Sept 6

Recall Prob Axiom 3

if A & B are disjoint : $P(A \cup B) = P(A) + P(B)$

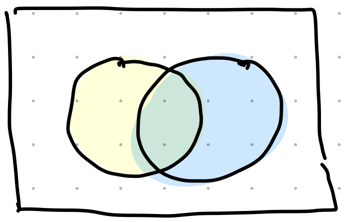
$P(A \cap B) = \emptyset$

Now For any A, B events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Inclusive-Exclusive Prop

subtract overlap



$\left(\text{OR } P(A \cap B) = P(A) + P(B) - P(A \cup B)\right)$

ex roll a die, 6 outcomes

A: roll a 6 $= \{6\}$ $\qquad P(A) = \frac{1}{6}$

B: roll an even # $= \{2,4,6\}$ $\qquad P(B) = \frac{3}{6} = \frac{1}{2}$

$$P(A \cap B) = P(\{6\}) = \frac{1}{6}$$

$$P(A \cup B) = P(\{2,4,6\}) = \frac{3}{6}$$

same

Inc-Exc

$$= P(A) + P(B) - P(A \cap B) = \frac{1}{6} + \frac{1}{2} - \frac{1}{6} = \frac{1}{2}$$

ex UMD is playing Villanova this year & from past experience:

- UMD 50% chance winning home
- " 40% " " away
- " 30% " " both

What is the prob that UMD loses both games?

Define events: H: winning home Given: $P(H)=0.5$

A: winning away $P(A)=0.4$

$P(H\cap A)=0.3$

DeMorgan

Want: $P(H^c\cap A^c) = P((H\cup A)^c)$

$= 1-P(H\cup A)$ Complement Rule

$= 1-[P(H)+P(A)-P(H\cap A)]$ Inc-Exc

$= 1-[.5+.4-.3] = .4$

Inc-Exc for 3 events:

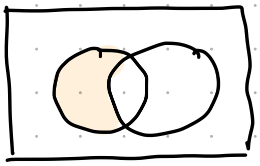
$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(C\cap B) + P(A\cap B\cap C)$$

Inc-Exc for 4 events:

$P(A\cup B\cup C\cup D)$ = add singletons − pair intersections + triple intersection − quadruple intersection

One last property:

$$P(A \cap B^c) = P(A-B) = P(A) - P(A \cap B)$$



<u>ex</u> Suppose

- 60% chance rains today $\longrightarrow A$ $\quad P(A) = .6$
- 50% " " tomorrow $\rightarrow B$ $\quad P(B) = .5$
- 30% it does not rain either day $\quad P(A^c \cap B^c) = .3$

no rain today & tomorrow

<u>Find</u> the prob

a) rain today OR tomorrow

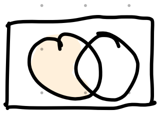
$$P(A \cup B) = 1 - P((A \cup B)^c) = 1 - P(A^c \cap B^c) = 1 - .3 = .7$$

b) rain today AND tomorrow

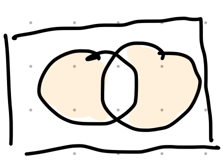
$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = .6 + .5 - .7 = .4$$

c) rain today but not tomorrow

$$P(A \cap B^c) = P(A) - P(A \cap B) = .6 - .4 = .2$$



d) either rain today or tomorrow but not both



$$P(A-B) + P(B-A)$$

$$\underbrace{.2}_{\text{(part c)}} + P(B) - P(B \cap A)$$

$$.2 + .5 - .4 = .3$$

Sample Spaces having equally likely outcomes

ex: roll a die
toss a coin
pick a card from deck

Not ex: chance of rain tomorrow
" " passing exam

For any event E,

$$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } \Omega}$$

But counting can get difficult.

