $F_Y(y)$, which is

$$
\begin{aligned}
f(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{d}{dy} F_X(\ln y) \\
&= \frac{1}{2y} \ln y
\end{aligned}
$$

Therefore, the pdf of Y is

$$
f(y) = \begin{cases} \frac{1}{2y} \ln y, & y \in (1, e^2) \\ 0, & \text{otherwise} \end{cases}
$$

(g) According to the definition of expectation and Y,

$$
\begin{aligned}
E[e^X] &= E[Y] \\
&= \int_{-\infty}^{\infty} y f(y) \; dy \\
&= \int_{1}^{e^2} y \frac{1}{2y} \ln y \; dy \\
&= \frac{1}{2} \int_{1}^{e^2} \ln y \; dy \\
&= \frac{1}{2} (y \ln y \Big|_1^{e^2} - y \Big|_1^{e^2}) \\
&= \frac{1}{2} (e^2 + 1)
\end{aligned}
$$

2. A store-owner buys up to 100 liters of milk from a wholesaler at the beginning of the day with the price per liter equal to $2 - (x/400)$ dollars, where x is the total amount (in liters) that he buys. He then sells it during the day at 3 dollar per liter. Any unsold milk is wasted. The daily demand (in liters) is random, uniformly distributed on the interval $[0, 100]$. What amount of milk should the store-owener buy to maximize his expected profit?

Answer:

Denote the daily demand as Y, $Y \sim \text{Uniform}(0, 100)$. Denote the milk that are sold daily is M, where

$$M = \min(x, y)$$

The daily profit is thus

$$\text{Profit} = 3M - x[2 - (x/400)] = 3\min(x, y) - x[2 - (x/400)]$$

Firstly, we are going to find the expected profit given $Y \sim \text{Uniform}(0, 100)$, where $f(y) = \frac{1}{100}, y \in [0, 100]$.

Consider two cases: (1) $y \leq x$, then $M = min(x, y) = y$ (2) $y > x$, then $M = min(x, y) = x$. So, the expected profit is

$$\begin{aligned}
E[\text{Profit}] &= 3E[min(x, y)] - x[2 - (x/400)] \\
&= 3(\int_0^x y\frac{1}{100}\, dy + \int_x^{100} x\frac{1}{100}\, dy) - x[2 - (x/400)] \\
&= 3[\frac{1}{200}x^2 + \frac{x}{100}(100 - x)] - x[2 - (x/400)] \\
&= x - \frac{5}{400}x^2
\end{aligned}$$

where $x \in [0, 100]$. After than, to find the maximum of expected profit, we set its first derivative to 0

$$\begin{aligned}
\frac{d}{dx}(x - \frac{5}{400}x^2) &= 0 \\
\Rightarrow x &= 40
\end{aligned}$$

To check whether $x = 40$ is the maximum point, we calculate its second derivative

$$\frac{d^2}{dx^2}(x - \frac{5}{400}x^2) = -\frac{1}{40} < 0$$

Therefore, $x = 40$ is the point to maximize expected profit. The store-owner should buy 40 liters of milk.

3. The density of a random variable X is

$$f(x) = \begin{cases} a + bx^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find a and b if you know that $EX = \frac{5}{8}$.
Answer:
According to the definition of expectation, we have $\int_{-\infty}^{\infty} xf(x)\ dx = \frac{5}{8}$. Thus,

$$\begin{aligned} \int_0^1 x(a + bx^2)\ dx &= \frac{b}{4}x^4\Big|_0^1 + \frac{a}{2}x^2\Big|_0^1 \\ &= \frac{b}{4} + \frac{a}{2} = \frac{5}{8} \end{aligned}$$

Also, since $\int_{-\infty}^{\infty} f(x)\ dx = 1$, we have

$$\begin{aligned} \int_0^1 a + bx^2\ dx &= ax\Big|_0^1 + \frac{b}{3}x^3\Big|_0^1 \\ &= a + \frac{b}{3} = 1 \end{aligned}$$

Combine these two equations we have

$$\begin{cases} 4a + 2b = 5 \\ 3a + b = 3 \end{cases} \Rightarrow \begin{cases} a = \dfrac{1}{2} \\ b = \dfrac{3}{2} \end{cases}$$

Therefore, $a = \frac{1}{2}, b = \frac{3}{2}$.