$= 1 - \Phi(2)$

chart

$= 1 - .9772$

$= 0.0228$

close!

Oct 18

→ Finish Binomial + Normal

→ Exp R.V.

→ Joint Distributions

Binomial discrete $\xrightarrow{\text{approx}}$ Normal continuous

$P(X=n)$	$P(n-\frac{1}{2} < X < n+1/2)$
$P(X>n)$	$P(X > n+1/2)$
$P(X<n)$	$P(X < n-1/2)$
$P(X\geq n)$	$P(X > n-1/2)$
$P(X\leq n)$	$P(X < n+1/2)$

Continuity Correction

ex Let X be the # of heads shown for 40 coin flips (fair $\to p = 1/2$). Find $P(X=20)$.

Binomial: $P(X=20) = \binom{40}{20}\left(\frac{1}{2}\right)^{20}\left(\frac{1}{2}\right)^{40-20} = .1254$

Normal dist: $np(1-p) = 40 \cdot \frac{1}{2} \cdot \frac{1}{2} = 10$ ✓

$\mu = np = 20$

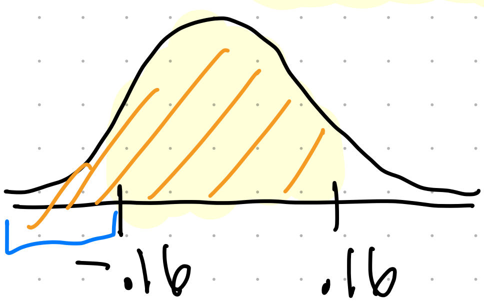
$\sigma = \sqrt{np(1-p)} = \sqrt{10}$

$$P(19.5 < X < 20.5) = P\left(\frac{19.5-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{20.5-\mu}{\sigma}\right)$$

$$= P(-.16 < Z < .16) = \Phi(.16) - \underbrace{\Phi(-.16)}$$

standard normal

CDF



$$= \underset{\text{chart}}{\Phi(.16)} - (1 - \underset{\text{chart}}{\Phi(.16)})$$

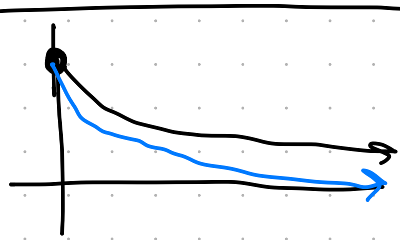
$$= .5636 - 1 + .5636$$

$$= .1272$$

Exponential Cont R.V

λ: parameter $\lambda > 0$

how long to wait until something occurs



pdf: $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & x < 0 \end{cases}$

Models the amt of <u>time</u> until a specific event occurs (starting from now)

- → earthquakes
- → telephone calls
- → someone arrives
- → light bulb burns out

skip work

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx = \frac{1}{\lambda} = E[X]$$

$$Var(X) = \frac{1}{\lambda^2}$$

<u>ex</u> Suppose length of phone call in min is an exp R.V. w/ $\lambda = \frac{1}{10}$. If someone arrives immediately before you at a public phone booth, find the prob that you will have to wait:

a) more than 10 min

b) btwn 10 & 20 min

CDF: $\int_{-\infty}^{y} \lambda e^{-\lambda x}\,dx = \int_{0}^{y} \lambda e^{-\lambda x}\,dx$

$$= \frac{\cancel{\lambda} e^{-\lambda x}}{-\cancel{\lambda}}\Bigg|_0^y = -e^{-\lambda y} - -e^{0}$$

