Probability and Statistics Homework 8

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1. Suppose that X is an exponential random variable with parameter $\lambda = 1$. Let $Y = [X]$ (i.e., Y is the integer part of X).

(a) Write a formula for the probability mass function of Y.

(b) Calculate $E(Y)$.

Answer:

(a) The pdf of X is

$$f(x) = e^{-x}, \quad x \geq 0$$

Since Y is the integer part of X, the value of Y is $Y = n$ for $n \leq x \leq n+1, n \in \mathcal{N}$. To find the probability that $Y = n$, we need to compute the probability that X falls in $[n, n+1]$, which is

$$\begin{aligned} P(Y = n) &= P(n \leq X \leq n+1) \\ &= \int_n^{n+1} e^{-x}\, dx \\ &= (1 - e^{-1})e^{-n} \end{aligned}$$

Therefore, the probability mass function of Y is

$$P(y = n) = (1 - e^{-1})e^{-n}, \quad n \in \mathcal{N}$$

(b) The expectation of Y is given by

$$\begin{aligned} E[Y] &= \sum_{n=0}^{\infty} n(1 - e^{-1})e^{-n} \\ &= (1 - e^{-1}) \sum_{n=0}^{\infty} ne^{-n} \end{aligned}$$

Let $S_n = \sum_{i=0}^{n} ie^{-i} = 1 \cdot e^{-1} + 2 \cdot e^{-2} + ... + n \cdot e^{-n}$. Subtract S_n and $e^{-1}S_n$, we have

$$S_n - e^{-1}S_n = e^{-1} + e^{-2} + ... + e^{-n} - n \cdot e^{-(n+1)}$$

Since $e^{-1} + e^{-2} + ... + e^{-n} = \frac{e^{-1}(1-e^{-n})}{1-e^{-1}}$, after simplification, S_n is given by

$$S_n = \frac{e^{-1}(1 - e^{-n})}{(1 - e^{-1})^2} - \frac{ne^{-(n+1)}}{1 - e^{-1}}$$

When n goes to infinity, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} ne^{-n} &= \lim_{n\to\infty} S_n \\
&= \lim_{n\to\infty} \frac{e^{-1}(1-e^{-n})}{(1-e^{-1})^2} - \frac{ne^{-(n+1)}}{1-e^{-1}} \\
&= \frac{e^{-1}}{(1-e^{-1})^2}
\end{aligned}$$

Therefore, the expectation of Y is given by

$$\begin{aligned}
E[Y] &= (1-e^{-1}) \sum_{n=0}^{\infty} ne^{-n} \\
&= (1-e^{-1}) \frac{e^{-1}}{(1-e^{-1})^2} \\
&= \frac{e^{-1}}{1-e^{-1}} \\
&= \frac{1}{e-1}
\end{aligned}$$

2. In the following problem, you should use the normal approximation to the binomial. Use a calculator, a computer program, or an online resource to evaluate (approximately) the required integrals. The answers will be approximate.

Supposed that a game is played where you win each round with probability equal to 1/4.

(a) If you play 1,000,000 rounds, what is your probability (approximately) to win at least 250,100 rounds?

(b) If you play 1,000,000 rounds, find such n that the probability of winning at least n rounds is around 90 percent?

(c) How many rounds should you play in order to have your chances of winning 24 percent of the rounds equal to 90 percent?

Answer:

(a) Check if approximation works gives by

$$np(1-p) = 1000000 \times \frac{1}{4} \times \frac{3}{4} = 187500 \geq 10$$

Therefore, the approximate probability to win at least 250100 rounds is

$$\begin{aligned}
P_{Binomial}(S_n \geq 250100) &= P_{Normal}(S_n > 250100 - 0.5) \\
&= P(\frac{S_n - np}{\sqrt{np(1-p)}} \geq \frac{250100 - 0.5 - np}{\sqrt{np(1-p)}}) \\
&= P(\frac{S_n - 1000000 \times \frac{1}{4}}{\sqrt{1000000 \times \frac{1}{4}(1-\frac{1}{4})}} \geq \frac{250100 - 0.5 - 1000000 \times \frac{1}{4}}{\sqrt{1000000 \times \frac{1}{4}(1-\frac{1}{4})}}) \\
&\approx P(Z \geq 0.22979) \\
&= 1 - P(Z \leq 0.22979) \\
&= 1 - \Phi(0.22979) \\
&= 0.409
\end{aligned}$$

(b) The probability that at least n rounds is 90 percent is given by

$$P(Z \geq z) = 0.9$$

From code, the corresponding z-score $z = -1.2816$. Therefore, the formula is given by

$$\frac{x - 0.5 - np}{\sqrt{np(1-p)}} = z = -1.2816$$

The above formula gives result $x = 249445.5719$. So, the probability of winning at least 249445 rounds is around 90%.

