Sept 20th

Today Gambler's Ruin

2 players → A & B compete until one person has all the money (T = total)

each round players A & B bet \$1 each and one player wins \$2

Let p = prob of player A winning round

$q = 1-p$ = " " " B " "

T is total amt of \$

K be the current amt A has (B would have $T-K$)

⇒ We want the prob player A will win all T amt of \$ starting from \$$K$

let $P(K) = P_K$ = prob that A wins the game starting from \$$K$

either A OR B wins → Total Law of Prob

$$\underbrace{P(K)}_{P_K} = \underbrace{P(A \text{ wins} \mid A \text{ wins next round})}_{P_{K+1}} \cdot \underbrace{P(A \text{ win next round})}_{p} + \underbrace{P(A \text{ wins} \mid A \text{ loses next round})}_{P_{K-1}} \cdot \underbrace{P(A \text{ loses next round})}_{q}$$

$$\Longrightarrow \underbrace{P_K = p \cdot P_{K+1} + q \cdot P_{K-1}}_{\text{solve this for } P_K}$$

difference eq'n

Step 1 make a guess : $P_K = (r)^K \leftarrow$ power & plug in & check

$\uparrow$ constant $\longrightarrow$ solve for r

$$\frac{r^K}{r^{K-1}} = p \cdot \frac{r^{K+1}}{r^{K-1}} + q \cdot \frac{r^{K-1}}{r^{K-1}}$$

$$\underbrace{r^{K-(K-1)}}_{r^1} = p \cdot \underbrace{r^{K+1-(K-1)}}_{r^2} + q \cdot \underbrace{r^{(K-1)-(K-1)}}_{r^0=1}$$

$$r = p \cdot r^2 + q$$

$$pr^2 - r + q = 0$$

$$r = \frac{1 \pm \sqrt{1-4pq}}{2p} \longrightarrow$$

$1-4pq$

$1-4p(1-p)$

$1-4p+4p^2$

$(1-2p)(1-2p)$

$(1-2p)^2$

$$r = \frac{1 \pm \sqrt{(1-2p)^2}}{2p}$$

$$= \frac{1 \pm (1-2p)}{2p}$$

$$r_1 = \frac{1+1-2p}{2p} = \frac{2-2p}{2p} = \frac{1-p}{p} = \frac{q}{p}$$

$$r_2 = \frac{1-(1-2p)}{2p} = \frac{1-1+2p}{2p} = \frac{2p}{2p} = 1$$

$\Rightarrow$ General sol'n: $r = \frac{q}{p}$ $r = 1$

$\Rightarrow$ Guess: $P_K = r^K$, two roots

$$P_K = C_1(1)^K + C_2\left(\frac{q}{p}\right)^K$$

$$P(K) = P_K = C_1 + C_2\left(\frac{q}{p}\right)^K$$

(2) C_1, C_2? use "extremes"/defaults/boundary condition

$\hookrightarrow K = 0 \longrightarrow P_0 = 0$

$$\Rightarrow P_0 = C_1 + C_2\underbrace{\left(\frac{q}{p}\right)^0}_{1} = 0$$

$$C_1 + C_2 = 0 \longrightarrow C_1 = -C_2$$

system of eq'ns

$\hookrightarrow K = T$ (total) $P_T = 1$

$$P_T = C_1 + C_2\left(\frac{q}{p}\right)^T = 1$$

Plug

$$-C_2 + C_2\left(\frac{q}{p}\right)^T = 1$$

$$C_2\left(\left(\frac{q}{p}\right)^T - 1\right) = 1$$

$$C_2 = \frac{1}{\left(\frac{q}{p}\right)^T - 1}$$

$$\Rightarrow C_1 = -C_2 = \frac{-1}{\left(\frac{q}{p}\right)^T - 1}$$

③ Plug C_1 & C_2 back into P_K:

$$P_K = C_1 + C_2\left(\frac{q}{p}\right)^K$$

$$= \frac{-1}{\left(\frac{q}{p}\right)^T - 1} + \frac{1}{\left(\frac{q}{p}\right)^T - 1}\left(\frac{q}{p}\right)^K$$

$$P(K) = P_K = \frac{1-\left(\frac{q}{p}\right)^K}{1-\left(\frac{q}{p}\right)^T}$$

Total amt of \$

chance of winning each round

↳ if $p \neq q$ (unfair game)

$$P_K = \frac{K}{T} \quad \text{if } p = q \text{ (fair game)}$$

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ex Olivia & James are rolling a fair die. If the roll results in an even #, James gives Olivia \$1.

