① $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

② $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$

③ $\mathrm{Cov}(aX,Y) = a\,\mathrm{Cov}(X,Y)$

④ $\mathrm{Cov}(X+c,Y) = \mathrm{Cov}(X,Y)$ $\qquad \mathrm{Cov}(X,Y+c) = \mathrm{Cov}(X,Y)$

⑤ $\mathrm{Cov}(X \pm Y, Z) = \mathrm{Cov}(X,Z) \pm \mathrm{Cov}(Y,Z)$

$$\mathrm{Cov}(2X-Y, X+3Y) = \mathrm{Cov}(2X, X+3Y) - \mathrm{Cov}(Y, X+3Y)$$

$$= \mathrm{Cov}(2X,X) + \mathrm{Cov}(2X,3Y) - \left[\mathrm{Cov}(Y,X) + \mathrm{Cov}(Y,3Y)\right]$$

$$= 2\underbrace{\mathrm{Cov}(X,X)}_{\mathrm{Var}(X)} + \underbrace{2\cdot 3\,\mathrm{Cov}(X,Y) - \mathrm{Cov}(X,Y)}_{5\,\mathrm{Cov}(X,Y)} - 3\underbrace{\mathrm{Cov}(Y,Y)}_{\mathrm{Var}(Y)}$$

Nov 1 **Conditional Expectation** (of X, Y R.V.s)

$$E[X \mid Y=y] = \begin{cases} \sum_x x \cdot P_{X|Y}(x) & \text{discrete} \\ \int x \cdot f_{X|Y}(x)\,dx & \text{continuous} \end{cases}$$

ex) $f(x,y) = \dfrac{e^{-x/y} \cdot e^{-y}}{y}$ $\quad 0 < x < \infty,\ 0 < y < \infty$

Compute $E[X \mid Y=y]$

first get: $f_{X|Y}(x) = \dfrac{f(x,y)}{f_Y(y)}$ ✓ $\quad f_Y(y) \rightarrow$? marginal

$$f_Y(y) = \int_0^\infty \frac{e^{-x/y} e^{-y}}{y}\,dx = \frac{e^{-y}}{y}\int_0^\infty e^{-\frac{x}{y}}\,dx$$

$$= \frac{e^{-y}}{y} \left. \frac{e^{-x/y}}{-1/y} \right|_0^\infty = -e^{-\frac{x}{y}} e^{-y} \Big|_{x=0}^{x=\infty}$$

$$= -\underbrace{e^{\text{"}-\infty\text{"}}}_{0} \cdot e^{-y} - -\underbrace{e^0}_{1} e^{-y} = e^{-y}$$

$$\Rightarrow f_{X|Y} = \frac{\frac{e^{-x/y} e^{-y}}{y}}{e^{-y}} = \frac{e^{-x/y}}{y}$$

Then,

$$E[X|Y=y] = \int_0^\infty x \cdot \underbrace{\frac{e^{-x/y}}{y}}_{f_{X|Y}} dx = \frac{1}{y}\int_0^\infty x e^{-x/y} dx$$

$u = x \qquad dv = e^{-x/y} dx$

$du = dx \qquad v = \frac{e^{-x/y}}{-1/y}$

$uv - \int v\,du$

$$= \frac{1}{y}\left[-xy e^{-x/y}\Big|_0^\infty + \int_0^\infty y e^{-x/y} dx\right]$$

$$= \frac{1}{\cancel{y}}\left[-x\cancel{y} e^{-x/y} + \cancel{y} \frac{e^{-x/y}}{-1/y}\Big|_0^\infty\right] = 0 + \frac{e^{\text{"}-\infty/y\text{"}}}{-1/y} - 0 + y e^0$$

$e^{\text{"}-\infty/y\text{"}} \to 0$, $e^0 \to 1$

$$= y$$

$\lim_{x\to\infty} x e^{-x/y} \to \infty \cdot 0$

$= \lim_{x\to\infty} \frac{x}{e^{x/y}} \to \frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{y}e^{x/y}} \to \frac{1}{\infty} = 0$

$$E[X|Y=y] = y$$

R.V. itself!