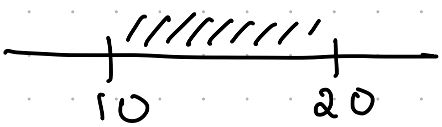
$$= 1 - e^{-\lambda y} = F(y)$$

$\lambda = \frac{1}{10}$

a) $P(X > 10) = 1 - P(X \le 10) = 1 - F(10)$

$$= 1 - \left(1 - e^{-\frac{1}{10}\cdot 10}\right)$$

$$= \frac{1}{e} = .368$$



b) $P(10 < X < 20) = F(20) - F(10)$

CDF

$$= 1 - e^{-\frac{1}{10}\cdot 20} - \left(1 - e^{-1}\right)$$

$$= 1 - e^{-2} - 1 + e^{-1} = e^{-1} + e^{-2}$$

$$= .233$$

"Memoryless" Property of Exp R.V.

times: s , t

$$P(X > s+t \overset{\text{cond}}{\underset{\text{given that}}{\Big|}} X > t) = P(X > s)$$

Proof $P(X > \underset{5}{s} + \underset{3}{t} \mid X > \underset{3}{t}) = \dfrac{P(X > \overset{8}{s+t} \text{ and } X > \overset{3}{t})}{P(X > t)}$

$$= \frac{P(X > s+t)}{P(X > t)}$$

$$= \frac{1 - P(X \le s+t)}{1 - P(X \le t)} \quad \text{CDF above}$$

$$= \frac{1 - \left(1 - e^{-\lambda(s+t)}\right)}{1 - \left(1 - e^{-\lambda t}\right)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = \frac{e^{-\lambda s} \cancel{e^{-\lambda t}}}{\cancel{e^{-\lambda t}}}$$

$$= P(X > s)$$

ex Phone life w/ mean life is 4 years

Given phone has lasted 3 years, what is the prob it will last 5 more years?

$$E[X] = 4 = \frac{1}{\lambda} \longrightarrow \lambda = \frac{1}{4}$$

$$P(X > \underset{s}{5} + \underset{t}{3} \mid X > 3) = P(X > 5)$$

$$= 1 - P(X \leq 5)$$

$$= e^{-\frac{1}{4} \cdot 5}$$

Ch. 6 (Ross) Joint Distributed R.V.

↳ depends on more than one R.V.

ex: height & weight of a person

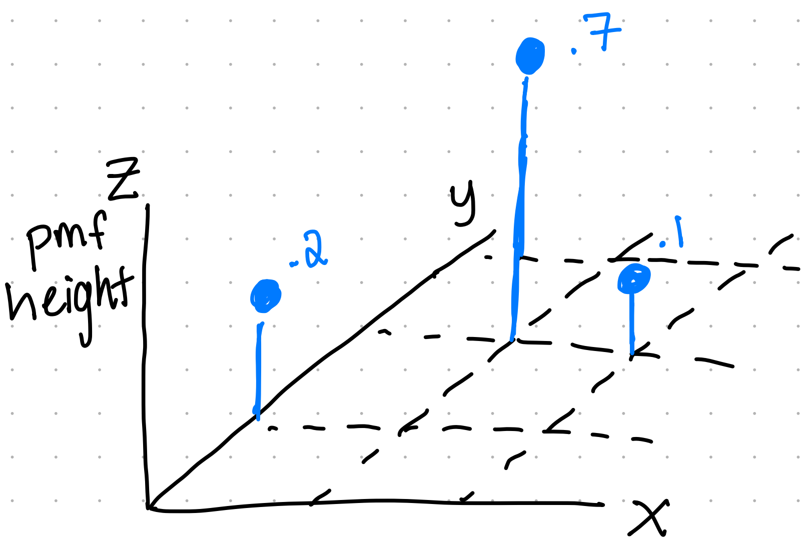
interested in several characteristics at once

