School of Mathematics and Statistics

MATH2111 - Higher Several Variable Calculus

2. Elementary Analysis

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2025 Term 1

Functions from $\mathbb{R}^n \to \mathbb{R}^m$

- Paths/curves: $\mathbf{c} : I \to \mathbb{R}^n$; trajectory of a rocket.
- Surfaces: graph of $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ or level set of $F : \mathbb{R}^3 \to \mathbb{R}$.
- Functions $f : D \to \mathbb{R}$ are known as **scalar fields**.

  Functions $\mathbf{F} : D \to \mathbb{R}^n$ with $n \geqslant 2$ are known as **vector fields**.

  The components of a vector field are scalar fields.
- Temperature $T : \mathbb{R}^3 \to \mathbb{R}$ is a scalar field, and wind $\mathbf{W} : \mathbb{R}^3 \to \mathbb{R}^2$ is a vector field.

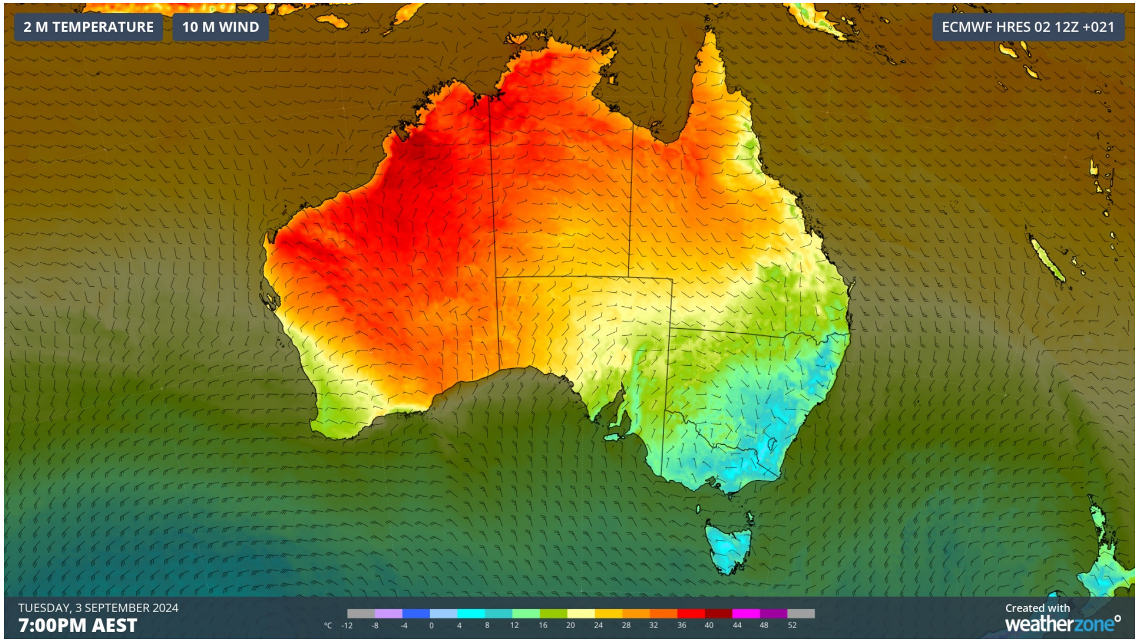


Figure: A weather map of Australia from Weatherzone showing temperature and wind.

