$$\vec{s}^{(1)} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \longrightarrow \vec{s}^{(2)} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3} & \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{2}{3} \end{bmatrix}$$

$$\frac{1}{4}+\frac{1}{6} \qquad \frac{1}{4}+\frac{1}{3}$$

$$\vec{s}^{(3)} = \vec{s}^{(2)} Q = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

adds to 1 ✓

$$\begin{bmatrix} \frac{5}{12}\cdot\frac{1}{2}+\frac{7}{12}\cdot\frac{1}{3} & \frac{5}{12}\cdot\frac{1}{2}+\frac{7}{12}\cdot\frac{2}{3} \end{bmatrix}$$

$$\frac{5}{24}+\frac{7}{36} \qquad \frac{5}{24}+\frac{14}{36}$$

$$\begin{bmatrix} \frac{29}{72} & \frac{43}{72} \end{bmatrix}$$

adds to 1 ✓

Nov 15th What happens as $n \to \infty$? More time steps. Does it reach an equilibrium?

What does $\vec{s}Q^n$ approach as $n \to \infty$?

↳ It depends on the structure

↳ Not every Markov chain will

"stabilize"

Stationary / Invariant Distribution

$$\vec{\pi} P = \vec{\pi} \qquad (\vec{S} Q = \vec{S})$$

P ↓ matrix Q ↓ matrix

if exists: Questions

1. does sol'n exist?
2. Is it unique?
3. Does it converge to $\vec{\pi}$?
4. How to compute

First: Classify Markov Chains

Chain is irreducible ("connected / communicate") if it is possible (w/ pos prob) to get from anywhere to anywhere (states) w/ finite # of steps

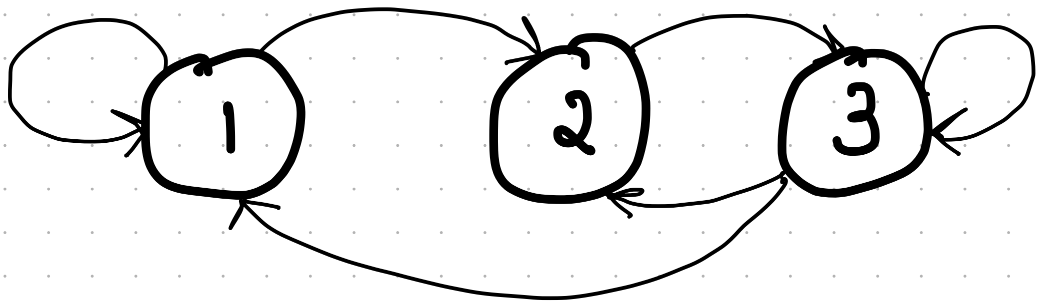
↳ all states communicate w/ each other

↳ No isolated states

Not irreducible → reducible → can always try to split into irreducible components & study them & bring together

A state is recurrent if starting at a state, the chain has prob 1 of returning to that state
↳ otherwise, "transient"

