MATH2621 — Higher Complex Analysis. XIII
Inverses of exponential and related functions

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In this lecture, we discuss the inverse functions for the exponential, hyperbolic and trigonometric functions, such as

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2} \quad \text{and} \quad \cos(z) = \frac{\exp(iz) + \exp(-iz)}{2}.$$

We will also use square roots: remember that

$$\mathrm{PV}\,\sqrt{z} = \exp(\tfrac{1}{2}\,\mathrm{Log}(z));$$

equivalently, $\mathrm{PV}\,\sqrt{z}$ is the choice of square root with argument in $(-\pi/2, \pi/2]$.

Exercise 1

Fix w in $\mathbb{C}$. Find all z in $\mathbb{C}$ such that

(a) $\exp(z) = w$, (b) $\cosh(z) = w$, (c) $\sinh(z) = w$.

Answer.

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Answer. (a) If $\exp(z) = w$, then we may write

$$z = \mathrm{Log}\,(w) + 2\pi i k,$$

where $k \in \mathbb{Z}$.

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Answer. (a) If $\exp(z) = w$, then we may write

$$z = \mathrm{Log}\,(w) + 2\pi i k,$$

where $k \in \mathbb{Z}$.

Alternatively, we may write

$$z = \log\,(w)\,,$$

where log is multi-valued.

Answer to Exercise 1

(b) If $\cosh(z) = w$, then $\exp(z) + \exp(-z) = 2w$, so

$$(\exp(z))^2 - 2w(\exp(z)) + 1 = 0,$$

whence

$$\exp(z) = \frac{2w \pm \mathrm{PV}\sqrt{(2w)^2 - 4}}{2} = w \pm \mathrm{PV}\sqrt{w^2 - 1}.$$

Noting that $(w + \mathrm{PV}\sqrt{w^2 - 1})(w - \mathrm{PV}\sqrt{w^2 - 1}) = 1$, we conclude that

$$z = \pm \mathrm{Log}(w + \mathrm{PV}\sqrt{w^2 - 1}) + 2\pi i k$$

for some $k \in \mathbb{Z}$.

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Each step of the argument is reversible, so all such z solve $\cosh(z) = w$.

Answer to Exercise 1 (continued)

We could also write

$$z = \ln \left| w \pm \mathrm{PV}\sqrt{w^2 - 1} \right| + i \,\mathrm{Arg}\left(w \pm \mathrm{PV}\sqrt{w^2 - 1} \right) + 2\pi i k,$$

where $k \in \mathbb{Z}$ and we take the same choice of $\pm$ in both ln and Arg,

Answer to Exercise 1 (continued)

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where $k \in \mathbb{Z}$ and we take the same choice of $\pm$ in both ln and Arg, or

$$z = \log\left(w + \sqrt{w^2 - 1}\right),$$

where both log and square root are multi-valued.

Answer to Exercise 1 (continued)

Note that cosh is even, so if $\cosh(z) = w$, then $\cosh(-z) = w$ too. We can see that more clearly in solutions such as

$$z = \pm \operatorname{Log}\left(w + \mathrm{PV}\sqrt{w^2 - 1}\right) + 2\pi i k,$$

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where log is multi-valued.

(c) Is similar to (b). $\triangle$

The exponential function

The principal candidate for an inverse for the exponential is ...

Definition

The *principal branch* of the complex logarithm is the function Log from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$, given by

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z),$$

where $\mathrm{Arg}(z)$ takes values in the range $(-\pi, \pi]$.

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By properties of the exponential, if $z = re^{i\theta}$, then

$$e^{\operatorname{Log}(z)} = e^{\ln(r)+i\theta} = re^{i\theta} = z;$$

however, if $z = x + iy$, then

$$\operatorname{Log}(e^z) = \ln|e^z| + i\operatorname{Arg}(e^z) = x + i\operatorname{Arg}(e^{iy}) = x + iy + 2\pi ik,$$

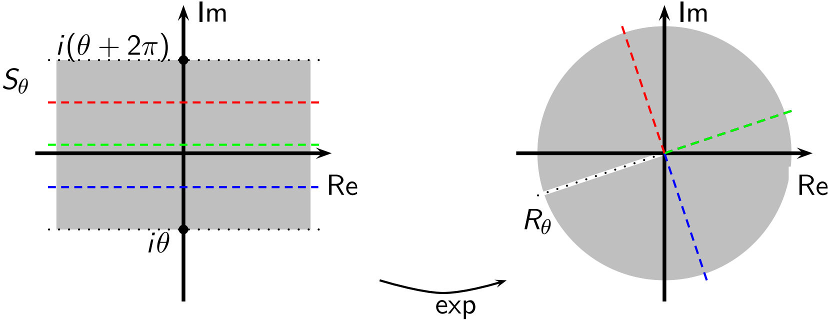
for some $k \in \mathbb{Z}$; it may be that $\operatorname{Log}(e^z) \neq z$.