Functions from $\mathbb{R}^n \to \mathbb{R}^m$

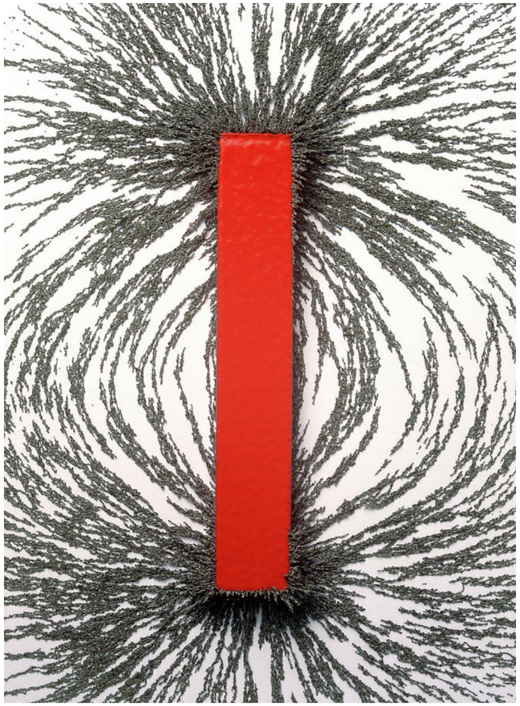


Figure: Magnetic field around a bar magnet

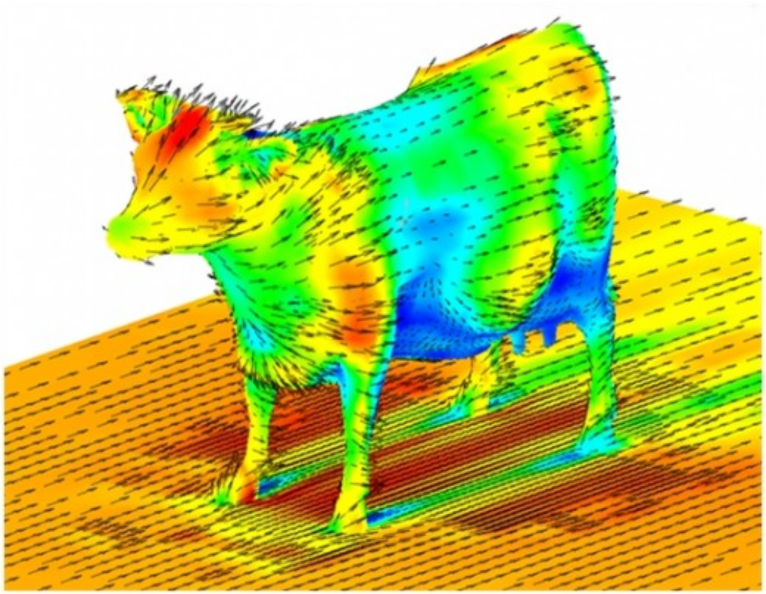


Figure: The aerodynamics of a cow

Recall...

How do we formally say that temperature and wind are continuous functions? How do we say they vary smoothly? We need to generalise our notion of limits.

In first-year mathematics, we defined what it meant for:

- sequences of numbers to converge;
- functions from $\mathbb{R}$ to $\mathbb{R}$ to be continuous or differentiable.

A sequence of real numbers $\{a_k\}_{k=1}^{\infty}$ converges to $L \in \mathbb{R}$ if and only if

$$\forall \varepsilon > 0 \qquad \exists K \in \mathbb{Z}^{+} \qquad \forall k > K \qquad |a_k - L| < \varepsilon.$$

Here, the absolute value measures the distance between a_k and L. To generalise this to sequences of points in $\mathbb{R}^n$, we will need a way to measure distance between points in $\mathbb{R}^n$.

Metrics

Definition

A **metric on** $\mathbb{R}^n$ is a function $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ such that

1. d is positive definite. That is, $d(\mathbf{x}, \mathbf{y}) \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$ and $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.
2. d is symmetric. That is, $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
3. d satisfies the triangle inequality. That is, $d(\mathbf{x}, \mathbf{y}) \leqslant d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.

Canonical Example: The **Euclidean distance on** $\mathbb{R}^n$ defined by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

is a metric. The only tricky axiom to check is the triangle inequality. (Board Tutorial).

If $n = 1$, then $d(x, y) = |x - y|$ is the standard distance on $\mathbb{R}$.

In future lectures, we will primarily use the Euclidean distance.

Examples of Metrics

- The 'taxicab' or 'Manhattan' distance is defined by

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|.$$

- For $p \geqslant 1$, the "p-metric" is defined by $d_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}$.

- For $p = \infty$, the Chebyshev distance is defined by

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \leqslant i \leqslant n} |x_i - y_i| = \lim_{p \to \infty} d_p(\mathbf{x}, \mathbf{y}).$$

- The discrete metric is defined by $d(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{x} \neq \mathbf{y}, \\ 0 & \text{if } \mathbf{x} = \mathbf{y}. \end{cases}$

- Given a metric d, we can construct another metric δ by

$$\delta(\mathbf{x}, \mathbf{y}) = \frac{d(\mathbf{x}, \mathbf{y})}{1 + d(\mathbf{x}, \mathbf{y})}.$$

Convergence in $\mathbb{R}^n$

Definition

the limit

Suppose that d is a metric on $\mathbb{R}^n$. A sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ is said to be **convergent** (with respect to d) if there exists $\mathbf{x} \in \mathbb{R}^n$ such that for all $\varepsilon > 0$, there exists $K \in \mathbb{Z}^+$ such that if $k > K$, then $d(\mathbf{x}_k, \mathbf{x}) < \varepsilon$.

$$\exists \mathbf{x} \in \mathbb{R}^n \qquad \forall \varepsilon > 0 \qquad \exists K \in \mathbb{Z}^+ \qquad \forall k > K \qquad d(\mathbf{x}_k, \mathbf{x}) < \varepsilon.$$

We may also say that the sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ **converges to** $\mathbf{x}$ (with respect to d), written $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$, and its **limit is** $\mathbf{x}$.

