THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

November 2016

MATH2621
Higher Complex Analysis

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER EACH QUESTION IN A SEPARATE BOOKLET

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) ONLY CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED.

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked "Q1"

1. i) For the following three regions in the complex plane

α) $2 < |z+4| < 4$, β) $\mathrm{Re}(z) + \mathrm{Im}(z) < 1$, γ) $\mathrm{Re}(z^2) > 0$,

a) sketch each region in a separate diagram;

b) state, with reasons, whether or not each region is a domain;

c) for those regions which are domains, state, with reasons, whether or not they are simply connected.

ii) Find all values of the expression

$$i^{1+i}$$

in $x + iy$ form. Which of these is the principal value?

iii) Let

$$f(x+iy) = x^3 + iy^3$$

where $x, y \in \mathbb{R}$.

a) Determine the set of points where f is differentiable, justifying every assertion.

b) Where is f analytic? Give a reason for your answer.

c) Find $f'(x+iy)$ in terms of x and y, where it exists.

iv) Suppose that Ω is a domain, that $f : \Omega \to \mathbb{C}$ is analytic in Ω, and that $|f(z)| = 1$ for all $z \in \Omega$. By using the Cauchy–Riemann equations, show that f is constant.

v) Define the function $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x, y) = e^x \sin(y) + 4xy$.

a) Show that u is harmonic on $\mathbb{R}^2$.

b) Find the harmonic conjugate v for u that satisfies $v(0, 0) = 1$.

c) Let $f(x+iy) = u(x, y) + iv(x, y)$ for all $x, y \in \mathbb{R}$ (for the v found in the previous part). Find $f(z)$ as a function of z alone.

Use a separate book clearly marked "Q2"

2. i) Find the image of the line $x = y$ under the fractional linear transformation

$$w = \frac{2z}{z+1-i}\,.$$

ii) Suppose that

$$f(z) = \frac{2}{(z+2)^2} - \frac{5}{z-4}\,.$$

Find the Laurent series for f in powers of $(z-2)$ that converges when $z = 1$.

iii) Evaluate the integral $\displaystyle\int_\gamma \bar{z}\,dz$, where $\gamma(t) = t + it^2$ for $t \in [0,1]$.

iv) Suppose that

$$g(z) = \frac{z\,e^z}{z^2+\pi^2}\,.$$

Let Γ denote the circle centre 0 with radius 10, traversed in the anti-clockwise direction.

a) Find all the singular points of g, classify them, and find the residue of g at each pole.

b) Hence calculate the integral

$$\int_\Gamma g(z)\,dz.$$

v) Suppose that

$$h(z) = \sum_{n=1}^{\infty} n(z+3)^n$$

for all z for which the sum converges.

a) Find the centre and radius of convergence of the power series.

b) Find an expression for h as an elementary function.

c) Where is the function h analytic?

Use a separate book clearly marked "Q3"

3. i) Define

$$k(z) = \mathrm{pv}(1 - z^2)^{1/2}$$

for all $z \in \mathbb{C}$. Find where k is continuous, and where k is differentiable, giving reasons for your answer.

ii) a) State Cauchy's generalised integral formula.

b) Using the previous part, prove that if f is a bounded entire function, then f is constant.

iii) Consider the function f, defined by

$$f(z) = \frac{z}{2z^4 + 5z^2 + 2}.$$

a) Find all the singularities of f, and classify these.

b) Find the residues of f at the singularities that lie inside the unit circle in $\mathbb{C}$.

c) Hence find $\displaystyle\int_0^{2\pi} \frac{d\theta}{8\cos^2\theta + 1}$.

