MATH2621 — Higher Complex Analysis. VII
Connections with calculus in the plane†

This lecture?

In this lecture, we link limits, continuity and differentiability of complex functions with their analogues for functions of two real variables.

[P]

The definitions and properties of limits and continuity for functions of two real variables are essentially the same as limits for functions of one complex variable. To save space, we represent vectors in $\mathbb{R}^2$ as row vectors rather than column vectors for the moment.

Limits

Definition

Suppose that u is a real-valued function of two real variables, and that $(x_0, y_0) \in \text{Domain}(u)^{-}$. We say that $u(x, y)$ tends to ℓ as (x, y) tends to (x_0, y_0), or that ℓ is the limit of $u(x, y)$ as (x, y) tends to (x_0, y_0), and we write $u(x, y) \to \ell$ as $(x, y) \to (x_0, y_0)$, or

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = \ell,$$

if, for every $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|u(x, y) - \ell| < \varepsilon$ provided that $(x, y) \in \text{Domain}(u)$ and $0 < |(x, y) - (x_0, y_0)| < \delta$.
[P]
The same definition applies to a vector-valued function F, provided that we interpret $|F(x, y) - \ell|$ as a vector length.

Connection for limits

Recall that we associate to a complex function f the real-valued functions u and v of two real variables by the formula

$$f(x+iy) = u(x,y) + iv(x,y).$$

We will also associate to f an $\mathbb{R}^2$-valued function F of two real variables as follows:

$$F(x,y) = (u(x,y), v(x,y)).$$

Connection

Let functions f, u, v and F be related as above. Then the following are equivalent:

(1) $f(z) \to \ell$ *as* $z \to z_0$;

(2) $u(x,y) \to \operatorname{Re}\ell$ *and* $v(x,y) \to \operatorname{Im}\ell$ *as* $(x,y) \to (x_0,y_0)$;

(3) $F(x,y) \to (\operatorname{Re}\ell, \operatorname{Im}\ell)$ *as* $(x,y) \to (x_0,y_0)$.

Consequences

This means that to most theorems about limits of real functions such as u and v, there is a corresponding theorem about complex-valued functions f, and vice versa.

[P] For example, a theorem about vector-valued functions states that a vector-valued function tends to a limit ℓ if and only if each component of the function tends to the corresponding component of ℓ. This is the analogue of the theorem that a complex-valued function tends to a complex limit if and only if the real and imaginary parts of the function tend to the real and imaginary parts of the limit.

[P]

By the way, vector-valued functions are often called vector fields.

Continuity

Continuity for functions of two real variables is defined much as for functions of a complex variable.
[P]

Definition

Suppose that u is a real-valued function of two real variables, and that $(x_0, y_0) \in \text{Domain}(u)$. We say that u is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u(x_0, y_0).$$

We say that u is continuous if it is continuous at all points of $\text{Domain}(u)$.
[P]
The same definition applies to vector-valued functions.

Connection for continuity

Recall that we write

$$f(x+iy) = u(x,y) + iv(x,y).$$

and

$$F(x,y) = (u(x,y), v(x,y)).$$

[P]

Connection

Let functions f, u, v and F be related as above. Then the following are equivalent:

(1) *f is continuous;*

(2) *u and v are continuous;*

(3) *F is continuous.*

Differentiability

For functions of two real variables, we cannot define the derivative as for functions of a real variable, because this would involve dividing by a vector. But we note that an equivalent definition of the derivative of a function of one (real or complex) variable is that f is differentiable at z_0 and $f'(z_0) = D$ if

$$\lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - Dh|}{|h|} = 0.$$

To see this, note that

$$\begin{aligned}\lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - Dh|}{|h|} &= \lim_{h \to 0} \left| \frac{f(z_0 + h) - f(z_0) - Dh}{h} \right| \\ &= \lim_{h \to 0} \left| \frac{f(z_0 + h) - f(z_0)}{h} - D \right|.\end{aligned}$$

We define the derivative by extending this modified definition.

Definition of differentiability

Definition

A real- or vector-valued function u of two real variables is differentiable at (x_0, y_0), and its derivative is the linear transformation D, if

$$\lim_{(h,k)\to(0,0)} \frac{|u((x_0, y_0) + (h, k)) - u(x_0, y_0) - D(h, k)|}{|(h, k)|} = 0,$$

or equivalently, if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|u(x, y) - u(x_0, y_0) - D(x - x_0, y - y_0)|}{|(x - x_0, y - y_0)|} = 0.$$

The linear transformation D sends $\mathbb{R}^2$ to $\mathbb{R}$ if u is real-valued, and sends $\mathbb{R}^2$ to $\mathbb{R}^2$ if u is $\mathbb{R}^2$-valued.