Similar Prop:

$$E[g(x)|Y=y] = \sum_x g(x)\, p(x|y)$$

$$\int_{-\infty}^{\infty} g(x)\, f(x|y)\, dx$$

$$Var(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2$$

Expectations by conditioning:

$$E[X] = E[\underbrace{E[X|Y]}_{\text{random variable that depends on } Y}]$$

$$= \sum_y E[X|Y=y] \cdot P(Y=y) \quad \text{discrete}$$

$$= \int_{-\infty}^{\infty} E[X|Y=y]\, f_Y(y)\, dy$$

ex A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours. The second leads to a tunnel that will return to the mine after 5 hours. Third door → tunnel back to mine after 7 hours. If we assume miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?

X = amt of time (hours) until miner reaches safety

Let Y = door he initially chooses

$$E[X] = E[X|Y=1]\underbrace{P(Y=1)}_{1/3} + E[X|Y=2]\underbrace{P(Y=2)}_{1/3}$$
$$+ E[X|Y=3]\underbrace{P(Y=3)}_{\frac{1}{3}}$$

But $E[X|Y=1] = 3$

$E[X|Y=2] = 5 + E[X]$ ← door 2 leads to a 5 hr detour to his expected time to safety

$E[X|Y=3] = 7 + E[X]$

$$E[X] = 3\cdot\frac{1}{3} + (5+E[X])\frac{1}{3} + (7+E[X])\cdot\frac{1}{3}$$

$\Rightarrow$ Solve for $E[X] \longrightarrow E[X] = 15$ hours

ch 7 end

Ch 8 (Ross)

Inequalities associated w/ Random Variables

Want to compute

$$P(X>a) = P(X \text{ takes on values bigger than } a)$$

To calculate exact value:

$$P(X>a) = \int \quad \sum_{x>a} p_X(x) \qquad X \text{ discrete}$$

$$\Big] \int_a^\infty f(x)dx \quad X \text{ cont}$$

In both cases, we need the pdf/pmf to be known.

What if pmf/pdf is not available?

$\downarrow$

Can we estimate this probability still?

Yes $\longrightarrow$ inequalities

Markov's Inequality If X is a R.V. that only take non-negative values of x, then for any value $a>0$

$$P(X \geq a) \leq \frac{E[X]}{a} = \frac{\mu}{a}$$

Chebyshev's Inequality

If X is a R.V. with finite $\mu = E[X]$ and variance σ^2, then for any value $a>0$

$$P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

Derivation (Chebyshev from Markov)

$$P(|X-\mu| \ge a) = P(\underbrace{|X-\mu|^2}_{>0} \ge a^2) \underset{\text{Markov}}{\le} \frac{E[|X-\mu|^2]}{a^2}$$

$$= \frac{Var(X)}{a^2}$$

$$= \frac{\sigma^2}{a}$$

One-sided Chebyshev (Cantelli's) Inequality

$$\underbrace{P(X \ge \mu + a)}_{P(X-\mu \ge a)} \le \frac{\sigma^2}{\sigma^2 + a^2}$$

Ex Suppose that it is known that the number of items produced by a factory during a week is a random variable with mean 50.

a) What can be said about the probability that this week's production will exceed 75?

$X = \#$ of items produced in a week

Markov's Inequality : $P(X>75) \le \frac{\mu}{a} = \frac{50}{75} = \frac{2}{3}$

a

b) If the variance of a week's production is known to be 25, what can be said about the prob that this week's production will be btwn 40 & 60?