Use a separate book clearly marked "Q4"

4. i) Consider the the equation $z^9 + 2z^3 - 8z + 1 = 0$. Giving reasons for your answers, find:

a) how many roots of this equation lie inside the circle with centre 0 and radius 1?

b) how many roots of this equation lie outside the circle with centre 0 and radius 2?

ii) Suppose that $A \in \mathbb{R}$, and that $f : [0, \infty) \to \mathbb{R}$ is a function of *exponential type A*.

a) Define the *Laplace transform* $\mathcal{L}f$ of f, specifying its domain.

b) Use the Laplace transformation to solve the Volterra integral equation

$$f(t) = t + \tfrac{1}{2} \int_0^t (t-u)^2 f(u)\, du$$

for the unknown function f.

iii) In this question, no distinction is made between a subset of $\mathbb{C}$ and the corresponding subset of $\mathbb{R}^2$. Define

$$f(z) = \frac{1+z}{1-z}$$

for all $z \in \mathbb{C} \setminus \{1\}$.

a) Show that f maps $\{z \in \mathbb{C} : |z| < 1\}$ onto $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$.

b) Explain why $H \circ f$ is harmonic in $\{z \in \mathbb{C} : |z| < 1\}$ if H is harmonic in $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$.

Now suppose that $H(w) = \mathrm{Re}(w)$ when $\mathrm{Re}(w) > 0$.

c) Compute $H \circ f$ as a function of (x, y) in the unit disc.

d) Find a harmonic conjugate for $H \circ f$ as a function of (x, y) in the unit disc.

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

November 2017

MATH2621
Higher Complex Analysis

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(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

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Use a separate book clearly marked "Q1"

1. i) For the following three regions in the complex plane

α) $2 < |z-3| < 3$, β) $\mathrm{Re}(z)\,\mathrm{Im}(z) < 1$, γ) $\mathrm{Re}(z^3) > 0$,

a) sketch each region in a separate diagram;

b) state, with reasons, whether or not each region is a domain;

c) for those regions which are domains, state, with reasons, whether or not they are simply connected.

ii) Let

$$f(x+iy) = x^4 + iy^4$$

where $x, y \in \mathbb{R}$.

a) Determine the set of points where f is differentiable, justifying every assertion.

b) Where is f analytic? Give a reason for your answer.

c) Find $f'(x+iy)$ in terms of x and y, where it exists.

iii) Suppose that Ω is a domain, that $f : \Omega \to \mathbb{C}$ is analytic in Ω, and that $|f(z)| = 1$ for all $z \in \Omega$. By using the Cauchy–Riemann equations, show that f is constant.

iv) Find all values of the expression

$$i^{2-i}$$

in Cartesian $(x+iy)$ form. Which of these is the principal value?

v) Define the function $u : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x,y) = x\cosh(x)\cos(y) - y\sinh(x)\sin(y) + y.$$

a) Show that u is harmonic on $\mathbb{R}^2$.

b) Find the harmonic conjugate v for u that satisfies $v(0,0) = 0$.

c) Let $f(x+iy) = u(x,y) + iv(x,y)$ for all $x, y \in \mathbb{R}$, for the function v found in the previous part. Find $f(z)$ as a function of z alone.

Use a separate book clearly marked "Q2"

2. i) Find the image of the line $y = 4 - x$ under the fractional linear transformation

$$w = \frac{8}{z - 2 - 2i}\,.$$

ii) Suppose that

$$f(z) = \frac{2}{(z+1)^2} - \frac{5}{z-5}\,.$$

Find the Laurent series for f in powers of $(z-1)$ that converges when $z = 4$.

iii) Evaluate the integral $\displaystyle\int_\gamma |z|^2\,dz$, where $\gamma(t) = 1 + it^2$ for $t \in [0,1]$.

iv) Suppose that

$$g(z) = \frac{z\sin(\pi z)}{(z+1)(z-1)^2}\,.$$

Let Γ denote the circle centre 0 with radius 3, traversed in the anti-clockwise direction.

a) Find all the singular points of g, classify them, and find the residue of g at each pole.

b) Hence calculate the integral

$$\int_\Gamma g(z)\,dz.$$

v) Suppose that

$$h(z) = \sum_{n=1}^{\infty} \frac{(z-2)^n}{n}$$

for all z for which the sum converges.

a) Find the centre and radius of convergence of the power series.

b) Find an expression for h as an elementary function.

c) Where is the function h analytic?

