Nov 8th

Law of Large Numbers → if you repeat an experiment independently a large number of times, we obtain something to the expected value

↳ two versions — weak law of large numbers (WLLN)
— strong " " " (SLLN)

↳ difference is mostly theoretical

Let: X_i $i=1,\dots,n$ i.i.d. (independent, identical distributions)

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$ (sample mean)

WLLN Let X_i $i=1,\dots,n$ be i.i.d. random variables ($\mu < \infty$, finite mean). Then for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\bar{X} - \mu| \geq \epsilon) = 0$$

the prob that the diffenence btwn mean of the sample ($\bar{X}$) and the expected value of r.v. (μ) tends to zero

Proof assume $Var(X_i) = \sigma^2$ finite

$$P(|\bar{X} - \mu| \geq \epsilon) \leq \frac{Var(\bar{X})}{\epsilon^2}$$ (Chebyshev)

$$= \frac{Var(X)}{n \cdot \epsilon^2}$$ #

#

as $n \to \infty$: $\lim_{n \to \infty} P(|\bar{X} - \mu| \geq \epsilon) = 0$ $\blacksquare$

Where does $Var(\bar{X}) = \frac{Var(X)}{n}$ come from?

linearity

$$E[\bar{X}] = E\left[\frac{X_1 + \dots + X_n}{n}\right] = \frac{E[X_1] + E[X_2] + \dots + E[X_n]}{n}$$

i.i.d

$$= \frac{\overbrace{E[X] + E[X] + \dots + E[X]}^{n \text{ times}}}{n}$$

$$= \frac{n \cdot E[X]}{n} = E[X]$$

$$Var(\bar{X}) = Var\left(\frac{X_1 + \dots + X_n}{n}\right) = \frac{Var(X_1 + \dots + X_n)}{n^2}$$

$$= \frac{Var(X_1) + Var(X_2) + \dots Var(X_n)}{n^2}$$

$$= \frac{Var(X) + Var(X) + \dots Var(X)}{n^2}$$

$$= \frac{n\,Var(X)}{n^2} = \frac{Var(X)}{n}$$

CLT Suppose X_i $i=1,\ldots,n$ are i.i.d r.v with $E[X_i]=\mu<\infty$ $Var(X_i)=\sigma^2<\infty$ will tend to the standard normal as $n\to\infty$.

$$P\left(\frac{X_1+X_2+\ldots+X_n-n\mu}{\sqrt{n}\,\sigma}<c\right)\longrightarrow \Phi(c)$$

standard normal

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ex There are 100 people on a plane. Let X_i be the weight continuous of the ith person. Suppose X_i's i.i.d and $E[X_i]=\mu=170$ and $\sigma=30$. Find the prob that the total weight exceeds 18,000 pounds.

Let $W=X_1+X_2+\ldots+X_{100}$

$$E[W]=n\mu=100\cdot 170=17{,}000$$

$$\sqrt{n}\,\sigma=10\cdot 30=300$$

$$P(W>18000)=1-P(W\leq 18000)$$

$$= 1 - P\left(\frac{W-17000}{300} \le \frac{18000-17000}{300}\right)$$

$$= 1 - P\left(Z \le \frac{10}{3}\right)$$

$$= 1 - \Phi\left(\frac{10}{3}\right) \quad \frac{10}{3} \to 3.33$$

$$= 1 - .9996 = .0004$$

Let $X_1, X_2, \ldots, X_{25}$ be i.i.d with pmf

$$P_X(k) = \begin{cases} 0.6 & k=1 \\ 0.4 & k=-1 \\ 0 & \text{otherwise} \end{cases} \quad \Big\} \text{ discrete}$$

estimate $P(4 \le Y \le 6)$ if $Y = X_1 + \cdots + X_{25}$

We have $E[X_i] = 1(0.6) + (-1)(0.4) = \frac{1}{5} = \mu$

$$E[X_i^2] = 1(0.6) + (-1)^2(0.4) = 1$$

$$Var(X_i) = 1 - \left(\frac{1}{5}\right)^2 = \frac{24}{25} = \sigma^2$$

$$\sigma = \frac{\sqrt{24}}{5}$$

$\Rightarrow n\mu = 25 \cdot \frac{1}{5} = 5 \qquad \sqrt{n} \cdot \sigma = \sqrt{25} \cdot \frac{\sqrt{24}}{5} = \sqrt{24}$

