Aug 30

## Introduction Prob vs Stat

↳ "chance"
"randomness"

**Applications:**
- physics
- finance
- computer science

A typical problem from probability: you toss a coin (fair 50/50) 100 times. How likely is it to land heads up at least 60 times?

- know system
- predict future

A typical problem from stat: You tossed a coin 100 times & it landed heads up 64/100 times. How certain are you that the coin is biased? (unfair?)

- use known data
- want to understand system

↳ applied prob.

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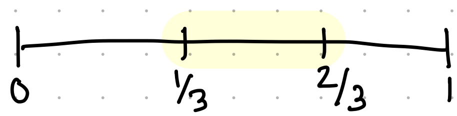
ex What is the prob of '5' coming up when we roll a die.

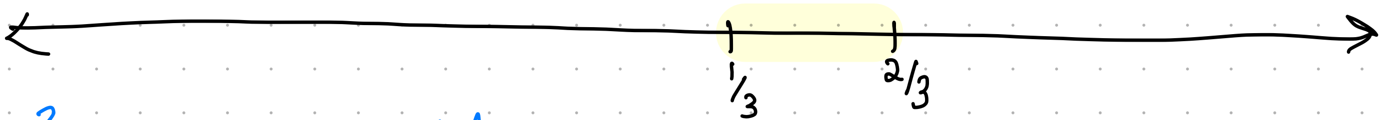
ans: $\frac{1}{6}$

ex What is the prob of when we drop a point randomly on the segment $[0,1]$ & it lands in $[\frac{1}{3}, \frac{2}{3}]$?

ans: $\frac{1}{3}$



ex What is the prob of dropping a point randomly on the entire real line $\mathbb{R}$ & it lands in $[\frac{1}{3}, \frac{2}{3}]$?




? Not well-defined

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## Sample Space (Ch 2)

Def'n sample space: space for all elementary outcomes

↳ denoted by $S$ OR $\Omega$

ex toss one coin

$S = \{H, T\}$ 2 outcomes

finite

$S = \{1, 0\}$

ex One one die: $S = \{1, 2, 3, 4, 5, 6\}$ 6 outcomes

finite

ex 3 tosses of a coin

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

8 outcomes

ex Someone wants to pass driving test. How many attempts?

$S = \{1, 2, 3, 4, \ldots\}$ infinite countable

ex a student is chosen & height is recorded. (no rounding)

$S = \mathbb{R}$ infinite uncountable

Def'n Event → subset of the sample space
↳ characteristic/property
→ denoted by capital letters $A, B, C, E, F$

ex Coin toss 3 times

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

$A = \{\text{first toss is head}\} = \{HHH, HHT, HTH, HTT\}$

$B = \{\text{exactly 2 heads}\} = \{HHT, HTH, THH\}$

$C = \{\text{the \# of heads is 3}\} = \{HHH\}$

$D = \{\text{the \# of heads is 4}\} = \{\ \} = \emptyset$ empty set

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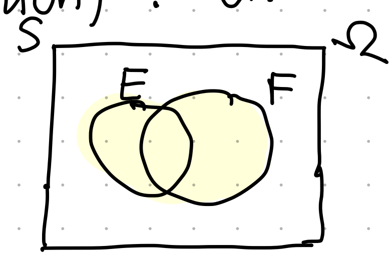
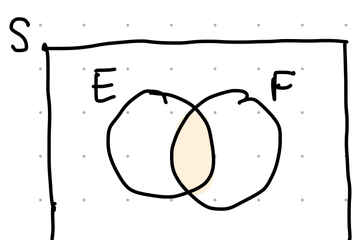
Operations with Events (Sets) Events $E$ & $F$.

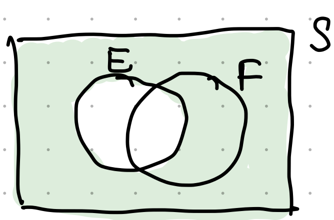
- $E \cup F$ (union) : either $E$ happens OR $F$ happens OR both



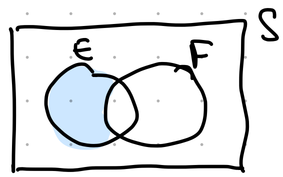

- $E \cap F$ (intersection) happens both in $E$ and $F$
  ($EF$)