Use a separate book clearly marked "Q3"

3. i) Define

$$k(z) = \mathrm{pv}(z-1)^{1/2}\,\mathrm{pv}(z+1)^{1/2}$$

for all $z \in \mathbb{C}$. Find where k is continuous and where k is differentiable, giving reasons for your answer.

ii) a) State Cauchy's generalised integral formula.

b) Hence show that if f is an entire function and $|f(z)| \le |z| + 1$, then $f^{(k)}(0) = 0$.

c) Deduce that there are complex constants a, b such that $f(z) = az + b$ and moreover $|a| \le 1$ and $|b| \le 1$.

iii) Consider the function f, defined by

$$f(z) = \frac{z}{2z^4 - 5z^2 + 2}.$$

a) Find all the singularities of f, and classify these.

b) Find the residues of f at the singularities that lie inside the unit circle in $\mathbb{C}$.

c) Hence find $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + 8\sin^2\theta}$.

Use a separate book clearly marked "Q4"

4. i) Consider the equation $-z^7 + 2z^5 + 8z + 1 = 0$. Giving reasons for your answers, find:

a) how many roots of this equation lie outside the circle with centre 0 and radius 2?

b) how many roots of this equation lie on the imaginary axis?

c) how many roots of this equation lie in the right half plane?

ii) Suppose that $A \in \mathbb{R}$, and that $f : [0, \infty) \to \mathbb{R}$ is a function of *exponential type* A.

a) Define the *Laplace transform* $\mathcal{L}f$ of f, specifying its domain.

b) Use the Laplace transformation to find the unknown function $f : [0, \infty) \to \mathbb{R}$ in the integral equation

$$f(t) = t + \frac{1}{6} \int_0^t (t-u)^3 f(u)\, du.$$

iii) In this question, no distinction is made between a subset of $\mathbb{C}$ and the corresponding subset of $\mathbb{R}^2$. Define

$$f(z) = \frac{1+z}{1-z}$$

for all $z \in \mathbb{C} \setminus \{1\}$.

a) Show that f maps $\{z \in \mathbb{C} : |z| < 1\}$ onto $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$.

b) Explain why $H \circ f$ is harmonic in $\{z \in \mathbb{C} : |z| < 1\}$ if H is harmonic in $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$.

c) Find a harmonic function H in $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$ such that

$$\lim_{\substack{w \to it \\ \mathrm{Re}(w) > 0}} H(w) = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0. \end{cases}$$

d) Find a harmonic function G in $\{z \in \mathbb{C} : |z| < 1\}$ such that

$$\lim_{\substack{z \to e^{i\theta} \\ |z| < 1}} G(z) = \begin{cases} 1 & \text{if } 0 < \theta < \pi \\ -1 & \text{if } -\pi < \theta < 0. \end{cases}$$

e) Describe the level curves of the function G in the unit disc.

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Term Three 2020

MATH2621
Higher Complex Analysis

(1) TIME ALLOWED – THREE (3) HOURS

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) START EACH QUESTION ON A NEW PAGE.

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

YOU ARE TO COMPLETE THE TEST UNDER STANDARD EXAM CONDITIONS, WITH HANDWRITTEN SOLUTIONS.

YOU WILL THEN SUBMIT ONE OR MORE FILES CONTAINING YOUR SOLUTIONS. **MAKE SURE YOU SUBMIT ALL YOUR ANSWERS.**

ONE OF THE SUBMITTED FILES MUST INCLUDE A PHOTOGRAPH OF YOUR **STUDENT ID CARD** WITH THE **SIGNED**, HANDWRITTEN STATEMENT:
"**I declare that this submission is entirely my own original work.**"

YOU CAN DELETE AND/OR RELOAD FILES UNTIL THE DEADLINE.

Start a new page clearly marked Question 1.

1) (i) Consider the map

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, f(z) := z^{10} + \frac{1}{z^{10}}.$$

Find the image of the unit circle $C = \{z \in \mathbb{C} : \ |z| = 1\}$ by the map f.