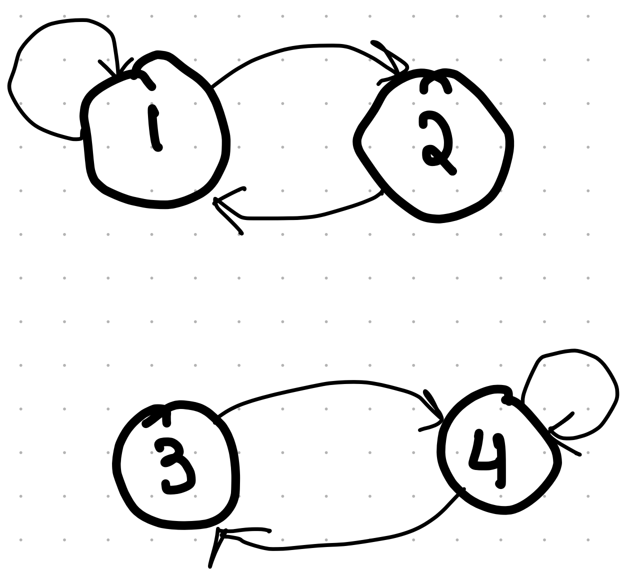
ex A



irreducible → since we can get from one state to another in a sequence in tim

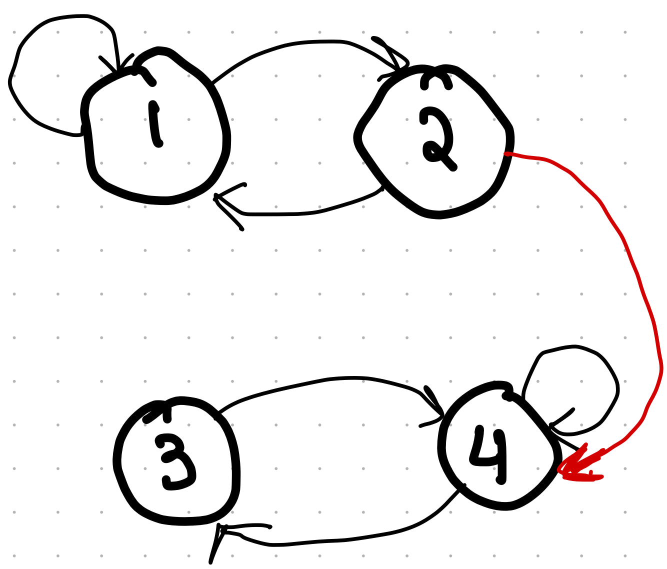
all recurrent as well

ex B



Not irreducible since we can never get from states 1-2 to 3-4 all states are recurrent since all states have a prob 1 of returning to that state

What if we add an edge?



Still is not irreducible since from 4 (or 3) we cannot reach 1 or 2

However: 1-2 transcient

3-4 recurrent

<u>ex C</u>

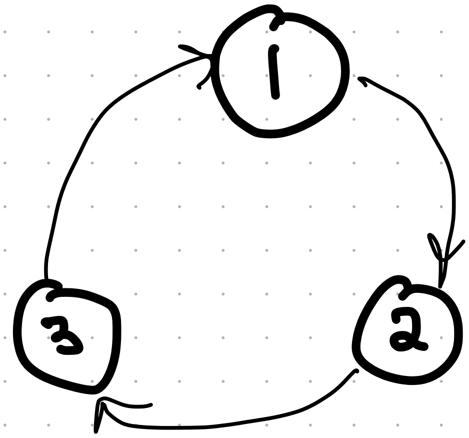
Gambler's Ruin

- Not irreducible, cannot go from 3 to 2 or 1 or 0

- States 0,3 recurrent

  States 1,2 transcient

ex D



irreducible
periodic chain
all recurrent

Back to stationary/invariant sol'n

$\vec{\pi} P = \vec{\pi}$ $\quad (\vec{s} Q = \vec{s})$

Existence: For any finite Markov chain w/ transition matrix P

↳ there is always at least one stationary sol'n $\vec{\pi}$, satisfing $\vec{\pi} P = \vec{\pi}$

Uniqueness For any irreducible Markov chain, then

↳ stationary sol'n is unique

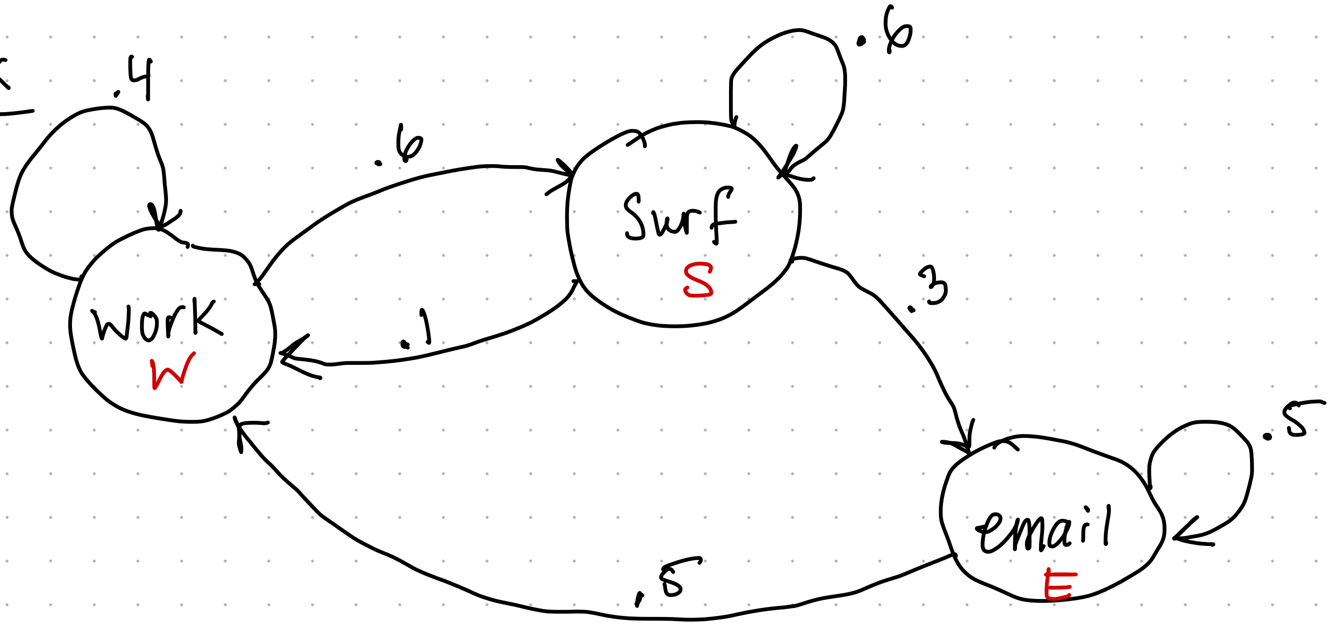
Convergence (Ergodic Theorem)

