Oct 25

Conditional Prob

recall: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ and

Discrete Joint: $P_{X|Y}(x|y) = \dfrac{P_{XY}(x,y)}{P_Y(y)}$ and $\quad P_Y(y) \neq 0$

Cont Joint: $f_{X|Y}(x|y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$ ← Joint, ← marginal

* If X, Y are independent R.V.:

$$f_{X|Y}(x|y) = f_X(x)$$ (y gives you no new info)

Suppose $p(x,y)$ is a joint pmf of X, Y,

$p(0,0) = .4 \qquad p(1,0) = .1 \qquad X = \{0,1\}$

$p(0,1) = .2 \qquad p(1,1) = .3 \qquad Y = \{0,1\}$

Calculate the conditional prob mass function of X given Y = 1.

$$P_{X|Y}(X|Y=1) = \frac{P_{XY}(x,1)}{}$$

$P_Y(1)$

sum x

$$P_Y(1) = p(0,1) + p(1,1) = .2 + .3 = .5$$

pmf: $$\underset{x=0}{P_{X|Y}(0|1)} = \frac{p(0,1)}{P_Y(1)} = \frac{.2}{.5} = \frac{2/10}{5/10} = \frac{2}{5}$$

$$\underset{x=1}{P_{X|Y}(1|1)} = \frac{p(1,1)}{P_Y(1)} = \frac{.3}{.5} = \frac{3}{5}$$

---

ex

$$f_{XY}(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & 0<x<1,\ 0<y<1 \\ 0 & \text{Otherwise} \end{cases}$$

Compute the conditional density (pdf) of X given $Y=y$ where $0<y<1$.

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$ ← $f_{XY}(x,y)$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{12}{5}x(2-x-y)\,dx$$

$$= \frac{12}{5}\int_0^1 (2x - x^2 - xy)\,dx$$

$$= \frac{12}{5}\left(\frac{2x^2}{2} - \frac{x^3}{3} - \frac{x^2}{2}y\right)\Bigg|_{x=0}^{x=1}$$

$$= \frac{12}{5}\left(1 - \frac{1}{3} - \frac{y}{2}\right)$$

$$= \frac{12}{5}\left(\frac{2}{3} - \frac{y}{2}\right)$$

$$\Rightarrow f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\cancel{\frac{12}{5}}\, x(2-x-y)}{\cancel{\frac{12}{5}}\left(\frac{2}{3} - \frac{y}{2}\right)}$$

$$= \frac{6x(2-x-y)}{4-3y}$$

Sums of R.V.

X, Y be continuous, independent R.V.,

the pdf of $X+Y$ is given by

the convolution

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

(derivation: $P(X+Y \le a) = \iint_{\{x+y \le a\}} f_{XY}(x,y)\,dx\,dy$

$f_{XY} \xrightarrow{\text{Ind}} f_X \cdot f_Y$

---

ex Suppose we choose 2 #s at random from $[0, \infty)$ w/ an exp density w/ parameter λ. What is the pdf of sum?

$$f_X(x) = \lambda e^{-\lambda x} \quad x \ge 0$$

$$f_Y(y) = \lambda e^{-\lambda y} \quad y \ge 0$$

$$\underbrace{f_{X+Y}}_{Z}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\,dy$$

$$= \int_0^{\infty} \lambda e^{-\lambda(a-y)} \lambda e^{-\lambda y}\,dy$$

λ, a constants

$$= \lambda^2 \int_0^{\infty} e^{-\lambda a} \underbrace{e^{\lambda y} e^{-\lambda y}}_{e^{\lambda y - \lambda y} = e^0 = 1} dy$$

$$= \lambda^2 e^{-\lambda a} \int_0^{\infty} dy$$

$$= \lambda^2 e^{-\lambda a} \int_0^a dy \qquad 0 \le a - y < \infty$$

$$= \lambda^2 e^{-\lambda a} \, y \Big|_0^a = \underbrace{\lambda^2 e^{-\lambda a} \cdot a}_{\text{sum of 2 exp R.V. } \lambda_1 = \lambda_2}$$

$$\underline{f_{X+Y}(x,y) = \lambda^2 a e^{-\lambda a}}$$

Sums of Normal R.V.