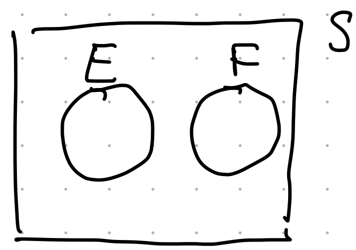

- $E^c$ (complement) when $E$ does NOT happen



- $E \setminus F$ $(E-F)$ Difference, happens in E but not in F




- If $E \cap F = \emptyset$, then E & F are called <u>disjoint</u> OR <u>Mutually</u> <u>exclusive</u>

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<u>ex</u> 3 coin toss

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$A = \{\text{first toss } H\} = \{HHH, HHT, HTH, HTT\}$

$B = \{\text{exactly } 2\ H\} = \{HHT, HTH, THH\}$

$C = \{\text{second toss } T\} = \{HTH, HTT, TTH, TTT\}$

Want: $A \cup B$ $\quad A \cap B$ $\quad A \cap C$ $\quad A^c$

$A \cup B = \{HHH, HHT, HTH, HTT, THH\}$

$A \cap B = \{HHT, HTH\}$

$A \cap C = \{HTH, HTT\}$ $\qquad A^c = \{THH, THT, TTH, TTT\}$

Useful :

$$E \cup F = F \cup E$$

$$E \cap F = F \cap E$$

$$E \cup (F \cup G) = (E \cup F) \cup G$$

$$E \cap (F \cap G) = (E \cap F) \cap G$$

$$\left.\begin{aligned} E \cap (F \cup G) &= (E \cap F) \cup (E \cap G) \\ E \cup (F \cap G) &= (E \cup F) \cap (E \cup G) \end{aligned}\right\} \text{Distributative Prop}$$

DeMorgan's Law: $(E \cup F)^c = E^c \cap F^c$ *

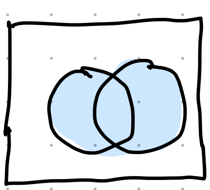
$$(E \cap F)^c = E^c \cup F^c$$

$$\left( \bigcup_{i=1}^{n} E_i \right)^c = \left( E_1 \cup E_2 \cup E_3 \cup E_4 \cup \dots \cup E_n \right)^c = \bigcap_{i=1}^{n} E_i^{\,c}$$

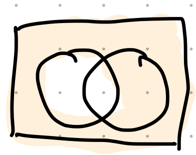
$$\left( \bigcap_{i=1}^{n} E_i \right)^c = \bigcup_{i=1}^{n} E_i^{\,c}$$

Proof : $(E \cup F)^c \stackrel{?}{=} E^c \cap F^c$

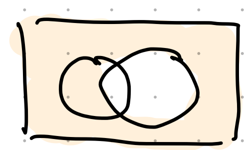
$E \cup F$



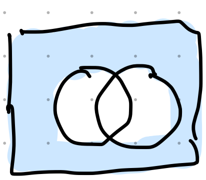
$E^c$



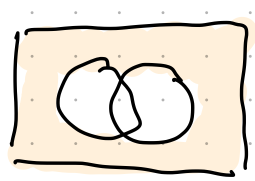
$F^c$



$(E \cup F)^c$



$E^c \cap F^c$



* Sometimes, not every subset of $S$, sample space, deserves to be called an event.

<u>$\sigma$-algebra</u> A collection $\mathcal{F}$ of subsets of $S$ is called a $\sigma$-alg if these 3 properties:

1. $S \in \mathcal{F}$ ($S$ is in $\mathcal{F}$)
2. If $E \in \mathcal{F}$ (if $E$ is in $\mathcal{F}$), then $E^c \in \mathcal{F}$.
3. If $E_1, E_2, E_3, \ldots \in \mathcal{F}$, then $E_1 \cup E_2 \cup \ldots \in \mathcal{F}$

<u>ex</u> $\sigma$-alg.

1. $S$ = any sample $\qquad \mathcal{F}$ = collection of all subsets of $S$
2. $S$ = any sample space $\qquad \mathcal{F} = \{S, \emptyset\}$
3. $S = \{1,2,3,4,5,6\}$ $\qquad \mathcal{F} = \{S, \emptyset, \{1,2,3\}, \{4,5,6\}\}$

$$= \{\{1,2,3,4,5,6\}, \emptyset, \{1,2,3\}, \{4,5,6\}\}$$

Not $\sigma$-alg: $\mathcal{F} = \{S, \emptyset, \{1,2\}, \{1,2,4,5\}\}$

fails: $\{1,2\}^c$ is not in $\mathcal{F}$

* From now on, we will have sample space $S$ & $\sigma$-alg $\mathcal{F}$.