Another candidate

Given $\theta \in \mathbb{R}$, we define the ray R_θ and the horizontal strip S_θ in the complex plane by

$$R_\theta = \{w \in \mathbb{C} : \mathrm{Arg}(w) - \theta \in 2\pi\mathbb{Z}\}$$
$$S_\theta = \{z \in \mathbb{C} : \theta < \mathrm{Im}(z) < \theta + 2\pi\}.$$

Another candidate



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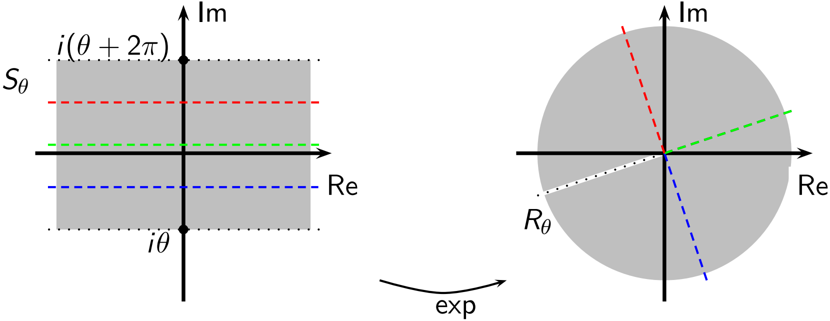

The exponential map exp takes horizontal lines to rays, and is one-to-one and onto from the open horizontal strip S_θ to $\mathbb{C} \setminus R_\theta$.

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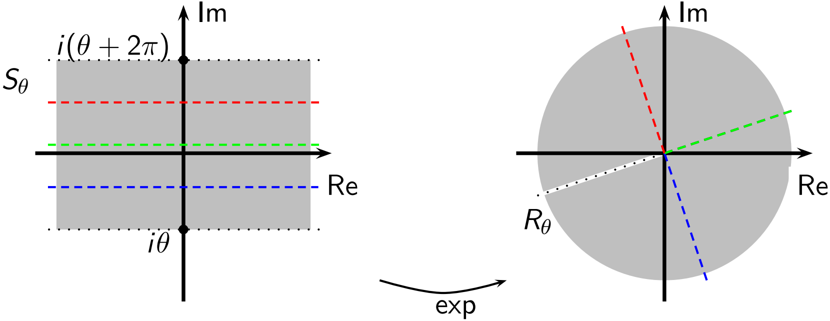

The exponential map exp takes horizontal lines to rays, and is one-to-one and onto from the open horizontal strip S_θ to $\mathbb{C} \setminus R_\theta$. We may define an inverse function $\log_\theta$ from $\mathbb{C} \setminus R_\theta$ to S_θ:

$$\log_\theta(w) = \ln|w| + i \arg_\theta(w),$$

where $\arg_\theta(w)$ is the argument in the range $(\theta, \theta + 2\pi)$.

A zoo of inverse functions

As θ varies, we get different inverse functions. These inverse functions are *branches* of the complex logarithm, and the rays R_θ are *branch cuts*.

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Different branches of the logarithm differ by a constant in connected open sets where they are both defined.

Differentiability

Lemma

For any branch $\log_\theta$ *of the complex logarithm,*

$$\log'_\theta(w) = \frac{1}{w}$$

for all $w \in \mathbb{C} \setminus R_\theta$*.*

Proof. We may use the Cauchy–Riemann equations.

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Alternatively, we can use the formula for the derivative of an inverse function to prove this. $\square$

Differentiability

Lemma

For any branch $\log_\theta$ *of the complex logarithm,*

$$\log_\theta'(w) = \frac{1}{w}$$

for all $w \in \mathbb{C} \setminus R_\theta$*.*

Proof. We may use the Cauchy–Riemann equations.

Alternatively, we can use the formula for the derivative of an inverse function to prove this. □

The notation $\log_\theta$ is not standard, and we will not use it any more. Rather, we use the expression "the branch of the logarithm with imaginary part in $(\theta, \theta + 2\pi)$".

