MATH2621 — Higher Complex Analysis. XI
Exponential and related functions

This lecture?

In this lecture, we define exponential, hyperbolic and trigonometric functions using power series.

The exponential function

Definition

We define the exponential series by the formula

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \forall z \in \mathbb{C}.$$

This is the only possible power series extension of the real exponential into the whole complex plane. Indeed, we saw in the previous lecture that a complex power series $\sum_{n=0}^{\infty} a_n z^n$ is determined by its values on any real interval $(-\varepsilon, \varepsilon)$.

Properties of the exponential series

Theorem

1. $\exp(0) = 1$;
2. $\exp(z + w) = \exp(z)\exp(w)$ *for all* $z, w \in \mathbb{C}$;
3. $\exp(-z) = \exp(z)^{-1}$ *for all* $z \in \mathbb{C}$;
4. $\exp(z) \neq 0$ *for all* $z \in \mathbb{C}$;
5. $\exp'(z) = \exp(z)$ *for all* $z \in \mathbb{C}$;
6. *if a function* $f : \mathbb{C} \to \mathbb{C}$ *satisfies* $f(0) = 1$ *and* $f'(z) = f(z)$ *for all* $z \in \mathbb{C}$, *then* $f(z) = \exp(z)$ *for all* $z \in \mathbb{C}$;
7. $\exp(x + iy) = e^x(\cos(y) + i\sin(y))$ *for all* $x, y \in \mathbb{R}$.

Proof of properties

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$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(z+w)^n}{n!} &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}z^j w^{n-j}\\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{z^j}{j!}\,\frac{w^{n-j}}{(n-j)!}\\ &= \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{z^j}{j!}\,\frac{w^m}{m!}.\end{aligned}$$

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$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{z^j}{j!} \frac{w^{n-j}}{(n-j)!}$$
$$= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{z^j}{j!} \frac{w^m}{m!}.$$

From items (2) and (1), $\exp(z)\exp(-z) = \exp(z+(-z)) = 1$, and so $\exp(z)^{-1} = \exp(-z)$, proving item (3).

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From items (2) and (1), $\exp(z)\exp(-z) = \exp(z + (-z)) = 1$, and so $\exp(z)^{-1} = \exp(-z)$, proving item (3).

Since $\exp(z)$ is invertible, it cannot be 0, proving item (4).

Proof of exponential series properties. 2

Item (5) follows from the calculus of power series.

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Now consider the derivative of the quotient function $f\,/\,\exp$:

$$\begin{aligned}\left(\frac{f}{\exp}\right)'(z) &= \frac{f'(z)\exp(z) - f(z)\exp'(z)}{\exp^2(z)} \\ &= \frac{f(z)\exp(z) - f(z)\exp(z)}{\exp^2(z)} \\ &= 0.\end{aligned}$$

Thus $f\,/\,\exp$ is constant; when $z = 0$ its value is 1, so it is identically 1, and $f(z) = \exp(z)$ for all $z \in \mathbb{C}$, proving item (6).

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Finally, we prove (7). From item (2), $\exp(z) = \exp(x)\exp(iy)$; using results for real power series, we see that

$$\exp(x) = e^x \quad \text{and} \quad \exp(iy) = \cos(y) + i\sin(y). \qquad \square$$

Periodicity of the exponential mapping

Corollary

The exponential exp *maps* $\mathbb{C}$ *onto* $\mathbb{C} \setminus \{0\}$*, and* $\exp(z_1) = \exp(z_2)$ *if and only if* $z_1 - z_2 \in 2\pi i\mathbb{Z}$*.*

Proof. From the theorem, $\exp(z) \neq 0$ for all $z \in \mathbb{C}$. If $w \neq 0$, then take $z = \ln(|w|) + i \operatorname{Arg}(w)$; clearly, $\exp(z) = w$. Hence exp maps $\mathbb{C}$ *onto* $\mathbb{C} \setminus \{0\}$. If $\exp(z_1) = \exp(z_2)$, then $\exp(z_1 - z_2) = 1$, and so $\operatorname{Re}(z_1 - z_2) = 0$ and $\operatorname{Im}(z_1 - z_2) \in 2\pi\mathbb{Z}$, by trigonometry. $\square$

The fact that $\exp(z) = \exp(z + 2\pi ik)$ for all $k \in \mathbb{Z}$ is called the *periodicity* of exp. Often, we write e^z rather than $\exp(z)$.