Start w/ 2 but extends to more

Discrete Case: pmf

$$P_{XY}(x,y) = P(X=x, Y=y)$$

and



Note:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_{XY}(x_i, y_j) = 1$$

ex

Y \ X	1	2	3
1	0	1/6	1/6
2	1/6	0	1/6
3	1/6	1/6	0

$P(X=3, Y=2) = 1/6$

$P(X=1, Y=1) = 0$

$P(X \le 2, Y=3) = 1/6 + 1/6 = \frac{2}{6}$

$P(X \le 2, Y \ge 2) = 1/6 + 0 + 1/6 + 1/6 = 3/6$

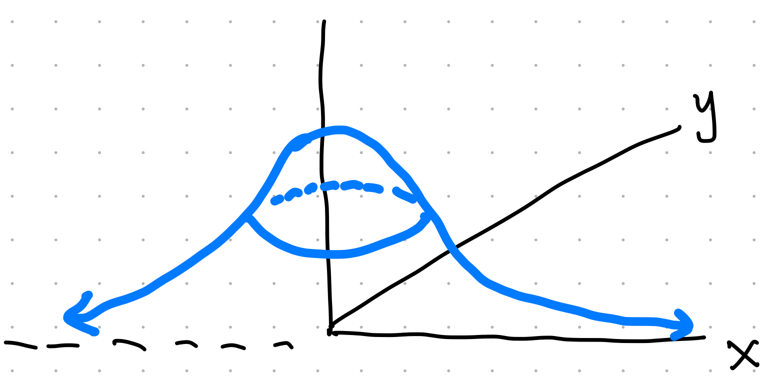
Continuous jointly distributed X & Y if there is a non-negative $f_{XY}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that for any $A \subseteq \mathbb{R} \times \mathbb{R}$

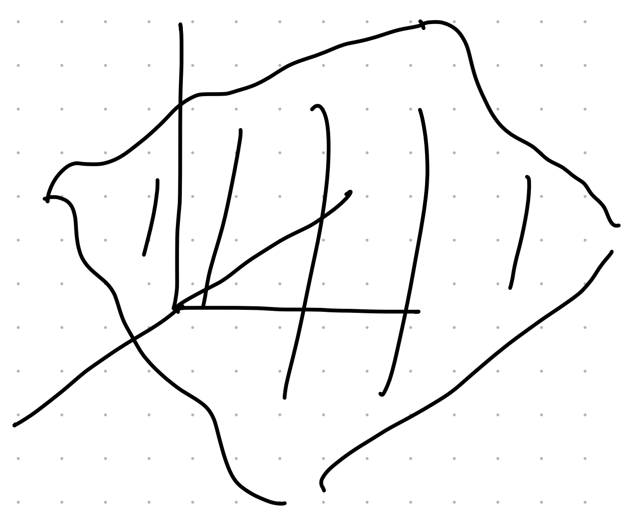
$$P\big((x,y) \in A\big) = \iint_A f_{XY}(x,y)\,dx\,dy$$

A could be a rectangle: $A = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

↓

$$P(a \le X \le b,\ c \le Y \le d) = \int_c^d \int_a^b f_{XY}(x,y)\,dx\,dy$$





surface

$\Rightarrow$ Similarly: $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,\underbrace{dx\,dy}\, = 1$

order does not matter

iterated integrals

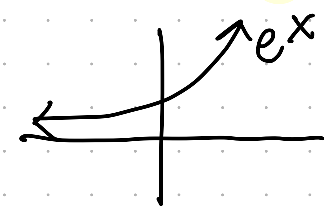
ex) $f_{XY}(x,y) = Ce^{-x}e^{-2y}$ $\quad 0<x<\infty$

$x<y<\infty$

What is c?

$$1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ce^{-x}e^{-2y}\,dy\,dx = \int_{0}^{\infty}\int_{x}^{\infty} Ce^{-x}e^{-2y}\,dy\,dx$$

$$1=\int_{0}^{\infty} Ce^{-x}\left[\int_{x}^{\infty} e^{-2y}\,dy\right]dx$$



$$1=\int_{0}^{\infty} Ce^{-x}\,\frac{e^{-2y}}{-2}\Big|_{y=x}^{y=\infty}dx = \int_{0}^{\infty}\frac{Ce^{-x}}{-2}\left(\cancelto{0}{e^{-\infty}}-e^{-2x}\right)dx$$

