Oct 11th Continuous Random Variables (Ch 5 in Ross)

uncountable infinite values for the RV

↳ $\mathbb{R}$

↳ interval of $\mathbb{R}$ ex: $[a,b]$, $[0,1]$

examples "exact" weight, time, temp, share price } if we can divide the units w/o stopping

A random variable X is said to have a continuous distribution if there exists a non-negative $f = f_X$ such that

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

area under the curve of $f(x)$

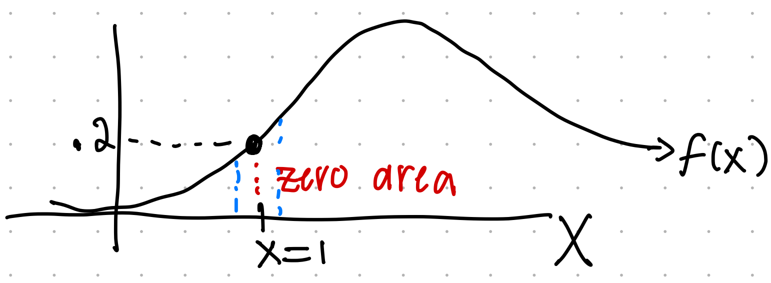
for every a and b.

f: prob density function (pdf)

* we consider intervals for continuous R.V.

* prob at one specific point $= 0$

$$P(1 \le X \le 1) = \int_1^1 f(x)\,dx = 0$$

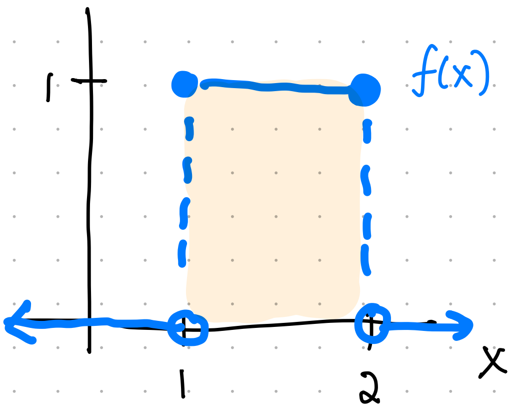


$$P(.99999 \le X \le 1.000001) = \int_{.99999}^{1.000001} f(x)\,dx > 0$$

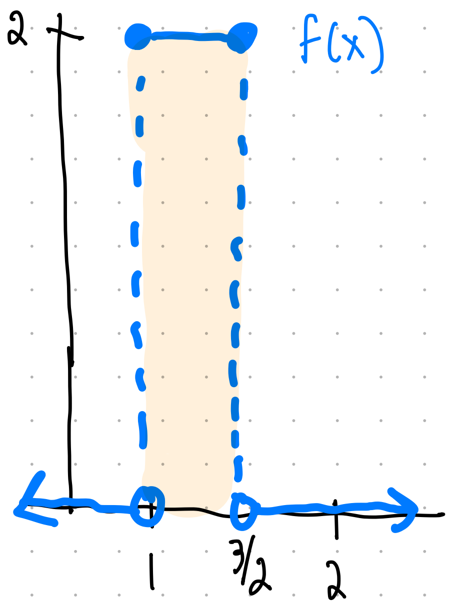
Note: $f(x)$ is <u>not</u> the prob

↳ the prob is the integral of $f(x)$ of a certain interval

<u>Imagine two examples</u>



$$P(1 \le X \le 2) = \text{area of rectangle}$$
$$= \text{base} \cdot \text{height}$$
$$= (2-1) \cdot 1 = 1 \cdot 1 = 1$$



$$P\left(1 \le X \le \tfrac{3}{2}\right) = \text{area of rectangle}$$
$$= \text{base} \cdot \text{height}$$
$$= \left(\tfrac{3}{2} - 1\right) \cdot 2 = \tfrac{1}{2} \cdot 2 = 1$$

* $f(x) > 1$ can occur for some values of $x \in \mathbb{R}$

→ If we integrate over the entire domain → should get 1

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

<u>ex</u> Let X be a cont R.V. with PDF: c constant

$$f(x) = \begin{cases} c(4x - 2x^2) & 0 < x < 2 \quad (0 \leq x \leq 2) \\ 0 & \text{otherwise} \end{cases}$$

a) What is c?

b) $P(X > 1)$?

