MATH2621 — Higher Complex Analysis. I
Inequalities and sets of complex numbers

# Information

Lecturers:

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1. Complex functions
2. Differentiation
3. Integration
4. Applications

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1. Complex functions
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Are we all in the right place? . . . . Harder exercises, theory and all that . . .

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- We look at some equalities and inequalities.
- We discuss different types of regions.
- We consider some examples.

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- We discuss different types of regions.
- We consider some examples.

This will let us talk about domains and ranges of complex functions.

# An equality

**Lemma**

*For all complex numbers $w$ and $z$,*

$$|w+z|^2 = |w|^2 + 2\,\mathrm{Re}(w\bar{z}) + |z|^2\,.$$

**Proof.**

# An equality

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$$|w+z|^2 = |w|^2 + 2\,\mathrm{Re}(w\bar{z}) + |z|^2\,.$$

Proof. Observe that

$$\begin{aligned}
|w+z|^2 &= (w+z)(\bar{w}+\bar{z}) \\
&= w\bar{w} + w\bar{z} + z\bar{w} + z\bar{z} \\
&= |w|^2 + w\bar{z} + (w\bar{z})^{-} + |z|^2 \\
&= |w|^2 + 2\,\mathrm{Re}(w\bar{z}) + |z|^2.
\end{aligned}$$

□

# An equality. 2

We have just seen that

$$|w+z|^2 = |w|^2 + 2\,\mathrm{Re}(w\bar{z}) + |z|^2\,.$$

For vectors,

$$\begin{aligned} \|w+z\|^2 &= \langle w+z, w+z\rangle \\ &= \langle w,w\rangle + \langle w,z\rangle + \langle z,w\rangle + \langle z,z\rangle \\ &= \|w\|^2 + 2\,\langle w,z\rangle + \|z\|^2\,. \end{aligned}$$

Thus $\mathrm{Re}(w\bar{z})$ corresponds to the inner product of the vectors corresponding to the complex numbers $w$ and $z$.

# The triangle inequality

The triangle inequality states:

$$|w + z| \leq |w| + |z| \qquad \forall w, z \in \mathbb{C}.$$

# The circle inequality

## Lemma

*For all complex numbers $w$ and $z$,*

$$\big||w| - |z|\big| \le |w - z|.$$

Proof. Observe that $w = (w - z) + z$, so $|w| \le |w - z| + |z|$ by the triangle inequality, and hence

$$|w| - |z| \le |w - z|.$$

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Interchanging the roles of $w$ and $z$, and recalling that $|w - z| = |z - w|$, we see that

$$|z| - |w| \leq |z - w| = |w - z|.$$

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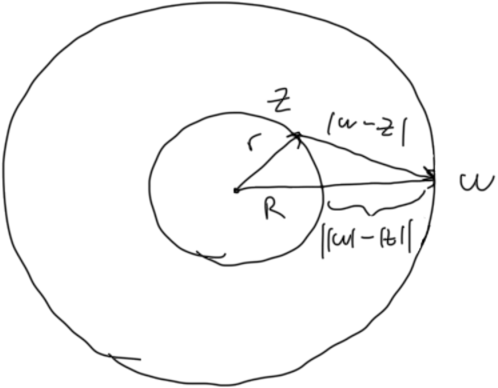
Combining these inequalities, we deduce that

$$\big||w| - |z|\big| = \max\{|w| - |z|, |z| - |w|\} \leq |w - z|. \qquad \square$$

# Alternatively . . .

We may also consider points on circles of radii $r$ and $R$, and compare the distance between the points with the difference of the radii.



# Complex exponentials

If $z = x + iy$, then we define

$$e^z = e^x(\cos y + i \sin y).$$

Then $e^w e^z = e^{w+z}$ for all complex numbers $w$ and $z$.

