MATH2621 — Higher Complex Analysis. XXX
The Laplace transformation

Today?

In this lecture, we introduce the Laplace transformation $\mathcal{L}$. The first step is to define the class of functions on which $\mathcal{L}$ will act.

Then we define $\mathcal{L}$ and give some of its properties.

We also look at some examples.

Functions of exponential type

Definition

Suppose that $A \in \mathbb{R}$. A function $f : [0, \infty) \to \mathbb{C}$ is said to be of *exponential type A* if there exists a constant C such that

$$|f(t)| \leq C\, e^{At} \qquad \forall t \in [0, \infty).$$

We will say that a function $f : [0, \infty) \to \mathbb{C}$ is of *exponential type $A+$* if it is exponential type $A + \varepsilon$ for all $\varepsilon \in \mathbb{R}^+$.

Exercise 1

In this exercise, $a, b \in \mathbb{C}$ and p denotes a polynomial. Show that:

(a) $t \mapsto a\, e^{bt}$ is of exponential type $\mathrm{Re}(b)$;

(b) $t \mapsto p(t)\, e^{bt}$ is of exponential type $\mathrm{Re}(b)+$;

(c) $t \mapsto e^{t^2}$ is *not* of exponential type A for *any* $A \in \mathbb{R}$.

Answer. [P] For part (a), we observe that

$$\left|e^{bt}\right| = e^{\mathrm{Re}(bt)} = e^{\mathrm{Re}(b)t} \qquad \forall t \in [0, +\infty);$$

we then take $C = |a|$ in the definition of exponential type.

Answer to Exercise 1

For (b), observe first that if $\varepsilon > 0$, then $t \mapsto p(t)\, e^{-\varepsilon t}$ is continuous on $[0, +\infty)$ and

$$\lim_{t \to +\infty} p(t)\, e^{-\varepsilon t} = \lim_{t \to +\infty} \frac{p(t)}{e^{\varepsilon t}} = 0$$

from l'Hôpital's rule. This implies that there is a constant C such that

$$\left| \frac{p(t)}{e^{\varepsilon t}} \right| \leq C \qquad \forall t \in [0, +\infty),$$

that is,

$$|p(t)| \leq C\, e^{\varepsilon t} \qquad \forall t \in [0, +\infty),$$

and so

$$\left| p(t)\, e^{bt} \right| \leq C\, e^{\varepsilon t} \left| e^{bt} \right| = C\, e^{(\mathrm{Re}(b) + \varepsilon) t} \qquad \forall t \in [0, +\infty).$$

Answer to Exercise 1

Finally, for part (c), suppose with a view to a contradiction that there is a real number A such that

$$e^{t^2} \leq C\, e^{At} \qquad \forall t \in [0, +\infty).$$

Then taking logarithms, we see that

$$t^2 \leq \ln(C) + At \qquad \forall t \in [0, +\infty),$$

which is impossible, and so A cannot exist. $\triangle$

[P]
Note that if f is of exponential type A, then it is of exponential type $A+$ and also of exponential type B for all $B \in [A, \infty)$. For computing exponential type, the following lemma may be useful.

A lemma

Lemma

Suppose that the functions $f, g : [0, +\infty) \to \mathbb{C}$ *are of exponential types* A *and* B *respectively. Then the functions* $af + bg$ *and* afg *are of exponential type* $\max\{A, B\}$ *and* $A + B$ *respectively, for all* $a, b \in \mathbb{C}$*. Similarly, if the functions* f *and* g *are of exponential types* $A+$ *and* $B+$*, then the functions* $af + bg$ *and* afg *are of exponential types* $\max\{A, B\}+$ *and* $(A + B)+$*.*

Proof.

We leave this as an exercise. $\square$

Locally integrable functions and half-planes

Definition

A function $f : [0, +\infty) \to \mathbb{C}$ is said to be *locally integrable* if it is Riemann integrable on all intervals $[0, R]$ where $R \in \mathbb{R}^+$.

[P]

We denote the half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > A\}$ by H_A.