$\mu = 50 \quad \sigma^2 = 25$

$|X-50| \le 10$

$-10 \le X-50 \le 10$

$40 \le X \le 60$

$$P(40 \le X \le 60) = P(|X-50| \le 10)$$

$$= 1 - P(|X-50| \ge 10)$$

μ a

$P(|X-50| \ge 10) \le \frac{\sigma^2}{a^2}$

$-P(|X-50| \ge 10) \ge -\frac{\sigma^2}{a^2}$

$$\ge 1 - \frac{\sigma^2}{a^2}$$

$$= 1 - \frac{25}{100} = \frac{3}{4}$$

$$\Rightarrow P(40 \le X \le 60) \ge \frac{3}{4}$$

ex If the # of items produced by factory in a week has $\mu = 100$, $\sigma^2 = 400$, compute an upper bound on probability this week, where production will be at least 120.

$$P(X \geq 120) \leq \frac{\mu}{a} = \frac{100}{120} = \frac{5}{6}$$ Markov's inequality

$a = 120$

$$P(X \geq 120) = P(X - 100 \geq 120 - 100) = P(\underset{\mu}{X - 100} \geq \underset{a=20}{20})$$

$$\leq \frac{\sigma^2}{\sigma^2 + a^2} = \frac{400}{400 + 20^2}$$

$$= \frac{400}{400+400} = \frac{1}{2}$$

*Notice: Markov is a weaker bound.

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ex A coin is tossed 100 times. Estimate the prob that there are at least 67 heads using all 3 inequalities.

binomial

$$P(X \geq 67) = \sum_{x=67}^{100} \cdots$$

long to compute

$$X \sim Bin(\overset{n}{100}, \overset{p}{\tfrac{1}{2}})$$

$$\mu = np = 100 \cdot \tfrac{1}{2} = 50$$

$$\sigma^2 = np(1-p) = 100 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 25$$

a) Markov $P(X \geq 67) \leq \frac{\mu}{a} = \frac{50}{67} \approx 0.75$

b) Chebyshev

$$P(X \geq 67) = P(X - \mu \geq 67) = P(X - 50 \geq 17)$$

$$\leq \frac{\sigma^2}{a^2} = \frac{25}{17^2} \approx .086$$

b) $P(X \geq 67) = P(X - 50 \geq 17) \leq \frac{\sigma^2}{\sigma^2 + a^2} = \frac{25}{25 + 17^2} \approx .079$

Central Limit Thm (CLT)

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent, and identically distributed (i.i.d) R.V., each having mean μ and variance σ^2. Then the distribution

$$\frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$

$$P\left(\frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sigma\sqrt{n}} \leq a\right) \xrightarrow{n \to \infty} \Phi(a)$$

CDF of normal

Another way to write this : $S_n = X_1 + X_2 + \dots + X_n$

$$P\left(a \le \frac{S_n - n\mu}{\sigma\sqrt{n}} \le b\right) \xrightarrow[n\to\infty]{} P(a \le Z \le b)$$

standard normal

$N(0,1)$

for any a and b.

ex An instructor has 1000 large n exams to grade. The time required to grade exams are all i.i.d. with a mean 20 min & standard deviation of 4 min. Approximate the prob that the instructor will be able to grade at least 25 exams in the first 450 min of work.

Let X_i be the time it takes to grade exam i

Then $X = X_1 + X_2 + \dots + X_{25}$ is the time it takes to grade 25 exams

($25 \to n$)

CLT: $n\mu = 25 \cdot 20 = 500$

$\sigma\sqrt{n} = 4 \cdot \sqrt{25} = 20$

$$P(X \le 450) = P\left(\frac{X - 500}{20} \le \frac{450 - 500}{20}\right)$$

$$\underset{CLT}{\cong} P(Z \le -2.5) = \Phi(-2.5)$$

$$= 1 - \Phi(2.5) = 1 - .9938$$

<u>ex</u> the # of students who enroll in a course is a Poisson R.V. w/ mean 100.

↳ discrete

If the # enrolled $\ge 120 \to$ the course splits into 2

If " " " $< 120 \to$ stays as one class.