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**Lemma**

*If $z \in \mathbb{C}$, then*

$$|e^z| = e^{\mathrm{Re}(z)}.$$

*Proof.* See Exercise Sheet. $\square$

# Another inequality

**Lemma**
*For all real numbers $\theta$,*

$$\left| e^{i\theta} - 1 \right| \leq |\theta|.$$

**Proof.** See Exercise Sheet. $\square$

# Open balls

## Definition

The *open ball* with centre $z_0$ and radius $\varepsilon$, written $B(z_0, \varepsilon)$, is the set $\{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$. Sometimes these sets are called discs rather than balls.

The *punctured open ball* with centre $z_1$ and radius $\varepsilon$, written $B^\circ(z_1, \varepsilon)$, is the set $\{z \in \mathbb{C} : 0 < |z - z_1| < \varepsilon\}$.

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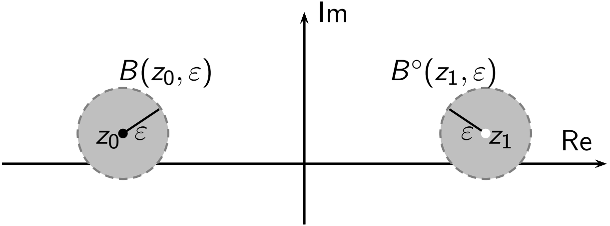



The ball $B(z_0, \varepsilon)$ and the punctured ball $B^\circ(z_1, \varepsilon)$

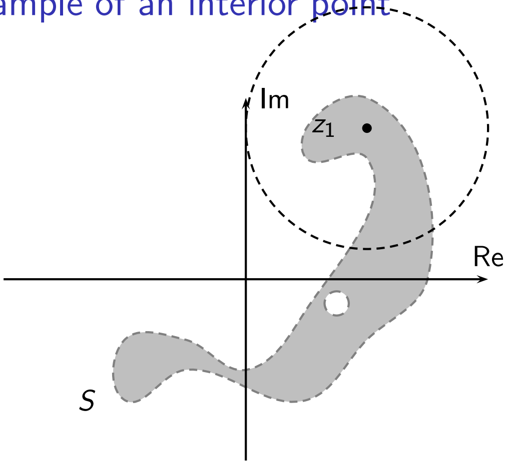
# Types of points

## Definition

Suppose that $S \subseteq \mathbb{C}$. For any point $z_0$ in $\mathbb{C}$, there are three mutually exclusive and exhaustive possibilities.

1. Provided that $\varepsilon$ is a sufficiently small positive real number, $B(z_0, \varepsilon)$ is a subset of $S$, that is, $B(z_0, \varepsilon) \cap S = B(z_0, \varepsilon)$. In this case, $z_0$ is an *interior point* of $S$.
2. Provided that $\varepsilon$ is a sufficiently small positive real number, $B(z_0, \varepsilon)$ does not meet $S$, that is, $B(z_0, \varepsilon) \cap S = \emptyset$. In this case, $z_0$ is an *exterior point* of $S$.
3. No matter how small the positive real number $\varepsilon$ is, neither of the above holds, that is, $\emptyset \subset B(z_0, \varepsilon) \cap S \subset B(z_0, \varepsilon)$. In this case, $z_0$ is a *boundary point* of $S$.

# Example of an interior point




If the radius of the ball centred at $z_1$ is small enough, then the ball lies inside the set $S$, and $B(z_1, \varepsilon) \cap S = B(z_1, \varepsilon)$. Thus $z_1$ is an interior point.

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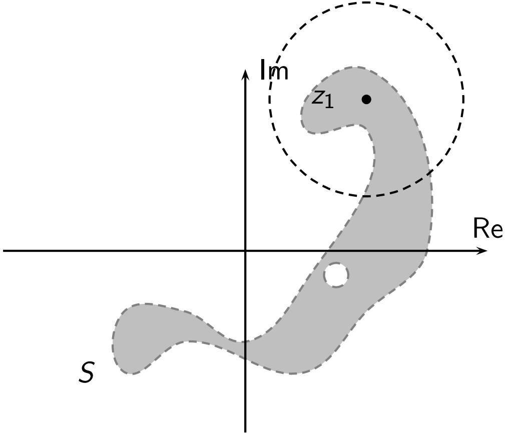