For any irreducible & aperiodic chain, then

Ergodic

the chain is "ergodic" and the distribution converges to the stationary sol'n regarding initial state

ex



Stationary sol'n? irreducible & aperiodic ✓
sol'n exists & is unique

Compute stationary sol'n: $\vec{\pi} P = \vec{\pi}$

$$\begin{bmatrix} \pi_W & \pi_S & \pi_E \end{bmatrix} \begin{array}{c} \begin{array}{ccc} W & S & E \end{array} \\ \begin{bmatrix} .4 & .6 & 0 \\ .1 & .6 & .3 \\ .5 & 0 & .5 \end{bmatrix} \end{array} = \begin{bmatrix} \pi_W & \pi_S & \pi_E \end{bmatrix}$$

all rows add to 1 $\longrightarrow$ $\pi_W + \pi_S + \pi_E = 1$

$\rightarrow \quad 0.4\pi_W + 0.1\pi_S + 0.5\pi_E = \pi_W$

$0.6\pi_W + 0.6\pi_S = \pi_S$

$0.3\pi_S + 0.5\pi_E = \pi_E$

$\pi_W + \pi_S + \pi_E = 1$

Solve for π_W, π_S, π_E

$0.6\pi_W = 0.4\pi_S$

$\pi_W = \frac{4}{6}\pi_S$

$0.5\pi_E = 0.3\pi_S$

$\pi_E = \frac{3}{5}\pi_S$

$\frac{4}{6}\pi_S + \pi_S + \frac{3}{5}\pi_S = 1$

$\pi_S = \frac{15}{34}$

$\pi_W = \frac{10}{34}$

$\pi_E = \frac{9}{34}$

$$\longrightarrow \vec{\pi} = \begin{array}{c} \begin{bmatrix} \frac{10}{34} & \frac{15}{34} & \frac{9}{34} \end{bmatrix} \\ \begin{matrix} \quad W & \quad\; S & \quad\; E \end{matrix} \end{array}$$

LLN for Markov Chains

For an ergodic Markov Chain (irreducible, and aperiodic), the fraction of time the Markov chain spends in each state approximates the stationary probability of that state $\longrightarrow$ regardless of starting point

example above: spend $\frac{9}{34} \approx \frac{1}{3}$ time emailing

—— end Markov Chain

Random Process (in general)

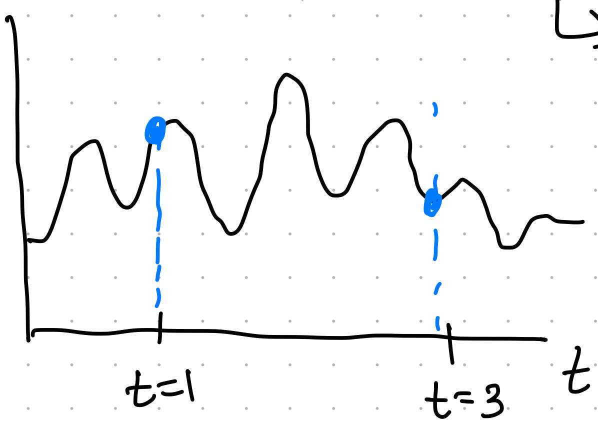
↳ RV over a period of time → dependent on time

