Probability and Statistics Homework 13

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1. Let $X_1, ..., X_n$ be a random sample from a distribution X with the density function

$$f(x;\theta) = \begin{cases} \frac{1}{2}(1+\theta x), & x \in [-1,1] \\ 0, & x \notin [-1,1] \end{cases}$$

that depends on the parameter θ. Find k such that $k\bar{X}_n$ is an unbiased estimator for θ, where

$$\bar{X}_n = \frac{X_1 + ... + X_n}{n}$$

Answer:
Since $k\bar{X}_n$ is an unbiased estimator for θ, we have

$$E[k\bar{X}_n] - \theta = 0$$

$$\begin{aligned} E[k\bar{X}_n] - \theta &= kE[\frac{X_1 + ... + X_n}{n}] - \theta \\ &= \frac{k}{n} \times nE[X_1] - \theta \quad (i.i.d.) \\ &= k\int_{-1}^{1} \frac{1}{2}x(1+\theta x)\ dx - \theta \\ &= \frac{\theta}{3}k - \theta = 0 \Rightarrow k = 3 \end{aligned}$$

Therefore, $k = 3$ if $k\bar{X}_n$ is an unbiased estimator for θ.

2. Let $X_1, ..., X_n$ be a random sample from a distribution X with the density function (that depends on (λ, θ))

$$f(x;\theta) = \begin{cases} \lambda e^{-\lambda(x-\theta)}, & x > \theta \\ 0, & x \leq \theta \end{cases}$$

with $\lambda > 0$ and $\theta \in \mathcal{R}$.

1. Find the methods of maximum likelihood estimator for (λ, θ).
2. Find the corresponding estimate of (λ, θ) when $n = 10$ and $x_1 = 3, x_2 = 0.5, x_3 = 2.5, x_4 = 2, x_5 = 5, x_6 = 3.5, x_7 = 10, x_8 = 9, x_9 = 18, x_{10} = 1.5$.

Answer:

1. Given the random sample $X_1, ..., X_n$, the likelihood function is

$$L(\lambda, \theta) = \prod_{i=1}^{n} f(X_i; \lambda, \theta)$$

Since $f(x; \lambda, \theta) = 0$ for $x \leq \theta$, we require $\theta \leq min(X_i)$. For $x_i > \theta$, we have

$$\begin{aligned} L(\lambda, \theta) &= \prod_{i=1}^{n} f(X_i; \lambda, \theta) \\ &= \lambda^n e^{-\lambda \sum_{i=1}^{n}(X_i - \theta)} \end{aligned}$$

Taking the log:

$$\begin{aligned} l(\lambda, \theta) &= n \ln \lambda - \lambda \sum_{i=1}^{n} (X_i - \theta) \\ &= n \ln \lambda + \lambda n \theta - \lambda \sum_{i=1}^{n} X_i \end{aligned}$$

Since $\lambda > 0$ and $n > 0$, increase θ will increase the above formula. For that we have the constrain of the max value of θ, thus $\theta^* = min(X_i)$.
Then we maximize the likelihood with respect to λ given $\theta^* = min(X_i)$. Consider the derivative of λ and set it to zero

$$\begin{aligned} \frac{\partial l}{\partial \lambda} &= \frac{n}{\lambda} + n\theta^* - \sum_{i=1}^{n} X_i = 0 \\ \Rightarrow \lambda^* &= \frac{n}{\sum_{i=1}^{n}(X_i - \theta^*)} \end{aligned}$$

Therefore, the maximum likelihood estimator for (λ, θ) is $\theta^* = min(X_1, ..., X_n), \lambda^* = \frac{n}{\sum_{i=1}^{n}(X_i - \theta^*)}$.

2. First, for θ, with the answer of part 1, we have

$$\theta^* = min(X_i) = 0.5$$

Then we can find λ^* as

$$\begin{aligned}
\lambda^* &= \frac{n}{\sum_{i=1}^{n}(X_i - \theta^*)} \\
&= \frac{10}{\sum_{i=1}^{10} X_i - 5} \\
&= \frac{10}{55 - 5} \\
&= 0.2
\end{aligned}$$

Therefore, the MLE for (λ, θ) given samples is $\theta^* = 0.5, \lambda^* = 0.2$.

3. Let $x_1, x_2, ..., x_n$ be independent and identically disputed samples from uniform distribution on the set $[0, \theta]$. (These values might look like $x_1 = 2.325, x_2 = 1.1242, x_3 =$ 9.262,etc...) What is the MLE of θ?

Answer:

The uniform distribution on $[0, \theta]$ has PDF

$$f(x;\theta) = \begin{cases} \frac{1}{\theta}, & 0 \leq x \leq \theta \\ 0, & otherwise \end{cases}$$

where $\theta > 0$. Given $x_1, ..., x_n$ i.i.d. samples, the likelihood function is

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} f(x_i;\theta) \\ &= \prod_{i=1}^{n} \frac{1}{\theta} \times \mathbf{1}(0 \leq x_i \leq \theta) \\ &= \frac{1}{\theta^n} \mathbf{1}(\theta \geq max(x_1, ..., x_n)) \end{aligned}$$

To let $L(\theta) > 0$, θ must be large or equal than $max(x_1, ..., x_n)$. So that the formula turns to

$$L(\theta) = \frac{1}{\theta^n}$$

The formula increase with a decreasing θ, since we have a constrain of $\theta \geq max(x_1, ..., x_n)$, $\theta^* = max(x_1, ..., x_n)$.

Therefore, the MLE of θ is $\theta^* = max(x_1, ..., x_n)$