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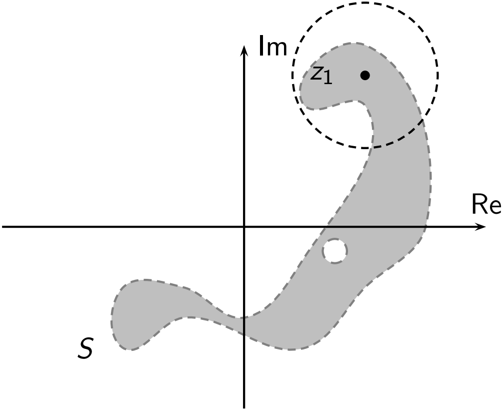


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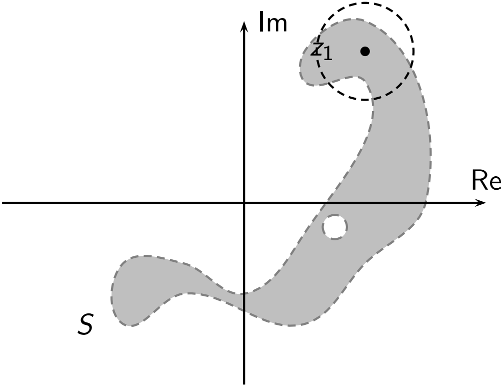


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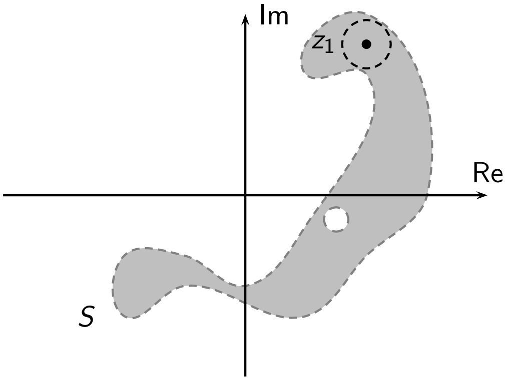


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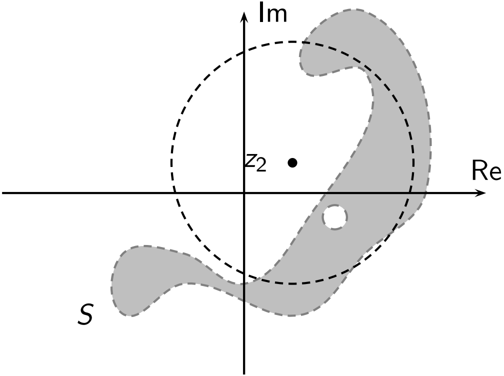
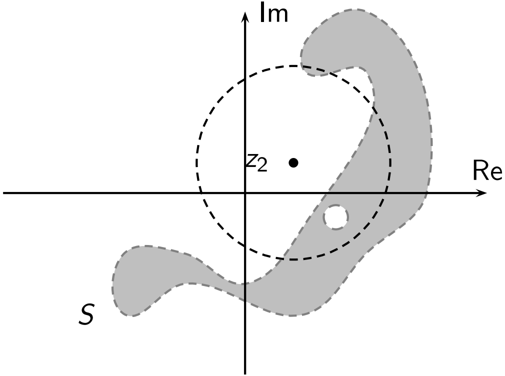
# Example of an exterior point




If the radius of the ball centred at $z_2$ is small enough, then the ball lies outside the set $S$, and $B(z_2, \varepsilon) \cap S$ is empty. Thus $z_2$ is an exterior point.