if X_i $i = 1, \ldots, n$ are independent normal R.V.

w/ respective μ_i, σ_i^2, then

$\sum_{i=1}^{n} X_i$ is normally distributed with

mean $= \sum_{i=1}^{n} \mu_i$ & var $= \sum_{i=1}^{n} \sigma_i^2$

ex

distribution

$X \sim \mathcal{N}(\mu_X, \sigma_X^2)$

$Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$

then $X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$

$X - Y \sim \mathcal{N}(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2)$

ex class 1 score on midterm $C_1 \sim \mathcal{N}(95, 25)$

class 2 score on " $C_2 \sim \mathcal{N}(65, 36)$

a) What is the prob that the combined average grades will be less than 90?

$$C_1 + C_2 \sim \mathcal{N}(95+65,\ 25+36) = \mathcal{N}(160, 61)$$

$$P\left(\frac{C_1+C_2}{2} < 90\right) = P(C_1+C_2 < 180)$$

$$= P\left(\frac{C_1+C_2-160}{\sqrt{61}} < \frac{180-160}{\sqrt{61}}\right)$$

$$= P(Z \leq 2.56) = \Phi(2.56)$$

$$= .9961$$

b) What is the prob that someone's grade from C_2 will have scored higher than someone in C_1?

$C_2 - C_1 \sim \mathcal{N}(65-95,\ \overset{25+36}{61}) = \mathcal{N}(-30, 61)$

$$P(C_2 > C_1) = P(C_2 - C_1 > 0)$$

$$= P\left(\frac{C_2 - C_1 - (-30)}{\sqrt{61}},\ \frac{0-(-30)}{\sqrt{61}}\right)$$

$$P\left(Z > \frac{30}{\sqrt{61}}\right) = P(Z > 3.84)$$

$$= 1 - P(Z \leq 3.84) = 1 - \Phi(3.84)$$

$$= 0.00006$$

Discrete (Sums)

$$P_{X+Y}(a) = \sum_{y} P_X(x) P_Y(a-x)$$

ex: 2 Poisson λ_1 & λ_2

↳ Sum: $P(X+Y=K) = \dfrac{e^{-(\lambda_1+\lambda_2)} (\lambda_1+\lambda_2)^K}{K!}$

ex: 2 Binomial $X \sim Bin(n, p)$

$Y \sim Bin(m, p)$

$$(X+Y) \sim Bin(n+m, p)$$

$$P_{X+Y}(X+Y=K) = \binom{n+m}{K} p^K (1-p)^{(n+m)-K}$$

Ch. 7 (Ross) Expectations for Joint R.V.

Previously: $\left.\begin{array}{l} E[X] \\ E[Y] \end{array}\right\}$ get marginal pdf/pmf first

compute as usual

Now: $E\left[g(X,Y)\right] = \sum_x \sum_y g(x,y)\, p(x,y)$

$= \iint g(x,y) f(x,y)\, dx\, dy$

if: $g(X,Y) = X+Y$: $E[X+Y] = \iint (x+y) f(x,y)\, dx\, dy$

$= \iint x f(x,y)\, dx\, dy$

$+ \iint y f(x,y)\, dx\, dy$

$E[X+Y] = \sum \sum (x+y)\, p(x,y)$

$= \sum \sum x\, p(x,y)$

$+ \sum \sum y\, p(x,y)$

$\Rightarrow E[X+Y] = E[X] + E[Y]$

* careful $E[XY] \neq E[X]E[Y]$ in all cases

ex the pmf for X & Y:

		$Y=0$	$Y=1$
X	0	0.2	0.7
	1	0.0	0.1

Find $E[XY] = \sum_{\substack{x \\ i}} \sum_{\substack{y \\ j}} x_i y_j \, P(x_i, y_j)$

$\vee$
g

$$= \overset{x=0 \;\; y=0}{0 \cdot 0 \, p(0,0)} + \overset{x=0 \;\; y=1}{0 \cdot 1 \, p(0,1)}$$

$$+ \underset{x=1,\, y=0}{1 \cdot 0 \cdot p(1,0)} + \underset{x=1,\, y=1}{1 \cdot 1 \, p(1,1)}$$

$$= p(1,1) = 0.1$$

ex $f(x,y) = \begin{cases} 10xy^2 & 0<x<y \quad 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