(c) Similar to question (2), the z-score given $P(Z \geq z) = 0.9$ is $z = -1.2816$. Let the number of rounds be N. Then the formula is given by

$$\frac{0.24N - 0.5 - Np}{\sqrt{Np(1-p)}} = -1.2816$$

Given the value $p = \frac{1}{4}$, after calculation, the value of $N = 2978.61$.
Therefore, it should play 2978 rounds to have 90% chance of winning 24% of rounds.

3. X and Y are independent random variables uniformly distributed over (0,1). Compute the probability that the larger of the two is at least three times as large as the other one.

Answer:
The probability that the larger of the two is at least three times larger that the other one is given by

$$P = P(X \geq 3Y) + P(Y \geq 3X)$$

Since X, Y are symmetric, the probability can be simplified to

$$P = 2P(X \geq 3Y)$$

Given X, Y are independent uniform R.V., $f(x) = f(y) = 1, x \in (0,1), y \in (0,1)$, the probability of $P(X \geq 3Y)$ is thus

$$\begin{aligned}
P(X \geq 3Y) &= \int_0^1 \int_0^{\frac{1}{3}x} f(x)f(y)\ dydx \\
&= \int_0^1 \int_0^{\frac{1}{3}x} 1\ dydx \\
&= \int_0^1 \frac{1}{3}x\ dx \\
&= \frac{1}{6}x^2\Big|_0^1 \\
&= \frac{1}{6}
\end{aligned}$$

Therefore, the probability that larger is at least three times than the other is

$$P = 2P(X \geq 3Y) = \frac{1}{3}$$

4. The random variables X an Y have joint density

$$p(x,y) = \begin{cases} cxy(1-x) & \text{if } 0 < x < 1,\ 0 < y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where c is a positive constant.

(a) Find c.
(b) Are X and Y independent?
(c) Find $\mathrm{E}Y$.
(d) Find $\mathrm{Var}(X)$.

Answer:
(a) The joint density function must integrate to 1. So,

$$\int_0^1 \int_0^1 cxy(1-x)\ dydx = 1$$

Compute the integral as follows:

$$\begin{aligned} \int_0^1 \int_0^1 cxy(1-x)\ dydx &= \frac{1}{2}c \int_0^1 x(1-x)\ dx \\ &= \frac{1}{2}c(\int_0^1 x\ dx - \int_0^1 x^2\ dx) \\ &= \frac{1}{2}c(\frac{1}{2} - \frac{1}{3}) \\ &= \frac{1}{12}c = 1 \end{aligned}$$

Therefore, $c = 12$.

(b) If $p(x,y) = p(x)p(y)$, then X, Y are independent.
Firstly, compute $p(x)$.

$$\begin{aligned} p(x) &= \int_Y p(x,y) \\ &= \int_0^1 12xy(1-x)\ dy \\ &= 6x(1-x) \end{aligned}$$

Then, compute p(y)

$$\begin{aligned} p(y) &= \int_X p(x,y) \\ &= \int_0^1 12xy(1-x)\ dx \\ &= 2y \end{aligned}$$

Since $p(x)p(y) = 12xy(1-x) = p(x,y)$, X, Y are independent.

(c) The expected value of Y is computed from its marginal pdf in question (b)

$$\begin{aligned}
E[Y] &= \int_0^1 yf(y)\ dy \\
&= \int_0^1 2y^2\ dy \\
&= \frac{2}{3}
\end{aligned}$$

(d) The variance value of X can be computed from its marginal pdf in question (b)

$$\begin{aligned}
Var(X) &= E[X^2] - (E[X])^2 \\
&= \int_0^1 x^2 6x(1-x)\ dx - (\int_0^1 x6x(1-x)\ dx)^2 \\
&= 6(\int_0^1 x^3\ dx - \int_0^1 x^4\ dx) - 36(\int_0^1 x^2\ dx - \int_0^1 x^3\ dx)^2 \\
&= 6 \times \frac{1}{20} - 36(\frac{1}{12})^2 \\
&= \frac{3}{10} - \frac{1}{4} \\
&= \frac{1}{20}
\end{aligned}$$