1. Consider 2×2 matrix $\mathbf{B}$ given by

$$\mathbf{B} = \begin{bmatrix} 2.25 & -0.433 \\ -0.433 & 2.75 \end{bmatrix}.$$

(a) Find the eigenvalues of $\mathbf{B}$.

Ans: The eigenvalues are the solution to the characteristic equation.

$$(\lambda - 2.25)(\lambda - 2.75) - (-0.433)(-0.433) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0$$

Therefore, the two eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

(b) Find the eigenvectors corresponding to the eigenvalues and show that they are orthogonal. (Note that the eigenvectors are not unique.)

Ans: Recall that an eigenvalue λ and the corresponding eigenvector $\mathbf{v}$ satisfy $\mathbf{Bv} = \lambda\mathbf{v}$.

$$\mathbf{B}\begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = \begin{bmatrix} 2v_1^1 \\ 2v_2^1 \end{bmatrix} \quad \text{and} \quad \mathbf{B}\begin{bmatrix} v_1^2 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} 3v_1^2 \\ 3v_2^2 \end{bmatrix}$$

Thus, $2.25v_1^1 - 0.433v_2^1 = 2v_1^1$ or, equivalently, $v_1^1 = 1.732v_2^1$. Hence, we can choose $\mathbf{v}^1 = [1.7432 \;\; 1]^T$. Similarly, $2.25v_1^2 - 0.433v_2^2 = 3v_1^2$ or, equivalently, $v_2^2 = -1.732v_1^2$, and we can select $\mathbf{v}^2 = [1 \;\; -1.732]^T$. Since the inner product $\langle \mathbf{v}^1, \mathbf{v}^2 \rangle = 1.7432 \times 1 + 1 \times (-1.732) = 0$, the eigenvectors are orthogonal.

(c) Compute the determinant of $\mathbf{B}$ and verify that the determinant is equal to the product of the eigenvalues.

Ans: The determine of $\mathbf{B}$ is equal to $\det(\mathbf{V}) = 2.25 \times 2.75 - (-0.433)(-0.433) = 6 = \lambda_1 \times \lambda_2$.

(d) Verify that the trace of $\mathbf{B}$ (i.e., the sum of diagonal elements) is equal to the sum of the eigenvalues.

Ans: The trace $\text{Tr}(\mathbf{B}) = 2.25 + 2.75 = 5 = \lambda_1 + \lambda_2$.

2. Suppose that the joint distribution of the network state and the response time is shown in the following table.

	response time		
	fast	normal	slow
under attack	0.05	0.1	0.25
normal operation	0.2	0.3	0.1

(a) Compute the probability that the network is under attack.

Ans: Let $B = \{\text{network is under attack}\}$.

$$\mathbb{P}(B) = \mathbb{P}(B \cap \text{fast}) + \mathbb{P}(B \cap \text{normal}) + \mathbb{P}(B \cap \text{slow}) = 0.05 + 0.1 + 0.25 = 0.4\,.$$

(b) Determine if the events $A = \{\text{response time is slow}\}$ and $B = \{\text{network is under attack}\}$ are independent.

Ans: $\mathbb{P}(A) = 0.25 + 0.1 = 0.35$ and $\mathbb{P}(B) = 0.4$ from part (a). Since $\mathbb{P}(A \cap B) = 0.25 \neq 0.35 \times 0.4 = 0.14$, these two events are not independent.

(c) Compute the conditional probability that the response time is slow given that the network is under attack.

Ans: From the definition of conditional probability,

$$\mathbb{P}(\text{slow}|B) = \frac{\mathbb{P}(\text{slow} \cap B)}{\mathbb{P}(B)} = \frac{0.25}{0.4} = \frac{5}{8} .$$

(d) Compute the conditional probability that the network is under attack given that the response time is slow, using the Bayes' rule (and the answer to part (c) above).

Ans: Using the Bayes' rule,

$$\mathbb{P}(B|\text{slow}) = \frac{\mathbb{P}(\text{slow}|B)\,\mathbb{P}(B)}{\mathbb{P}(\text{slow})} = \frac{\frac{5}{8} \times \frac{2}{5}}{\frac{7}{20}} = 0.7143 .$$

3. **[Suggested Problem - will not be graded]** Show that the eigenvalues of positive semidefinite matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are nonnegative.

Ans: Since a positive semidefinite matrix is symmetric, we know that its eigenvalues are real. Suppose that λ is an eigenvalue of $\mathbf{A}$ with a corresponding eigen vector $\mathbf{v}$. Then, from the definition of a positive semidefinite matrix, it must satisfy $\mathbf{v}^T \mathbf{A} \mathbf{v} = \mathbf{v}^T \lambda \mathbf{v} = \lambda \|\mathbf{v}\|^2 \geq 0$. Since $\|\mathbf{v}\| > 0$, it implies $\lambda \geq 0$.