↳ Markov chain is an example

Random Process is a collection of random variables usually indexed by time (or sometimes by space/position)

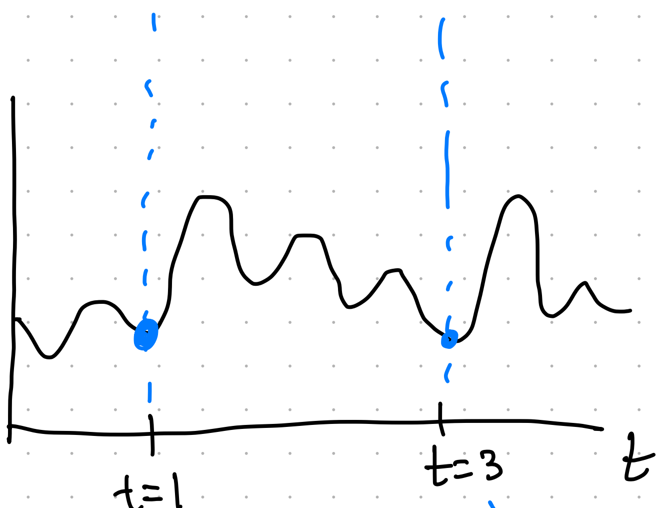
ex measure phone waves of phone calls

↳ amplitude



one phone call

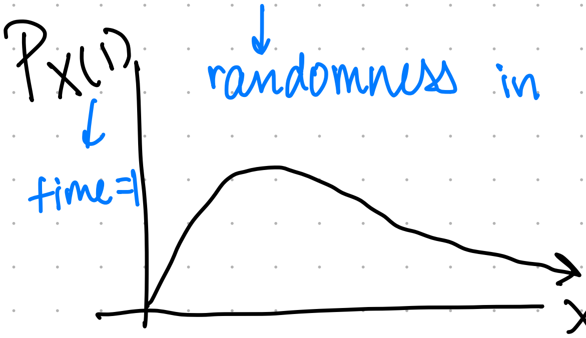
Sample function

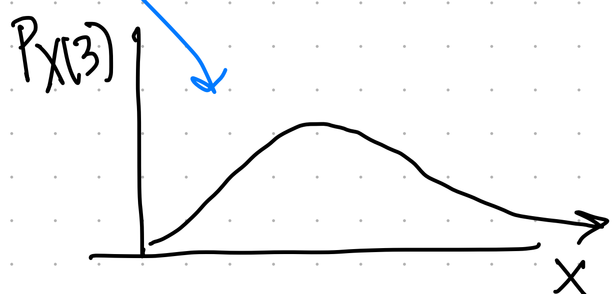


another call

Sample function

randomness in time=1





↑ ↑

probability distributions

Random process looks at these sample function and tries to characterize them in terms of random variables

↳ Random process is a lot of R.V. next to each other in time

Consider a random experiment with Sample Space S

If a time function $X(t,s)$ is assigned to each outcome $s \in S$, and $t \in [0,\infty)$ then the collection of such functions, $X(t,s)$, is called a <u>random process.</u>

if t is fixed $t = T$, $X(t,s) = X(T,s)$ is a random variable

However, if s is fixed, $s = s_i$

$X(t, s_i) = X_i(t)$

is a single function of time t, is called sample function of the random process

ex Let $\{X(t) \mid t \in [0,\infty)\}$ be defined as

$$X(t) = A + Bt \qquad \text{for all } t \in [0,\infty)$$

$A, B \sim N(1,1)$ independent

a) find all possible functions

b) define the R.V. $Y = X(1)$. Find the pdf of Y.

a) $X(t) = \underbrace{A + Bt}_{\text{random}}$ $\longrightarrow$ once A & B are known, we have $X(t)$

