Probability and Statistics Homework 9

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1. Suppose that there are two identically-looking batteries in a box. The first one should last for a time that is exponentially distributed with parameter $\lambda_1 = 1$ (in months). The second one should last for a time that is exponentially distributed with parameter $\lambda_2 = 2$. A battery has been picked randomly and is still working after two months. Given this information, what is the probability that the first battery was picked?

Answer:
Denote the event that the first battery is picked to be A_1, the second one is picked to be A_2, the battery works longer than 2 months to be L. The problem is to find the probability that the first battery was picked given its working after two months, which is

$$P(A_1|L)$$

According to Bayes Rule, the formula can be rewritten as

$$\begin{aligned} P(A_1|L) &= \frac{P(A_1 \cap L)}{P(L)} \\ &= \frac{P(L|A_1)P(A_1)}{P(L|A_1)P(A_1) + P(L|A_2)P(A_2)} \\ &= \frac{P(T > 2|\lambda_1 = 1)\frac{1}{2}}{P(T > 2|\lambda_1 = 1)\frac{1}{2} + P(T > 2|\lambda_2 = 2)\frac{1}{2}} \end{aligned}$$

Given the cdf of exponential distribution is $F(x) = 1 - e^{-\lambda x}$, the above formula is given be

$$\begin{aligned} P(A_1|L) &= \frac{e^{-2}}{e^{-2} + e^{-4}} \\ &= \frac{e^2}{1 + e^2} \end{aligned}$$

Therefore, the probability that the first battery was picked given it's still working after two months is $\frac{e^2}{1+e^2}$.

2. Suppose that $\mathrm{Var}(X) = 3$, $\mathrm{Var}(Y) = 4$, and $\mathrm{Cov}(X, Y) = 1$.
(a) Find $\mathrm{Cov}(2X - Y, X + 3Y)$.
(b) Find $\rho(2X - Y, X + 3Y)$ (the correlation between $2X - Y$ and $X + 3Y$).

Answer:

(a) With the property of Covirance calculation, we have

$$\begin{aligned}
Cov(2X - Y, X + 3Y) &= Cov(2X - Y, X) + Cov(2X - Y, 3Y) \\
&= 2Cov(X, X) - Cov(X, Y) + 6Cov(X, Y) - 3Cov(Y) \\
&= 2Var(X) + 5Cov(X, Y) - 3Var(Y) \\
&= 2 \times 3 + 5 \times 1 - 3 \times 4 \\
&= -1
\end{aligned}$$

(b) According to the definition of correlation ρ,

$$\begin{aligned}
\rho(2X - Y, X + 3Y) &= \frac{Cov(2X - Y, X + 3Y)}{\sqrt{Var(2X - Y)}\sqrt{Var(X + 3Y)}} \\
&= \frac{Cov(2X - Y, X + 3Y)}{\sqrt{4Var(X) - 4Cov(X, Y) + Var(Y)}\sqrt{Var(X) + 6Cov(X, Y) + 9Var(Y)}} \\
&= \frac{-1}{\sqrt{4 \times 3 - 4 \times 1 + 4}\sqrt{3 + 6 \times 1 + 9 \times 4}} \\
&= -\frac{\sqrt{15}}{90}
\end{aligned}$$

3. Suppose that X and Y are independent random variables. Both are exponential with parameter 1. Find the density of the random variable $X + Y$.

Answer:
Given two independent random variable X, Y, the pdf of $X + Y$ is the convolution

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy$$

the density of the random variable $Z = X + Y$ is

$$\begin{aligned} f_Z(z) &= \int_0^z f_X(z-y) f_Y(y)\, dy \\ &= \int_0^z e^{-(z-y)} e^{-y}\, dy \\ &= e^{-z} \int_0^z dy \\ &= z e^{-z} \end{aligned}$$

Therefore, the pdf of Z is given by

$$f_Z(z) = \begin{cases} z e^{-z}, & z \geq 0 \\ 0, & z < 0 \end{cases}$$