Combinatorial Analysis

Basic Counting Principle: (Multiply)

Suppose 2 experiments

Exp 1 has m outcomes
Exp 2 " n "

$\Rightarrow$ then there $m \cdot n$ total possible outcomes.

ex Suppose 4 shirts diff color R, G, B, Y
3 pants diff color R, G, B

How many diff outfits possible? 12 $\longrightarrow$ $4 \cdot 3$

$$\left.\begin{matrix} RR & GR & BR & YR \\ RG & GG & BG & YG \\ RB & GB & BB & YB \end{matrix}\right\} 12$$

ex How many license plates possible → 3 letters followed by 3 numbers
(can repeat)

$\underline{26}\cdot\underline{26}\cdot\underline{26}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10} = 26^3\cdot 10^3 = 17{,}576{,}000$

no repeats: $26\cdot 25\cdot 24\cdot 10\cdot 9\cdot 8$

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ex Amy & Joe want to split. Their household has 20 valuable objects. How many ways are there to divide the objects? For each object there 2 options: A or J

$$\underbrace{\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\underline{2}\cdot\ldots\underline{2}}_{\text{20 objects/slots}} = 2^{20}$$

Permutations: different <u>ordered</u> arrangements
→ each arrangement is called a permutation
↳ order/position matters (no repeats)

ex: $\{a, b, c\}$ as choices for an ordered permutation

6 permutations total:

abc bac cab
acb bca cba

$\underline{3}\ \underline{2}\ \underline{1} = 3! = 6$

ex 5 friends have 5 diff movie tickets. How many ways to distribute the tickets? 1 per person

$\underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 5! = 120$

ex 3 friends have 5 diff movie tickets. How many ways to distribute the tickets? 1 per person

$$\underbrace{\underline{5}\ \underline{4}\ \underline{3}}_{\text{almost a factorial}} = 60$$

$\Rightarrow$ If you want to find # of permutations of k objects chosen a set of n objects, the formula is $_nP_k = P(n,k) = \frac{n!}{(n-k)!}$

ex prev: $n=5$ $k=3$ $P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$

$= \frac{5\cdot4\cdot3\cdot\cancel{2}\cdot\cancel{1}}{\cancel{2}\cdot\cancel{1}}$

ex 5 couples have 10 seats at movie theater together (10 seats in a row). How many arrangements possible if each couple wants to sit together.

$$\underbrace{\_\ \_}\ \underbrace{\_\ \_}\ \underbrace{\_\ \_}\ \underbrace{\_\ \_}\ \underbrace{\_\ \_}$$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ ways to arrange the couples

However, for each couple, there are 2 possibilities

$$5! \cdot 2\cdot2\cdot2\cdot2\cdot2 = 5!\cdot 2^5 = 3840$$

Combination A way of selecting items from a collection such that order does not matter

"card hand"

ex $\{a, b, c, d\}$ "words" w/ 2 letters (no repeats)

permutation (order matters): $\underline{4}\ \underline{3} = 12$

$$P(4,2) = \frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2}{2} = 4 \cdot 3 = 12$$

ab	ba	ca	da
ac	bc	cb	db
ad	bd	cd	dc

combination:

$$\left.\begin{array}{lll} ab & & \\ ac & bc & \\ ad & bd & cd \end{array}\right\} 6$$

Combination (binomial coefficient) "n choose k" $_nC_k$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

In general, for same n, k:

\# of perm > # of combo

ex How many different 3 card hands are possible from 20 cards?

↳ order does not matter
↓
combo

$$\binom{20}{3} = \frac{20!}{(20-3)!\,3!} = \frac{20!}{17!\,3!}$$

$$= \frac{20\cdot 19\cdot 18\cdot \cancel{17!}}{\cancel{17!}\, 3!} = \frac{20\cdot 19\cdot 18}{3\cdot 2} = 1140$$

ex 8 men 8 women

want form a committee that 2 women & 2 men. How many ways?

order does not matter → Combo

$\underbrace{\_\ \_}_{M}\ \underbrace{\_\ \_}_{W}$

$$\binom{8}{2}\cdot\binom{8}{2}$$

$$28\cdot 28 = 784$$

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ex 4 Americans, 6 Canadians to pick from

a) how many ways can we arrange them in a line?

order matters

$\underline{10}\ \underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

$= 10! = 3{,}628{,}800$

$$\frac{10!}{(10-10)!} = \frac{10!}{0!} = \frac{10!}{1} = 10!$$

b) " " " if all Americans have to stand together?

