QUIZ4

条件概率Conditional Probability：

$$f_{X|Y}(x|y)=\frac{f_{XY}(x,y)}{f_Y(y)}$$

连续随机变量求和的分布：X,Y是两独立的随机变量，$X+Y$的pdf写作卷积

$$f_{X+Y}(a)=\int_{-\infty}^{\infty}f_X(a-y)f_Y(y)dy$$

- 正态随机变量的求和：如果$X_i,i=1,...,n$是相互独立且其对应的均值和方差为μ,σ^2，则$\sum_{i=1}^{n}X_i$的分布是均值为$\sum_{i=1}^{n}\mu_i$且方差为$\sum_{i=1}^{n}\sigma_i^2$的正态分布

离散随机变量求和的分布：

$$P_{X+Y}(a)=\sum_x P_X(x)P_Y(a-x)$$

- Poisson求和：已知两个Poisson分布$X,Y\lambda_1,\lambda_2$，求和为$P(X+Y=k)=\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^K}{K!}$
- 二项式分布求和：$X\sim\mathrm{Bin}(n,p),Y\sim\mathrm{Bin}(m,p),(X+Y)\sim\mathrm{Bin}(n+m,p)$

联合随机变量的期望：

$$E[g(X,Y)]=\sum_X\sum_Y g(x,y)p(x,y)=\int_X\int_Y g(x,y)f(x,y)\,dxdy$$

- 期望的线性：$E[X+Y]=E[X]+E[Y]$
- 当X,Y相互独立时，$E[XY]=E[X]E[Y]$

协方差Covariance：两个随机变量之间的信息量

$$\begin{aligned}Cov(X,Y)&=E[(X-E[X])(Y-E[Y])]\\&=E[XY]-E[X]E[Y]\end{aligned}$$

- 当$Cov(X,Y)>0$，X,Y向相同方向移动；当$Cov(X,Y)<0$，X,Y向相反方向移动
- 当$Cov(X,Y)=0$，X,Y不会move together，但**不代表二者独立**
- 性质：
  - $Cov(X,X)=Var(X)$
  - $Cov(X,Y)=Cov(Y,X)$
  - $Cov(aX,Y)=aCov(X,Y)$
  - $Cov(X+c,Y)=Cov(X,Y)$
  - $Cov(X+Y,Z)=Cov(X,Z)+Cov(Y,Z)$
  - $Var(aX+bY)=a^2VarX+2abCov(X,Y)+b^2VarY$

Correlation Coefficient：

$$\rho(X,Y)=\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Cauchy-Schwarz不等式：

$$(E[XY])^2\le E[X^2]E[Y^2]$$

条件期望Conditional Expectation：

$$E[X|Y=y]=\int x\cdot f_{X|Y}(x)\,dx$$

Expectation by Conditioning：

$$E[X]=E[E[X|Y]]=\int_{-\infty}^{\infty}E[X|Y=y]f_Y(y)\,dy$$

Markov's Inequality：如果X是一个非负的随机变量，对于任意的$a>0$，

$$P(X\ge a)\le\frac{E[X]}{a}$$

Chebyshev's Inequality：如果X是一个随机变量，有限均值为$\mu = E[X]$且方差为σ^2，对于任意的$a > 0$

$$P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

One-sided Chebysheb (Cantelli's) Inequality：

$$P(X \geq \mu + a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

中心极限定理Central Limit Thm (CLT)：令$X_1, X_2, ..., X_n$是一系列独立同分布 (i.i.d.) 的随机变量，每个变量的均值为μ，方差为σ^2，则当$n \to \infty$

$$\frac{X_1 + X_2 + ... + X_n - n\mu}{\sigma\sqrt{n}} \sim N(1,0)$$

且

$$P(\frac{X_1 + X_2 + ... + X_n - n\mu}{\sigma\sqrt{n}} \leq a) \to \Phi(a)$$

或者可以写作，令$S_n = X_1 + X_2 + ... + X_n$，有

$$P(a \leq \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq b) \to_{n\to\infty} P(a \leq Z \leq b)$$

WLLN：令$x_i, i = 1, ..., n$是i.i.d.随机变量，对于任意$\epsilon > 0$

$$lim_{n\to\infty} P(|\bar{X} - \mu| \geq \epsilon) = 0$$

基本初等函数求导公式

(1) $(C)' = 0$ (2) $(x^{\mu})' = \mu x^{\mu-1}$

(3) $(\sin x)' = \cos x$ (4) $(\cos x)' = -\sin x$

(5) $(\tan x)' = \sec^2 x$ (6) $(\cot x)' = -\csc^2 x$

(7) $(\sec x)' = \sec x \tan x$ (8) $(\csc x)' = -\csc x \cot x$

(9) $(a^x)' = a^x \ln a$ (10) $(e^x)' = e^x$

(11) $(\log_a x)' = \dfrac{1}{x \ln a}$ (12) $(\ln x)' = \dfrac{1}{x}$,

(13) $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$ (14) $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

(15) $(\arctan x)' = \dfrac{1}{1+x^2}$ (16) $(\operatorname{arc}\cot x)' = -\dfrac{1}{1+x^2}$

函数的和、差、积、商的求导法则

设 $u = u(x)$， $v = v(x)$ 都可导，则

(1) $(u \pm v)' = u' \pm v'$ (2) $(Cu)' = Cu'$（ C 是常数）

(3) $(uv)' = u'v + uv'$ (4) $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

反函数求导法则

若函数 $x = \varphi(y)$ 在某区间 I_y 内可导、单调且 $\varphi'(y) \neq 0$，则它的反函数 $y = f(x)$ 在对应区间 I_x 内也可导，且

$$f'(x) = \frac{1}{\varphi'(y)} \quad 或 \quad \frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

1. $\int a\,\mathrm{d}x = ax + C$

2. $\int x^a\,\mathrm{d}x = \frac{1}{a+1}x^{a+1} + C$，其$a$是常数$a \neq -1$

3. $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$

4. $\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C$，其$a > 0$，$a \neq 1$

5. $\int \sin x\,\mathrm{d}x = -\cos x + C$

6. $\int \cos x\,\mathrm{d}x = \sin x + C$

7. $\int \tan x\,\mathrm{d}x = -\ln|\cos x| + C$

8. $\int \cot x\,\mathrm{d}x = \ln|\sin x| + C$

9. $\int \sec x\,\mathrm{d}x = \mathrm{ReArth}\tan\frac{x}{2} + C = \ln|\sec x + \tan x| + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$

10. $\int \csc x\,\mathrm{d}x = \mathrm{ReLn}\tan\frac{x}{2} + C = \ln|\csc x - \cot x| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C$

11. $\int \sec^2 x\,\mathrm{d}x = \tan x + C$

12. $\int \csc^2 x\,\mathrm{d}x = -\cot x + C$

13. $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + C$

14. $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + C$

15. $\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C$

16. $\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C$

17. $\int \sinh\ x\,\mathrm{d}x = \cosh\ x + C$

18. $\int \cosh\ x\,\mathrm{d}x = \sinh\ x + C$

19. $\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x = \ln(x + \sqrt{x^2+a^2}) + C$

20. $\int \frac{1}{\sqrt{x^2-a^2}}\mathrm{d}x = \ln|x + \sqrt{x^2-a^2}| + C$