a) $$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 c(4x - 2x^2)\,dx = c\int_0^2 (4x - 2x^2)\,dx$$

$$= c\left(\frac{4x^2}{2} - \frac{2x^3}{3}\right)\Bigg|_{x=0}^{x=2}$$

$$= c\left(\frac{4(2)^2}{2} - \frac{2(2)^3}{3} - \left(\cancelto{0}{\frac{4(0)^2}{2}} - \cancelto{0}{\frac{2(0)^3}{3}}\right)\right)$$

$$1 = c\left(8 - \frac{16}{3}\right) = c\left(\frac{24-16}{3}\right) = c\,\frac{8}{3}$$

$$c = \frac{3}{8}$$

b) $$P(X>1) = P(X \geq 1) = \int_1^{\infty} f(x)\,dx$$

$$= \int_1^2 \frac{3}{8}(4x - 2x^2)\,dx$$

$$= \frac{3}{8}\left(2x^2 - \frac{2}{3}x^3\right)\Bigg|_1^2$$

$$= \frac{3}{8}\left(8 - \frac{2}{3}\cdot 8 - \left(2 - \frac{2}{3}\right)\right)$$

$$= \frac{3}{8}\left(6 - \frac{16}{3} + \frac{2}{3}\right)$$

$$= \frac{3}{8}\left(6 - \frac{14}{3}\right) = \frac{3}{8}\left(\frac{18}{3} - \frac{14}{3}\right)$$

$$= \frac{3}{8}\left(\frac{4}{3}\right) = \frac{1}{2}$$

$\lambda = 2$

Assume you receive an average of 2 phone calls per hour. You make the Poisson assumption so the time between calls follows an exponential density,

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \in [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

Assume you just hung up the phone

a) what is the prob that you are going to have to wait more than an hour for the next call?

b) You'll recieve a call w/in 10 min?

c) Wait more than 10 min but less than an hour?

a) $P(X>1) = P(X \geq 1)$

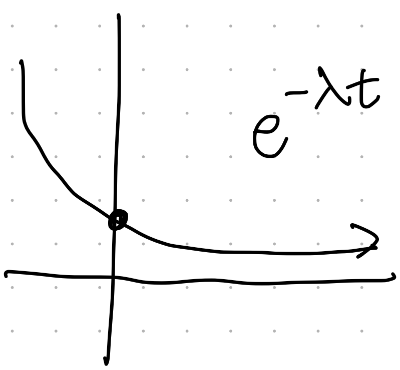
$$= \int_1^\infty f(x)\,dx$$

λ: constant

$$= \int_1^\infty \lambda e^{-\lambda x}\,dx$$

$$= \cancel{\lambda}\,\frac{e^{-\lambda x}}{-\cancel{\lambda}}\Bigg|_1^\infty$$

$$= \lim_{t\to\infty} -e^{-\lambda x}\Big|_1^t$$



$$= \lim_{t\to\infty} -\left(e^{-\lambda t} - e^{-\lambda \cdot 1}\right)$$

$\lambda=2$

$$= 0 + e^{-\lambda} = e^{-\lambda} = e^{-2} \approx 0.135$$

b) $P\left(X < \frac{10}{60}\right) = P\left(X \leq \frac{1}{6}\right)$

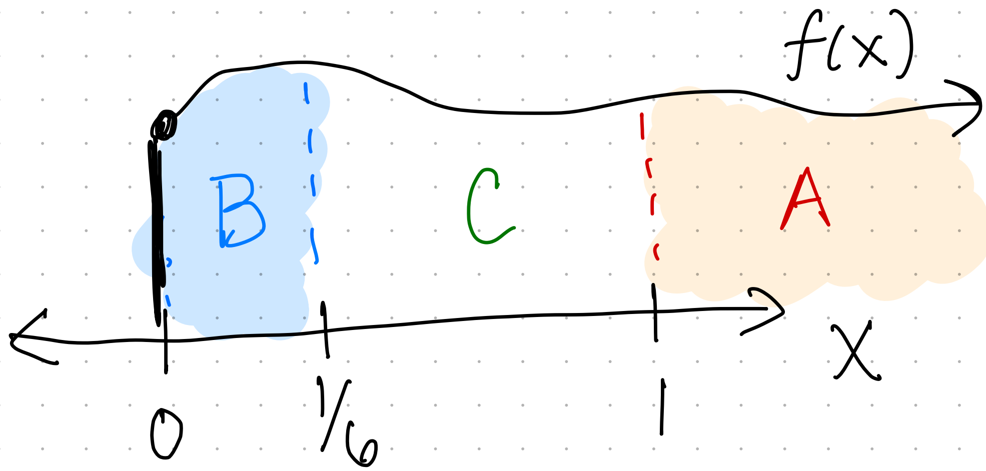
$$= \int_0^{1/6} 2e^{-2x}\,dx$$

$$= \frac{\cancel{2} e^{-2x}}{-\cancel{2}} \Big|_0^{1/6}$$

$$= -e^{-1/3} - -e^{0} \quad {=1}$$

$$= 1 - e^{-1/3} = 0.2834$$

c) more than 10 min but less than 1hr



$$P\left(\tfrac{1}{6} \le X \le 1\right) = 1 - .135 - .2834$$

$$= .5816$$

OR $\int_{1/6}^{1} \lambda e^{-\lambda x}\, dx$

The <u>expected value</u> of a cont R.V. X is

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

More generally: $E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

Linearity: $E[\underbrace{a}_{\text{constants}} X + \underbrace{b}_{\text{constants}}] = a E[X] + b$ ↳ $E[b] = b$

$$E[X+Y] = E[X] + E[Y]$$

$Var(x) = E[(X-\mu)^2]$ $\quad \mu = E[X]$

$= E[X^2] - (E[X])^2$

ex X R.V. with density:

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

find $E[X]$ & $Var(X)$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 2x^2\,dx = \frac{2x^3}{3}\Big|_0^1$$

$$= \frac{2}{3}(1^3 - 0^3) = \frac{2}{3}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 2x^3\,dx = \frac{2x^4}{4}\Big|_0^1 = \frac{2}{4}(1-0) = \frac{1}{2}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9}$$

$$= \frac{9}{18} - \frac{8}{18} = \frac{1}{18}$$

ex The density of X is given by:

$$f(x) = \begin{cases} \frac{1}{2} & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

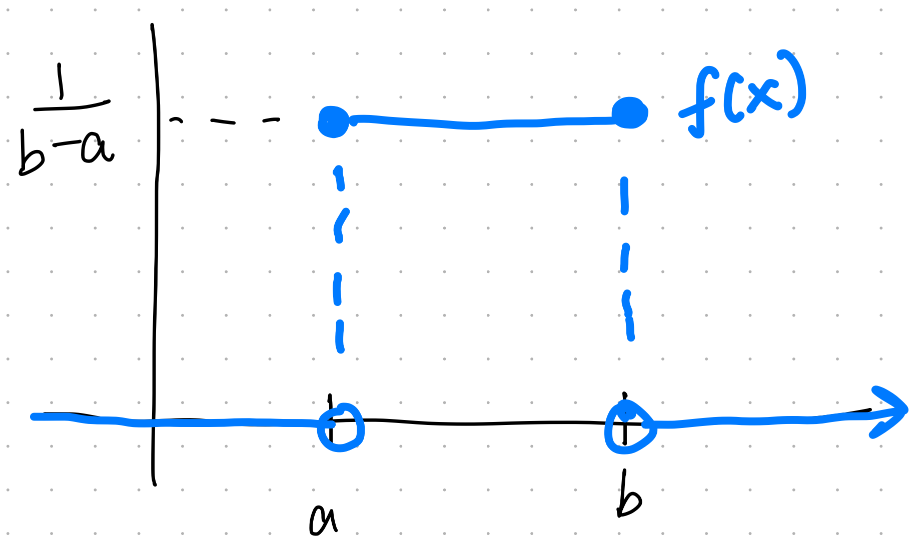
find $E[e^x] = \int_0^2 e^x \cdot \frac{1}{2}\, dx = \frac{1}{2}\int_0^2 e^x\, dx$

$$= \frac{1}{2} e^x \Big|_0^2 = \frac{1}{2}\left(e^2 - e^0\right) = \underline{\frac{1}{2}(e^2 - 1)}$$

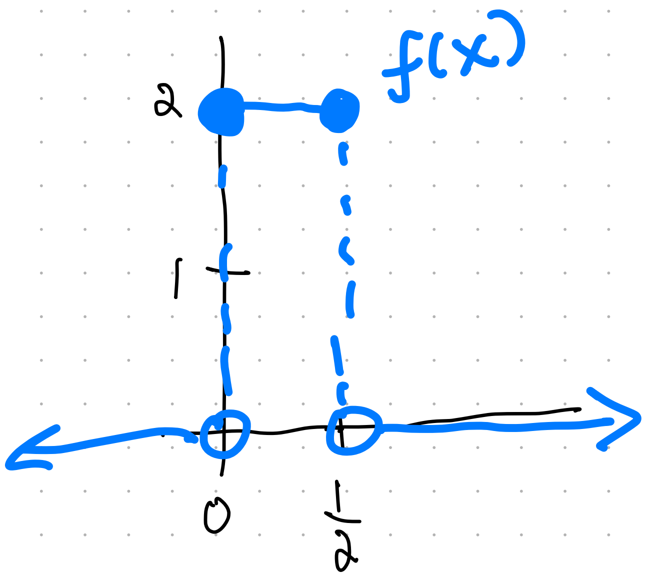
We say that a R.V. X has a <u>uniform distribution</u> on $[a,b]$ if

constant

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



ex Uniform$\left(\underset{a}{\underset{\downarrow}{0}}, \underset{b}{\underset{\downarrow}{\tfrac{1}{2}}}\right)$ $\qquad \dfrac{1}{b-a} = \dfrac{1}{\frac{1}{2}-0} = 2$



$$E[X] = \frac{a+b}{2} \text{ (midpoint)}$$

Derivation:

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\int_a^b x\, dx$$

$$= \frac{1}{b-a} \frac{x^2}{2}\Big|_a^b$$

$$= \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right)$$

$$= \frac{1}{2(b-a)}\left(b^2 - a^2\right) = \frac{1\,\cancel{(b-a)}(b+a)}{2\,\cancel{(b-a)}}$$

diff of 2 □

$$= \frac{a+b}{2}$$

$$E[x^2] = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\frac{x^3}{3}\Bigg|_a^b$$

$$= \frac{1}{3(b-a)}(b^3-a^3) \leftarrow \text{diff of cubes}$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$Var(x) = E[x^2] - (E[x])^2$$

clean up algebra

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(a-b)^2}{12} \text{ OR } \frac{(b-a)^2}{12}$$