$C_1 C_2 C_3 C_4 C_5 C_6 A$ $\begin{cases} A_1 A_2 A_3 A_4 \\ A_2 A_1 A_3 A_4 \\ \vdots \end{cases}$

$C_1\ C_2 C_3\ C_4\ C_5\ A\ C_6$

4! arrange w/in American

$\underbrace{\underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}}_{7!}$

$\Rightarrow 7!\,4! = 120{,}960$

c) How many ways if <u>not</u> all Americans are together?

$$10! - 7!4! = 3,507,840$$

order does not matter

d) Committee of 3 w/ only all Amer OR all Canadian. How many ways?

$$\underset{A}{\binom{4}{3}} + \underset{C}{\binom{6}{3}} = 24$$

e) How many ways for a committee of 3 that is not all Amer OR all Canadian

$$\binom{10}{3} - 24 = 96$$

<u>ex</u> How many different 8 letter arrangements can be made from the letters NONSENSE?

8 places/slots

N: 3 identical

S: 2 "

E: 2 "

O: 1

<u>ex</u> $N_1ON_2SENSE = N_2ON_1SENSE$

→ First: $\underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = \dfrac{8!}{\underbrace{3!\,2!\,2!\,1!}_{\text{deletes duplicates}}} = 1680$

N S E O

deletes duplicates

In general, there are

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

(Multinomial Coefficient)

↳ "mixture" of perm & combo

different permutations (arrangements) of n objects where

n_1 are alike

n_2 are alike

$\vdots$

n_k are alike

order matters for the line/arrangement but the individual identical components the order does not matter (identical)

Back to prob:

ex A committee of 5 selected from a group of 6 men and 9 women. What is the prob that it consists of 3 men & 2 women?

order does not matter

equally likely outcomes

E: Event that committee has 3 men & 2 women

$$P(E) = \frac{\text{the \# of groups w/ 3 men \& 2 women}}{\text{the total \# of groups of 5}}$$

$$= \frac{\overbrace{\_\ \_\ \_}^{M}\ \overbrace{\_\ \_}^{W}}{\binom{15}{5}} = \frac{\binom{6}{3}\cdot\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}$$

ex On the exam, 3 topics out of 50 to be tested. Bob knows 30 of them. If all topics on the exam are familiar to him, he gets an A. ↪ event A

- Knows 2 topics → B → event B
- Knows 1 topic → C → event C
- Know 0 topics → D → event D

Compute the probability of each grade.

Bob knows topics 1-30, does not know $\underbrace{31\text{-}50}_{20}$

$$P(A) = \frac{\binom{30}{3}}{\binom{50}{3}} = \frac{4060}{19600} \approx 0.21$$

$$P(B) = \frac{\binom{30}{2}\cdot 20}{\binom{50}{3}} = \frac{8700}{19600} \approx .44$$

$$P(C) = \frac{\binom{30}{1}\cdot\binom{20}{2}}{\binom{50}{3}} = \frac{5700}{19600} \approx .29$$

$$P(D) = \frac{\binom{20}{3}}{\binom{50}{3}} = \frac{1140}{19600} \approx .06$$

ex What is the probability that in a poker hand (5 cards out of 52) we get exactly 4 of a kind?

Deck of cards: 4 suits
13 kinds/rank
2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

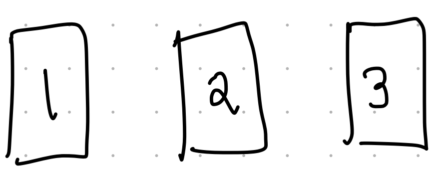
→ the probability of 4 aces & 1 king

$$\frac{\overset{A}{\binom{4}{4}}\overset{K}{\binom{4}{1}}}{\binom{52}{5}}$$

→ however the same prob as 4 jacks & one 3
↳ adjust for this

$$\frac{\binom{4}{4}\binom{4}{1}}{\binom{52}{5}} \cdot 13 \cdot 12 = \frac{624}{2,598,960} = 0.00024$$

Monty Hall Problem : 1 host, 1 player
3 doors ⟶ behind 1 car, 2 goats



Player chooses a door (door #1) then host opens one w/ a goat. (ex : door #2)
Should the player switch?