ex What is the probability that in a poker hand (5 cards out of 52) we get exactly 4 of a kind?

Deck of cards: 4 suits
13 kinds/rank
2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

→ the probability of 4 aces & 1 king

$$\frac{\overset{A}{\binom{4}{4}}\overset{K}{\binom{4}{1}}}{\binom{52}{5}}$$

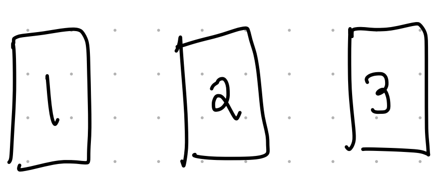
→ however the same prob as 4 jacks & one 3

↳ adjust for this

$$\frac{\binom{4}{4}\binom{4}{1}}{\binom{52}{5}} \cdot 13 \cdot 12 = \frac{624}{2,598,960} = 0.00024$$

Monty Hall Problem : 1 host, 1 player

3 doors → behind
1 car
2 goats



Player chooses a door (door #1) then host opens one w/ a goat. (ex: door #2)
Should the player switch?

Sept 13^{th}

choose #2	1	2	3	Switch
scenario 1	c	g	g	car ✓
2	g	c	g	goat :(
3	g	g	c	car ✓

switch

host opens

$\frac{2}{3}$ car

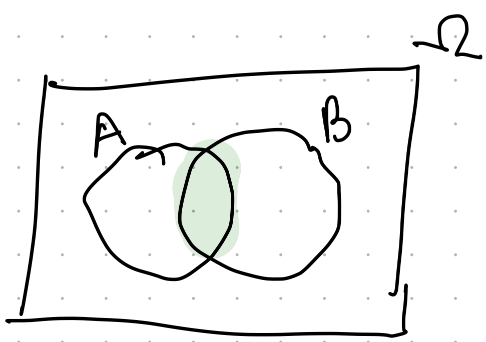
w/o switching: P(win car) $= \frac{1}{3}$

w/ switching: P(win car) $= \frac{2}{3}$

Conditional prob answers the question "if I already know event A is going to happen, how should I update my beliefs about event B?

Idea: $P(B|A) = \frac{P(B \cap A)}{P(A)}$ Conditional Prob

prob B given A



→ replace Ω with A

→ replace B with $B \cap A$

ex flip 3 coins. What is the prob of 2 out 3 heads (B) given that the first flip is heads (A)

Ω

HHH		A
HHT	B	A
HTH	B	A
THH	B	
TTH		
THT		
HTT		A
TTT		

8 outcomes

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\frac{2}{8}}{\frac{4}{8}} = \frac{2}{4} = \frac{1}{2}$$

ex A couple has 2 children. Assume having a boy or girl is equally likely. What is the prob that the 2 children are girls if

a) no additional info given

b) know older one is a girl

c) " one of the two is a girl

a) $\Omega = \{ \overset{O\ Y}{GG}, \overset{O\ Y}{GB}, \overset{O\ Y}{BG}, \overset{O\ Y}{BB} \}$ $\quad P(A) = \frac{1}{4}$

b) $\underset{\text{event}}{B} = \{ GG, GB \}$ $\quad P(B) = \frac{1}{2}$

c) $\underset{\text{event}}{C} = \{ GG, GB, BG \}$ $\quad P(C) = \frac{1}{3}$

Note $P(B \cap A) = P(A \cap B)$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A|B)\,P(B)}{P(A)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \cap A)}{P(B)} \Rightarrow P(B \cap A) = P(A|B)\,P(B)$$

$$\Rightarrow \boxed{P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}}$$

know $P(A|B)$
want $P(B|A)$

ex 36% of families own a dog
30% " " " " cat
22% of " that have a dog, also have a cat

A family is chosen at random & found to have a cat. What is the prob they also have a dog?

event: D: family w/ dog
C: " " cat

given
$P(D) = .36$
$P(C) = .3$
$P(C|D) = .22$

Want: $P(D|C) = \dfrac{P(C|D)\,P(D)}{P(C)} = \dfrac{(.22)(.36)}{(.3)}$

$= .264$

26.4%

Tower/Multiplication Rule

Events: A, B, C

$$P(A\cap B) = P(A)\,P(B|A) \longrightarrow P(B|A) = \frac{P(A\cap B)}{P(A)}$$

same

$$P(A\cap B\cap C) = P(A)\,P(B|A)\,P(C|B\cap A)$$

$\vdots$

$$P(E_1\cap E_2\cap\ldots\cap E_n) = P(E_1)\,P(E_2|E_1)\,P(E_3|E_2\cap E_1)\cdot\ldots\cdot P(E_n|E_{n-1}\cap\cdots\cap E_1)$$

ex In a factory there are 100 units of a certain product, 5 of which are defective. We pick 3 units from the 100 at random. What is the prob that none of them are defective?