Uniform Distribution (a,b) interval

$$E[x] = \frac{a+b}{2} \quad \text{(midpt)}$$

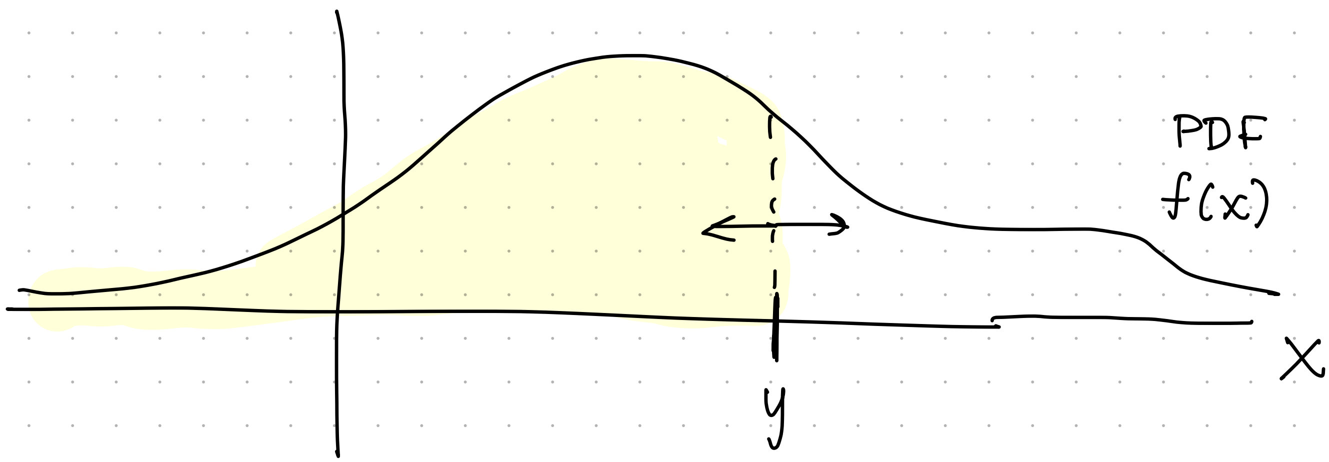
$$Var(X) = \frac{(b-a)^2}{12} \qquad \frac{(\text{length of interval})^2}{12}$$

The Cumulative Distribution Function (CDF) of X

is defined as

pdf

$$F(y) = F_X(y) = P(-\infty < X \le y) = \int_{-\infty}^{y} f(x)\,dx$$



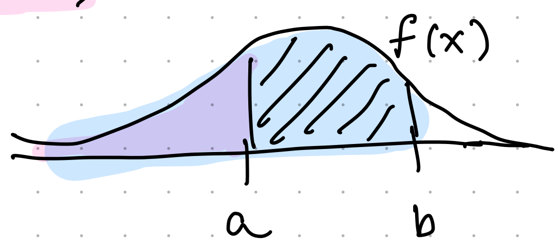
$$\begin{tikzcd} \text{PDF } f(x) \arrow[r, bend left, "\text{integrate}"] & \text{CDF } F_X(y) \arrow[l, bend left, "\text{differentiate}"] \end{tikzcd}$$

Thm $\dfrac{d}{dy} F_X(y) = f(x)$

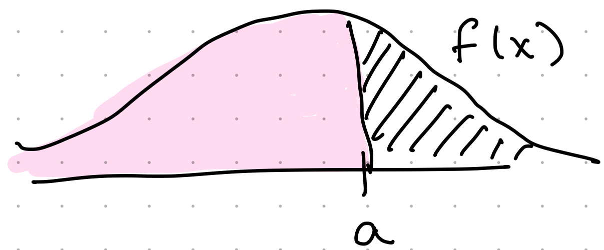
Prop

- $\lim_{y \to \infty} F_X(y) = \lim_{y \to \infty} \int_{-\infty}^{y} f(x)\,dx = 1$
- $\lim_{y \to -\infty} F_X(y) = 0$