A property of differentiability

It turns out that a vector-valued function is differentiable if and only if each of its components is differentiable, so that much of the theory can be developed for real-valued functions and then extended to vector-valued functions component by component. [P]

One of the first steps in the development of calculus in several real variables is the identification of the linear transformation D.

What is the derivative

If u is real-valued and differentiable, then the partial derivatives exist and

$$\begin{aligned} D(h,k) &= \frac{\partial u}{\partial x}(x_0,y_0)h + \frac{\partial u}{\partial y}(x_0,y_0)k \\ &= \left(\frac{\partial u}{\partial x}(x_0,y_0), \frac{\partial u}{\partial y}(x_0,y_0)\right) \cdot (h,k) \\ &= \left(\frac{\partial u}{\partial x}(x_0,y_0) \quad \frac{\partial u}{\partial y}(x_0,y_0))\right) \begin{pmatrix} h \\ k \end{pmatrix}, \end{aligned}$$

where the second line is a dot product of vectors, and the third is a product of two matrices.

[P]

The plane $z = u(x_0,y_0) + D(x - x_0, y - y_0)$ in $\mathbb{R}^3$ is the tangent plane to the surface $z = u(x,y)$ in $\mathbb{R}^3$ at $(x_0, y_0, u(x_0,y_0))$.

A Theorem

To avoid having to deal with limits all the time, most treatments of multivariable calculus prove the next result as soon as possible. For ease of notation, we state it for a real-valued function, but it also holds for vector-valued functions.
[P]

Theorem

Suppose that u is a real-valued function of two real variables, and that the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ exist and are continuous in an open set Ω. Then u is differentiable in Ω.

Proof of Theorem. 1

Proof. Take $(x_0, y_0) \in \Omega$. To show that u is differentiable at (x_0, y_0), we need to make the quotient

$$\frac{|u(x,y) - u(x_0,y_0) - D(x-x_0, y-y_0)|}{|(x-x_0, y-y_0)|}$$

small, by taking (x, y) close to (x_0, y_0). To make this quantitative, we take $\varepsilon \in \mathbb{R}^+$, and make the quotient less than ε. Recall that

$$D(x-x_0, y-y_0) = \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0).$$

To understand what comes next, look at the numerator of the quotient, and then go to page 3 of the proof first.

Proof of Theorem. 2

Choose δ such that $B((x_0, y_0), \delta) \subset \Omega$ and

$$\left|\frac{\partial u}{\partial x}(x_1, y_1) - \frac{\partial u}{\partial x}(x_0, y_0)\right| < \frac{\varepsilon}{2}$$

and

$$\left|\frac{\partial u}{\partial y}(x_1, y_1) - \frac{\partial u}{\partial y}(x_0, y_0)\right| < \frac{\varepsilon}{2}$$

whenever $(x_1, y_1) \in B((x_0, y_0), \delta)$. This is possible because Ω is open and because both the partial derivatives are continuous at (x_0, y_0). If $(x, y) \in B((x_0, y_0), \delta)$, then the line segments joining (x_0, y_0) to (x, y_0) and joining (x, y_0) to (x, y) lie in $B((x_0, y_0, \delta)$, by the geometry of balls in $\mathbb{R}^2$.

Proof of Theorem. 3

The fundamental theorem of calculus and the mean value theorem for integrals implies that there exist y_1 between y and y_0 and x_1 between x and x_0 such that

$$\begin{aligned} u(x,y) - u(x_0,y_0) &= u(x,y) - u(x,y_0) + u(x,y_0) - u(x_0,y_0) \\ &= \int_{y_0}^{y} \frac{\partial u}{\partial y}(x,t)\,dt + \int_{x_0}^{x} \frac{\partial u}{\partial x}(s,y_0)\,ds \\ &= \frac{\partial u}{\partial y}(x,y_1)(y-y_0) + \frac{\partial u}{\partial x}(x_1,y_0)(x-x_0) \end{aligned}$$

while

$$D(x-x_0, y-y_0) = \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0)$$

and this in turn implies that ...

Proof of Theorem. 4

$$
\begin{aligned}
&\left|u(x,y) - u(x_0,y_0) - D(x-x_0, y-y_0)\right| \\
&= \left|\frac{\partial u}{\partial y}(x,y_1)(y-y_0) + \frac{\partial u}{\partial x}(x_1,y_0)(x-x_0)\right. \\
&\qquad \left. - \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0) - \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0)\right| \\
&\le \left|\frac{\partial u}{\partial y}(x,y_1) - \frac{\partial u}{\partial y}(x_0,y_0)\right| |y-y_0| \\
&\qquad + \left|\frac{\partial u}{\partial x}(x_1,y_0) - \frac{\partial u}{\partial x}(x_0,y_0)\right| |x-x_0| \\
&< \frac{\varepsilon}{2}|y-y_0| + \frac{\varepsilon}{2}|x-x_0|,
\end{aligned}
$$

...