<u>Probability</u> (probability measure) is a <u>function</u> from $\mathcal{F}$ to $\mathbb{R}$. Assigns a real # labeled $P(A)$ to every event $A$ in $S$.

$P(A)$: "prob of event A"

<u>Axiom 1</u> (Unit Interval Axiom)

For all events $A$, we have $0 \le P(A) \le 1$

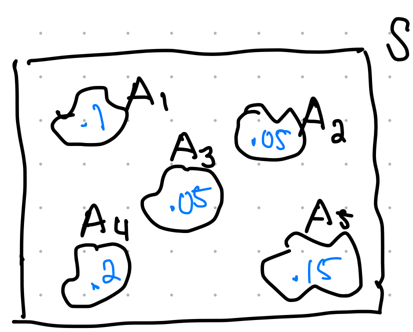
<u>Axiom 2</u> (Axiom of Certainty): $P(S) = 1$

$P(\Omega) = 1$

<u>Axiom 3</u> (Additivity Axiom for Mutually Ex. Events)

For any sequence of mutually ex. (disjoint) events $A_1, A_2, \ldots$ ($A_i \cap A_j = \emptyset$ when $i \neq j$)

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$




$P\left(\bigcup_{i=1}^{5} A_i\right) = .55$

<u>ex</u> Toss coin 2 times

$S = \{HH, HT, TH, TT\} \rightarrow$ 4 outcomes

$A = \{\text{at least } 1\ H\} = \{HH, HT, TH\} \longrightarrow P(A) = \frac{3}{4}$

3 outcomes

$B = \{\text{at least } 1\ T\} = \{TT, HT, TH\} \longrightarrow P(B) = \frac{3}{4}$

$C = \{\text{only } 1\ H\} = \{HT, TH\} \longrightarrow P(C) = \frac{2}{4}$

$D = \{\text{both the same}\} = \{TT, HH\} \longrightarrow P(D) = \frac{2}{4}$

$C \cap D = \{\ \} = \emptyset$ disjoint $\quad P(C \cap D) = 0$

$C \cup D = \{HH, HT, TH, TT\} \longrightarrow P(C \cup D) = 1$

Check Axiom 3: $P(C \cup D) = P(C) + P(D)$

$= \frac{2}{4} + \frac{2}{4} = 1$ ✓

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<u>ex</u> We can define <u>biased</u> coin. $S = \{H, T\}$ one toss

Define

$$\left.\begin{aligned} P(\{H\}) &= 0.4 \\ P(\{T\}) &= 0.6 \\ P(\emptyset) &= 0 \\ P(S) &= 1 \end{aligned}\right\} \text{Valid prob function}$$

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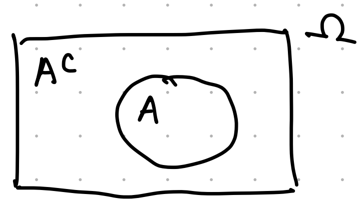
Let $A$ be an event. What $P(A^c)$ in terms of $P(A)$?

$A \cup A^c = \Omega$

$A \cap A^c = \emptyset$, mut. ex (disjoint)



$$\Rightarrow \quad A \cup A^c = \Omega \Rightarrow P(A \cup A^c) = P(\Omega) \underset{A.2}{=} 1$$

Since $A \cap A^c = \emptyset \Rightarrow P(A \cup A^c) \underset{A.3}{=} P(A) + P(A^c)$

$$1 = P(A) + P(A^c)$$

$$\Rightarrow P(A^c) = 1 - P(A)$$

$$P(A) = 1 - P(A^c)$$

*quite useful

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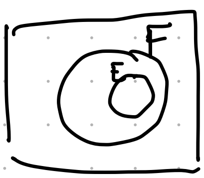
If $E \subseteq F$, then $P(E) \leq P(F)$




ex roll a die

6 outcomes

event $E$: toss a multiple of 3

$= \{3, 6\}$

$F$: toss a multiple of 6

$= \{6\}$

$F \subset E \qquad \{6\} \subset \{3, 6\}$

$P(F) \leq P(E)$

$\frac{1}{6} \leq \frac{2}{6}$ ✓