The hyperbolic functions

Definition

We define the complex hyperbolic cosine and sine by the formulae

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2} = \sum_{n \in \mathbb{N}} \frac{z^{2n}}{(2n)!}$$

and

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2} = \sum_{n \in \mathbb{N}} \frac{z^{2n+1}}{(2n+1)!}$$

for all $z \in \mathbb{C}$.

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These are the only power series extensions of the real functions cosh and sinh into the whole complex plane.

Properties of the hyperbolic sine and cosine

Theorem

$$\begin{array}{rlrl}
\text{(i)} & \cosh(-z) = \cosh(z) & \text{(ii)} & \sinh(-z) = -\sinh(z) \\
\text{(iii)} & \cosh'(z) = \sinh(z) & \text{(iv)} & \sinh'(z) = \cosh(z) \\
\text{(v)} & \cosh(z + 2\pi ik) = \cosh(z) & \text{(vi)} & \sinh(z + 2\pi ik) = \sinh(z)
\end{array}$$

$$\begin{array}{rl}
\text{(vii)} & \cosh(z + w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w) \\
\text{(viii)} & \sinh(z + w) = \sinh(z)\cosh(w) + \cosh(z)\sinh(w) \\
\text{(ix)} & \cosh^2(z) - \sinh^2(z) = 1 \\
\text{(x)} & \cosh(x + iy) = \cosh(x)\cos(y) + i\sinh(x)\sin(y) \\
\text{(xi)} & \sinh(x + iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y).
\end{array}$$

for all $w, z \in \mathbb{C}$, *all* $k \in \mathbb{Z}$ *and all* $x, y \in \mathbb{R}$,

Proof of properties

Proof. Items (i) and (ii) follow from the definition. Items (iii) and (iv) follow from the definition and the calculus of power series. Items (v) and (vi), often called the "periodicity of cosh and sinh", follow from the periodicity of exp.

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To prove item (vii), observe that

$$\begin{aligned}
&4(\cosh(z)\cosh(w) + \sinh(z)\sinh(w))\\
&= (e^z + e^{-z})(e^w + e^{-w}) + (e^z - e^{-z})(e^w - e^{-w})\\
&= 2(e^{z+w} + e^{-z-w}) = 4\cosh(z+w),
\end{aligned}$$

and divide by 4. Items (viii) and (ix) may be proved similarly.

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\end{aligned}$$

and divide by 4. Items (viii) and (ix) may be proved similarly.

Items (x) and (xi) follow from the identifications $\sinh(iy) = i\sin(y)$ and $\cosh(iy) = \cos(y)$, which may be seen using power series.
$\square$

Exercise 1

Expand $\cosh(z + i\pi/2)$ and $\sinh(z + i\pi/2)$ in terms of $\cosh(z)$ and $\sinh(z)$. Hence find $\cosh(z + i\pi)$.

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Answer. From the theorem,

$$\begin{aligned}
\cosh(z + i\pi/2) &= \cosh(z)\cosh(i\pi/2) + \sinh(z)\sinh(i\pi/2) \\
&= \cosh(z)\cos(\pi/2) + i\sinh(z)\sin(\pi/2) \\
&= i\sinh(z).
\end{aligned}$$

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Similarly, $\sinh(z + i\pi/2) = i\cosh(z)$.

Hence

$$\cosh(z+i\pi) = i\sinh(z+i\pi/2) = i^2\cosh(z) = -\cosh(z). \quad \triangle$$

The trigonometric functions

Definition

We define the complex cosine and sine by the formulae

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} = \sum_{n\in\mathbb{N}} \frac{(-1)^n\, z^{2n}}{(2n)!}$$

and

$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i} = \sum_{n\in\mathbb{N}} \frac{(-1)^n\, z^{2n+1}}{(2n+1)!}$$

for all $z \in \mathbb{C}$.

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for all $z \in \mathbb{C}$.

These are the only possible power series extensions of the real functions cos and sin into the whole complex plane.

Exercise 2

Show that the complex sine and cosine have the following properties:

$$
\begin{array}{rlrl}
\text{(i)} & \cos(-z) = \cos(z) & \text{(ii)} & \sin(-z) = -\sin(z) \\
\text{(iii)} & \cos'(z) = -\sin(z) & \text{(iv)} & \sin'(z) = \cos(z) \\
\text{(v)} & \cos(z + 2\pi k) = \cos(z) & \text{(vi)} & \sin(z + 2\pi k) = \sin(z)
\end{array}
$$

for all $z \in \mathbb{C}$ and all $k \in \mathbb{Z}$. Show also that

$$\text{(vii)} \quad \cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$

$$\text{(viii)} \quad \sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

$$\text{(ix)} \quad \cos^2(z) + \sin^2(z) = 1$$

for all $w, z \in \mathbb{C}$.