The Laplace transform

Definition

Suppose that $f : [0, \infty) \to \mathbb{C}$ is locally integrable and of exponential type A. The *Laplace transform* $\mathcal{L}f : H_A \to \mathbb{C}$ of f is the function given by

$$\mathcal{L}f(z) = \int_0^\infty f(t)\, e^{-zt}\, dt = \lim_{R\to\infty} \int_0^R f(t)\, e^{-zt}\, dt.$$

The integral in the definition converges by comparison with $\int_0^\infty C\, e^{(A-\mathrm{Re}(z))t}\, dt$.
[P]
Notice that $\mathcal{L}$ is linear: if f and g are locally integrable and of exponential types A and B, and $a, b \in \mathbb{C}$, then $af + bg$ is also locally integrable and is of exponential type C, where $C = \max\{A, B\}$. Moreover, if $\mathrm{Re}(z) > C$, then

$$\mathcal{L}(af + bg)(z) = a\mathcal{L}f(z) + b\mathcal{L}g(z).$$

Exercise 2

Find the Laplace transforms of the functions
(a) $t \mapsto a$ (b) $t \mapsto p(t)$
(c) $t \mapsto e^{at}$ (d) $t \mapsto p(t)\, e^{at}$.

Answer. [P] Parts (a) to (c) are all particular cases of part (d), and so we solve (d) only. Suppose that $p(t) = \sum_{k=0}^{K} c_k\, t^k$.

Answer to Exercise 2

If $f : [0, \infty) \to \mathbb{R}$ is given by $f(t) = p(t)\, e^{at}$, then f is locally integrable and of exponential type $\mathrm{Re}(a)+$. Hence the Laplace transform of f is defined in the half-plane $H_{\mathrm{Re}(a)}$ and, when $z \in H_{\mathrm{Re}(a)}$,

$$\begin{aligned}\mathcal{L}f(z) &= \int_0^\infty p(t)\, e^{at}\, e^{-zt}\, dt = \lim_{R\to\infty} \int_0^R \sum_{k=0}^K c_k\, t^k\, e^{(a-z)t}\, dt \\ &= \sum_{k=0}^K c_k \lim_{R\to\infty} \int_0^R t^k\, e^{(a-z)t}\, dt = \sum_{k=0}^K c_k \frac{k!}{(z-a)^{k+1}}\,;\end{aligned}$$

we computed the integral by parts and induction on k. $\triangle$

Laplace transforms are holomorphic

Theorem

If $f : [0, \infty) \to \mathbb{C}$ is locally integrable and of exponential type $A+$, then $\mathcal{L}f$ is holomorphic on H_A. Further,

$$\frac{d}{dz}\mathcal{L}f(z) = \mathcal{L}g(z) \qquad \forall z \in H_A,$$

where $g : [0, +\infty) \to \mathbb{C}$ is given by $g(t) = -t\, f(t)$.

Proof. We give the proof at the end of this lecture. $\square$

Properties of the Laplace transformation

Proposition

Suppose that $f : [0, \infty) \to \mathbb{C}$ *is locally integrable and of exponential type* $A+$. *Then*

(a) *if* $a \in \mathbb{C}$ *and* $g(t) = e^{-at} f(t)$ *for all* $t \in [0, +\infty)$, *then*

$$\mathcal{L}g(z) = \mathcal{L}f(z + a) \qquad \forall z \in H_{A-\mathrm{Re}(a)};$$

(b) *if* $a \in \mathbb{R}^+$ *and* $g(t) = f(t/a)$ *for all* $t \in [0, +\infty)$, *then*

$$\mathcal{L}g(z) = a\mathcal{L}f(az) \qquad \forall z \in H_{A/a};$$

(c) *if* $g(t) = tf(t)$ *for all* $t \in [0, +\infty)$, *then*

$$\mathcal{L}g(z) = -\frac{d}{dz}\mathcal{L}f(z) \qquad \forall z \in H_A;$$

Properties of the Laplace transformation. 2

(d) if f is differentiable and f' is also of exponential type $A+$, then

$$\mathcal{L}(f')(z) = z\mathcal{L}f(z) - f(0) \qquad \forall z \in H_A.$$

Proof. To prove (a), observe that

$$\begin{aligned}\mathcal{L}g(z) &= \int_0^\infty g(s)\, e^{-zs}\, ds = \int_0^\infty e^{-as}\, f(s)\, e^{-zs}\, ds \\ &= \int_0^\infty f(s)\, e^{-(z+a)s}\, ds = \mathcal{L}f(z+a).\end{aligned}$$

Properties of the Laplace transformation. 3

To prove (b), we make the change of variables $s = at$:

$$\begin{aligned} \mathcal{L}g(z) &= \int_0^\infty g(s)\, e^{-zs}\ ds = \int_0^\infty f(s/a)\, e^{-zs}\ ds \\ &= \int_0^\infty f(t)\, e^{-azt}\, a\, dt = a\mathcal{L}f(az). \end{aligned}$$

[P]
We prove something very close to (c) to show that $\mathcal{L}f$ is holomorphic, so we omit this.