(ii) Consider a fractional linear transformation of the form

$$f(z) = \lambda \frac{z + \overline{\mu}}{\mu z + 1}$$

where $\lambda \in \mathbb{C}, |\lambda| = 1$ and $\mu \in \mathbb{C}, |\mu| \neq 1$.

a) Prove that $f(C) = C$ where $C = \{z \in \mathbb{C} : \ |z| = 1\}$. (To answer completely this question you need to prove that $f(C)$ is equal to C and not only contained in C.)

b) Prove that if T is a fractional linear transformation satisfying $\lim_{z\to\infty} T(z) = \infty$, then T is affine (i.e. there exists $a, b \in \mathbb{C}$ such that $T(z) = az + b$ for all $z \in \mathbb{C}$).

c) Determine for which $\lambda \in \mathbb{C}, |\lambda| = 1$ and $\mu \in \mathbb{C}, |\mu| \neq 1$ we have that the function f above maps the unit disc to itself.

d) Are there other fractional linear transformation g satisfying $g(C) = C$ but not of the form described above? If yes provide an explicit example, if not provide a proof.

(iii) Consider the polynomial $P(z) = z^5 + 1$.

a) Find all the complex roots of P.

b) Give a complex factorisation of P.

c) Give a real factorisation of P.

(iv) Consider the rational map

$$r(z) = \frac{z^4 + 3z - 1}{z^2 + 2}.$$

Let $R > 3$ and fix $z \in \mathbb{C}$ satisfying $|z| = R$.

a) Prove that

$$|r(z)| \geq \frac{R^4 - 3R - 1}{R^2 + 2}.$$

b) Deduce that

$$|r(z)| \geq \frac{R}{2} - 1.$$

Start a new page clearly marked Question 2.

2) (i) Consider the map in two variables

$$u(x, y) = e^x \cos(y)(x^2 - y^2) - 2e^x \sin(y)xy.$$

a) Find an entire function f satisfying that $Re(f(x + iy)) = u(x, y)$ for all $x, y \in \mathbb{R}$.

b) Explain why u is harmonic on $\mathbb{R}^2$ and find a harmonic conjugate v for u satisfying $v(0, 0) = 0$.

(ii) Consider the map

$$f(x + iy) = x + x^2y + ixy^2 + iy,\ \ x, y \in \mathbb{R}.$$

(a) Compute the Cauchy-Riemann system of equations for this function.

(b) Determine where the map $f : \mathbb{C} \to \mathbb{C}$ of above is differentiable.

(iii) Consider the following two subsets of $\mathbb{C}^3$:

$$A := \{(z_1, z_2, z_3) \in \mathbb{C}^3 :\ z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3\}$$

and

$$B := \{(z_1, z_2, z_3) \in \mathbb{C}^3 :\ |z_1 - z_2| = |z_1 - z_3| = |z_2 - z_3|\}.$$

(a) Show that if $(z_1, z_2, z_3) \in A$, then $(f(z_1), f(z_2), f(z_3)) \in A$ for all affine maps $f(z) = az + b, a, b \in \mathbb{C}, a \neq 0$.

(b) Show that if $(z_1, z_2, z_3) \in B$, then $(f(z_1), f(z_2), f(z_3)) \in B$ for all affine maps $f(z) = az + b, a, b \in \mathbb{C}, a \neq 0$.

(c) Prove that $(0, 1, z) \in A$ if and only if $(0, 1, z) \in B$ if and only if $z = e^{i\pi/3}$ or $z = e^{-i\pi/3}$.

(d) Deduce that $A = B$.

(iv) Consider the power series

$$S(z) = \sum_{n \geq 0} (n + 1)(z + i)^{n+1}$$

a) Find the centre and radius of convergence of this power series.

b) Write down $S(z)$ in the form of a rational function on its disc of convergence.

Start a new page clearly marked Question 3.

3) (i) Consider the principal branch of the square root:

$$\mathbb{C} \to \mathbb{C}, z \mapsto PV\sqrt{z} := \begin{cases} |z|^{1/2} \exp(i \operatorname{Arg}(z)/2) \text{ if } z \neq 0 \\ 0 \text{ otherwise} \end{cases},$$

where $\operatorname{Arg}(z)$ is the principal branch of the argument function. Define the map

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, f(z) := PV\sqrt{z^2 + z^{-2}}.$$

a) Where is f holomorphic? Justify your answer.

b) Where is f continuous? Justify your answer.