A multi-valued inverse function

A different sort of "inverse function" of the exponential function is the "multi-function" (resp. function) log from $\mathbb{C} \setminus \{0\}$ to (resp. the power set of) $\mathbb{C}$ given by

$$\log(z) = \ln|z| + i \arg(z).$$

This is a "multifunction" in the sense that it takes multiple values, because $\arg(z)$ takes multiple values.

Complex powers

We define complex powers of complex numbers using exponentials and logarithms.

Definition

Given $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$, we define

$$z^\alpha = \exp(\alpha \log(z)).$$

The *principal branch of* z^α is found by using Log, the principal branch of the logarithm:

$$\mathrm{PV}\, z^\alpha = \exp(\alpha\, \mathrm{Log}(z)).$$

The possible values of the multi-valued function z^α are $\exp(\alpha\, \mathrm{Log}(z) + 2\pi i k \alpha)$ where $k \in \mathbb{Z}$. Different values of k may give very different values of z^α.

A lemma

Lemma

The function $z \mapsto \text{PV}\, z^\alpha$ *is differentiable in* $\mathbb{C} \setminus (-\infty, 0]$*, and its derivative is* $\alpha\, \text{PV}\, z^\alpha / z$*.*

Proof. It suffices to differentiate $\exp(\alpha \operatorname{Log}(z))$, which is differentiable where Log is differentiable. $\square$

Exercise 2

Compute the possible values of i^i; which is the principal value?

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Answer. By definition, the possible values are

$$i^i = \exp(i \log(i)) = \exp(i(i\frac{\pi}{2} + 2\pi ik)) = \exp(-\frac{\pi}{2} - 2\pi k),$$

where $k \in \mathbb{Z}$. When $k = 0$ we get the principal value:
PV $i^i = e^{-\pi/2}$. $\triangle$

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For what a and b is a^b single-valued?

The inverse hyperbolic sine

A little while ago, we defined the hyperbolic sine and cosine, and established some of their properties. Now we consider the inverse function(s) of sinh.

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Find all z in $\mathbb{C}$ such that $\sinh(z) = w$.

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$$(e^z)^2 - 2we^z - 1 = 0,$$

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$$(e^z)^2 - 2we^z - 1 = 0,$$

whence

$$e^z = \frac{2w \pm \mathrm{PV}(4w^2 + 4)^{1/2}}{2} = w \pm \mathrm{PV}(w^2 + 1)^{1/2}$$

$$\begin{aligned} z &= \mathrm{Log}(w \pm \mathrm{PV}(w^2 + 1)^{1/2}) + 2\pi ik \\ &= \ln \left| w \pm \mathrm{PV}(w^2 + 1)^{1/2} \right| + i \,\mathrm{Arg}(w \pm \mathrm{PV}(w^2 + 1)^{1/2}) + 2\pi ik \end{aligned}$$

where $k \in \mathbb{Z}$; we must take the same choice of $\pm$ in both the ln part and in the Arg part.

Answer to Exercise 3

Alternatively, we may note that

$$(w + \mathrm{PV}(w^2+1)^{1/2})(w - \mathrm{PV}(w^2+1)^{1/2}) = w^2 - (w^2+1) = -1;$$

hence e^z is $w + \mathrm{PV}(w^2+1)^{1/2}$ or $-(w + \mathrm{PV}(w^2+1)^{1/2})^{-1}$, and either

$$z = \mathrm{Log}(w + \mathrm{PV}(w^2+1)^{1/2}) + 2\pi i k$$

or

$$z = -\mathrm{Log}(w + \mathrm{PV}(w^2+1)^{1/2}) + 2\pi i k + \pi i.$$

This is a better answer because there is less room for ambiguity.

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We may also write

$$z = \log(w + \mathrm{PV}(w^2+1)^{1/2}) \quad \text{or} \quad -\log(w + \mathrm{PV}(w^2+1)^{1/2}) + \pi i,$$

where log is multi-valued. $\triangle$

The inverse hyperbolic sine

The principal branch of the inverse hyperbolic sine function is given by

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however, it need not be true that $\operatorname{PV}\sinh^{-1}\sinh z = z$.

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however, it need not be true that $\mathrm{PV}\sinh^{-1}\sinh z = z$.
What are the possible values of $\mathrm{PV}\sinh^{-1}\sinh z - z$?

