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Question 1

(a) This is a combination problem.

We select 3 women from 8 and 3 men from 6. The number of combinations is

$$C_8^3 \times C_6^3 = 56 \times 20 = 1120.$$

(b) We can count the combinations that these two men serve together, and use answer in (a) to subtract that number.

Suppose these two men serve together, so there are only one man chosen from the rest 4.

Serve Together $= C_8^3 \times C_4^1 = 56 \times 4 = 224$.

Two men don't serve Together $= 1120 - 224 = 896$.

(c) This question is similar to (b), so we need to count combination that 2 women serve together.

2 Women Serve Together $= C_6^1 \times C_6^3 = 6 \times 20 = 120$.

Two women don't serve together $= 1120 - 120 = 1000$.

(d) Similarly, we choose 2 man from rest 5 and 2 women from rest 7 to count the combinations that 1 man and 1 woman serve together.

1 man and 1 woman serve together $= C_5^2 \times C_7^2 = 10 \times 21 = 210$

They don't serve together $= 1120 - 210 = 910$

Question .2

(a) To check if they are independent, We need to justify whether

$$f(x,y) = f(x) f(y).$$

$$f(x) = \int_0^1 12xy(1-x)\,dy$$
$$= 6x(1-x)\,y^2\Big|_0^1$$
$$= 6x(1-x)$$

$$f(y) = \int_0^1 12xy(1-x)\,dx$$
$$= 6y\,x^2\Big|_0^1 - 4y\,x^3\Big|_0^1$$
$$= 2y.$$

Since $f(x)f(y) = 6x(1-x)\,2y = 12xy(1-x) = f(x,y)$;

X, Y are independent.

(b) $$E[X] = \int_0^1 x f(x)\,dx = \int_0^1 6x^2(1-x)\,dx$$
$$= 2x^3\Big|_0^1 - \frac{3}{2}x^4\Big|_0^1$$
$$= \frac{1}{2}$$

(c) $$Var(Y) = E[Y^2] - (E[Y])^2$$
$$= \int_0^1 y^2 f(y)\,dy - \left(\int_0^1 y f(y)\,dy\right)^2.$$
$$= \int_0^1 2y^3\,dy - \left(\int_0^1 2y^2\,dy\right)^2$$
$$= \frac{1}{2}y^4\Big|_0^1 - \left(\frac{2}{3}y^3\Big|_0^1\right)^2$$
$$= \frac{1}{2} - \frac{4}{9}$$
$$= \frac{1}{18}$$

Question 3

(a) To find the $P_{\geq 1}$ that earns ≥ 1 points, we can find the $P_{=0}$ that earns 0, and $P_{\geq 1} = 1 - P_{=0}$.

$P_{=0}$ occurs when all adjacent flips are not same, there are only two cases, Thus,

$$P_{\geq 1} = 1 - P_{=0}$$
$$= 1 - \frac{2}{2^5}$$
$$= 1 - \frac{1}{2^4} = \frac{15}{16}$$

(b) Let X_i be the point she earns at i-th turn. For each X_i, it has $\frac{1}{2}$ chance the same as previous flip and $\frac{1}{2}$ chance not.

So $\begin{cases} p(X_i = 1) = \frac{1}{2} \\ p(X_i = 0) = \frac{1}{2} \end{cases} \Rightarrow E(X_i) = \frac{1}{2}$.

$$E(X) = E(X_2 + X_3 + X_4 + X_5)$$
$$= E(X_2) + E(X_3) + E(X_4) + E(X_5)$$
$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2.$$

(c). To check if events are independent, we need to justify whether.

$$P(X_2 = 1) P(X_3 = 1) = P(X_2 = 1, X_3 = 1)$$

$P(X_2 = 1) = P(X_3 = 1) = \frac{1}{2}$ as discussed in (b).

$P(X_2 = 1, X_3 = 1) = P(X_3 = 1 \mid X_2 = 1) P(X_2 = 1)$

Since the second flip is fixed, only $\frac{1}{2}$ chance for the third flip to get 1 point, Therefore. $P(X_3 = 1 \mid X_2 = 1) P(X_2 = 1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

For that $p(X_2 = 1) p(X_3 = 1) = P(X_2 = 1, X_3 = 1)$, they are independent.

Question 4

(a) Likelihood Function is defined as :

$$L(\theta) = \prod_{i=1}^{n} f(x_i) = f(x_1) f(x_2)$$

$$= (2\theta - \frac{2}{3} + \frac{2}{3} - \frac{4}{3}\theta)(2\theta - \frac{2}{3} + \frac{4}{3} - \frac{8}{3}\theta).$$

$$= \frac{2}{3}\theta(\frac{2}{3} - \frac{2}{3}\theta).$$

$$= \frac{4}{9}[-(\theta - \frac{1}{2})^2 + \frac{1}{4}].$$

To maximum $L(\theta)$, $\hat{\theta}_{MLE} = \frac{1}{2}$, which satisfies $\frac{1}{3} \leq \theta \leq \frac{2}{3}$.