- $P(a \le X \le b) = F_X(b) - F_X(a)$

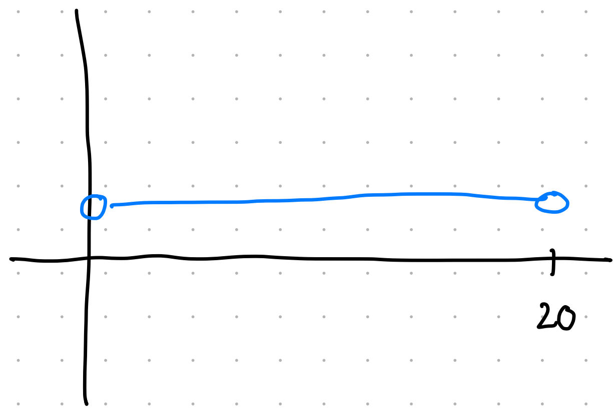


- $P(X > a) = 1 - F_X(a)$



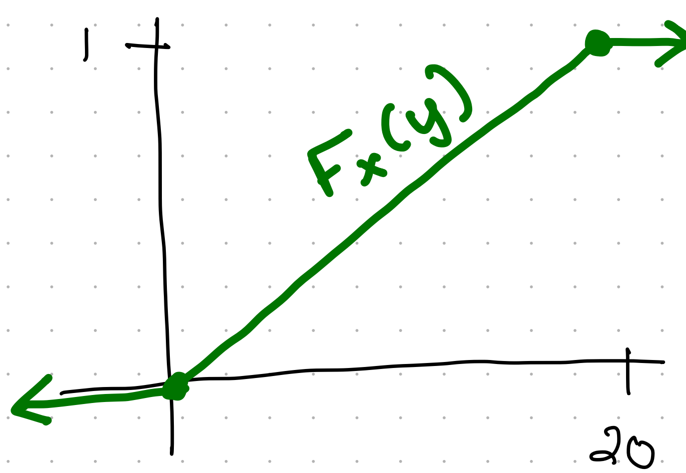
ex) Uniform(0,20)

$$f_X(y) = \begin{cases} \frac{1}{20} & \text{if } 0 < y < 20 \\ 0 & \text{otherwise} \end{cases}$$



CDF: $P(X \le y) = \int_{-\infty}^{y} f_X(y)\,dx = \int_0^y \frac{1}{20}\,dx = \frac{1}{20} x \Big|_0^y = \frac{1}{20} y$

$$F_X(y) = \begin{cases} 0 & y \le 0 \\ y/20 & 0 < y < 20 \\ 1 & y > 20 \end{cases}$$



<u>Given</u>: PDF of X (distribution of X)

<u>Want</u>: PDF of function of X $(g(X))$

ex) $X = \text{Uniform}(0,1) = f_X = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases}$

Want the distribution for $Y = X^n$.

$\hookrightarrow X^2, X^{10}$

trick: CDF, PDF

$$\begin{tikzcd} \begin{matrix}\text{PDF}\\ f_X(x)\end{matrix} \arrow[rr, bend left, "\text{int}"] & & \begin{matrix}\text{CDF}\\ F_X(x)\end{matrix} \arrow[ll, bend left, "\text{diff}"] \end{tikzcd}$$

For $0 \le y \le 1$:

$$\underbrace{F_Y(y)}_{\text{CDF for } Y} = \underbrace{P(Y \le y)}_{\text{def'n of CDF}} = P(X^n \le y) = P(X \le y^{1/n}) = \underbrace{F_X(y^{1/n})}_{\text{def'n of CDF}}$$

$$F_Y(y) = \underbrace{F_X(y^{1/n})}_{\text{CDF for } x}$$

$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 \cdot x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$

above (last page)

plug into CDF

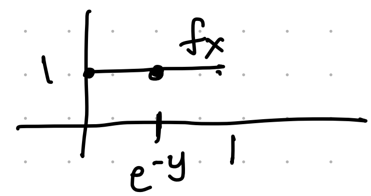
CDF $F_Y(y) = y^{1/n}$

PDF $\quad f_Y(y) = \frac{1}{n} y^{1/n - 1}$ $\swarrow$ take der

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{n} y^{1/n - 1} & 0 \le y \le 1 \\ 0 & \text{Otherwise} \end{cases}$$

<u>ex</u> $\quad X = \text{Uniform}(0,1) \qquad f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$Y = -\log(X)$



Want PDF of Y:

$$\begin{aligned} F_Y(y) = P(Y \le y) &= P(-\log(X) \le y) \\ &= P(\log(X) \ge -y) \\ &= P(X \ge e^{-y}) \\ &= 1 - P(X \le e^{-y}) \\ &= 1 - F_X(e^{-y}) \end{aligned}$$

$$F_Y(y) = 1 - F_X(e^{-y})$$

$\downarrow$ take der, chain rule

$$f_Y(y) = 0 - \underline{f_X(e^{-y})} \cdot (-e^{-y})$$

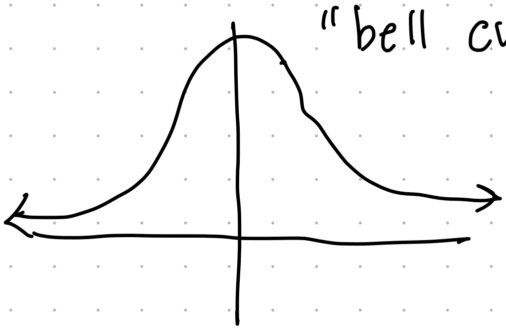
$$= 0 - 1 \cdot (-e^{-y})$$

$$= e^{-y}$$ distribution of Y

PDF

Normal Random Variable

"bell curve"