↳ 1, 2, 3, 4, 5, 6 $\quad p = \frac{1}{2} = .5$

Otherwise, odd → Olivia gives James \$1

The game continues until one of them loses their money.

a) James starts with \$4 and Olivia has \$6. $\quad T = 10$

What is prob Olivia wins?

$p = q = \frac{1}{2} \quad T = 10 \quad K = 6$

$$\Rightarrow P_K = \frac{K}{T} = \frac{6}{10} = .6 \longrightarrow 60\%$$

b) James \$4 & Olivia \$200 $\quad p=q=\frac{1}{2}$

$\underbrace{\text{James } \$4 \ \& \ \text{Olivia } \$200}_{T=204}$, $\$200 \to K$

$$\Rightarrow P_K = \frac{K}{T} = \frac{200}{204} = .98 \longrightarrow 98\%$$ ← large increase in chance of winning

c) repeat a–b, they play chess where James each game w/ prob $= \frac{3}{5} \simeq q \quad p = \frac{2}{5}$ ← slightly less than 50/50 chance of winning

a) $K=6$ Olivia's \$6, James \$4 $\longrightarrow T=10$

$$P_{10} = \frac{1-\left(\frac{q}{p}\right)^K}{1-\left(\frac{q}{p}\right)^T} = \frac{1-\left(\frac{3/5}{2/5}\right)^6}{1-\left(\frac{3/5}{2/5}\right)^{10}} = .1833 \quad 18.33\%$$

b) Olivia \$200, James \$4 $\longrightarrow T=204$

$\hookrightarrow K=200$

$$P_{200} = \frac{1-\left(\frac{3/5}{2/5}\right)^{200}}{1-\left(\frac{3/5}{2/5}\right)^{204}} = .1975 \quad 19.75\%$$

did not increase by a large %!

Gambler's Ruin

So even though Olivia was able to bet a large amount, having the game slightly less fair against her favor, it only increased her chances by ~1.5% (small!)

——— end Conditional Ch

Random Variables

When an experiment occurs, we are often interested in some function of an outcome, rather than actual outcome.

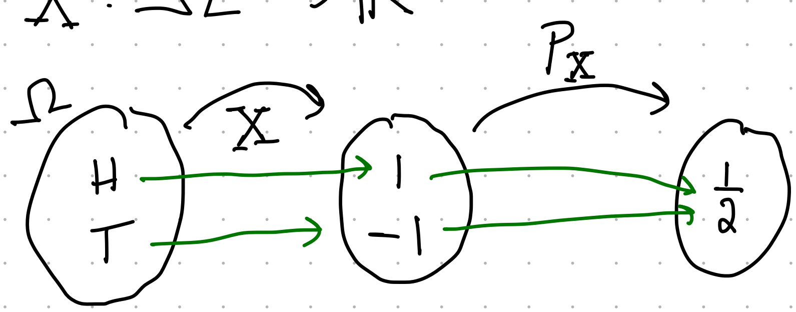
ex tossing 2 die: want the sum is 9, but we don't need all the outcomes ((3,6), (4,5), (5,4), (6,3))

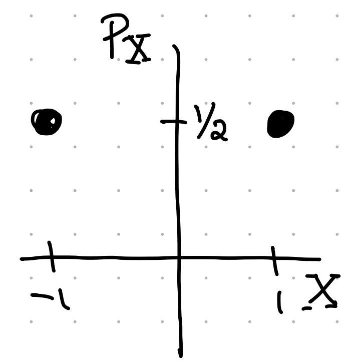
ex flip a coin 3 times → want the # of heads

A random variable (r.v.), usually written as X, Y, is a real-valued function whose domain is the sample space of an experiment.