$\therefore f(t) = a + bt \quad t \geq 0, \quad a, b \in \mathbb{R}$

sample function

b) $Y = X(1) = A + B(1) = A + B$

$Y = A + B \qquad A, B \sim N(1,1)$ indep

$Y \sim N(\underset{?}{\mu}, \underset{?}{\sigma^2})$ (sum of normal $\rightarrow$ normal)

$$E[Y] = E[A+B] = E[A] + E[B] = 1+1 = 2$$

$$Var(Y) = Var(A+B) = Var(A) + Var(B) = 1+1=2$$

b/c indep

$$Y \sim \mathcal{N}(2,2)$$

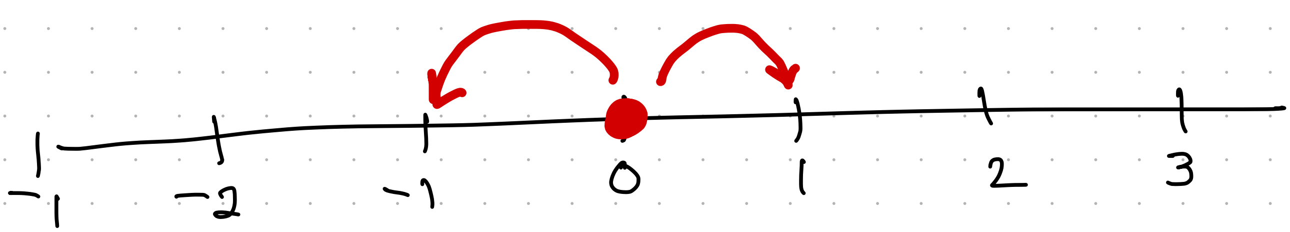
$$f_Y(y) = \frac{1}{\sqrt{2\pi \cdot 2}} e^{-\frac{(y-2)^2}{4}}$$

Random Walk applications → finance
→ web searching
→ biology

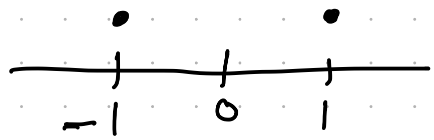
↳ very general type of random process

↳ series of steps, direction of steps is determined probabilistically

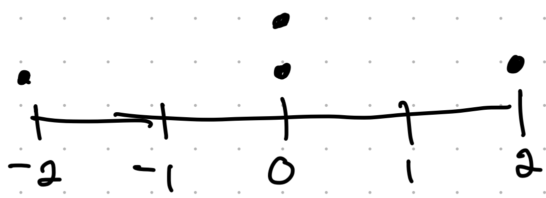
ex On integers: say you start at 0: flip a coin → heads +1 ("simple" $p = 1/2$)
tails −1



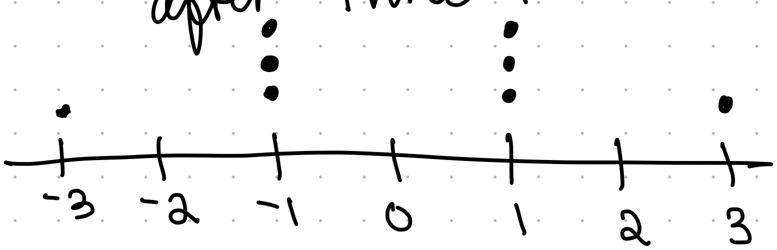
after one turn → ½ prob standing at 1
½ " " " −1