# Example of an exterior point

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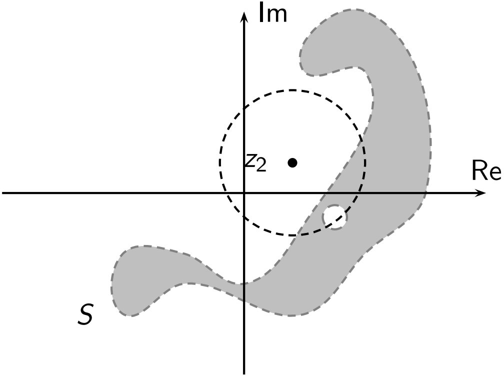


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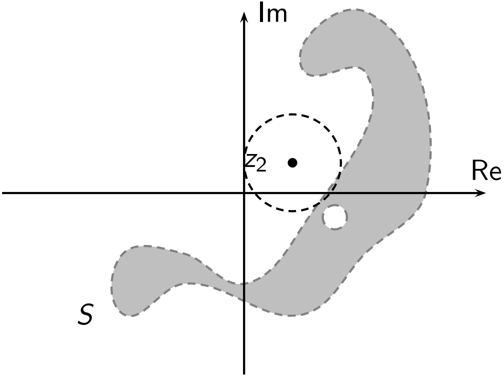


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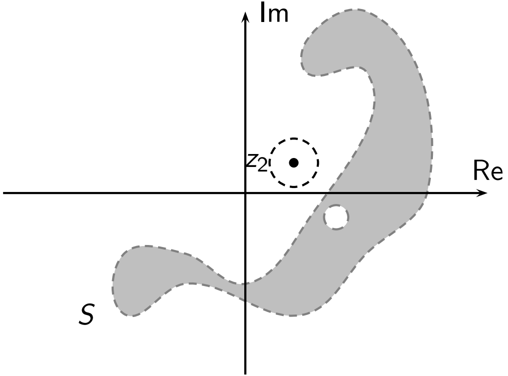


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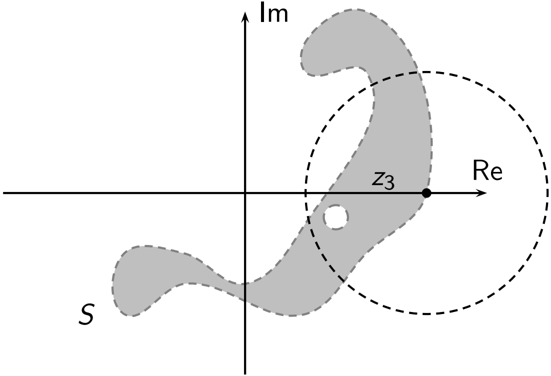
# Example of a boundary point




No matter how small the radius of the ball centred at $z_3$ is, part of the ball lies inside $S$, and part lies outside $S$, and $B(z_3, \varepsilon) \cap S$ is neither empty nor all of $B(z_3, \varepsilon)$. Thus $z_3$ is a boundary point.

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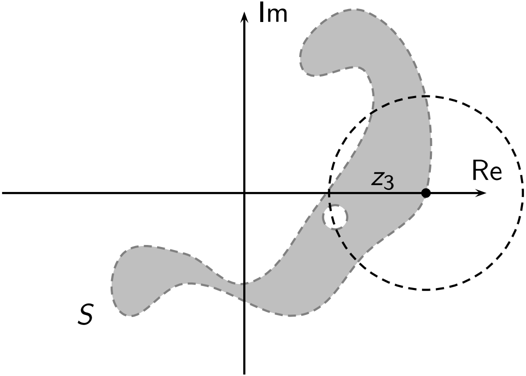


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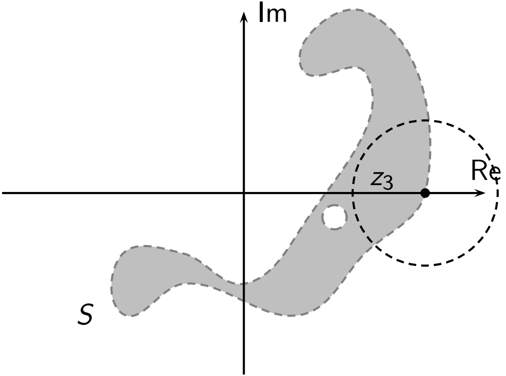