Properties of the Laplace transformation. 4

Finally, integration by parts shows that [P]

$$\begin{aligned}\mathcal{L}g(z) &= \int_0^\infty g(s)\, e^{-zs}\, ds = \int_0^\infty f'(s)\, e^{-zs}\, ds \\ &= [f(s)\, e^{-zs}]_0^\infty - \int_0^\infty f(s)\,(-z)\, e^{-zs}\, ds \\ &= -f(0) + z\mathcal{L}f(z),\end{aligned}$$

and (d) is proved. □

In the integration by parts, we really ought to deal with integrals over $[0, R]$ and then let R tend to $+\infty$.
[P]
Remembering these properties can save time; for example, by part (a), since the Laplace transform of t^3 is $6/z^4$, the Laplace transform of $t^3\, e^{-2t}$ is just $6/(z+2)^4$.

The inversion formulae for the Laplace transform

Theorem

If $f : [0, \infty) \to \mathbb{C}$ is continuous and of exponential type $A+$, then

$$f(t) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_\lambda \mathcal{L}f(z)\, e^{tz}\, dz \qquad \forall t \in \mathbb{R}^+, \tag{1}$$

where λ is the line segment from $\sigma - iR$ to $\sigma + iR$ and $\sigma \in (A, \infty)$. Suppose further that $\mathcal{L}f$ extends to a holomorphic function on $\mathbb{C} \setminus \{a_1, \ldots, a_n\}$, and that there are positive constants M and k such that

$$|\mathcal{L}f(z)| \le M\, |z|^{-k}$$

whenever $|z|$ is sufficiently large. Then for any $t \in \mathbb{R}^+$,

$$f(t) = \sum_{j=1}^{n} \operatorname{Res}(\mathcal{L}f(z)\, e^{zt}; z = a_j) \qquad \forall t \in \mathbb{R}^+. \tag{2}$$

Sketch proof of the inversion theorem

The first inversion formula follows from the inversion formula for the *Fourier* transformation. If we write $z = \sigma + iy$, then

$$\mathcal{L}f(z) = \int_0^\infty f(t)\, e^{-(\sigma+iy)t}\, dt = \int_{-\infty}^\infty g_\sigma(t)\, e^{-iyt}\, dt = \widehat{g}_\sigma(y),$$

where $g_\sigma(t) = f(t)\, e^{-\sigma t}$ if $t \geq 0$ and $g_\sigma(t) = 0$ if $t < 0$. By the inversion formula for the Fourier transform, if $t > 0$ then

$$\begin{aligned} f(t)\, e^{-\sigma t} &= \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{L}f(\sigma + iy)\, e^{iyt}\, dy \\ &= e^{-\sigma t} \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\sigma-iR}^{\sigma+iR} \mathcal{L}f(z)\, e^{zt}\, dz. \end{aligned}$$

Cancelling the factor of $e^{-\sigma t}$ now leads to formula (1).

Sketch proof of the inversion theorem. 2

For formula (2), we consider the semicircular contour $\gamma + \lambda$, where $\gamma(\theta) = \sigma + R\, e^{i\theta}$ and $\pi/2 \leq \theta \leq 3\pi/2$. The result will follow from the Cauchy Residue Theorem once we show that

$$\lim_{R\to\infty} \int_{\gamma} g(z)\, e^{tz}\ dz = 0.$$

This may be done using a Jordan's lemma-type argument. $\square$

Remarks

Strictly speaking, this is *not* a proof, because we proved the Fourier inversion formula for a more limited class of functions. However, with a little more work the Fourier inversion formula may be proved for more general functions, justifying the above argument. [P]

The key fact about the Laplace transformation is that it is *invertible*. Thus to find the inverse Laplace transform of g, it suffices to find a continuous function $f : [0, +\infty) \to \mathbb{C}$ (of exponential type) such that $\mathcal{L}f = g$; then f is the desired inverse transform.

Exercise 3

Find the continuous function $f : [0, \infty) \to \mathbb{C}$ of exponential type $1+$ for which $\mathcal{L}f(z) = \dfrac{1}{(z-1)^2}$.

Answer. [P] We have seen that the Laplace transform of $t \mapsto t$ is $z \mapsto \dfrac{1}{z^2}$. Hence the Laplace transform of $t \mapsto t\, e^t$ is $\dfrac{1}{(z-1)^2}$. $\triangle$

Exercise 4

Find the continuous function $f : [0, \infty) \to \mathbb{C}$ of exponential type $1+$ such that $\mathcal{L}f(z) = \dfrac{1}{z^2 - 1}$.