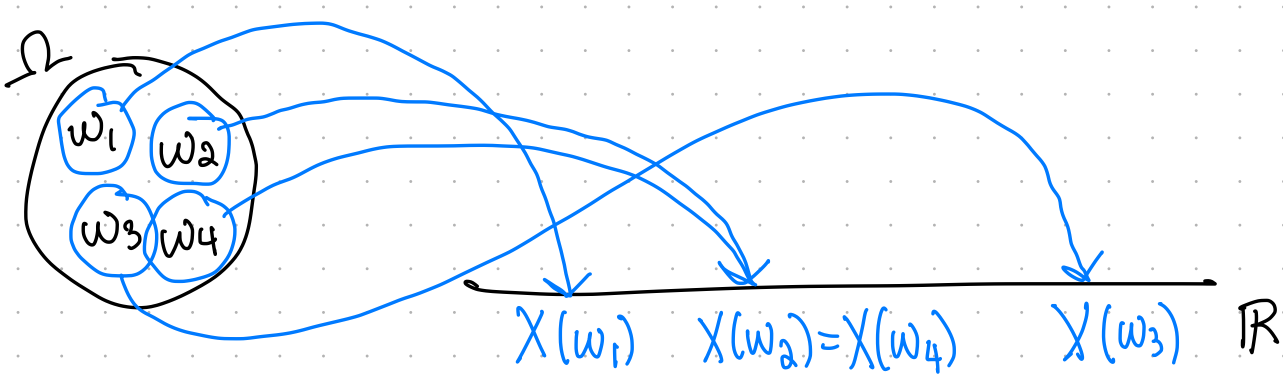
$$X : \Omega \to \mathbb{R}$$

ex coin flip





OR imagine



There are 2 types of r.v.s:

discrete   continuous

<u>Discrete</u> rv: can take on at most a countable number of values

countable: finite / infinite but can be counted

<u>Probability mass function</u> (pmf) (density of X)

$$P_X(x) = P(X=x) = P(\{\omega \in \Omega : X(\omega) = x\})$$

↳ a <u>list</u> of numbers in $[0,1]$ indexed by possible values x of X.

↳ answers "what's the prob I measure a certain value?"

<u>ex</u> coin flip, heads $=1$, tails $=-1$

$$P_X(1) = P(X=1) = \frac{1}{2}$$

$$P_X(-1) = P(X=-1) = \frac{1}{2}$$

$2^3 = 8$ outcomes

<u>ex</u> toss coin 3 times. X is the # of heads obtained

$$X = \{0, 1, 2, 3\}$$

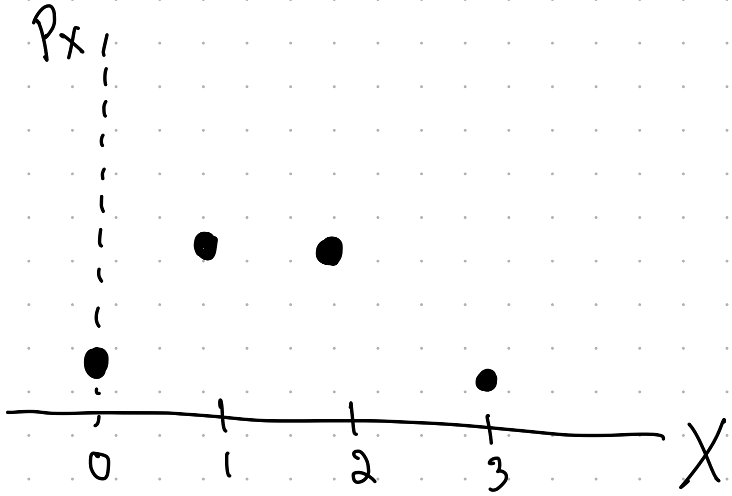
$$P_X(0) = P(X=0) = P(\{TTT\}) = \frac{1}{8}$$

$$P_X(1) = P(X=1) = P(\{TTH, THT, HTT\}) = \frac{3}{8}$$

$$P_X(2) = P(X=2) = 3/8$$

$P_X(3) = P(X=3) = 1/8$

*adds up to 1 ✓



$\{1,2,3,4,5,6\}$

$6 \cdot 6 = 36$ outcomes

ex suppose that a die is rolled twice.
Let X denote the <u>sum</u> of the values.

$X = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$P(X=2) = P(\{1,1\}) = \frac{1}{36}$