$\overline{5}$

cont. correction $\longrightarrow$ discrete

$$P(4 \le Y \le 6) = P(3.5 \le Y \le 6.5)$$

$$= P\left(\frac{3.5-5}{\sqrt{24}} \le Z \le \frac{6.5-5}{\sqrt{24}}\right)$$

$$\approx P(-.31 \le Z \le .31)$$

$$= P(Z \le .31) - P(Z \le -.31)$$

$$= P(Z \le .31) - (1 - P(Z \le .31))$$

$$= 2\Phi(.31) - 1$$

$$\approx 0.24$$

<u>Monte Carlo</u> $\longrightarrow$ named after casino in Monaco

randomness

chance

↳ simulation, computational method that uses sampling to estimate numerical results

↓

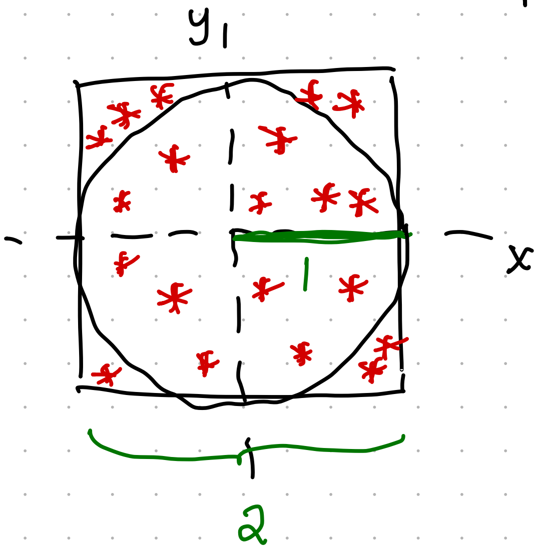
random

↳ by performing a large # of simulations with varying results, method provides probabistic outcomes close to what we hope for

↳ usually used in place of other methods that can be analytically hard

<u>Estimate π</u> (by using computer)

↳ pick a random point within a square *



Probability that the point * lies in the circle:

$$P = \frac{A_{circle}}{A_{square}} = \frac{\pi r^2}{2^2} = \frac{\pi}{4} \quad r=1$$

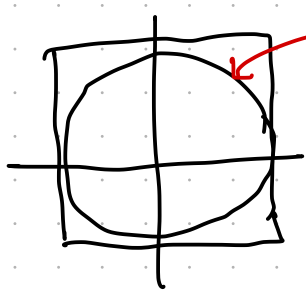
$$\text{Ratio} = \frac{N \text{ of pt inside circle}}{N \text{ of total pts used}} = \frac{\pi}{4}$$

$\longrightarrow \pi \approx 4 \cdot \text{Ratio}$

ex: if $N = 1{,}000{,}000$ (total) but inside the circle there are 785,398

$$\pi \approx 4 \cdot \frac{785{,}398}{1000000} \approx 4 \cdot 0.785398$$
$$\approx 3.141592$$

How to count/check if * is in the circle?



$x^2+y^2=1$

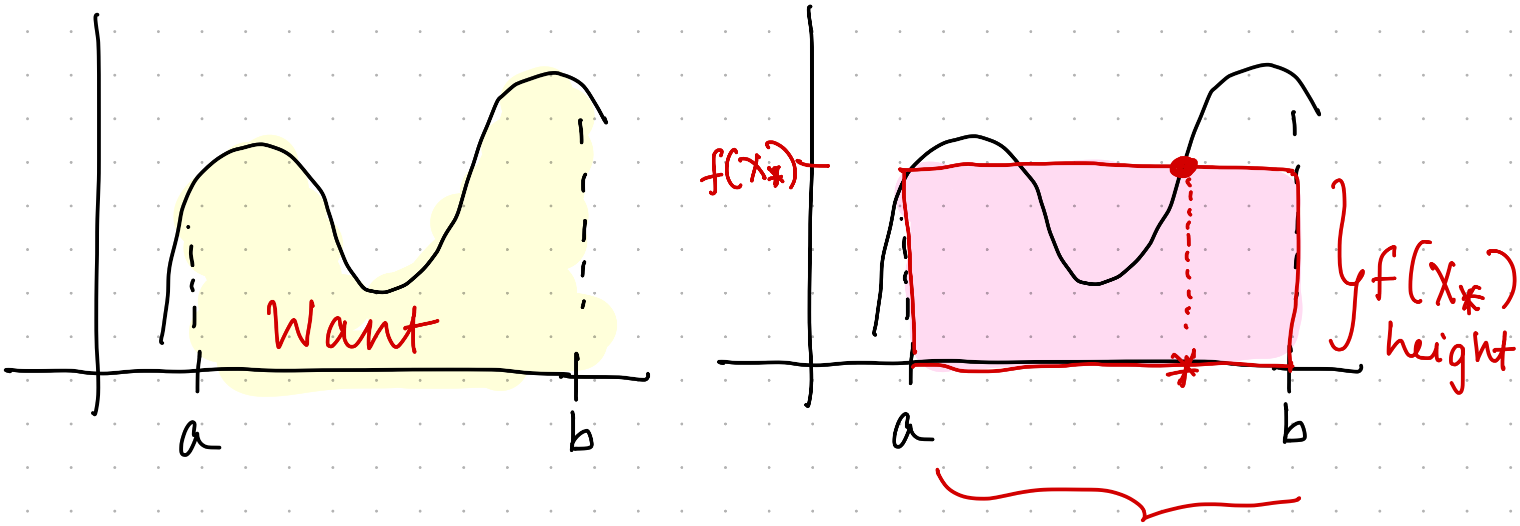
check if x^2+y^2 of * is ≤ 1. If so, count it.