(ii) Consider the function

$$f(z) = \frac{z^2 + 2}{z(z+i)^2}.$$

a) What are the maximal annuli centre at 0 in which f is holomorphic?

b) What are the maximal annuli centre at $-i$ in which f is holomorphic?

c) Find the three constant $A, B, C \in \mathbb{C}$ satisfying that:

$$f(z) = \frac{A}{z} + \frac{B}{z+i} + \frac{C}{(z+i)^2}.$$

d) Write down the Laurent series of f in power of $(z + i)$ in the the maximal annuli centred at $-i$ where f is holomorphic.

e) Compute

$$\int_\gamma f(z)dz$$

where $\gamma(t) = -i + 10e^{i2\pi t}, 0 \leq t \leq 1$. Justify your work.

(iii) Consider the integral:

$$I = \int_0^{2\pi} \frac{d\theta}{2 + \cos(\theta)}.$$

a) Write I as an integral $\int_C f(z)dz$ where C is the unit circle traversed once anticlockwise and f is a complex function.

b) Evaluate the integral I.

Start a new page clearly marked Question 4.

4) (i) Consider the real function

$$f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{\cos(x)}{(x^2+1)^2}.$$

We want to compute the improper integral $\int_{-\infty}^{+\infty} f(x)dx$. Define the map

$$g : \mathbb{C} \setminus \{i, -i\} \to \mathbb{C}, g(z) = \frac{\exp(iz)}{(z^2+1)^2}.$$

Consider $R > 1$. Let C_R be the half circle $\{z \in \mathbb{C} : \ |z| = R, \mathrm{Im}(z) \geq 0\}$ oriented anticlockwise and $[-R, R]$ the real interval oriented from left to right. We write $\Gamma_R = C_R \sqcup [-R, R]$ the join of those two contours.

a) Prove that $\lim_{R\to\infty} \int_{C_R} g(z)dz = 0$.

b) Evaluate the integral $\int_{\Gamma_R} g(z)dz$ for all $R > 1$. Justify your answer.

c) Evaluate the improper integral

$$\int_{-\infty}^{+\infty} f(x)dx.$$

Justify your answer.

(ii) Consider the sequence of functions $f_n(z) = \frac{z^{n^2}}{z^n-1}$ with $n \geq 1$ and let Γ be the closed contour equal to the circle centred at 0 of radius 2 having the standard orientation.

Fix $n \geq 1$.

a) Find all singularities of f_n and classify them.

b) Compute the residue at each of the singularities.

c) Compute the integral

$$I_n := \int_\Gamma f_n(z)dz.$$

Write your result in a simple way and justify your answer.

(iii) Let $\Delta \subset \mathbb{C}$ be a finite subset (possibly empty) and consider a function

$$f : \mathbb{C} \setminus \Delta \to \mathbb{C}$$

that is holomorphic and such that if $w \in \Delta$, then w is a pole of f.

a) Using Laurent series show that if $w \in \Delta$, then there exists a rational function r_w satisfying that $f - r_w$ is holomorphic on the complement of $\Delta \setminus \{w\}$. Justify carefully your answer.

b) Prove that $f = r + g$ where r is a rational function and g is entire (g is holomorphic on $\mathbb{C}$).

c) We now assume that there exists $\ell \in \mathbb{C}$ such that $\lim_{z\to\infty} f(z) = \ell$. Show that f is a rational function.

END OF EXAMINATION

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

November 2014

MATH2621
Higher Complex Analysis

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Use a separate book clearly marked "Q1"

1. i) For the following three regions in the complex plane

α) $|z-2| \leq 2$, β) $0 < \text{Arg}(z-i) < \pi/4$, γ) $\text{Re}(z^2) > 1$,

a) sketch each region in a separate diagram;
b) state, with reasons, whether or not each region is a domain;
c) for those regions which are domains, state, with reasons, whether or not they are simply connected.