$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

PDF

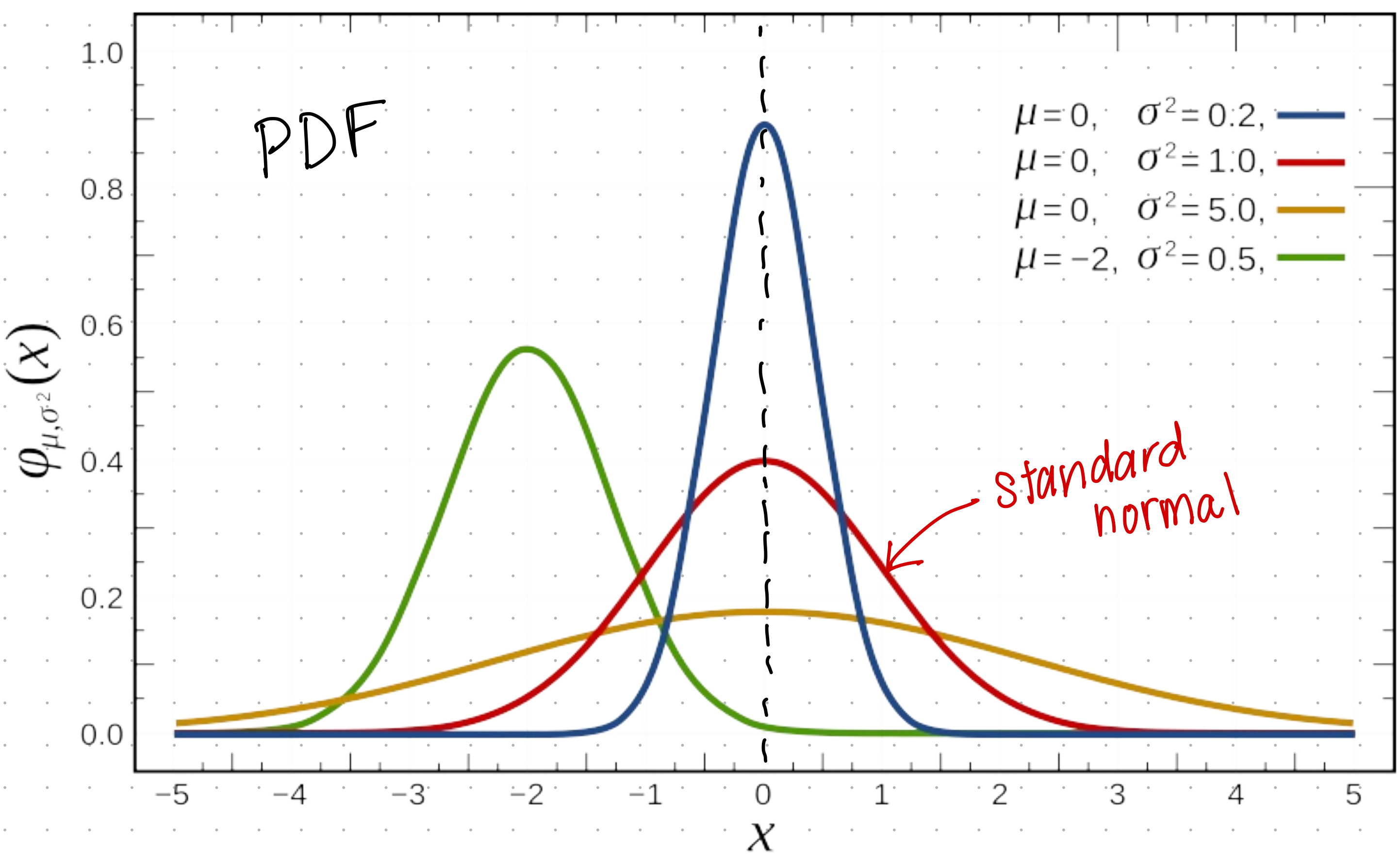
μ = mean = center

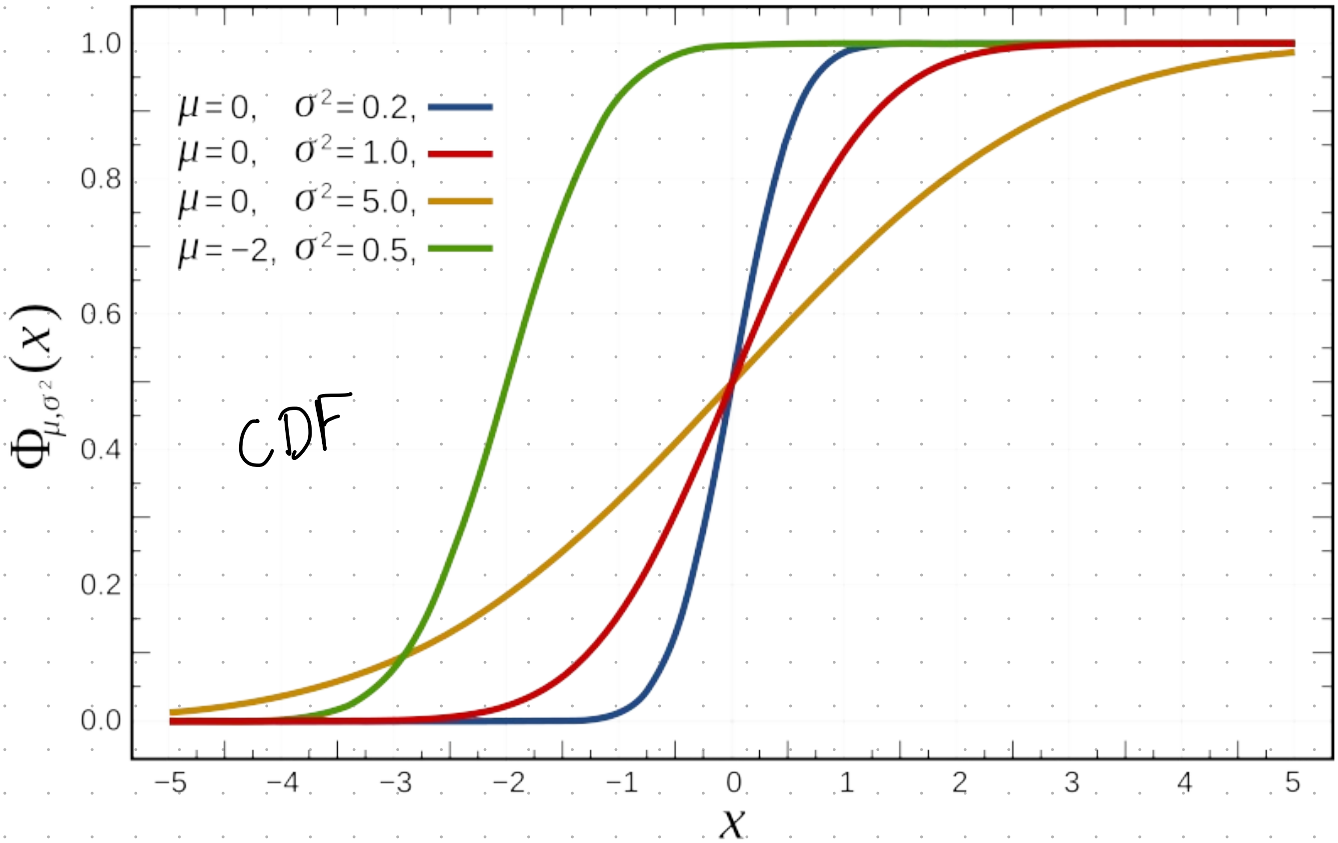
σ^2 = variance $\rightarrow$ spread

The "standard" normal with $\mu = 0$ and $\sigma^2 = 1$

$\sigma = 1$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$





Normal $f(x) \rightarrow$ tricky integral

$$\underbrace{\int_a^b \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu^2)}{2\sigma^2}} dx}_{\text{tricky to do}}$$

μ, σ, π constants

$$\sim \int_a^b e^{-x^2} dx$$

$\downarrow$

involves 2 integrals w/ polar coordinates