Proof of Theorem. 5

whence

$$\frac{|u(x,y)-u(x_0,y_0)-D(x-x_0,y-y_0)|}{|(x-x_0,y-y_0)|} < \frac{\frac{\varepsilon}{2}|y-y_0|+\frac{\varepsilon}{2}|x-x_0|}{|(x-x_0,y-y_0)|}$$
$$\leq \varepsilon,$$

as required. □

Another theorem

Here is another theorem about differentiation of functions of two real variables; we omit the proof.

Theorem

Suppose that u is a twice continuously differentiable real-valued function of two real variables. Then

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}\,.$$

Conservative vector fields and potentials

A consequence of this theorem is that the "derivative" of u, that is, the vector-valued function $(\partial u/\partial x, \partial u/\partial y)$, satisfies the condition that the derivative with respect to y of the first component is equal to the derivative with respect to x of the second component.
[P]
In general, a vector-valued function (p, q) is said to be *closed* or *conservative* if $\partial p/\partial y = \partial q/\partial x$; and we have have just seen that a derivative is conservative.
[P]
An important question in multivariable calculus is whether every conservative vector-valued function (p, q) is a derivative of a real-valued function called the *potential*; whether this is always true or not depends on whether the domain of definition of the function (p, q) is simply connected or not.
[P]
We will return to this later.

Differentiability in $\mathbb{R}^2$ and complex differentiability

Suppose that f is a complex function. As before, we associate to f the functions u, v and F:

$$\begin{aligned} f(x+iy) &= u(x,y) + iv(x,y) \\ F(x,y) &= (u(x,y), v(x,y)). \end{aligned}$$

[P] Then f is differentiable at z_0 with derivative $f'(z_0)$ if and only if

$$\lim_{h+ik\to 0} \frac{|f(z_0+h+ik) - f(z_0) - f'(z_0)(h+ik)|}{|h+ik|} = 0.$$

[P] And F is differentiable at (x_0, y_0) with derivative $F'(x_0, y_0)$ if and only if

$$\lim_{(h,k)\to 0} \frac{|F((x_0,y_0)+(h,k)) - F(x_0,y_0) - F'(x_0,y_0)(h,k)|}{|h+ik|} = 0.$$

A comparison. 1

Now $F'(x_0, y_0)$ is a linear transformation that may be identified with multiplication by the matrix

$$\begin{pmatrix} \dfrac{\partial u}{\partial x}(x_0, y_0) & \dfrac{\partial u}{\partial y}(x_0, y_0) \\[2ex] \dfrac{\partial v}{\partial x}(x_0, y_0) & \dfrac{\partial v}{\partial y}(x_0, y_0) \end{pmatrix}.$$

[P] Multiplication by $f'(z_0)$ may be identified with multiplication by the matrix

$$\begin{pmatrix} b & -c \\ c & b \end{pmatrix},$$

where $b = \operatorname{Re} f'(z_0)$ and $c = \operatorname{Im} f'(z_0)$.

A comparison. 2

These two operations correspond provided that

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0),$$

and then

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

[P]
These conditions are exactly the Cauchy–Riemann equations.

Connection for differentiation

In conclusion, we have established the following link between complex functions and $\mathbb{R}^2$-valued functions of two real variables.

Connection

A complex function f is complex differentiable at z_0 if and only if the associated vector-valued function of two real variables F is differentiable at (x_0, y_0) and the derivative of F corresponds to multiplication by a complex number.

[P]

A consequence of this connection and the theorem that continuous partial derivatives imply differentiability is that if the partial derivatives of u and v are continuous in an open set, then f is differentiable where the Cauchy–Riemann equations hold.

Another consequence of differentiability

If a function f is complex differentiable at z_0, then the derivative of the corresponding real function F is the matrix that corresponds to multiplying by the complex number $f'(z_0)$. Multiplication by the complex number c corresponds to scalar multiplication by the modulus $|c|$, which preserves angles, and then rotating through $\operatorname{Arg} c$, which also preserves angles.
[P]
It can be shown that each linear transformation of $\mathbb{R}^2$ that preserve angles is the composition of a scalar multiplication and a rotation.

End notes

Later we will see the noteworthy result that functions that are complex differentiable in open sets in $\mathbb{C}$ are infinitely differentiable. This contrasts with what happens in the theory of real functions, where a function can be differentiable k times but not $k+1$ times. Thus a small difference in definition can lead to very large differences in behaviour.