Machine Learning Homework 1

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1.

(a) Denote λ as the eigenvalue of $\mathbf{B}$. Then we have

$$\lambda^2 - (2.25 + 2.75)\lambda + (2.25 \times 2.75 - 0.433 \times 0.433) = 0$$

The result is $\lambda_1 = \frac{5 - \frac{2\sqrt{249989}}{10^3}}{2} \approx 2.000011, \lambda_2 = \frac{5 + \frac{2\sqrt{249989}}{10^3}}{2} \approx 2.999989$, they are the eigenvalues of $\mathbf{B}$.

(b) Consider $(\mathbf{B} - \lambda\mathbf{I})x = 0$, where x is the eigenvectors. Put $\lambda_1 = 2.000011, \lambda_2 = 2.999989$ into this equation as following:

$$(\mathbf{B} - \lambda_1\mathbf{I})v_1 = 0$$
$$(\mathbf{B} - \lambda_2\mathbf{I})v_2 = 0$$

Therefore, the result of eigenvectors are

$$v_1 = (-0.86602858, 0.4999945)$$
$$v_2 = (-0.4999945, -0.86602858)$$

Note that

$$\begin{aligned} < v_1, v_2 > &= -0.86602858 \times (-0.4999945) + 0.4999945 \times (-0.86602858) \\ &= 0 \end{aligned}$$

Since these two eigenvectors have inner product equal to 0, they are orthogonal.

(c)

$$\begin{aligned} det(\mathbf{B}) &= 2.25 \times 2.75 - (-0.433) \times (-0.433) \\ &= 6.000011 \end{aligned}$$

And,

$$\begin{aligned} \lambda_1 \times \lambda_2 &= \frac{5 - \frac{2\sqrt{249989}}{10^3}}{2} \times \frac{5 + \frac{2\sqrt{249989}}{10^3}}{2} \\ &= 6.000011 = det(\mathbf{B}) \end{aligned}$$

Therefore, the determinant is equal to the product of the eigenvalues.

(d)

$$\begin{aligned} Tr(\mathbf{B}) &= 2.25 + 2.75 \\ &= 5 \end{aligned}$$

And,

$$\begin{aligned} \lambda_1 + \lambda_2 &= \frac{5 - \frac{2\sqrt{249989}}{10^3}}{2} + \frac{5 + \frac{2\sqrt{249989}}{10^3}}{2} \\ &= 5 = Tr(\mathbf{B}) \end{aligned}$$

Therefore, the trace of $\mathbf{B}$ is equal to the sum of the eigenvalues.

2.

(a) Denote the network is under attack as A, normal operation as N, response fast as Rf, normal as Rn, slow as Rs, then

$$
\begin{aligned}
P(A) &= P(A, Rf) + P(A, Rn) + P(A, Rs) \\
&= 0.05 + 0.1 + 0.25 \\
&= 0.4
\end{aligned}
$$

Therefore, the probability that the network is under attack is 0.4.

(b)

$$
\begin{aligned}
P(A, B) &= 0.25 \\
P(A) &= 0.25 + 0.1 = 0.35 \\
P(B) &= 0.05 + 0.1 + 0.25 = 0.4
\end{aligned}
$$

Since $P(A) \times P(B) = 0.35 \times 0.4 = 0.14 \neq P(A, B)$, events A and B are not independent.

(c) Given the notation same as question (a), the conditional probability that the response time is slow given that the network is under attack is

$$
\begin{aligned}
P(Rs|A) &= \frac{P(Rs, A)}{P(A)} \\
&= \frac{0.25}{0.4} \\
&= 0.625
\end{aligned}
$$

Therefore, the conditional probability that the response time is slow given that the network is under attack is 0.625.

(d) Given the notation same as question (a), the conditional probability that the network is under attack given that the response time is slow is

$$
\begin{aligned}
P(A|Rs) &= \frac{P(Rs|A)P(A)}{P(Rs)} \\
&= \frac{0.625 \times 0.4}{0.35} \\
&= \frac{5}{7}
\end{aligned}
$$

Therefore, the conditional probability that the network is under attack given that the response time is slow is $\frac{5}{7}$.

3.

Proof. Denote λ as eigenvalues of $\mathbf{A}$, $\mathbf{v}$ as eigenvectors of $\mathbf{A}$, then

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

According to the definition of positive semidefinite matrix, $\forall \mathbf{x} \in R^n, \mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$. Replacing $\mathbf{x}$ with $\mathbf{v}$, then

$$\begin{aligned} \mathbf{x}^T\mathbf{A}\mathbf{x} &= \mathbf{v}^T\mathbf{A}\mathbf{v} \\ &= \mathbf{v}^T(\lambda\mathbf{v}) \\ &= \lambda\mathbf{v}^T\mathbf{v} \\ &= \lambda \geq 0 \end{aligned}$$

□