It is also common to write $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$ or, when the metric is important, $\mathbf{x}_k \xrightarrow{d} \mathbf{x}$ as $k \to \infty$.

If a sequences converges, its limit is unique!

Often we are lazy and won't refer to the metric when discussing convergence when the metric is obvious or clear from context. However, it is sometimes important to specify the metric since sequences may converge with respect to one metric but not another.

Ex. If d is the discrete metric, can you characterise the convergent sequences?

sequences that are eventually constant

Two Player Game Analogy

To prove that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}$ from the definition, you can think of playing a game against your lecturer Alan Stoneham.

1. Alan chooses some distance $\varepsilon > 0$.
   (You have no control over the value of ε.)
2. You now choose $K \in \mathbb{Z}^+$.
   (You have control over the value of K and it can depend on ε).
3. Alan now picks a $k > K$.
   (Again, you have no control over the value of k other than it is larger than K.)
4. If $d(\mathbf{x}, \mathbf{x}_k) < \varepsilon$ no matter Alan's choices, you WIN and have proven that

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}.$$

Convergence Example

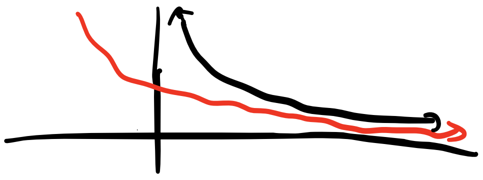
Exercise. Let $d = d_2$ and $\mathbf{x}_k = (2 - \frac{1}{k}, e^{-k})$ for $k \in \mathbb{Z}^+$. Show that $\mathbf{x}_k \to (2, 0)$ as $k \to \infty$.

Euclidean metric in $\mathbb{R}^2$

let $\varepsilon > 0$ choose $K = \lceil \frac{\sqrt{2}}{\varepsilon} \rceil$ s.t. if $k > K$

$$d_2(x_k, (2,0)) = \sqrt{\frac{1}{k^2} + e^{-2k}} < \sqrt{\frac{1}{k^2} + \frac{1}{k^2}} = \frac{\sqrt{2}}{k}$$

$$< \frac{\sqrt{2}}{\left(\frac{\sqrt{2}}{\varepsilon}\right)} = \varepsilon.$$



$$\sqrt{\frac{1}{k^2} + e^{-2k}} < \sqrt{\frac{1}{k^2} + \frac{1}{k^2}}$$

$$= \frac{\sqrt{2}}{k} < \varepsilon \iff k > \frac{\sqrt{2}}{\varepsilon}$$

Non-convergence Example

Exercise. Show that the sequence $\{\mathbf{y}_k\}_{k=1}^{\infty}$ defined by $\mathbf{y}_k = \left(\cos(\frac{k\pi}{2}), \sin(\frac{k\pi}{2})\right)$ does not converge with respect to $d = d_2$.

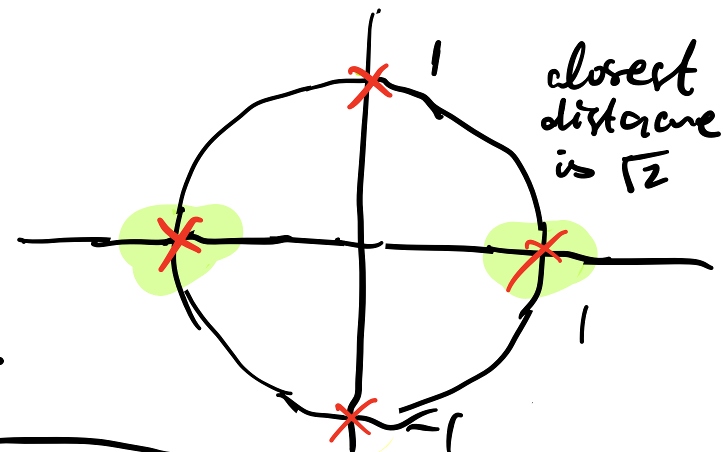
$\forall \mathbf{y} \in \mathbb{R}^2 \ \exists \varepsilon > 0 \ \forall K \in \mathbb{Z}^+ \ \exists k > K$ s.t.

$d_2(\mathbf{y}_k, \mathbf{y}) \geq \varepsilon$.

suppose $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$

if $\mathbf{y} \neq 0$, then choose $\varepsilon = |y_2|$.