What is the prob that there will be 2 courses?

exact sol'n: $\displaystyle\sum_{K=120}^{\infty} \frac{e^{-100}(100)^K}{K!}$

$\lambda = 100$

This can also be written as a sum of 100 Poisson R.V., each with a mean $= 1 = \lambda$

→ Sum of 100 Poisson R.V. (each i.i.d), then use CLT to approximate:

Poisson: $\mu = \lambda$ $\qquad n\mu = 100 \cdot 1 = 100$

$\sigma^2 = \lambda$ $\qquad \sigma\sqrt{n} = \sqrt{1} \cdot \sqrt{100} = 10$

$$P(X \geq 120) \approx P(X \geq 119.5)$$

continuity correction since Poisson discrete

$$\approx P\left(\frac{X-100}{10} \geq \frac{119.5-100}{10}\right)$$

CLT

$$\approx P(Z \geq 1.95) = 1 - P(Z \leq 1.95)$$

$$= 1 - \Phi(1.95)$$

$$= 1 - .9744$$

$$= 0.0256$$

Continuity Correction:

discrete	continuous
$P(X=n)$	$P(n-0.5 < X < n+0.5)$
$P(X>n)$	$P(X > n+0.5)$
$P(X<n)$	$P(X < n-0.5)$
$P(X \leq n)$	$P(X < n+0.5)$
$P(X \geq n)$	$P(X > n-0.5)$

ex If 10 fair die are rolled, find the approximate prob that the sum obtained is btwn 30 & 40, inclusive.

Let X_i be the value of the ith die $\{1,2,3,4,5,6\}$

want to use CLT $\longrightarrow$ get μ & σ^2

$$E[X_i] = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6} = \frac{21}{6}$$

$$= \frac{7}{2} = \mu$$

$$E[X_i^2] = 1\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 9\cdot\frac{1}{6} + 16\cdot\frac{1}{6} + 25\cdot\frac{1}{6} + 36\cdot\frac{1}{6} = \frac{91}{6}$$

$$\text{var}(x) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} = \sigma^2$$

Let $X = X_1 + \ldots + X_{10}$ CLT: $n\mu = 10\cdot\frac{7}{2} = 35$

discrete

$$\sigma\cdot\sqrt{n} = \frac{\sqrt{35}}{\sqrt{12}}\cdot\sqrt{10}$$

$$= \frac{\sqrt{350}}{\sqrt{12}}$$

$$P(30 \le X \le 40) = P(29.5 \le X \le 40.5)$$

$$\approx P\left(\frac{29.5-35}{\sqrt{\frac{350}{12}}} \le Z \le \frac{40.5-35}{\sqrt{\frac{350}{12}}}\right)$$

$$= P(-1.02 \le Z \le 1.02)$$

$$= \Phi(1.02) - \Phi(-1.02)$$

$$= \Phi(1.02) - (1-\Phi(1.02))$$

$$= 2\Phi(1.02) - 1 = .692$$

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ex Matt takes a bus to work 300 days a year. Suppose that bus waiting times are independent of one and each is uniformly distributed btwn 0 and 10 min. Let T be the total waiting time for a particular year.

Compute approximately the prob that T is ⓐ more than 26 hours and ⓑ it is between 24 & 25 hours.

each day: U_i waiting time during the i^{th} day then $T = \sum_{i=1}^{300} U_i$ $\qquad U_i \sim Unif(\overset{a}{0}, \overset{b}{10})$

$$= \begin{cases} \frac{1}{10} & 0 \le x \le 10 \\ 0 & \text{Otherwise} \end{cases}$$

$$\mu = \frac{b-a}{2} = \frac{10-0}{2} = 5$$

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{10^2}{12} = \frac{25}{3}$$

CLT: $n\mu = 300 \cdot 5 = 1500$

$$\sigma\sqrt{n} = \sqrt{300} \cdot \sqrt{\frac{25}{3}} = \sqrt{\frac{300}{3}} \cdot 5 = \sqrt{100} \cdot 5 = 10 \cdot 5 = 50$$

24 hr = 1440 min

25 hr = 1500 min

26 hr = 1560 min

a) $P(T \ge 1560) \overset{\text{CLT}}{\approx} P\left(Z \ge \frac{1560 - 1500}{50}\right)$

$$= 1 - P(Z \le 1.2) = 1 - \Phi(1.2)$$

$$= 1 - .8849 = .1151$$

b) $P(1440 \le X \le 1500) \approx P(-1.2 \le Z \le 0)$

$$= P(Z \leq 0) - (1 - P(Z \leq 1.2))$$

$$= .5 - 1 + .8849$$

$$= .3849$$