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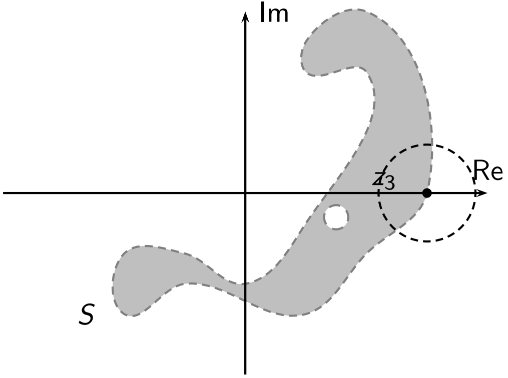


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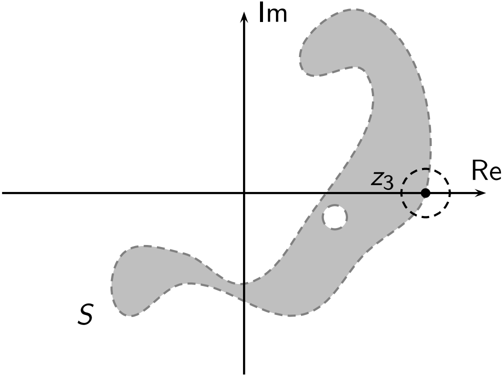


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# Types of sets

line segment (closed, not open)
infinite line (closed, not open)

## Definition

Suppose that $S \subseteq \mathbb{C}$.

1. The set $S$ is *open* if all its points are interior points.
2. The set $S$ is *closed* if it contains all of its boundary points, or equivalently, if its complement $\mathbb{C} \setminus S$ is open.
3. The *closure* of the set $S$ is the set consisting of the points of $S$ together with the boundary points of $S$. $c \in \mathbb{C}$
4. The set is *bounded* if $S \subseteq B(0, R)$ for some $R \in \mathbb{R}^+$. $S \subseteq B(c, R)$
5. The set $S$ is *compact* if it is both closed and bounded.
6. The set $S$ is a *region* if it is an open set together with none, some, or all of its boundary points.

For example, the dashed boundary lines of the set $S$ we considered before indicate that it does not contain any boundary points. Consequently, $S$ is open.

# Comments

Note that open and closed are neither mutually exclusive nor exhaustive: there are sets that are open and closed, such as the whole plane, and sets that are neither open nor closed, such as

$$\{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) > 0\}.$$

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In complex analysis, we focus on open sets, which we often write using the greek letter $\Omega$ (omega).

# Arcs

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A *simple closed polygonal arc* is a polygonal arc that does not cross itself, but the final point of the last segment is the initial point of the first segment.

The complement of a simple closed polygonal arc is made up of two pieces: one, the *interior of the arc*, is bounded, and the other, the *exterior*, is not.

# Polygonally path-connectedness

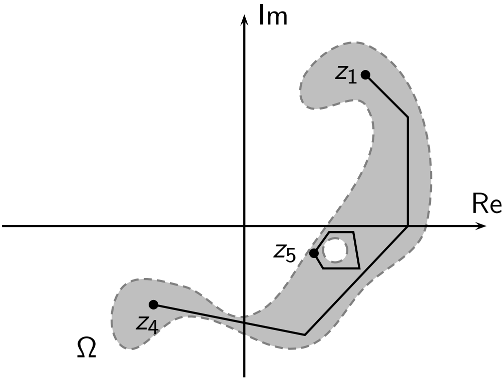
## Definition

Let $X \subseteq \mathbb{C}$ be a subset of the complex plane.

1. The set $X$ is *polygonally path-connected* if any two points of $X$ can be joined by a polygonal arc lying inside $X$.
2. The set $X$ is *simply polygonally connected* if it is polygonally path-connected and if the interior of every simple closed polygonal arc in $X$ lies in $X$, that is, if "$X$ has no holes".
3. The set $X$ is a *domain* if it is open and polygonally path-connected. doesn't apply to codom. $\mathbb{R}$ is not a domain in $\mathbb{C}$

2. Any loop you draw in $X$ has interior entirely inside $X$

# Example




The set $\Omega$ above is polygonally path-connected, because any two points in $\Omega$ (such as $z_1$ and $z_4$) can be joined by a polygonal arc. However, $\Omega$ is not simply polygonally connected, because part of the interior of the closed arc shown going through $z_5$ is not in $\Omega$.