ii) Find all complex numbers z such that

$$\sin(z) = 2i.$$

iii) Let

$$f(z) = \left(-xy^2 + \frac{y^4}{4} + \frac{x^4}{12}\right) + i\left(\frac{x^3y}{3} - \frac{y^3}{3}\right),$$

where $z = x + iy$, with $x, y \in \mathbb{R}$.

a) Determine the set of points where f is differentiable.
b) Where is f analytic? Give a reason for your answer.
c) Find $f'(z)$ in terms of x and y, where it exists.

iv) Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $u(x, y) = e^{-y}\sin(x) + x^2 - y^2 + 2y$.

a) Show that u is harmonic on $\mathbb{R}^2$.
b) Find a harmonic conjugate v for u.
c) Find an analytic function f such that $f(x + iy) = u(x, y) + iv(x, y)$ for all $x, y \in \mathbb{R}$.

Use a separate book clearly marked "Q2"

2. i) Suppose that

$$f(z) = \frac{5}{(z+1)(z-4)}.$$

Find the Laurent series for f in powers of $(z-2)$ that converges when $z = i$.

ii) State the definition of the residue $\mathrm{Res}(f, z_0)$ of a function f at a point $z_0 \in \mathbb{C}$.

iii) Suppose that

$$f(z) = \frac{z-1}{(z+1)^2(z-3)}.$$

a) Find all the singular points of f, classify them, and find the residue of f at each of them.

b) Hence calculate the integral

$$\int_\gamma f(z)\,dz,$$

where γ is the circle with centre -1 and radius 2, traversed in the anti-clockwise direction.

iv) Suppose that an analytic function f may be represented by the power series

$$\sum_{n=2}^{\infty} \frac{(-2)^n}{n(n-1)}(z-3)^n.$$

in a particular region.

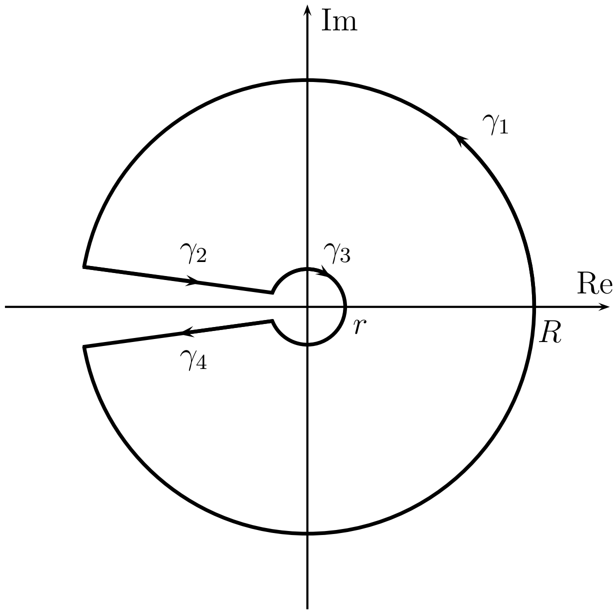
a) Find the centre and radius of convergence of the power series.

b) Write down the power series that represents the second derivative of f.

c) Find an expression for f as an elementary function.

Use a separate book clearly marked "Q3"

3. Consider the contour γ that is the sum of the contours γ_1, γ_2, γ_3, and γ_4 shown in the figure.



Here γ_1 and γ_3 are arcs of circles, with centre 0 and radii r and R respectively; γ_2 is the straight line segment $\{z \in \mathbb{C} : r < |z| < R,\ \mathrm{Arg}(z) = \pi - \delta\}$; and γ_4 is the straight line segment $\{z \in \mathbb{C} : r < |z| < R,\ \mathrm{Arg}(z) = \delta - \pi\}$. Further, $0 < \delta < \pi/2$, and $r < 1 < R$.