1. Computer generates random pt
2. Check if it is in the circle → count
   ↳ $x^2+y^2 \leq 1$?
3. $\pi \cong 4 \cdot \frac{N_{inside}}{N_{total}}$

Monte Carlo for Integrations → used to compute integrals that are high dimensional or complex to solve by hand

Given $f(x)$ on a domain (D)

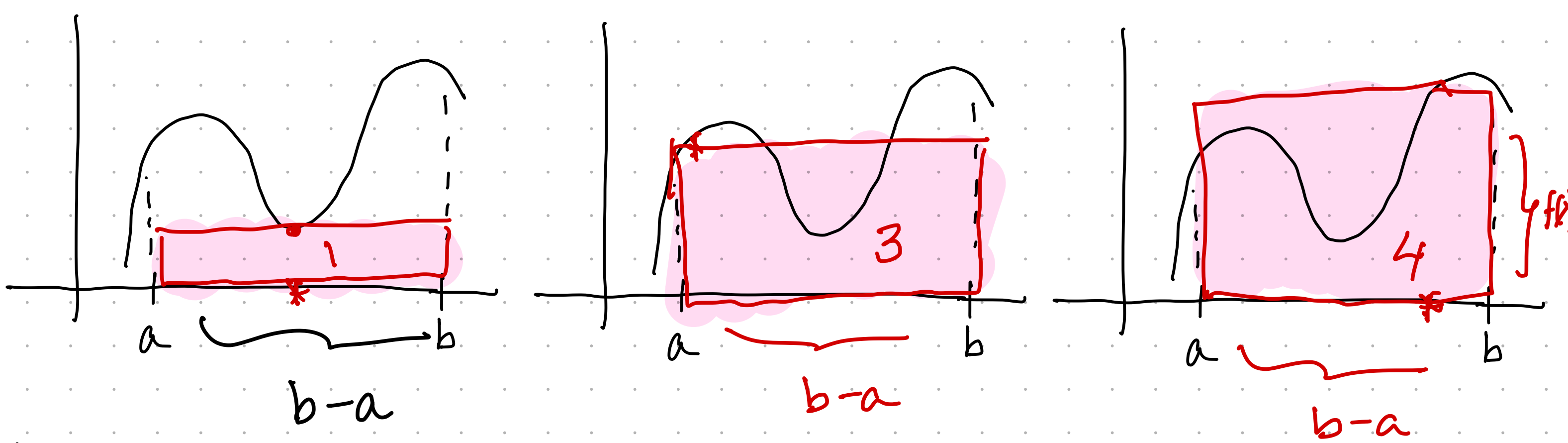
$$I = \int_D f(x)\,dx = \int_a^b f(x)\,dx$$



b-a width

area: $(b-a)f(x_*)$

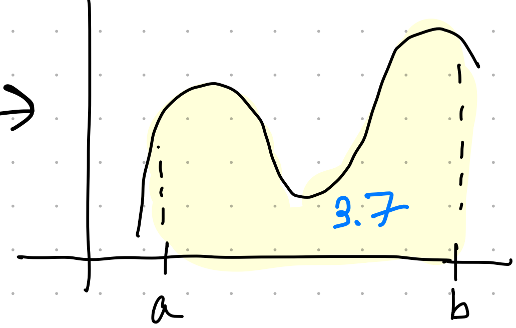
Repeat the process



Avg:

$$\underset{\uparrow N}{\frac{1}{3}}\left(\boxed{\quad 1 \quad} + \boxed{\quad 3 \quad} + \boxed{\quad 4 \quad}\right)$$