# Exercise 1

Suppose that $a, b, c, d \in \mathbb{C}$. Show that the set

$$\{z \in \mathbb{C} : |az + b| = |cz + d|\}$$

may be empty, a point, a line, a circle, or the whole complex plane, and all these possibilities occur for suitable values of $a, b, c, d$.

Answer.

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Answer. If $|az + b| = |cz + d|$, then $|az + b|^2 = |cz + d|^2$, so $(az + b)(az + b)^{-} = (cz + d)(cz + d)^{-}$, whence

$$(a\bar{a} - c\bar{c})z\bar{z} + (a\bar{b} - c\bar{d})z + (\bar{a}b - \bar{c}d)\bar{z} + (c\bar{c} - d\bar{d}) = 0.$$

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We may rewrite this in the form $e|z|^2 + fz + \bar{f}\bar{z} + g = 0$, where $e$ and $g$ are real, while $f$ may be complex.

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We may rewrite this in the form $e|z|^2 + fz + \bar{f}\bar{z} + g = 0$, where $e$ and $g$ are real, while $f$ may be complex. Now we pass to Cartesian coordinates:

$$ex^2 + ey^2 + (f + \bar{f})x + i(f - \bar{f})y + g = 0;$$

# Answer to Exercise 1

$$ex^2 + ey^2 + (f + \bar{f})x + i(f - \bar{f})y + g = 0;$$

all the coefficients are now real since $f + \bar{f} = 2\,\mathrm{Re}\,f$ while $i(f - \bar{f}) = -2\,\mathrm{Im}\,f$.

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The desired result follows from coordinate geometry. For instance, if $e \neq 0$ and $eg - |f|^2 < 0$, then we have a circle with centre $-\bar{f}/e$, while if $e = 0$ and $f \neq 0$, then we have a straight line.

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The other types of solutions arise when many of the coefficients are 0. $\triangle$

# Exercise 2

Sketch the set $\{z \in \mathbb{C} : |z - 3 - 2i| < 4,\ \mathrm{Re}(z) > 0\}$ in the complex plane. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region, or a domain?

Answer.

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Answer. If $|z - 3 - 2i| < 4$, then $z$ lies inside the circle with centre $3 + 2i$ and radius 4.

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Answer. If $|z - 3 - 2i| < 4$, then $z$ lies inside the circle with centre $3 + 2i$ and radius 4. Further, if $\mathrm{Re}(z) > 0$, then $z$ lies to the right of the imaginary axis.

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Answer. If $|z - 3 - 2i| < 4$, then $z$ lies inside the circle with centre $3 + 2i$ and radius 4. Further, if $\operatorname{Re}(z) > 0$, then $z$ lies to the right of the imaginary axis. The desired set is the region where both inequalities hold.

# Exercise 2

Sketch the set $\{z \in \mathbb{C} : |z - 3 - 2i| < 4,\ \mathrm{Re}(z) > 0\}$ in the complex plane. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region, or a domain?

Answer. If $|z - 3 - 2i| < 4$, then $z$ lies inside the circle with centre $3 + 2i$ and radius 4. Further, if $\mathrm{Re}(z) > 0$, then $z$ lies to the right of the imaginary axis. The desired set is the region where both inequalities hold.

The set does not include points on the circumference of the circle, nor does it include the points on the imaginary axis that also lie inside the circle.

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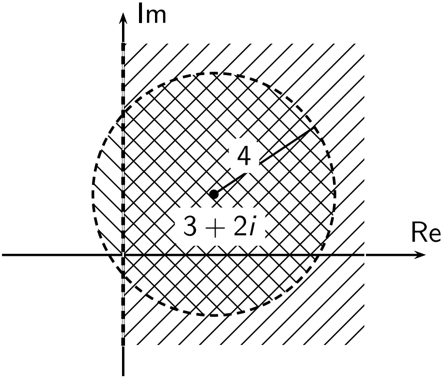
The set does not include points on the circumference of the circle, nor does it include the points on the imaginary axis that also lie inside the circle.