More on the inverse hyperbolic sine

Both the logarithm and the square root are possible causes of discontinuity. The function $\mathrm{PV}(w^2+1)^{1/2}$ is continuous as long as w^2+1 is not in the interval $(-\infty, 0]$, and the logarithm is continuous as long as $w + \mathrm{PV}(w^2+1)^{1/2}$ is not in the interval $(-\infty, 0]$.

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On the one hand, if w^2+1 is not in $(-\infty, 0]$, then w^2 is not in $(-\infty, -1]$. So one possible discontinuity is when $w = iv$, where $|v| \geq 1$.

Discontinuities of the inverse hyperbolic sine

On the other hand, we may try to solve the equation $w + \mathrm{PV}(w^2+1)^{1/2} = -t$ for $t \in [0, \infty)$; we get

$$\begin{aligned}\mathrm{PV}(w^2+1)^{1/2} &= -t - w\\ w^2 + 1 &= t^2 + w^2 + 2tw\\ w &= \frac{1-t^2}{2t},\end{aligned}$$

and so w is real.

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and so w is real. But if w is real, then

$$w + \mathrm{PV}(w^2+1)^{1/2} = w + (w^2+1)^{1/2} > 0,$$

and so $w + \mathrm{PV}(w^2+1)^{1/2}$ is not in the interval $(-\infty, 0]$. Thus the only possible discontinuities are when $w = iv$, where v is real and $|v| \geq 1$.

Differentiability

Lemma

The principal branch of the inverse hyperbolic sine function is differentiable in $\mathbb{C} \setminus ([i, +i\infty) \cup (-i\infty, -i])$*. Further,*

$$\left(\mathrm{PV}\sinh^{-1}\right)'(w) = \frac{1}{\mathrm{PV}\sqrt{w^2+1}}\,.$$

Proof

Proof. We compute the derivative:

$$\begin{aligned}\frac{d\sinh^{-1}(w)}{dw} &= \frac{d\,\mathrm{Log}(w + \mathrm{PV}(w^2+1)^{1/2})}{dw} \\ &= \frac{1 + w/\mathrm{PV}(w^2+1)^{1/2}}{w + \mathrm{PV}\sqrt{w^2+1}} \\ &= \frac{1}{\mathrm{PV}(w^2+1)^{1/2}},\end{aligned}$$

as required.
This is correct as long as we stay away from where Log is not differentiable, that is, z stays away from $[+i, +i\infty) \cup [-i, -i\infty)$.

$\square$

The inverse hyperbolic cosine

Similarly, we define

$$\mathrm{PV}\cosh^{-1}(w) = \mathrm{Log}(w + \mathrm{PV}(w+1)^{1/2}\,\mathrm{PV}(w-1)^{1/2}).$$

Exercise 4

Show that

$$\frac{d\,\mathrm{PV}\cosh^{-1}(w)}{dw} = \frac{1}{\mathrm{PV}(w+1)^{1/2}\,\mathrm{PV}(w-1)^{1/2}}$$

for most $w \in \mathbb{C}$. Where is PV $\cosh^{-1}$ not differentiable?

Answer.

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Answer. The calculation of the derivative is similar to that for PV $\sinh^{-1}$.

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Answer. The calculation of the derivative is similar to that for PV $\sinh^{-1}$.

The potential problems for differentiability are when $w - 1 \in (-\infty, 0]$, and when $w + 1 \in (-\infty, 0]$, and when $w + \mathrm{PV}(w-1)^{1/2}\,\mathrm{PV}(w+1)^{1/2} \in (-\infty, 0]$. Observe that $w + \mathrm{PV}(w-1)^{1/2}\,\mathrm{PV}(w+1)^{1/2} = -a$ if and only if $\mathrm{PV}(w-1)^{1/2}\,\mathrm{PV}(w+1)^{1/2} = -a - w$, that is if $w^2 - 1 = (w+a)^2$, or $w = -(a^2+1)/2a$. So the potential problems are when $w \in (-\infty, 1]$. This is the branch cut. $\triangle$

The inverse trigonometric functions

We may define the inverse trigonometric functions using the formulae $\cos(iz) = \cosh(z)$ and $\sin(iz) = i\sinh(z)$.

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For example, if $z = \cos^{-1}(w)$, then $w = \cos(z) = \cosh(iz)$, and so $iz = \cosh^{-1}(w)$.

Exercise 5

What are the ranges of $\sinh^{-1}$, $\cos^{-1}$, and $\sin^{-1}$? Where are the branch cuts for $\cos^{-1}$ and $\sin^{-1}$?