The function f is defined in $\mathbb{C} \setminus \{x + i0 : x \leq 0\}$ by

$$f(z) = \frac{z^{1/2}}{z^2 + 1},$$

where $z^{1/2}$ denotes the principal branch of the square root of z.

i) Find the singular points of f and the residues of f at these singular points.

ii) Give reasons why $\displaystyle\int_\gamma f(z)\, dz = i\pi\sqrt{2}$.

iii) Give reasons why $\displaystyle\int_{\gamma_1} f(z)\, dz \to 0$ as $R \to \infty$.

iv) Find $\displaystyle\lim_{r\to 0} \int_{\gamma_3} f(z)\, dz$.

v) Give reasons why $\displaystyle\lim_{\delta\to 0}\left(\int_{\gamma_2} f(z)\, dz + \int_{\gamma_4} f(z)\, dz\right) = 2i \int_r^R \frac{t^{1/2}}{t^2+1}\, dt.$

vi) Hence find $\displaystyle\int_0^\infty \frac{t^{1/2}}{t^2+1}\, dt$.

Use a separate book clearly marked "Q4"

4. i) Suppose that $A \in \mathbb{R}$, and that $f : [0, \infty) \to \mathbb{R}$ is a function of *exponential type* A.

a) Define the *Laplace transform* $\mathcal{L}f$ of f, specifying its domain.

b) Suppose that $\mathcal{L}f$ extends to a function that is holomorphic in $\mathbb{C}$ except for finitely many singularities $\lambda_1, \lambda_2, \ldots, \lambda_N$ in $\mathbb{C}$, and that $|\mathcal{L}f(z)| \leq |z|^{-1}$ when $|z|$ is big enough. Write down an expression for f in terms of residues at the points $\lambda_1, \lambda_2, \ldots, \lambda_N$.
Give a very brief outline of the proof of this formula.

c) Define the *convolution* $f * g$ of two functions f and g of exponential type on $[0, \infty)$, and compute the Laplace transform $\mathcal{L}(f * g)$.

d) Use the Laplace transformation to solve the Volterra integral equation

$$u(t) = 1 + \int_0^t (t - s)\, u(s)\, ds.$$

ii) Let $f(z) = \dfrac{1 + z}{1 - z}$ for all $z \in \mathbb{C} \setminus \{1\}$.

a) Show that f maps $\{z \in \mathbb{C} : |z| < 1\}$ onto $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$.

b) Explain why $H \circ f$ is harmonic in $\{z \in \mathbb{C} : |z| < 1\}$ if H is harmonic in $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$.

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

November 2015

MATH2621
Higher Complex Analysis

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER EACH QUESTION IN A SEPARATE BOOKLET

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) ONLY CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED.

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked "Q1"

1. i) For the following three regions in the complex plane

$$\alpha)\quad 1 < |z-3| < 3, \qquad \beta)\quad \mathrm{Re}(z)\,\mathrm{Im}(z) < 4, \qquad \gamma)\quad \mathrm{Im}(z^3) \geq 0,$$

a) sketch each region in a separate diagram;

b) state, with reasons, whether or not each region is a domain;

c) for those regions which are domains, state, with reasons, whether or not they are simply connected.

ii) Find all complex numbers z such that

$$\cosh(z) = i.$$

iii) Let

$$f(z) = \left(x^2 + \frac{x^4}{4} + \frac{y^4}{16}\right) + i\left(\frac{x^2y^2}{4} - \frac{y^2}{2}\right),$$

where $z = x + iy$, with $x, y \in \mathbb{R}$.

a) Determine the set of points where f is differentiable.

b) Where is f analytic? Give a reason for your answer.

c) Find $f'(z)$ in terms of x and y, where it exists.

iv) Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $u(x,y) = e^{-x}\cos(x) + 4xy + 3y$.

a) Show that u is harmonic on $\mathbb{R}^2$.

b) Find a harmonic conjugate v for u.

c) Find an analytic function f such that $f(x+iy) = u(x,y) + iv(x,y)$ for all $x, y \in \mathbb{R}$.

v) The function r is rational and satisfies $\lim_{z\to\infty} r(z) = \infty$. Find Range$(r)$.