The set is open, polygonally path-connected, simply polygonally connected and bounded, and hence it is also a domain and a region. It is not closed, so not compact.

# Sketch




$\triangle$

# Exercise 3

Sketch the set $\{z \in \mathbb{C} : 3 \leq |z - 3 - 2i| \leq 4\}$ in the complex plane. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region or a domain?

Answer.

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Sketch the set $\{z \in \mathbb{C} : 3 \le |z - 3 - 2i| \le 4\}$ in the complex plane. Is it open, closed, bounded, compact, polygonally path-connected, simply polygonally connected, a region or a domain?

Answer. The set includes points inside or on the circle with centre $3 + 2i$ and radius 4, but outside or on the circle with centre $3 + 2i$ and radius 3.

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The set $\{z \in \mathbb{C} : 3 < |z - 3 - 2i| < 4\}$ is open, because all points are interior points. The set we are considering is this set together will *all* its boundary points, so it is closed, and also a region.

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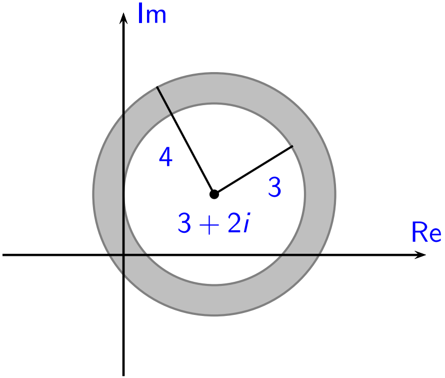
The set $\{z \in \mathbb{C} : 3 < |z - 3 - 2i| < 4\}$ is open, because all points are interior points. The set we are considering is this set together will *all* its boundary points, so it is closed, and also a region. It is not open, because some points are boundary points, and hence not a domain. It is closed and bounded, hence also compact. It is polygonally path-connected, but not simply polygonally connected, because there are points in the complement of the set inside a polygonal arc around the annulus.

# Answer to Exercise 3




$\triangle$

# Exercise 4

Sketch the set $\{z \in \mathbb{C} : |z + i| + |z - i| = 4\}$ in $\mathbb{C}$. Is it open or bounded? Describe the set $\{z \in \mathbb{C} : |z + i| + |z - i| < 4\}$.

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$$\begin{aligned} |z+i| + |z-i| &= 4 \\ |z+i| &= 4 - |z-i| \end{aligned}$$

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|z+i| + |z-i| &= 4 \\
|z+i| &= 4 - |z-i| \\
|z+i|^2 &= 16 - 8|z-i| + |z-i|^2
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8|z-i| &= 16 - 4y \\
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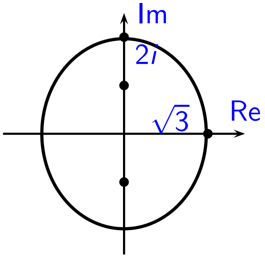
4\left(x^2 + (y-1)^2\right) &= 16 - 8y + y^2 \\
\frac{x^2}{3} + \frac{y^2}{4} &= 1.
\end{aligned}$$

# Answer to Exercise 4

This is the circumference of an ellipse, which is bounded and closed but not open.



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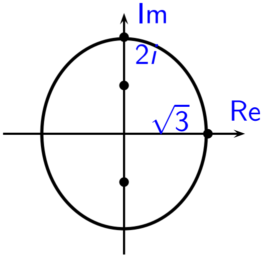


The ellipse is the set of points for which the sum of the distances from $i$ and $-i$ is exactly 4. The sets

$$\{z \in \mathbb{C} : |z+i| + |z-i| > 4\} \quad \text{and} \quad \{z \in \mathbb{C} : |z+i| + |z-i| < 4\}$$

are the exterior and interior of the ellipse. △