(b) 1st moment (Mean) is used.

$$E[X] = \int_0^3 x f(x)\, dx$$

$$= \int_0^3 x(2\theta - \frac{2}{3} + \frac{2}{3}x - \frac{4}{3}\theta x)\, dx.$$

$$= (\theta - \frac{1}{3}) x^2 \Big|_0^3 + \frac{1}{3}(\frac{2}{3} - \frac{4}{3}\theta) x^3 \Big|_0^3.$$

$$= -3\theta + 3.$$

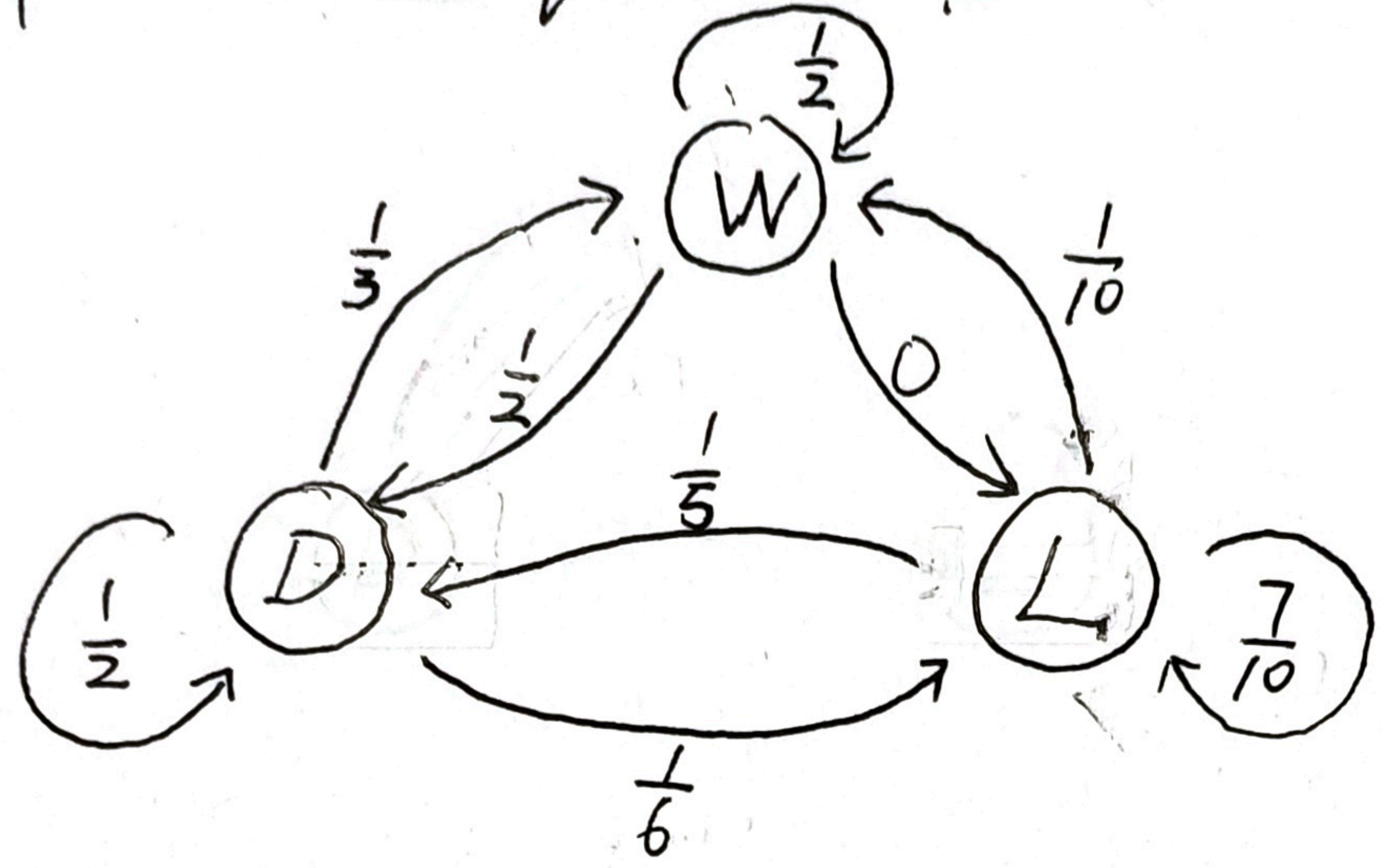
Sample Mean $= \bar{x} = \frac{1}{2}\sum x_i = \frac{1}{2}(1+2) = \frac{3}{2}$.

Solve for θ : $-3\theta + 3 = \frac{3}{2}$

$\hat{\theta} = \frac{1}{2}$, which satisfies $\frac{1}{3} \leq \theta \leq \frac{2}{3}$

Question 5

(a) Denote W, D, L as Win, Draw, Loss, the state diagram given information in question is



(b) Transition Matrix P is 3x3 and $P_{i,j}$ represents the probability of transitioning from i to j.

Use W, D, L as order

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{10} & \frac{1}{5} & \frac{7}{10} \end{bmatrix}$$

(c) According to Ergodic Theorm, since this Markov Chain is Ergodic, the fraction of time it spends in each state approximate the stationary probability of that state.

So we need to find $\vec{\pi}$, s.t. $\vec{\pi} P = \vec{\pi}$

$$\vec{\pi} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{10} & \frac{1}{5} & \frac{7}{10} \end{bmatrix} = \vec{\pi} \quad \& \quad \pi_W + \pi_D + \pi_L = 1$$

$$\Rightarrow \begin{cases} \pi_W = \frac{1}{3} \\ \pi_D = \frac{3}{7} \\ \pi_L = \frac{5}{21} \end{cases}$$

Therefore, Sid's long term win radio is $\frac{1}{3}$