Use a separate book clearly marked "Q2"

2. i) Suppose that
$$f(z) = \frac{2}{z+2i} - \frac{5}{z-4}.$$
Find the Laurent series for f in powers of $(z-2)$ that converges when $z = i$.

ii) State carefully the definition of the residue $\mathrm{Res}(f, z_0)$ of a function f at a point $z_0 \in \mathbb{C}$.

iii) Suppose that
$$f(z) = \frac{e^{1/z}}{(z-2)(z+2)};$$
let Γ_R denote the circle centre 0 with radius R, traversed in the anti-clockwise direction.

a) Find all the singular points of f, classify them, and find the residue of f at each *pole*.

b) Explain why
$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\, dz = 0.$$

c) Hence calculate the integral
$$\int_{\Gamma_1} f(z)\, dz.$$

iv) Suppose that
$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n}(z+1)^n.$$
in a particular region.

a) Find the centre and radius of convergence of the power series.

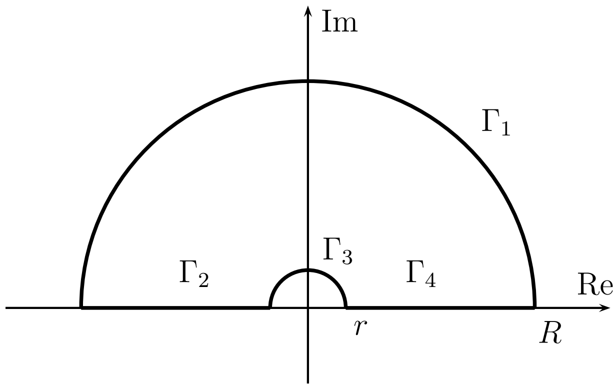
b) Write down the power series that represents $f'(z)$.

c) Find an expression for f as an elementary function.

Use a separate book clearly marked "Q3"

3. i) Suppose that f is an entire function and that $|f(z)| \leq 2|z| + 2$ for all $z \in \mathbb{C}$. Show that there are constants a and b such that $f(z) = az + b$.

ii) Consider the contour Γ that is the sum of the contours Γ_1, Γ_2, Γ_3, and Γ_4 shown in the figure, oriented in the usual anti-clockwise sense.



Here Γ_1 and Γ_3 are arcs of circles, with centre 0 and radii R and r respectively; Γ_2 is the straight line segment from $-R$ to $-r$; and Γ_4 is the straight line segment from r to R. Further, $r < 1 < R$.

The function f is defined in $\mathbb{C} \setminus \{0\}$ by

$$f(z) = \frac{1 - e^{iz}}{z^2}.$$

a) Give reasons why $\displaystyle\int_{\Gamma_1} f(z)\, dz \to 0$ as $R \to \infty$.

b) Find $\displaystyle\lim_{r \to 0} \int_{\Gamma_3} f(z)\, dz$.

c) Hence find $\displaystyle\int_0^\infty \frac{1 - \cos(t)}{t^2}\, dt$.

Use a separate book clearly marked "Q4"

4. i) Suppose that $A \in \mathbb{R}$, and that $f : [0, \infty) \to \mathbb{R}$ is a function of *exponential type* A.

a) Define the *Laplace transform* $\mathcal{L}f$ of f, specifying its domain.

b) Use the Laplace transformation to solve the Volterra integral equation

$$f(t) = t^2 + \int_0^t (t - s)\, f(s)\, ds$$

for the unknown function f.

ii) Consider the the equation $z^4 - 8z + 10 = 0$.

a) How many roots lie inside the annulus $\{z \in \mathbb{C} : 1 < |z| < 3\}$?

b) How many roots lie on the imaginary axis?

c) How many roots lie on the real axis?

d) How many roots lie in the first quadrant?

iii) Let $f(z) = \dfrac{1+z}{1-z}$ for all $z \in \mathbb{C} \setminus \{1\}$.

a) Show that f maps $\{z \in \mathbb{C} : |z| < 1\}$ onto $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$.

b) Explain why $H \circ f$ is harmonic in $\{z \in \mathbb{C} : |z| < 1\}$ if H is harmonic in $\{w \in \mathbb{C} : \mathrm{Re}(w) > 0\}$.

Now suppose that $H(w) = \mathrm{Re}(w)$ when $\mathrm{Re}(w) > 0$.

c) Compute $H \circ f$ as a function of (x, y) in the unit disc.

d) Find a harmonic conjugate for $H \circ f$ as a function of (x, y) in the unit disc.