$$=\frac{c}{-2}\int_{0}^{\infty} e^{-x}\cdot(-e^{-2x})\,dx$$

$$1=\frac{c}{2}\int_{0}^{\infty} e^{-3x}\,dx$$

$$1=\frac{c}{2}\,\frac{e^{-3x}}{-3}\Big|_{0}^{\infty}=\frac{c}{-6}\left[\cancelto{0}{e^{-\infty}}-e^{0}\right]$$

$$1=\frac{c}{6} \Rightarrow c=6$$

Cumulative functions

Discrete: $F(a,b) = P(X \le a, Y \le b)$

Marginals: sum up all of x OR
" " " " y

ex

Y \ X	1	2	3	4
4	0	1/20	1/20	1/20
3	1/20	2/20	3/20	1/20
2	1/20	2/20	3/20	1/20
1	1/20	1/20	1/20	0

$P(X,Y)$

Marginals:

X	1	2	3	4
$P_X(x)$	3/20	6/20	8/20	3/20

Sum Y

single variable pmfs!

Y	1	2	3	4
$P_Y(y)$	3/20	7/20	7/20	3/20

Sum X

Continuous CDF

$$F(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dy\,dx$$

$$\mathrm{PDF} \underset{\text{diff}}{\overset{\text{int}}{\rightleftarrows}} \mathrm{CDF} \longrightarrow f(a,b) = \frac{\partial^2}{\partial a\,\partial b} F(a,b)$$

partial derivative

Marginal average out one variable

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \longrightarrow \text{avg out } y$$

pdf
1-dim

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

— for both discrete & cont:

$$\left.\begin{array}{l} E[X],\ E[Y] \\ \mathrm{Var}(X),\ \mathrm{Var}(Y) \end{array}\right\}$$ marginals first then same method as prev lectures

ex $f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0<x<\infty \quad 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$

a) $P(X>1, Y<1) = \int_1^\infty \int_0^1 2e^{-x}e^{-2y}\,dy\,dx = e^{-1}(1-e^{-2})$

(same technique as when we solved for c above)

b) $P(\underbrace{X<Y}_{x<y}) = \int_0^\infty \int_0^y 2e^{-x}e^{-2y}\,dx\,dy = \frac{1}{3}$

c) $P(\underbrace{2 \le Y \le 3}_{\text{single variable}}) \rightarrow$ Need marginal first

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^\infty 2e^{-x}e^{-2y}\,dx$$

$$= 2e^{-2y}\int_0^\infty e^{-x}\,dx$$

$$= 2e^{-2y}\left.\frac{e^{-x}}{-1}\right|_{x=0}^{x=\infty}$$

$$= -2e^{-2y}\left(\cancel{e^{-\infty}}^{\to 0} - e^{0}\right)$$

$$= \underbrace{2e^{-2y}}_{\text{exp R.V. } \lambda=2}$$

$$P(2 \le Y \le 3) = \int_2^3 2e^{-2y}\,dy = \frac{2e^{-2y}}{-2}\Bigg|_2^3 = -e^{-6} + e^{-4}$$

d) $$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^{\infty} y \cdot 2e^{-2y}\,dy$$

IBP: $u = 2y \quad dv = e^{-2y}\,dy$

$du = 2\,dy \quad v = \frac{e^{-2y}}{-2}$

$$\int u\,dv = uv - \int v\,du$$

$$= 2y \cdot \frac{e^{-2y}}{-2}\Bigg|_0^{\infty} - \int_0^{\infty} \frac{2e^{-2y}}{-2}\,dy$$

$$= -ye^{-2y}\Big|_0^{\infty} + \frac{e^{-2y}}{-2}\Bigg|_0^{\infty}$$

$$= \lim_{t\to\infty}\left(-te^{-2t} + 0 + \frac{e^{-2t}}{-2} - \frac{e^{0}}{-2}\right)$$

$-\infty$ $\quad 0$

$$\lim_{t\to\infty} -te^{-2t} = \lim_{t\to\infty} \frac{-t}{e^{2t}} \to \frac{-\infty}{\infty}$$