Answer. [P] We rewrite $\dfrac{1}{z^2-1}$ in partial fractions:

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right).$$

By the properties of the Laplace transformation, this is the Laplace transform of

$$\frac{1}{2}\left(e^{t} - e^{-t}\right) = \sinh t,$$

and $f(t) = \sinh(t)$. $\triangle$

[P]
This also follows using the second inversion formula.

Proof that Laplace transforms are holomorphic

For convenience, we restate the result that we are going to prove.

Theorem

If $f : [0, \infty) \to \mathbb{C}$ is locally integrable and of exponential type $A+$, then $\mathcal{L}f$ is holomorphic on H_A. Further,

$$\frac{d}{dz}\mathcal{L}f(z) = \mathcal{L}g(z) \qquad \forall z \in H_A,$$

where $g : [0, +\infty) \to \mathbb{C}$ is given by $g(t) = -t\,f(t)$.

Proof. We do this by applying a theorem that states that we may exchange the order of differentiation with respect to a parameter and integration when several conditions are satisfied.

A useful theorem

Theorem

Suppose that $G : \mathbb{R} \times P \to \mathbb{R}$ *and* $t_0 \in P$*, where* $P \subseteq \mathbb{R}$*, and that the function* $s \mapsto G(s, t_0) \in L^1(\mathbb{R})$*. Suppose also that there exists* $\mu \in \mathbb{R}^+$ *such that*

(a) $(s, t) \mapsto \frac{\partial}{\partial t} G(s, t)$ *is continuous on* $\mathbb{R} \times B(t_0, \mu)$*, and*

(b) *there is a nonnegative function* $M \in L^1(\mathbb{R})$ *such that*

$$\left| \frac{\partial}{\partial t} G(s, t) \right| \leq M(s) \qquad \forall s \in \mathbb{R} \quad \forall t \in B(t_0, \mu).$$

Then

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} G(s, t)\, ds \Big|_{t=t_0} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} G(s, t) \Big|_{t=t_0} ds. \tag{3}$$

Proof that Laplace transforms are holomorphic. 2

By definition,

$$\mathcal{L}f(z) = \int_0^\infty f(s)\, e^{-zs}\; ds.$$

Consider the function $G : [0, +\infty) \times H_A \to \mathbb{C}$ given by $G(s, z) = f(s)\, e^{-zs}$. Observe that

$$\frac{dG}{dz}(s, z) = -s\, f(s)\, e^{-zs},$$

which is a continuous function on $[0, +\infty) \times \mathbb{C}$. Hence the continuity condition of the theorem is satisfied.

Proof that Laplace transforms are holomorphic. 3

Take $z \in H_A$, so that $\mathrm{Re}(z) > A$. Set $\varepsilon = (\mathrm{Re}(z) - A)/2$; then $\mathrm{Re}(z) - A = 2\varepsilon > \varepsilon$, that is, $z \in H_{A+\varepsilon}$. Since f is of exponential type $A+$, there exists C such that

$$|f(s)| \leq C\, e^{(A+\varepsilon/2)s} \qquad \text{and} \qquad |s\, f(s)| \leq C\, e^{(A+\varepsilon/2)s}$$

for all $s \in [0, \infty)$. If $z \in H_{A+\varepsilon}$, then

$$\begin{aligned} |G(s,z)| &\leq |f(s)|\, \left|e^{-zs}\right| = |f(s)|\, e^{-\mathrm{Re}(z)s} \\ &\leq C\, e^{(A+\varepsilon/2)s}\, e^{-(A+\varepsilon)s} = C\, e^{-\varepsilon s/2} \end{aligned}$$

and similarly

$$\left|\frac{d\,G}{dz}(s,z)\right| \leq C\, e^{-\varepsilon s/2},$$

and the domination conditions of the theorem are also satisfied.

Proof that Laplace transforms are holomorphic. 3

Thus we may differentiate under the integral and obtain

$$\begin{aligned}\frac{d}{dz}\mathcal{L}f(z) &= \int_0^{+\infty} \frac{d}{dz}\, f(s)\, e^{-zs}\ ds = \int_0^{+\infty} f(s)(-s)\, e^{-zs}\ ds \\ &= \mathcal{L}(-tf(t))\end{aligned}$$

for all $z \in H_{A+\varepsilon}$. $\square$

[P]
Note that, in the proof above, we wrote dG/dz; this denotes the complex derivative in the z variable. This is also a partial derivative, in the sense that it does not involve the s variable. However, we do not write $\partial G/\partial z$; this has a different meaning.