Exercise 3

Express the following as functions of z:

$$\text{(a)} \quad \sinh(iz), \quad \text{(b)} \quad \cosh(iz), \quad \text{(c)} \quad \sin(iz), \quad \text{(d)} \quad \cos(iz).$$

Answer.

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$$\text{(a)}\quad \sinh(iz), \quad \text{(b)}\quad \cosh(iz), \quad \text{(c)}\quad \sin(iz), \quad \text{(d)}\quad \cos(iz).$$

Answer. We do (a), using power series:

$$\begin{aligned} \sinh(iz) &= \frac{iz}{1!} + \frac{(iz)^3}{3!} + \frac{(iz)^5}{5!} + \frac{(iz)^7}{7!} + \dots \\ &= i\frac{z}{1!} - i\frac{z^3}{3!} + i\frac{z^5}{5!} - i\frac{z^7}{7!} + \dots \\ &= i\sin(z). \end{aligned}$$

The others are similar. $\triangle$

Exercise 4

Find the images of the lines $\mathrm{Re}(z) = c$ and $\mathrm{Im}(z) = d$ under the function cosh, and sketch these for various values of c and d.

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Answer. As usual, we write $w = \cosh(z)$ and split w and z into their real and imaginary parts: $w = u + iv$ and $z = x + iy$.

Exercise 4

Find the images of the lines $\mathrm{Re}(z) = c$ and $\mathrm{Im}(z) = d$ under the function cosh, and sketch these for various values of c and d.

Answer. As usual, we write $w = \cosh(z)$ and split w and z into their real and imaginary parts: $w = u + iv$ and $z = x + iy$.

If $\mathrm{Re}(z) = c$, then $z = c + iy$, for some $y \in \mathbb{R}$, and

$$\begin{aligned} w = \cosh(z) &= \cosh(c + iy) \\ &= \cosh(c)\cosh(iy) + \sinh(c)\sinh(iy) \\ &= \cosh(c)\cos(y) + i\sinh(c)\sin(y), \end{aligned}$$

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so $u = \cosh(c)\cos(y)$ while $v = \sinh(c)\sin(y)$.

Answer to Exercise 4

We can eliminate y from these two formulae, and obtain

$$\frac{u^2}{\cosh^2(c)} + \frac{v^2}{\sinh^2(c)} = \cos^2(y) + \sin^2(y) = 1.$$

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This curve is an ellipse, unless $c = 0$. The periodicity of cosh means that each time y changes by 2π, the point w moves once around the ellipse.

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This curve is an ellipse, unless $c = 0$. The periodicity of cosh means that each time y changes by 2π, the point w moves once around the ellipse.

In the exceptional case $c = 0$, we see that $w = \pm\cos(y)$, so w varies on the real axis between -1 and 1.

Answer to Exercise 4 (continued)

If $\mathrm{Im}(z) = d$, then $z = x + id$, for some $x \in \mathbb{R}$, and

$$\begin{aligned} w &= \cosh(z) = \cosh(x + id) \\ &= \cosh(x)\cosh(id) + \sinh(x)\sinh(id) \\ &= \cosh(x)\cos(d) + i\sinh(x)\sin(d), \end{aligned}$$

so $u = \cosh(x)\cos(d)$ while $v = \sinh(x)\sin(d)$.

Answer to Exercise 4 (continued)

We can eliminate x from these two formulae, and obtain

$$\frac{u^2}{\cos^2(d)} - \frac{v^2}{\sin^2(d)} = \cosh^2(x) - \sinh^2(x) = 1.$$

This curve is a hyperbola, unless $\cos(d) = 0$ or $\sin(d) = 0$.

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The periodicity of cosh means that horizontal lines $y = d$ and $y = d + 2k\pi$ are mapped to the same curve.

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If $\sin(d) = 0$, then $w = \pm\cosh(x)$, and so w varies on the real axis between $-\infty$ and -1, or between 1 and $+\infty$.