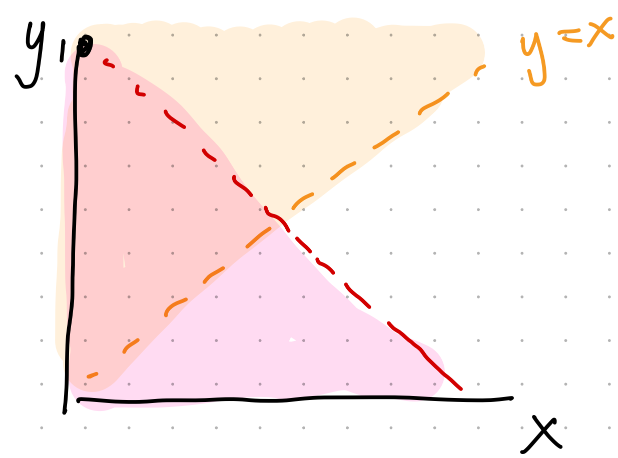
$$\overset{L'H}{=} \lim_{t\to\infty} \frac{-1}{2e^{2t}} = 0$$

$$= 0 + 0 + \cancelto{0}{e^{-\infty}} + \frac{1}{2} = \frac{1}{2}$$

---

ex) $f_{XY}(x,y) = e^{-y}$ $0 < x < y < \infty$

Calculate $P(X+Y \geq 1)$



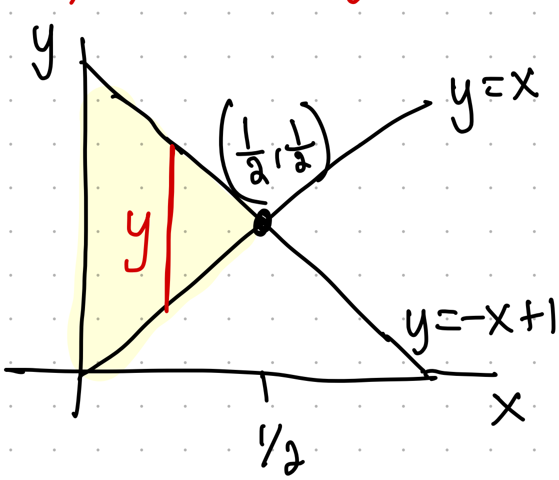
① $x<y$

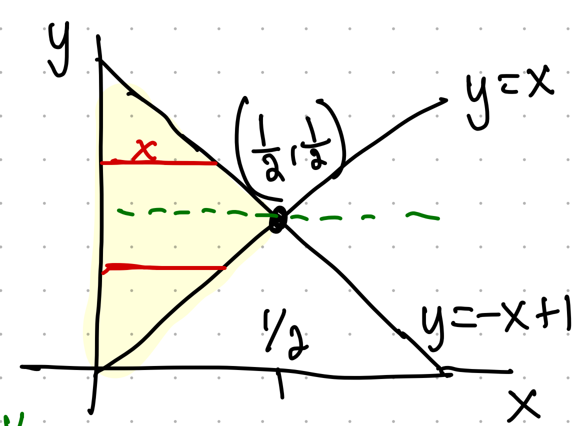
② $x+y \geq 1$

↳ where is $X+Y \leq 1 \to Y \leq 1-X$

$$P(X+Y\geq 1) = 1 - P(X+Y\leq 1)$$

together:





$$P(X+Y\leq 1) = \int_0^{1/2}\int_0^{y} e^{-y}\,dx\,dy + \int_{1/2}^{1}\int_0^{1-y} e^{-y}\,dx\,dy$$

$$P(X+Y\le 1) = \int_0^{1/2}\int_{x}^{-x+1} e^{-y}\,dy\,dx$$

$$= \int_0^{1/2} -e^{-y}\Big|_{y=x}^{y=-x+1} dx$$

$$= \int_0^{1/2} \left(-e^{x-1} + e^{-x}\right) dx$$

$$= \int_0^{1/2} \left(-e^{x}\underbrace{e^{-1}}_{\text{constant}} + e^{-x}\right) dx$$