$N \to \infty$



$$\int_a^b f(x)\,dx \approx \frac{1}{N}\sum_{i=1}^{N} \underbrace{(b-a)}_{\text{width}}\ \underbrace{f(x_i)}_{\text{height}}$$

random pts

$$= \frac{1}{N}(b-a)\sum_{i=1}^{N} f(x_i)$$

$$= \underbrace{(b-a)}_{\text{length of interval}} \cdot \underbrace{\frac{1}{N}\sum_{i=1}^{N} f(x_i)}_{\text{avg function}}$$

ex estimate $\int_0^1 e^{-x^2}\,dx$ * Note $\int_a^b e^{-x^2}dx$ does not have a closed-form solution

$D: [0,1] \to (b-a) = 1$

① Generate N random points in $[0,1]$

② Compute $f(x_i)$ (height)

③ $\int_0^1 e^{x^2}dx \approx 1 \cdot \frac{1}{N}\sum_{i=1}^{N} e^{-x_i^2}$

$\underbrace{\qquad\qquad}_{\text{mean/avg function}}$

→ exact value is approx 0.7468

<u>Markov Chains</u>: example of stochastic (random) process changing over time

<u>Before</u>: Sequence R.V. $X_1, X_2, X_3 \ldots$ and assumed to be i.i.d

↳ WLLN, CLT

<u>Now</u>: R.V. can have time dependence

↳ can get very complicated if we allowed for the dependence of an infinite amt of parameters

<u>Compromise</u>: Markov chain are "one step" beyond i.i.d

<u>Applications</u>: Weather
stock prices
genetics

computer science $\longrightarrow$ Google PageRank

* can extend to dependence of more than one dependence $\rightarrow$ need to look at single dependence

Let X_n be the <u>state</u> of the system at (discrete) time n & takes on finite # of states

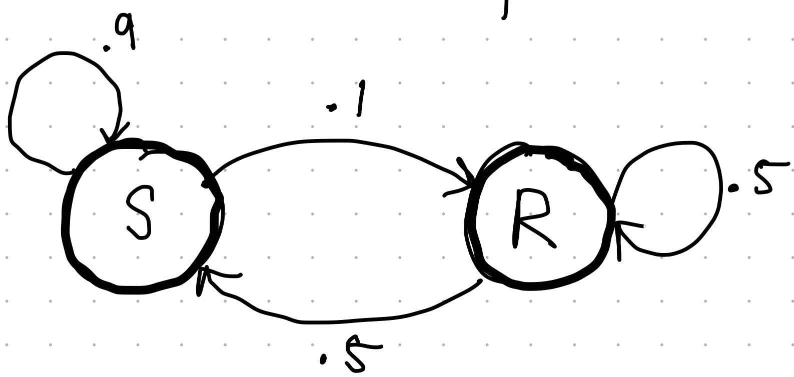
↳ process that is bouncing around randomly from state to state

<u>ex</u> Imagine a simple weather model where it is either $\underbrace{\text{sunny (S) or rainy}}_{\text{states}}$ (R)

The weather tomorrow only depends on the weather today.

- If it is sunny today, there is a 90% chance it is sunny tomorrow & 10% chance " " rainy "

- If it is rainy today, there is a 50% chance it is sunny tomorrow & 50%
  " " " rainy tomorrow



Transition Matrix (Q, OR P, OR M)

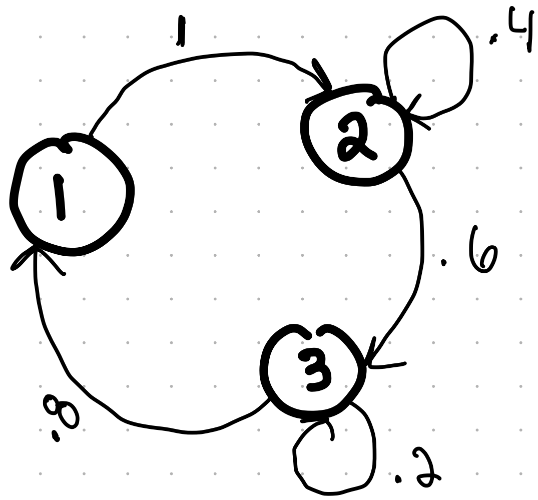
$$Q = \begin{array}{c c} & \begin{array}{cc} S & R \end{array} \\ \begin{array}{c} S \\ R \end{array} & \begin{bmatrix} .9 & .1 \\ .5 & .5 \end{bmatrix} \end{array}$$

Board Game Movement : 3 spaces $\{1,2,3\}$

The player's next position only depends on their current position.