$P(X=3) = P(\{(1,2),(2,1)\}) = \frac{2}{36}$

$P(X=4) = \qquad = \frac{3}{36}$

$P(X=5) \qquad = \frac{4}{36}$

$P(X=6) \qquad = \frac{5}{36}$

$P(X=7) \qquad = \frac{6}{36}$

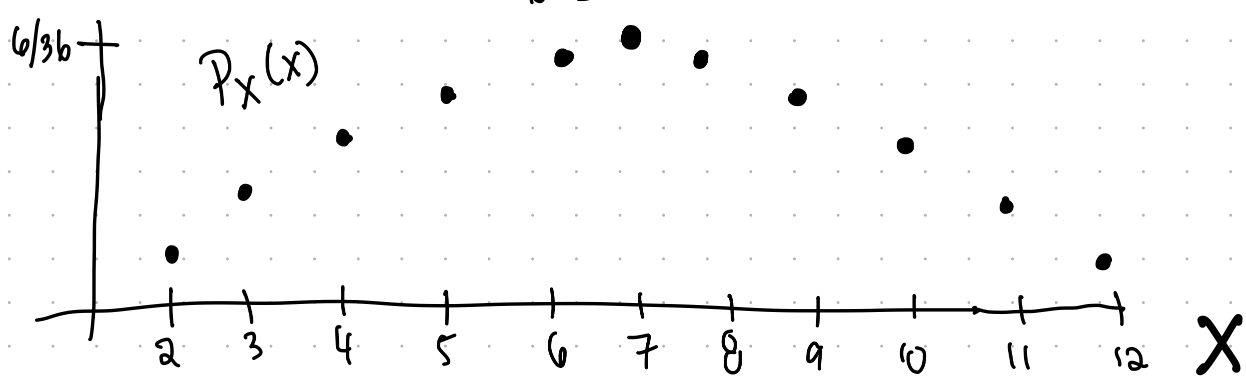
$P(X=8) = 5/36$

$P(X=9) = 4/36$

$P(X=10) = 3/36$

$P(X=11) = 2/36$

$P(X=12) = 1/36$



ex two balls randomly from urn/bag: 8 white, 4 black, 2 orange balls (14 total)

Win \$2 for black, lose \$1 for each white, neutral for orange

Let X denote the winnings.

W = white, B = black, O = orange

$$\Omega = \{\underbrace{WW}_{-1,\,-1\;\to\;-2},\ \underbrace{WB}_{-1,\,+2\;\to\;1},\ \underbrace{WO}_{-1,\,+0\;\to\;-1},\ \underbrace{BB}_{+2,\,+2\;\to\;4},\ \underbrace{BO}_{+2,\,+0\;\to\;2},\ \underbrace{OO}_{+0,\,+0\;\to\;0}\}$$ → Not equally likely

$X = \{-2, -1, 0, 1, 2, 4\}$

order does not matter → combination

$$P(X=-2) = P(\{WW\}) = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}$$

$$P(X=-1) = P(\{WO\}) = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$$

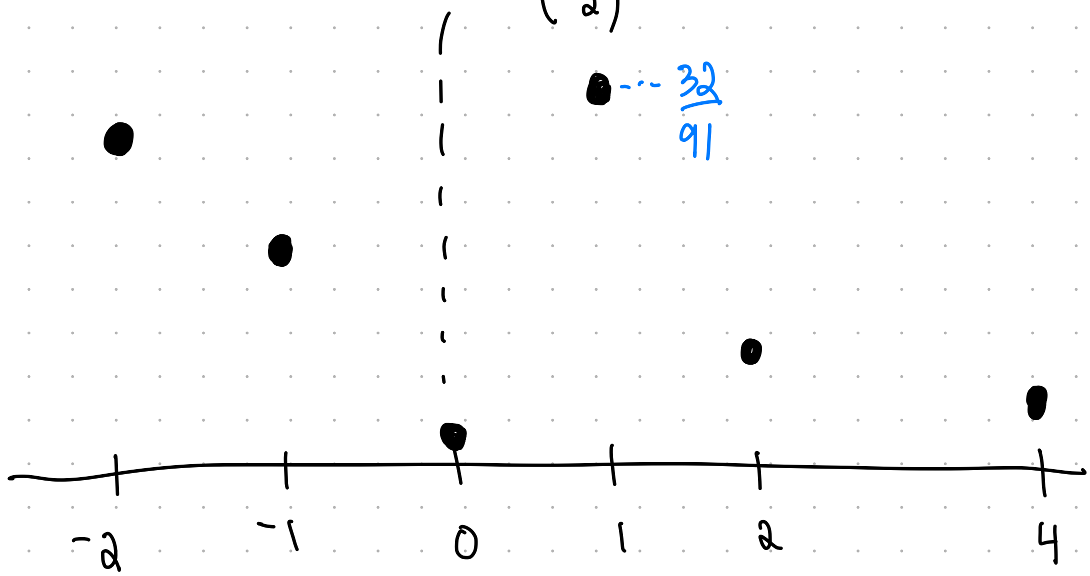
$$P(X=0) = P(\{OO\}) = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$$