$$= \left(\frac{-1}{e}e^{x} - e^{-x}\right)\Big|_0^{1/2}$$

$$= \frac{-1}{e}e^{1/2} - e^{-1/2} - \left(\frac{-1}{e}e^{\overset{=1}{0}} - e^{\overset{=1}{0}}\right)$$

$$= \frac{-1}{e}\sqrt{e} - \frac{1}{\sqrt{e}} + \frac{1}{e} + 1$$

$$= -\frac{1}{\sqrt{e}} - \frac{1}{\sqrt{e}} + \frac{1}{e} + 1 = \frac{-2}{\sqrt{e}} + \frac{1}{e} + 1$$

$= 0.155$

$$P(X+Y \geq 1) = 1 - 0.155 = 0.845$$

Independent R.V.

Recall: $P(A \cap B) = P(A)P(B)$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Joint Independence:

$$P(X \in A, Y \in B) = P(X \in A)\,P(Y \in B)$$

↑ and
intersection

ex $P(X \leq a, Y \leq b) = P(X \leq a)\,P(Y \leq b)$

$$F(a,b) = F_X(a)\,F_Y(b)$$

discrete: $p(x,y) = P_X(x)\,P_Y(y)$

continuous: $\iint_A f_{XY}(x,y)\,dy\,dx = \iint_A f_X(x)\,f_Y(y)\,dy\,dx$

Joint PDF: $f(x,y) = \begin{cases} 24xy & x>0,\ y>0,\ x+y<1 \\ 0 \end{cases}$

$x+y<1 \rightarrow y<1-x$

$x<1-y$

Are X & Y independent?

Does: $f(x,y) = f_X(x) \cdot f_Y(y)$?

marginal

$$f_X(x) = \int_0^{1-x} 24xy\,dy = 24x\int_0^{1-x} y\,dy$$

$$= 24x \cdot \frac{y^2}{2}\Big|_{y=0}^{y=1-x}$$

$$= 12x\left[(1-x)^2 - 0\right]$$

$$= 12x(1-x)^2$$

$$f_Y(y) = \int_0^{1-y} 24xy\,dx = 24y\int_0^{1-y} x\,dx$$

$$= 24y \cdot \frac{x^2}{2}\Big|_0^{1-y}$$

$$= 12y \cdot (1-y)^2$$

$f_X(x) \cdot f_Y(y) = 12x(1-x)^2 \cdot 12y(1-y)^2 \overset{?}{=} 24xy$

NO

Not independent

---

ex) A woman & man decide to meet at a certain location. If each of them independently arrives at a time uniformly distributed btwn 12pm & 1pm, find the prob that the first to arrive has to wait longer than 10 minutes.

X = time woman arrives (after 12) in min

Y = " man " " " "

X, Y : independent & uniform $(0, 60)$

Want $P(X+10<Y)$ and $P(Y+10<X)$

$= 2P(X+10<Y)$ by symmetry

$$= 2 \iint_{\{x+10<y\}} f(x,y)\,dx\,dy = 2 \iint_{\{x+10<y\}} f_X(x) f_Y(y)\,dx\,dy$$

$$= 2\int_{10}^{60}\int_{0}^{y-10} \frac{1}{60}\cdot\frac{1}{60}\,dx\,dy = \frac{2}{60^2}\int_{10}^{60}\int_{0}^{y-10} dx\,dy$$

$$= \frac{2}{60^2}\int_{10}^{60} x\Big|_{0}^{y-10} dy = \frac{2}{60^2}\int_{10}^{60} (y-10)\,dy$$

$$= \frac{2}{60^2}\left(\frac{y^2}{2} - 10y\right)\Big|_{10}^{60} = \frac{25}{36}$$