From space 1 → move to space 2 w/ prob 1
" space 2 → " " space 3 w/ prob .6
stays at space 2 w/ prob .4

From space 3 → move to space 1 w/ prob .8
stay at space 3 w/ prob .2



$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & .4 & .6 \\ .8 & 0 & .2 \end{bmatrix}$$

all rows add up to 1

* Not true for colums

pmf for each stage

each row is a pmf for the prob you will end up at the other stages in the next time step

<u>Markov Property</u> (mathematically)

Conditional prob statement if we think of X_n as the current time, we want to know one step in the future

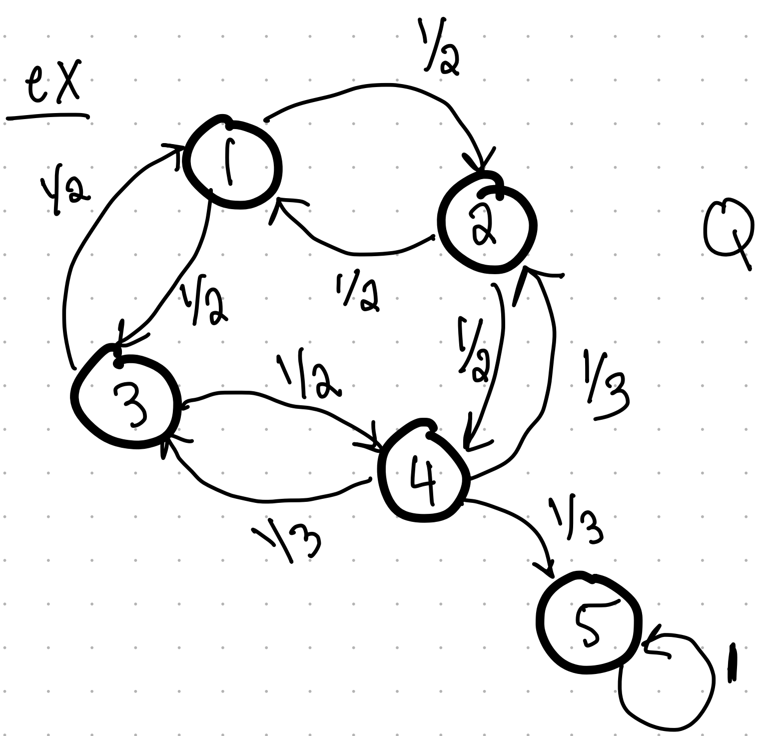
$$P\Big(X_{n+1}=j \;\Big|\; \underbrace{X_n=i_n,\ \overbrace{X_{n-1}=i_{n-1},\ X_{n-2}=i_{n-2},\ \ldots}^{\text{irrelevant}}}_{\text{all previous knowledge}}\Big)$$

and and

Markovian property

$$= P(X_{n+1} \mid X_n = i_n)$$

ex



$$Q = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Find $P(X_2=5 \mid X_0=3)$

Want to find prob we end up at state 5 after 2 timesteps, starting from state 3.

Intuitively $\rightarrow \frac{1}{2}\cdot\frac{1}{3}$ "follow" path

↳ this is just the $\underbrace{\text{total law of prob}}_{\text{TLOP}}$!

TLOP says if B_i's partition a sample space

$$P(A) = \sum_i P(A \mid B_i)\,P(B_i)$$

What if we need $P(A \mid C)$? We just condition everything on C to get

$$P(A \mid C) = \sum_i P(A \mid B_i \cap C)\,P(B_i \mid C)$$

So: $$P(X_2 = 5 \mid X_0 = 3) = \sum_{i=1}^{5} P(X_2 = 5 \mid \overset{\text{irrelevant}}{X_0 = 3}, X_1 = i)\,P(X_1 = i \mid X_0 = 3)$$