Define A_i as the event that the ith chosen unit is not defective $i=1,2,3$

Want $P(A_1\cap A_2\cap A_3) \underset{\text{tower}}{=} P(A_1)\,P(A_2|A_1)\,P(A_3|A_2\cap A_1)$

Know

$P(A_1) = \frac{95}{100}$

$P(A_2|A_1) = \frac{94}{99}$

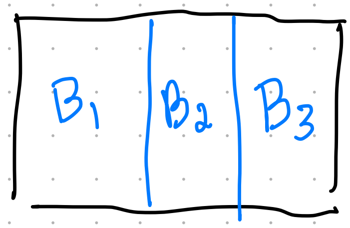
$P(A_3|A_2\cap A_1) = \frac{93}{98}$

$$= \frac{95}{100}\cdot\frac{94}{99}\cdot\frac{93}{98} = 0.8560$$

Law of Total Prob

$B_1, B_2, B_3, \ldots$ disjoint events that form a partition

$S = \Omega$

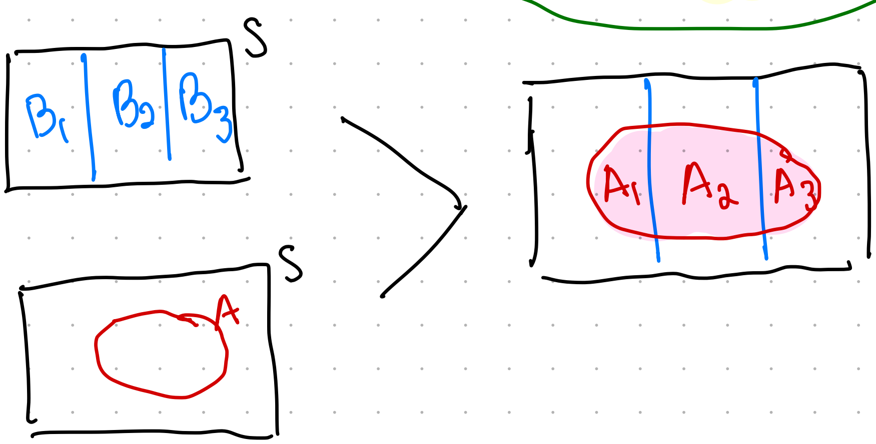


Let A be an event:

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A \mid B_i)\, P(B_i)$$

definition of cond prob

ex



Bayes' Rule #1 (total prob version)

Know: $P(A)$

$P(B \mid A)$

$P(B \mid A^c)$

Want $P(B)$

$$\Rightarrow P(B) = \underbrace{P(B \mid A)P(A) + P(B \mid A^c)\, P(A^c)}_{\text{cond prob def'n}}$$

$$= P(B \cap A) + P(B \cap A^c)$$

$$= P\big((B \cap A) \cup (B \cap A^c)\big)$$

Axiom 3

$B \cap A$ & $B \cap A^c$

$$= P(B \cap (A \cup A^c))$$ → distribute disjoint

$$= P(B \cap \Omega)$$

$$= P(B)$$ ✓

Bayes' Rule #2

Know: $P(A)$

$P(B|A)$

$P(B|A^c)$

Want $P(A|B)$

$$P(A|B) = \frac{P(B|A)P(A)}{\underbrace{P(B|A)P(A) + P(B|A^c)P(A^c)}}$$

cond defi'n

B.R. #1

$$= \frac{P(B \cap A)}{P(B)}$$

$$= P(A|B)$$ ✓

if we have more events (3 or more) add into sum ex A device is used for planes.

Company A makes 80% of devices

" B " 15% " "

" C " 5% " "

*disjoint & exhaustive
↓
partition

Some devices are defective:

4% from Company A are defective

6% " " B " "

9% " " C " "

If a randomly selected device is tested & is found to be defective, what is the prob it was made by company A?

disjoint, exhaustive

events

A: device from company A

B: " " B

C: " " C

$P(A)=.8$

$P(B)=.15$

$P(C)=.05$

D: device defective

D^c: device works

$P(D|A)=.04$

$P(D|B)=.06$

$P(D|C)=.09$

Want $P(A|D) = \dfrac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)}$

B.R. 2

$$= \frac{(.04)(.8)}{(.04)(.8) + (.06)(.15) + (.09)(.05)}$$

$$= .703$$

Try another way: Chart   10,000 devices

	defective	not defective	total
Comp A	4% 320	7680	8,000
Comp B	6% 90	1410	1,500
Comp C	9% 45	455	500
	455		10,000

$\Rightarrow \dfrac{320}{455} = .703$

Same ✓

<u>Independence</u> of events is a relationship assuming as one event happens, it does not affect the prob of the other.

$$\boxed{P(A \cap B) = P(A)P(B)} \quad \text{Independent}$$

$\neq$ not the same as disjoint

$P(A \cap B) = \emptyset$

Recall: Cond prob, if events A & B are independent:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \overset{\text{indep.}}{=} \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A)$$