Let $K \in \mathbb{Z}^+$ and choose $k = 2K$.



Then $d(\mathbf{y}_k, \mathbf{y}) = \sqrt{(\cos(K\pi) - y_1)^2 + (\sin(K\pi) - y_2)^2}$

$\geq \sqrt{0 + (0 - y_2)^2} = |y_2| = \varepsilon$

chosen $\mathbf{y}_k$

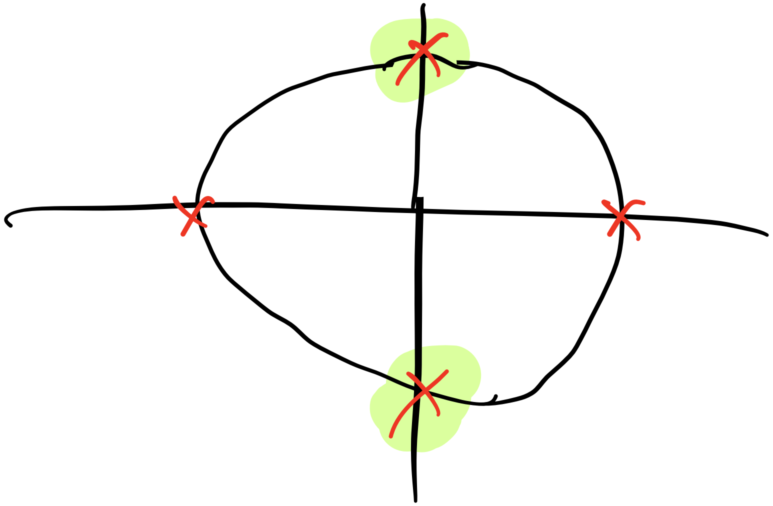
Non-convergence Example

Ex. cont. If $y_2 = 0$ let $\varepsilon = 1$, $K \in \mathbb{Z}^+$ and $k = 2K+1$

Then

$$d(y_{2k}, y) = \left[\left(\cos\left(\frac{2k+1}{2}\right) - y_1\right)^2 + \left(\sin\left(\frac{(2k+1)\pi}{2}\right) - 0\right)^2\right]^{1/2}$$

$$\geq \sqrt{0^2 + (\pm 1)^2} = 1 = \varepsilon$$



Cauchy Sequences

Definition

Suppose that d is a metric on $\mathbb{R}^n$. A sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ is **Cauchy** (with respect to d) if for all $\varepsilon > 0$, there exists $K \in \mathbb{Z}^+$ such that if $j, k > K$, then $d(\mathbf{x}_j, \mathbf{x}_k) < \varepsilon$. In quantifiers,

$$\forall \varepsilon > 0 \qquad \exists K \in \mathbb{Z}^+ \qquad \forall j, k > K \qquad d(\mathbf{x}_j, \mathbf{x}_k) < \varepsilon.$$

We could also say that $d(\mathbf{x}_j, \mathbf{x}_k) \to 0$ as $j, k \to \infty$.

Elements of a Cauchy sequence get arbitrarily close to each other and look like they are converging to something.

Proposition

If a sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ is convergent, then it is Cauchy.

Proof. Suppose that $\underset{\sim}{x}_k \to \underset{\sim}{x}$ and let $\varepsilon > 0$. Then there is $K \in \mathbb{Z}^+$ s.t. $d(\underset{\sim}{x}_k, \underset{\sim}{x}) < \frac{\varepsilon}{2}$ for all $k > K$.
Then if $j, k > K$, we have that
$d(\underset{\sim}{x}_j, \underset{\sim}{x}_k) \leq d(\underset{\sim}{x}_j, \underset{\sim}{x}) + d(\underset{\sim}{x}_k, \underset{\sim}{x}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Cauchy Sequences

Corollary

If a sequence is not Cauchy, then it is not convergent.

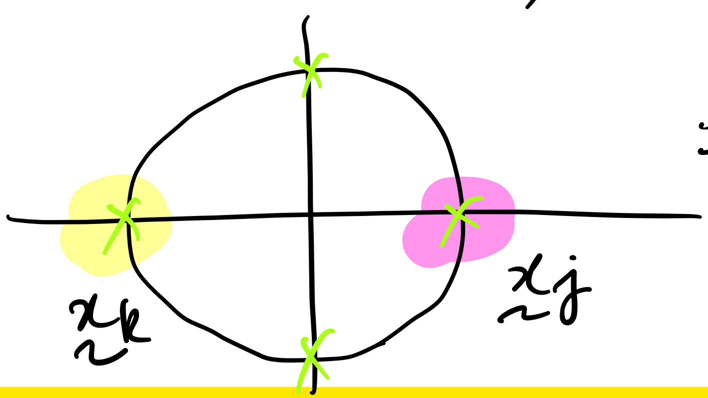
Exercise. Show that the sequence from before defined by $\mathbf{y}_k = \left(\cos(\frac{k\pi}{2}), \sin(\frac{k\pi}{2})\right)$ is not Cauchy.