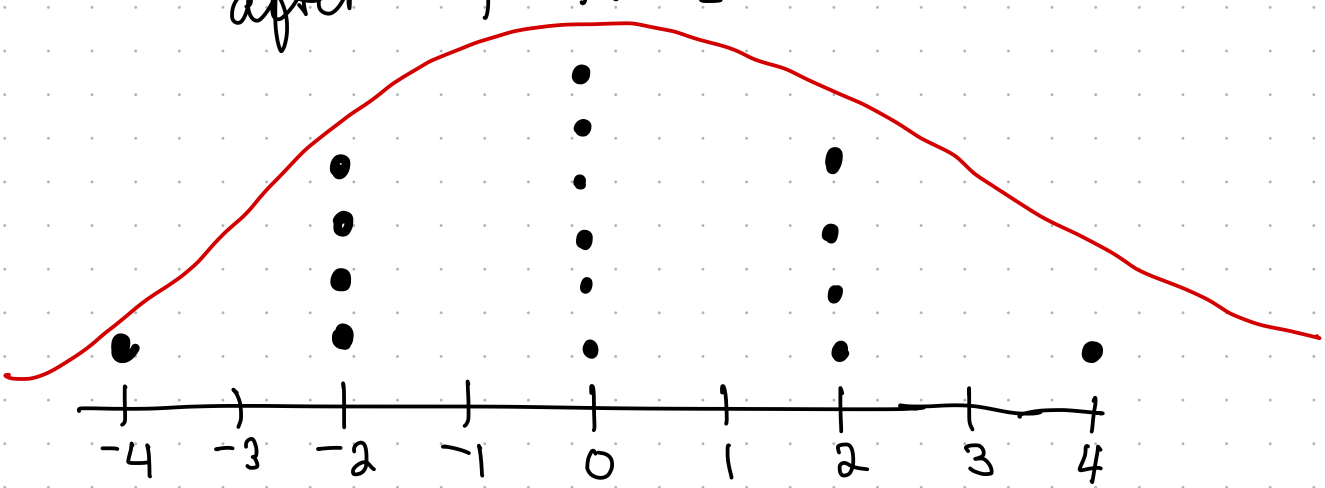
after two turns → ¼ " " " +2
½ " " " 0
¼ " " " −2



after three turns 1/8 " " " −3
3/8 " " " −1
3/8 " " " 1
1/8 " " " 3



after 4 turns



→ starts to spread out → Binomial

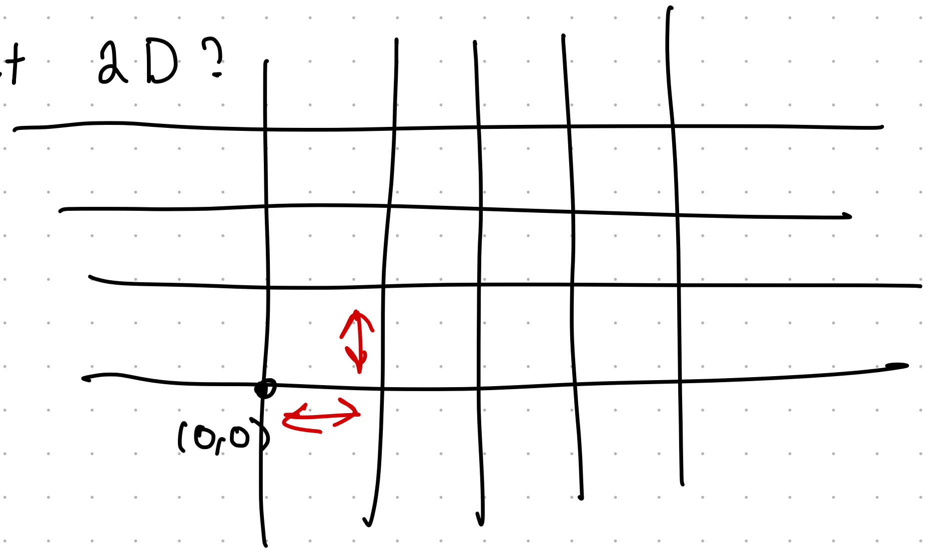
as $n \to \infty$ → approx by Normal

↳ center has most of distribution

Also pattern: odd turns → only odd # for random walk

even turns → " even #

What about 2D?



→ Still more likely to be at the origin

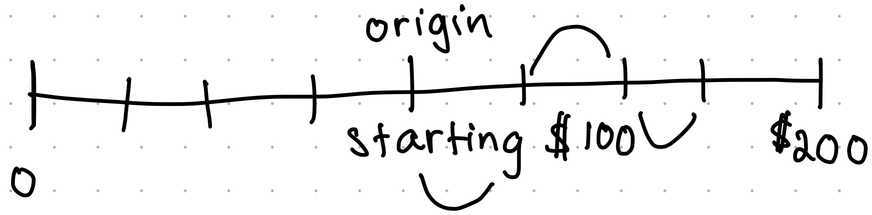
↳ Normal distribution

1D, 2D → <u>recurrent</u> → guaranteed to return to starting position (∞-many times)

<u>3D or higher</u> → chance of never returning "more space"

↳ <u>transcient</u>

ex



Gambler's Ruin (random walk)

We can have random walks w/ endpts

ex Markov chain → moving from one state another

ex Can have an unfair coin $p \neq \frac{1}{2}$

($p = 1/2$ "simple")