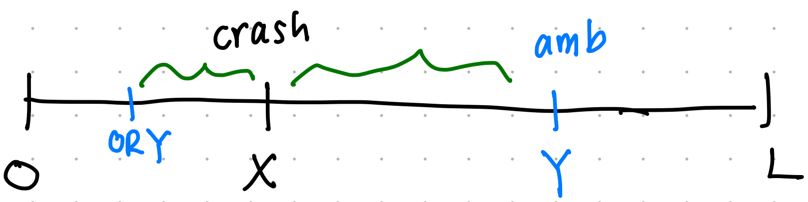
Find $E[XY] = \int_0^1 \int_0^y xy \cdot 10xy^2 \, dx \, dy$

$$= \int_0^1 10y^3 \int_0^y x^2 \, dx \, dy$$

$$= \int_0^1 10y^3 \cdot \frac{x^3}{3} \Big|_0^y \, dy$$

$$= \int_0^1 \frac{10}{3} y^6 \, dy = \frac{10}{3} \frac{y^7}{7} \Big|_0^1 = \frac{10}{3} \cdot \frac{1}{7} = \frac{10}{21}$$

Ex An accident occurs at a point X that is uniformly distributed on a road of length L. At the time of accident, ambulence is at a location Y that is also uniformly distributed. X & Y indep find the expected distance btwn ambulence & accident.



Want $E[|X-Y|]$

$$|X-Y| = \begin{cases} X-Y \\ -(X-Y) = Y-X \end{cases}$$

X, Y uni$(0,L)$, indep

$$f_X(x) = \begin{cases} \frac{1}{L} & 0<x<L \\ 0 & \text{other} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{L} & 0<y<L \\ 0 & \text{other} \end{cases}$$

$$f_{XY}(x,y) = f_X \cdot f_Y = \begin{cases} \frac{1}{L^2} & 0<x<L, \ 0<y<L \\ 0 & \text{Otherwise} \end{cases}$$

$$E[|X-Y|] = \int_0^L \int_0^L |x-y| \frac{1}{L^2} \, dy \, dx = \frac{1}{L^2} \int_0^L \int_0^L |x-y| \, dy \, dx$$

inner integral: $\displaystyle\int_0^L |x-y| \, dy = \int_0^x (x-y) \, dy + \int_x^L (y-x) \, dy$

$$= \left(xy - \frac{y^2}{2}\right)\Big|_0^x + \left(\frac{y^2}{2} - xy\right)\Big|_x^L$$

$$= x^2 - \frac{x^2}{2} - 0 + \left(\frac{L^2}{2} - Lx\right) - \left(\frac{x^2}{2} - x^2\right)$$

$$= \frac{L^2}{2} - Lx + x^2$$

$$= \frac{1}{L^2} \int_0^L \left(\frac{L^2}{2} - Lx + x^2\right) dx = \frac{1}{L^2}\left(\frac{L^2}{2} \cdot x - \frac{Lx^2}{2} + \frac{x^3}{3}\right)\Bigg|_{x=0}^{x=L}$$

$$= \frac{L}{3} = E[|X-Y|]$$

Covariance → tells us information btwn 2 R.V.

$$\text{Cov}(X,Y) = E\big[(X-E[X])(Y-E[Y])\big]$$

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

* if $\text{Cov}(X,Y) > 0$ then X & Y tend to move together (when X is high, then Y tends to be high)

$\text{Cov}(X,Y) < 0$ then X & Y tend to move opposite

$\text{Cov}(X,Y) = 0$ then X & Y do not consistently move together

→ If X, Y are indep → $E[XY] = E[X] \cdot E[Y]$

↳ $\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = 0$

↳ if X, Y are indep, then $\text{Cov}(X,Y) = 0$

* However, if $\text{Cov}(X,Y) = 0$, it doesn't automatically give indep

ex $P(X=0) = P(X=1) = P(X=-1) = \frac{1}{3}$

$$Y = \begin{cases} 0 & \text{if } X \neq 0 \\ 1 & \text{if } X = 0 \end{cases}$$

dependent

Y depends on X

$\text{Cov}(X,Y)$?