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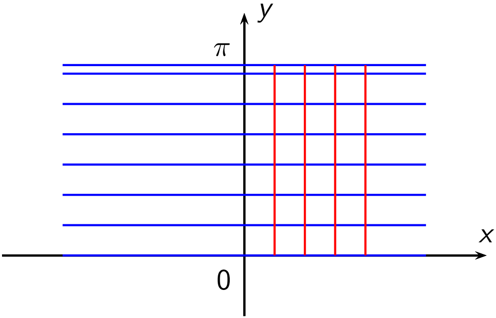
If $\sin(d) = 0$, then $w = \pm\cosh(x)$, and so w varies on the real axis between $-\infty$ and -1, or between 1 and $+\infty$.

If $\cos(d) = 0$, then $w = \pm i \sinh(x)$, and so w varies along the imaginary axis.

Answer to Exercise 4 (continued)



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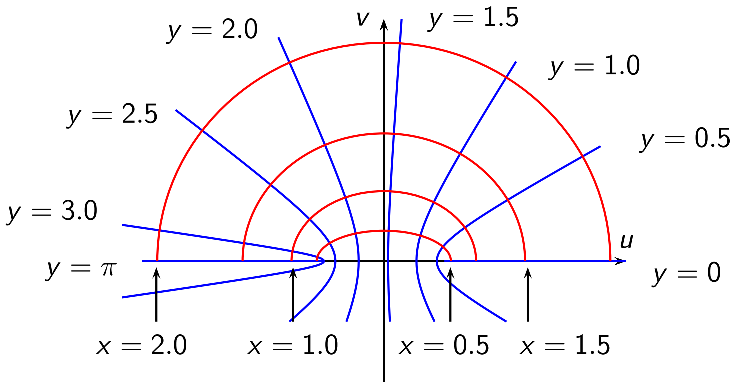

The blue curves fill the whole space. The red curves are the images of line segments where $x \geq 0$ and $0 < y < \pi$. We could complete the red ellipses by including segments of the lines $x = c$ where c is negative and $0 < y < \pi$, or by taking the line segments $x \geq 0$ and $-\pi < y < \pi$. $\triangle$

Exercise 5

Find the images of the lines $\mathrm{Re}(z) = c$ and $\mathrm{Im}(z) = d$ under the functions sinh, sin and cos, and sketch these for various values of c and d.

Exercise 6

Suppose that $u(x, y) = \cos(x)\sinh(y)$ for all $x, y \in \mathbb{R}$. Show that u is harmonic and find its harmonic conjugate.

Answer.

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Answer. First,

$$\begin{aligned}\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} &= \frac{\partial^2 \cos(x)}{\partial x^2}\sinh(y) + \cos(x)\frac{\partial^2 \sinh(y)}{\partial y^2} \\ &= -\cos(x)\sinh(y) + \cos(x)\sinh(y) \\ &= 0.\end{aligned}$$

Thus u is harmonic.

Answer to Exercise 6

The harmonic conjugate v of u must satisfy:

$$\frac{\partial v(x,y)}{\partial x} = -\frac{\partial u(x,y)}{\partial y} = -\cos(x)\cosh(y)$$

and

$$\frac{\partial v(x,y)}{\partial y} = \frac{\partial u(x,y)}{\partial x} = -\sin(x)\sinh(y).$$

It follows that

$$v(x,y) = -\sin(x)\cosh(y) + c(y),$$

where $c(y)$ depends only on y, and then

$$-\sin(x)\sinh(y) + c'(y) = \frac{\partial v(x,y)}{\partial y} = -\sin(x)\sinh(y),$$

so $c'(y) = 0$, that is, $c(y)$ is constant.

Answer to Exercise 6 (continued)

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As a check, observe that

$$\begin{aligned} u(x, y) + iv(x, y) &= \cos(x)\sinh(y) - i\sin(x)\cosh(y) + iC \\ &= \cosh(ix)\sinh(y) + \sinh(-ix)\cosh(y) + iC \\ &= \sinh(y - ix) + iC \\ &= \sinh(-iz) + iC \\ &= -i\sin(z) + iC, \end{aligned}$$

where $z = x + iy$. Since this is a differentiable function of z, the function v that we found must be correct. $\triangle$

Endnotes

The important formulae are:

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$$\begin{aligned} \sinh(iy) &= i\sin(y) \qquad & \cosh(iy) &= \cos(y) \\ \sin(iy) &= i\sinh(y) \qquad & \cos(iy) &= \cosh(y). \end{aligned}$$

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It may help to write series as $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$.