$\downarrow$

instead use a standard normal CDF chart!

X : any normal distribution "messy"

μ mean

σ^2 variance

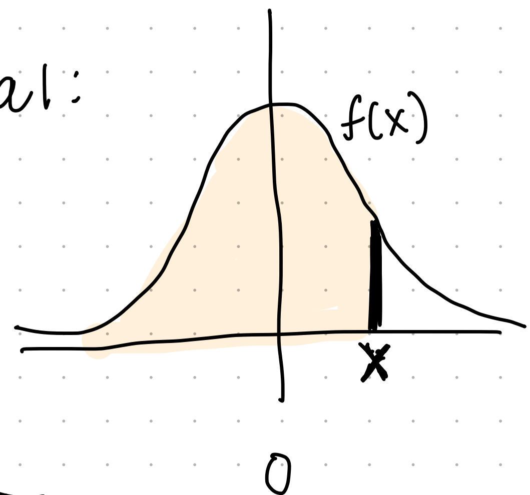
Z : Standard normal $\mu = 0$, $\sigma = 1$

$$\boxed{Z = \frac{X - \mu}{\sigma}}$$

standardize

CDF of a standard normal:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$



* for negative : $\Phi(-x) = 1 - \Phi(x)$

* CDF is only for standard normal

↳ standardize first! $Z = \frac{X - \mu}{\sigma}$

"messy"

standard

Area $\Phi(x)$ under the standard normal curve to the left of X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(1.27)$