X	Y
0	0
0	1
1	0
1	1
-1	0
-1	1

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$E[XY] = \sum_x \sum_y xy\, p(x,y)$$

$$= \cancel{0 \cdot 0 \cdot p(0,0)} + \cancel{0 \cdot 1\, p(0,1)} + \cancel{1 \cdot 0\, p(1,0)}$$

$$+ 1 \cdot 1\, \underbrace{p(1,1)}_{0} + \cancel{-1 \cdot 0\, p(-1,0)} + -1 \cdot 1\, \underbrace{p(-1,1)}_{0}$$

by design of Y

$= 0$

$$E[X] = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} - 1 \cdot \frac{1}{3} = \frac{1}{3} - \frac{1}{3} = 0$$

$$\mathrm{Cov}(X,Y) = \underbrace{\cancel{E[XY]}}_{0} - \underbrace{\cancel{E[X]}}_{0}E[Y] = 0$$

Properties of Covariance

X, Y, Z R.V.

a, c constants
b

1. $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$
2. $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$
3. $\mathrm{Cov}(aX,Y) = a\,\mathrm{Cov}(X,Y)$
4. $\mathrm{Cov}(X+c,Y) = \mathrm{Cov}(X,Y)$ $\quad \mathrm{Cov}(X, Y+c) = \mathrm{Cov}(X,Y)$
5. $\mathrm{Cov}(X+Y,Z) = \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)$
6. $\mathrm{Var}(aX+bY) = a^2\,\mathrm{Var}\,X + 2ab\,\mathrm{Cov}(X,Y) + b^2\,\mathrm{Var}\,Y$

* if X, Y are indep $\longrightarrow$ $\mathrm{Cov}(X,Y) = 0$

$$\mathrm{Var}(aX+bY) = a^2\,\mathrm{Var}\,X + b^2\,\mathrm{Var}\,Y$$

Correlation Coefficient (of X & Y R.V.)

greek

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} \qquad -1 \le \rho(X,Y) \le 1$$

$\rho(X,Y) > 0$ "positive correlation"

$\rho(X,Y) < 0$ " negative " "

$= 0$ "uncorrelated"

ex $f(x,y) = \begin{cases} x+y & 0<x<1 \quad 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

Find

a) $Cov(X,Y)$

b) $Var(X)$ & $Var(Y)$

c) $\rho(X,Y)$

$$\Rightarrow E[XY] = \int_0^1\int_0^1 xy\cdot(x+y)\,dx\,dy = \int_0^1\int_0^1 x^2y + xy^2\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^3}{3}y + \frac{x^2}{2}y^2\right)\Big|_{x=0}^{x=1} dy = \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy$$

$$= \left(\frac{y^2}{6} + \frac{y^3}{6}\right)\Big|_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$f_X(x) = \int_0^1 (x+y)\,dy = \left(xy + \frac{y^2}{2}\right)\Big|_0^1 = x + \frac{1}{2}$$

$$f_Y(y) = \int_0^1 (x+y)\,dx = y + \frac{1}{2}$$

$$E[X] = \int_0^1 x\left(x+\frac{1}{2}\right)dx = \int_0^1 \left(x^2 + \frac{x}{2}\right)dx = \left(\frac{x^3}{3} + \frac{x^2}{4}\right)\Big|_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

$$E[Y] = \int_0^1 y\left(y+\frac{1}{2}\right)dy = \frac{7}{12}$$

$$\Rightarrow Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$= \frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12} = -\frac{1}{144}$$

b) $Var(X)$, $Var(Y)$:

$$E[X^2] = \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = \int_0^1 \left(x^3 + \frac{1}{2}x^2\right)dx = \frac{5}{12}$$

$$E[Y^2] = \int_0^1 y^2\left(y+\tfrac{1}{2}\right)dy = \frac{5}{12}$$

$$\Rightarrow \mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144} = \mathrm{Var}(Y)$$

c) $$\rho(x,y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\cdot \mathrm{Var}(Y)}} = \frac{-1/144}{\sqrt{\frac{11}{144}\cdot\frac{11}{144}}} = \frac{-1}{11}$$

$$\simeq -.0909\ldots$$

<u>Cauchy-Schwarz</u> inequality

$$(E[XY])^2 \le E[X^2]\cdot E[Y^2]$$