1. Suppose the joint probability density function of two continuous random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & \text{if } x > 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the marginal probability density function f_X of random variable X.

   (b) Compute the expected value $\mathbb{E}[X]$.

   (c) Determine whether or not X and Y are independent.

2. A tumor can be either benign or malignant. The growth rate of a tumor, which we denote by X, can be modeled as a random variable, whose distribution depends on the type of tumor. The growth rate of a benign tumor has a probability density function

$$p_B(x) = \begin{cases} 5e^{-5x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, the growth rate of a malignant tumor has a probability density function

$$p_M(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

A doctor measures the size of a tumor on two different dates and computes the growth rate of the tumor in a patient. Based on the measured growth rate, the doctor has to tell a patient whether the tumor is benign or malignant. When the doctor makes a correct diagnosis, there is no cost. If the doctor tells a patient that he/she has a malignant tumor when it is benign, the doctor incurs a cost of 3. On the other hand, when the doctor tells a patient that he/she has a benign tumor when it is in fact malignant, the doctor suffers a cost of 20. Suppose that only 10 percent of all tumors are malignant. Design a Bayes classifier that minimizes the overall risk.

3. The distribution of observations $\mathbf{X} = (X_1, X_2)$, which are sensor measurements we take each day to detect a possible gas leak, depends on whether or not there is a gas leak at the time. When there is a gas leak, $\mathbf{X}$ is jointly Gaussian with mean $\boldsymbol{\mu}_{GL} = [10\ 10]^T$. When there is no gas leak, $\mathbf{X}$ is jointly Gaussian with mean $\boldsymbol{\mu}_{NL} = \mathbf{0}$. The probability that there is a gas leak on any given day is 0.1.

   (a) Assume that the covariance matrix $\Sigma = \sigma^2 \mathbf{I}_{2 \times 2}$ in both cases, where $\sigma = 10$. Design the Bayes classifier that selects either {gas leak} or {no gas leak}) and minimizes the probability of error, on the basis of the sensor measurements $\mathbf{X}$. Draw the two decision regions $\mathcal{R}_{GL}$ and $\mathcal{R}_{NL}$.

   (b) Suppose that the covariance matrix Σ is given by

$$\Sigma = \begin{bmatrix} 100 & 50 \\ 50 & 100 \end{bmatrix}.$$

   Determine the decision regions $\mathcal{R}_{GL}$ and $\mathcal{R}_{NL}$ for the Bayes classifier.