$$P(X=1) = P(\{WB\}) = \frac{\binom{8}{1}\binom{4}{1}}{\binom{14}{2}} = \frac{32}{91}$$

$$P(X=2) = P(\{BO\}) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}$$

* $\dfrac{28+16+1+32+8+6}{91} = 1$ ✓

$$P(X=4) = P(\{BB\}) = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$$

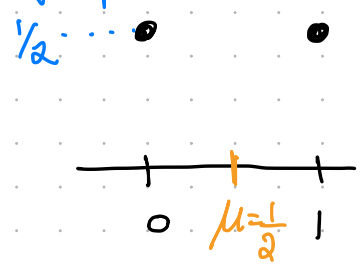


If X is a discrete r.v. with pmf $P_X(x)$ then the expectation of X (expected value) denoted by $E[X]$ OR EX OR μ is

$$\mu = E[X] = \sum_{x:p(x)>0} x \cdot p(x)$$ weighted avg

$$= \sum_{\text{values of } x} (\text{value of } x) \cdot P_X(\text{value of } x)$$

(provided this sum converges absolutely) ex given $P_X(0) = \frac{1}{2} = P_X(1)$ (coin flip)

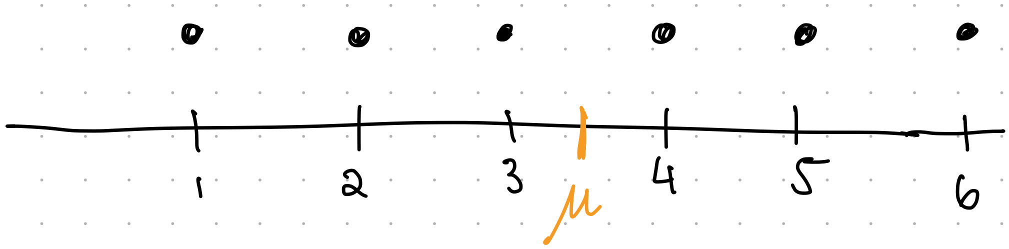


$$E[X] = \underbrace{0}_{x} \cdot \underbrace{\frac{1}{2}}_{\text{prob } x} + \underbrace{1}_{x} \cdot \underbrace{\frac{1}{2}}_{\text{prob } x} = 0 + \frac{1}{2} = \frac{1}{2}$$

ex X = the # rolled die face $= \{1,2,3,4,5,6\}$

$$E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= \frac{7}{2} = 3.5$$



ex X = the value (in \$) of the # of the die cubed.

$\{1,2,3,4,5,6\}$ equally likely

$X = \{\$1, \$8, \$27, \$64, \$125, \$216\}$

μ

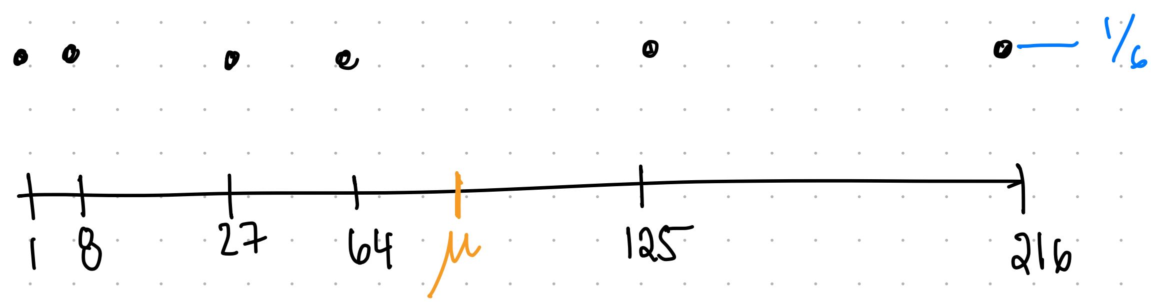
$$E[X] = 1 \cdot \frac{1}{6} + 8 \cdot \frac{1}{6} + 27 \cdot \frac{1}{6} + 64 \cdot \frac{1}{6} + 125 \cdot \frac{1}{6} + 216 \cdot \frac{1}{6}$$

$= \frac{441}{6} = 73.5$ ($73.5)

↳ "meaningless" in one round since I cannot win $73.5 exactly in one round

↓

after 100 tries, expect to win $7350



Linearity of Expectation X, Y r.v. on the same sample space and $c \in \mathbb{R}$ (constant)

i) $E[X \pm Y] = E[X] \pm E[Y]$

ii) $E[cX] = cE[X]$