$\Phi(.03)$

$\Phi(3.35)$

ex Find $P(1 \le x \le 4)$ if X is $\mathcal{N}(2, 25)$

normal $\quad \mu \quad \sigma^2$

not standard

First: standardize: $Z = \dfrac{X - \mu}{\sigma} \longrightarrow Z\sigma = X - \mu$

$Z\sigma + \mu = X$

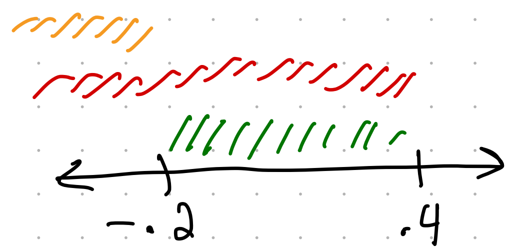
$$P(1 \le X \le 4) = P(1 \le Z\sigma + \mu \le 4)$$

$$= P(1 \le 5Z + 2 \le 4)$$

$$= P(-1 \le 5Z \le 2)$$

$$= P\left(\frac{-1}{5} \le Z \le \frac{2}{5}\right)$$

$$= P(-.2 \le Z \le .4)$$



$$= P(Z \le .4) - P(X \le -.2)$$

$$= \Phi(.4) - \Phi(-.2)$$

$$= \underbrace{\Phi(.4)}_{\text{chart}} - (1 - \underbrace{\Phi(.2)}_{\text{chart}})$$

$$= 0.6554 - (1 - 0.5793)$$

$$= 0.2347 \qquad 23.47\%$$

also: Standardize

$$P(1 \le X \le 4) = P(1-\mu \le X-\mu \le 4-\mu)$$

$\mu = 2$

$\sigma = 5$

$$= P\left(\frac{1-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{4-\mu}{\sigma}\right)$$

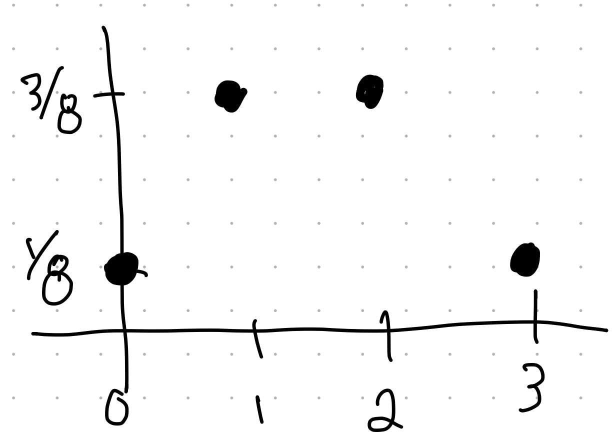
$$= P\left(\frac{1-2}{5} \le Z \le \frac{4-2}{5}\right)$$

The Normal Approximation to the Binomial Distribution

Recall Binomial distribution from discrete chapter

- n independent trials $\to$ prob p
- X = # of success after n trials

* $n = 3$ (coin flips)



symmetric!

$P(X=0) = 1/8$

$P(X=1) = 3/8$

$P(X=2) = 3/8$

$P(X=3) = 1/8$

Now if increase n (# of trials) the shape starts to look "normal"!

The DeMoivre-Laplace Limit Thm

If S_n = # of successes that can occur when n independent trials are performed (prob p each time) $\longrightarrow$ Binomial

then for any $a < b$,

$$P\left\{ a \le \underbrace{\frac{S_n - \overset{\mu = E[S_n]}{np}}{\sqrt{\underbrace{np(1-p)}_{\sigma^2 = var(S_n)}}}}_{\text{Standardize}} \le b \right\} \longrightarrow \underbrace{\Phi(b) - \Phi(a)}_{\text{CDF of standard normal chart}}$$

* this approximation is good if $np(1-p) \ge 10$
(gets better as $np(1-p)$ increases $\rightarrow$ n increases)

* this is a special case of the <u>Central Limit Thm</u> (we will see later)

ex Suppose a fair coin is tossed 100 times. What is the prob there will be more than 60 heads?

Binomial: $P(X>60) = \sum_{k=61}^{100} \binom{100}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{100-k}$

40 terms!

$= 0.028$

Normal? Can we approximate?

check: $np(1-p) \geq 10$?

$100 \cdot \frac{1}{2}\left(1-\frac{1}{2}\right) = 50 \cdot \frac{1}{2} = 25$ ✓

$$P(S_n > 60) = P\left(\frac{S_n - np}{\sqrt{np(1-p)}} > \frac{60-np}{\sqrt{np(1-p)}} \right)$$

Standardize

$$= P\left(\frac{X-50}{5} > \frac{60-50}{5} \right)$$

↓ standard

$$= P(Z > 2)$$

$$= 1 - P(Z \leq 2)$$

$$= 1 - \underbrace{\Phi(2)}_{\text{chart}}$$

$$= 1 - .9772$$

$$= 0.0228$$

close!

$\longrightarrow$ next exponential R.V.