Take $\varepsilon = 1$ and let $K \in \mathbb{Z}^+$ choose $j = 4K$ and $k = 4K+2$ Then

$$d(\underset{(1,0)}{\underset{\sim}{x}_j}, \underset{(-1,0)}{\underset{\sim}{x}_k}) = $$

$$= \sqrt{1-(-1)^2 + (0-0)^2}$$

$$= 2 > \varepsilon$$



Cauchy Sequences

Exercise. Show that the sequence $\left\{\left(\cos(\frac{1}{k}), \sin(\frac{1}{k})\right)\right\}_{k=1}^{\infty}$ is Cauchy w.r.t. d_1.

Cauchy Sequences

Depending on the metric, you might have Cauchy sequences that are not convergent! Consider the following (silly) metric on $\mathbb{R}$:

$$d(x, y) = \begin{cases} |x - y| & \text{if } x - y \in \mathbb{Q}, \\ |x - y| + 1 & \text{otherwise.} \end{cases}$$

Ex. Prove that this is indeed a metric.

Then the sequence $\{x_k\}_{k=1}^{\infty} = \{3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots\}$ is Cauchy but $d(\pi, x_k) \geqslant 1$ for all $k \in \mathbb{Z}^+$.

Could the sequence converge to something else? If it did, then the limit would have to be a rational number...

Theorem

If d is the Euclidean distance on $\mathbb{R}^n$, then every Cauchy sequence is convergent.

Equivalent Metrics

Definition

Two metrics δ and d defined on $\mathbb{R}^n$ are said to be **(strongly) equivalent** if there are constants $c, C > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$c\, d(\mathbf{x}, \mathbf{y}) \leqslant \delta(\mathbf{x}, \mathbf{y}) \leqslant C\, d(\mathbf{x}, \mathbf{y}).$$

Note: There is another notion of equivalent metrics, hence the "strongly" in parentheses. We will not be using the other notion in this course.

- d_p is equivalent to d_q for all $p, q \in [1, \infty]$ on $\mathbb{R}^n$ since

$$d_\infty(\mathbf{x}, \mathbf{y}) \leqslant d_p(\mathbf{x}, \mathbf{y}) \leqslant n^{1/p}\, d_\infty(\mathbf{x}, \mathbf{y}).$$

- The discrete metric is not equivalent to d_p for any $p \in [1, \infty]$ on $\mathbb{R}^n$.

Equivalent Metrics

- $\delta(\mathbf{x}, \mathbf{y}) = \dfrac{d(\mathbf{x}, \mathbf{y})}{1 + d(\mathbf{x}, \mathbf{y})}$ is
  - not equivalent to d if d is the Euclidean distance;
  - equivalent to d if d is the discrete metric.

Theorem

Suppose that δ and d are equivalent metrics on $\mathbb{R}^n$ and that $\{\mathbf{x}\}_{k=1}^{\infty} \subseteq \mathbb{R}^n$. Then $\{\mathbf{x}_k\}_{k=1}^{\infty}$ converges to $\mathbf{x} \in \mathbb{R}^n$ with respect to δ if and only if $\{\mathbf{x}_k\}_{k=1}^{\infty}$ converges to $\mathbf{x} \in \mathbb{R}^n$ with respect to d.

So when we have equivalent metrics, it doesn't matter which metric we use when discussing the convergence of a sequence!

Equivalent metrics also preserve whether a sequence is Cauchy.

Convergence of Components

From here on in the lectures, d will refer to the Euclidean distance d_2 and convergence will be with respect to d_2, unless stated otherwise.

Theorem

A sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ converges to $\mathbf{x} \in \mathbb{R}^n$ if and only if all of the components of $\mathbf{x}_k$ converge to the respective components of $\mathbf{x}$.

Proof. Suppose $\{ \underset{\sim}{x}_k \}_{k=1}$ converges to $\underset{\sim}{x}$ and let the ith component of x_k and $\underset{\sim}{x}$ be $x_k^{(i)}$ and $\underset{\sim}{x}^{(i)}$

Convergence of Components

Proof cont.