Markov prop and

$$= \sum_{i=1}^{5} P(X_2 = 5 \mid X_1 = i)\,P(X_1 = i \mid X_0 = 3)$$

$$= \cancel{\underbrace{P(X_2 = 5 \mid X_1 = 1)}_{=0}\,\underbrace{P(X_1 = 1 \mid X_0 = 3)}_{=1/2}} + \cancel{\underbrace{P(X_2 = 5 \mid X_1 = 2)}_{=0}\,\underbrace{P(X_1 = 2 \mid X_0 = 3)}_{=0}}$$
$$+ \cancel{\underbrace{P(X_2 = 5 \mid X_1 = 3)}_{=0}\,\underbrace{P(X_1 = 3 \mid X_0 = 3)}_{=0}} + \underbrace{P(X_2 = 5 \mid X_1 = 4)}_{=1/3}\,\underbrace{P(X_1 = 4 \mid X_0 = 3)}_{=1/2}$$
$$+ \cancel{\underbrace{P(X_2 = 5 \mid X_1 = 5)}_{=1}\,\underbrace{P(X_1 = 5 \mid X_0 = 3)}_{=0}}$$

$$= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} \checkmark$$

General Pattern:

$$P(X_{n+1}=j \mid X_n=i) = q_{ij}$$

$(i,j)^{th}$ entry in Q

What about 2 step?

$P(X_{n+2}=j \mid X_n=i)$

missing link/info X_{n+1}

$$= \sum_k P(X_{n+2}=j \mid X_{n+1}=k, X_n=i)\, P(X_{n+1}=k \mid X_n=i)$$

previously: $j=5$, $i=3$ $\quad k=1-5$

irrelevant

Markov prop

$$= \sum_k P(X_{n+2}=j \mid X_{n+1}=k)\, P(X_{n+1}=k \mid X_n=i)$$

$$= \sum_k q_{kj} \cdot q_{ik} = \underbrace{\sum_k q_{ik} q_{kj}}_{(i,j)^{th} \text{ entry of } Q^2}$$

transition matrix multiplied twice

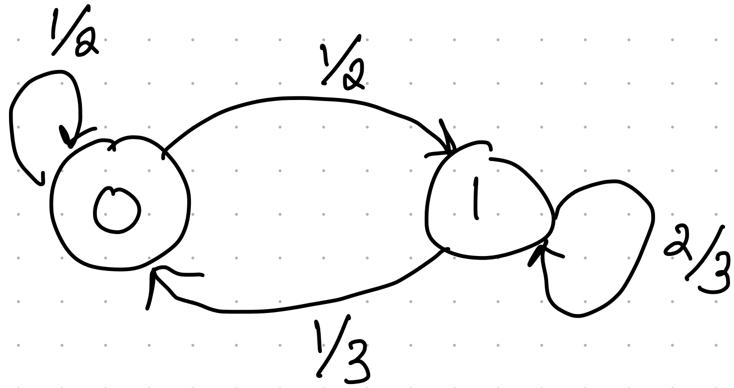
$$P(X_{n+m}=j \mid X_n=i) = (i,j)^{th} \text{ entry of } Q^m$$

ex Consider a system w/ 2 states $\{0, 1\}$

w/ $Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$

Suppose system is in state 0 at time $n=0$.

Find the prob that the system is in state 1 at $n=3$.



initial starting $\vec{S}^{(0)} = [P(X_0=0) \quad P(X_0=1)]$

OR $\vec{\pi}^{(0)} = [1 \quad 0]$

$$\vec{S}^{(1)} = \vec{S}^{(0)} Q = [1 \quad 0] \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$= \left[1 \cdot \tfrac{1}{2} + 0 \cdot \tfrac{1}{3} \quad 1 \cdot \tfrac{1}{2} + 0 \cdot \tfrac{2}{3}\right]$$

$$= [1/2 + 0 \quad 1/2 + 0] = [1/2 \quad 1/2]$$

$$\vec{S}^{(1)} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \longrightarrow \vec{S}^{(2)} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{3} & \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{2}{3} \end{bmatrix}$$

$$\frac{1}{4} + \frac{1}{6} \qquad \frac{1}{4} + \frac{1}{3}$$

$$\vec{S}^{(3)} = \vec{S}^{(2)} Q = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}$$

adds to 1 ✓

$$\begin{bmatrix} \frac{5}{12}\cdot\frac{1}{2} + \frac{7}{12}\cdot\frac{1}{3} & \frac{5}{12}\cdot\frac{1}{2} + \frac{7}{12}\cdot\frac{2}{3} \end{bmatrix}$$

$$\frac{5}{24} + \frac{7}{36} \qquad \frac{5}{24} + \frac{14}{36}$$

$$\begin{bmatrix} \frac{29}{72} & \frac{43}{72} \end{bmatrix}$$

adds to 1 ✓