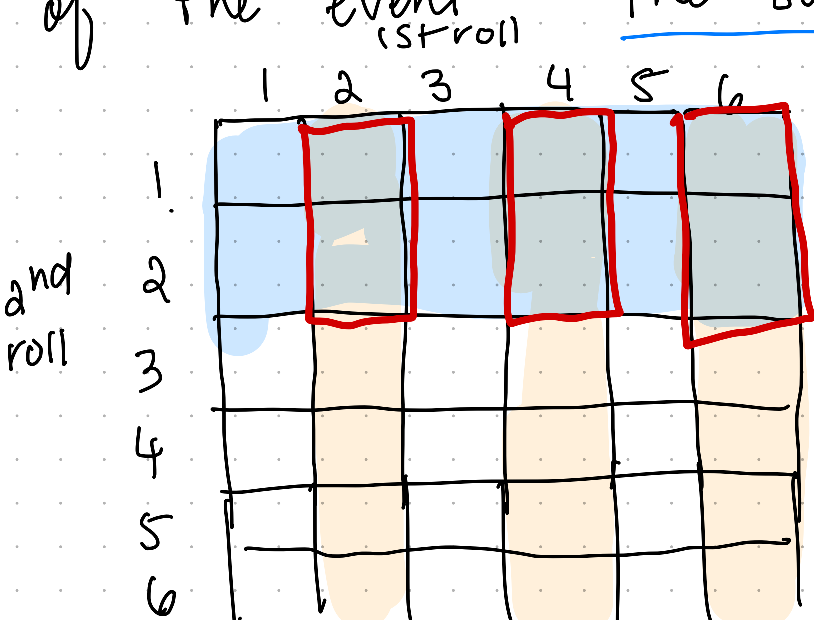
$$P(B|A) = P(B)$$

<u>ex</u> two fair dice rolled

Is the event "the first is even" independent of the event "the second is ≤ 2"

A: "the first is even"

B: "the second is ≤ 2"



A: first is even

$$P(A) = \frac{18}{36} = \frac{1}{2}$$

B: second ≤ 2

$$P(B) = \frac{12}{36} = \frac{1}{3}$$

Check $P(B \cap A) \stackrel{?}{=} P(B)P(A)$ for independence

$$\frac{6}{36} = \frac{1}{3} \cdot \frac{1}{2}$$

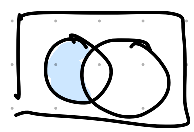
$$\frac{1}{6} = \frac{1}{6} \checkmark \quad \text{Independent}$$

If A & B independent then so are

A & B^c

A^c & B

A^c & B^c



$$P(A \cap B^c) = P(A) - P(B \cap A) \longrightarrow \text{venn diagram}$$
$$= P(A) - P(B)P(A)$$
$$= P(A)(1 - P(B))$$
$$= P(A)P(B^c) \checkmark$$

<u>Joint Independence</u> is the extension to a collect of sets (ex: 3)

$$\left.\begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(C \cap B) &= P(C)P(B) \end{aligned}\right\} \text{"pairwise" independent}$$

and $P(A \cap B \cap C) = P(A)P(B)P(C)$

all 4 need to be satisfied

ex Throw 2 fair die.

A: $\{\text{sum is } 7\} = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

B: $\{\text{first is } 3\} = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$

C: $\{2^{nd} \text{ is a } 4\} = \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$

Jointly Indep?

- $\underbrace{P(A \cap B)}_{\{(3,4)\}} \overset{?}{=} P(A)P(B)$

$$\frac{1}{36} = \frac{6}{36}\,\frac{6}{36} = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36} \quad \checkmark$$

- $\overset{\{(3,4)\}}{P(B \cap C)} \overset{?}{=} P(B)P(C)$

$$\frac{1}{36} = \frac{6}{36}\,\frac{6}{36} \quad \checkmark$$

- $\overset{\{(3,4)\}}{P(A \cap C)} = P(A)P(C)$

$$\frac{1}{36} = \frac{6}{36}\,\frac{6}{36} \quad \checkmark$$

pairwise Independent

But: $\underbrace{P(A \cap B \cap C)}_{\{(3,4)\}} = P(A)P(B)P(C)$

$$\frac{1}{36} \neq \frac{6}{36}\,\frac{6}{36}\,\frac{6}{36} = \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}$$

Not jointly independent

ex A system has 3 parallel components, each of which has a 30% chance of failing w/in a year. Assuming the failures are jointly independent, what is the prob the system functions for an entire year?

A: functions a whole year

B_i: component i fails in a year $\quad i=1,2,3$

$P(B_i) = .3$

parallel: $A^c = B_1 \cap B_2 \cap B_3$

$$P(B_1 \cap B_2 \cap B_3) = P(B_1)P(B_2)P(B_3)$$

$$= \frac{3}{10}\frac{3}{10}\frac{3}{10} = \frac{27}{1000} = .027$$

$$P(A) = 1 - \frac{27}{1000} = .973$$