MATH2621 — Higher Complex Analysis. X
Power series

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(b) Holomorphic functions can be expressed in power series, and, as we will see later, *vice versa*.

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(b) Holomorphic functions can be expressed in power series, and, as we will see later, *vice versa*.

The proofs of many theorems about complex power series are almost identical to the proof of the corresponding theorem for real power series, and so we omit most proofs.

Definition

A (complex) power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{1}$$

where the *centre* z_0 and the *coefficients* a_n are all fixed complex numbers, and the *variable* z is complex. We take $(z - z_0)^0$ to be 1 for all z, even when $z = z_0$.

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The first problem is whether the sum (1) makes sense. If $z = z_0$, then the sum converges trivially, to a_0; for other values of z, the series may or may not converge.

Radius of convergence

Theorem

Every power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has a radius of convergence ρ, given by the formulae

$$\rho = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1} = \left(\lim_{k\to\infty} \sup_{n\geq k} |a_n|^{1/n}\right)^{-1}.$$

The radius of convergence $\rho \in [0, +\infty]$ satisfies:

(a) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ **converges** *if* $|z-z_0| < \rho$

(b) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ **does not converge** *if* $|z-z_0| > \rho$

(c) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *may converge for no, some or all z such that* $|z-z_0| = \rho$.

Comments

If $\rho = 0$, then the series converges only when $z = z_0$, while if $\rho = +\infty$, then the series converges for all $z \in \mathbb{C}$.

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We often write $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $B(a, r)$ to mean that the sum converges for all z in $B(a, r)$. When $a = z_0$, this means that $r \leq \rho$.

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There are several tests for convergence that carry over to complex power series from the theory of real power series.

The ratio test

Lemma

The radius of convergence is given by

$$\rho = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

as long as the limit exists or is $+\infty$.

The root test

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In order to use the root test when the coefficients involve factorials, the following fact, known as *Stirling's formula*, or one of its variants, is sometimes useful:

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n/e} = 1$$

or more precisely:

$$n! \sim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{e}\right)^n .$$

Exercise 1

Find the centre and radius of convergence of $\displaystyle\sum_{n=0}^{\infty} 3^{n-1}(z+1)^n$.

Answer.

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Answer. The centre is -1. Take $a_n = 3^{n-1}$ for all $n \in \mathbb{N}$. Now $|a_n|/|a_{n+1}| = 1/3$, so $\lim_{n\to\infty} |a_n|/|a_{n+1}| = 1/3$, and $\rho = 1/3$ by the ratio test. $\triangle$

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This *geometric series* converges to $\left[3(1-3(z+1))\right]^{-1}$.

Exercise 2

Find the centre and radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(z-2)^n}{n!}$.

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Answer. The centre is 2. Take $a_n = 1/n!$ for all $n \in \mathbb{N}$. Then

$$\frac{|a_n|}{|a_{n+1}|} = \frac{1/n!}{1/(n+1)!} = \frac{(n+1)!}{n!} = n+1 \to \infty \qquad \text{as } n \to \infty,$$

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This *exponential series* converges to e^{z-2}.

Exercise 3

Find the centre and radius of convergence of $\sum_{n=0}^{\infty} a_n(z-1)^n$, where $a_0 = 1$ and, for some fixed $\alpha \in \mathbb{C}$,

$$a_n = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \qquad \forall n \in \mathbb{Z}^+.$$

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$$a_n = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} \qquad \forall n \in \mathbb{Z}^+.$$

Answer. The centre is 1. For $\alpha \notin \mathbb{N}$, by the ratio test,

$$\begin{aligned}\frac{|a_n|}{|a_{n+1}|} &= \frac{|\alpha(\alpha-1)\dots(\alpha-n+1)|\,(n+1)!}{|\alpha(\alpha-1)\dots(\alpha-n+1)(\alpha-n)|\,n!} \\ &= \frac{n+1}{|\alpha-n|} = \frac{n+1}{n}\,\frac{1}{|\alpha/n-1|} \to 1\end{aligned}$$

as $n \to \infty$. Thus $\rho = 1$, and the series converges in $B(1,1)$.

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as $n \to \infty$. Thus $\rho = 1$, and the series converges in $B(1,1)$.

If $\alpha \in \mathbb{N}$, then $a_n = 0$ if $n > \alpha$, and so all terms are eventually 0, and the sum converges to all $z \in \mathbb{C}$. $\triangle$

This is tricky

This *binomial series* converges to $(1 + (z - 1))^\alpha$, that is, z^α. If $\alpha \notin \mathbb{N}$, then z^α is not well defined. Later we will define $\mathrm{PV}\, z^\alpha$; this is the least ambiguous way to write the sum.

Exercise 4

Find the centre and radius of convergence of

$$\sum_{k=1}^{\infty} \frac{k(k+1)(k^2+2)}{2^k} (z+3)^k.$$

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Answer. The centre is -3. Take $a_k = k(k+1)(k^2+2)/2^k$. Now

$$\frac{|a_k|}{|a_{k+1}|} = \frac{k(k+1)(k^2+2)2^{k+1}}{(k+1)(k+2)((k+1)^2+2)2^k} = \frac{2k}{k+2}\,\frac{k^2+2}{(k+1)^2+2}$$

$$= \frac{2k}{k+2}\,\frac{k^2+2}{k^2+2k+3} \to 2$$

as $k \to \infty$, and the radius of convergence is 2, by the ratio test. $\triangle$

Exercise 5

Find the centre and radius of convergence of the series

$$\sum_{j=1}^{\infty} \frac{(-j)^j}{j!} (z-5)^j.$$

Answer.

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Answer. The centre is 5. Observe that

$$\begin{aligned} \frac{|a_j|}{|a_{j+1}|} &= \frac{(j+1)!}{j!} \frac{j^j}{(j+1)^{j+1}} \\ &= \frac{j+1}{j+1} \left(\frac{j}{j+1}\right)^j \\ &= \left(1 + \frac{1}{j}\right)^{-j} \to \frac{1}{e} \qquad \text{as } j \to \infty, \end{aligned}$$

whence the radius of convergence is $1/e$, by the ratio test.

$\triangle$

Answer to Exercise 5

Alternatively, . . .

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By the root test, and Stirling's formula,

$$\lim_{n\to\infty} \frac{1}{|a_n|^{1/n}} = \lim_{n\to\infty} \frac{n!^{1/n}}{n} = \lim_{n\to\infty} \frac{n!^{1/n}}{n/e} \frac{n/e}{n} = \frac{1}{e}.$$

Again, the radius of convergence is $1/e$.

Exercise 6

Find the centre and radius of convergence of the series

$$\sum_{\substack{m \in \mathbb{N} \\ m \text{ even}}} \frac{z^m}{2^m}.$$

Answer.

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$$\sum_{\substack{m \in \mathbb{N} \\ m \text{ even}}} \frac{z^m}{2^m}.$$

Answer. The centre is 0. Take $a_m = 0$ when m is odd while $a_m = 2^{-m}$ when m is even.

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Answer. The centre is 0. Take $a_m = 0$ when m is odd while $a_m = 2^{-m}$ when m is even. We cannot use the ratio test, as a_m/a_{m+1} is not defined when m is odd.

Exercise 6

Find the centre and radius of convergence of the series

$$\sum_{\substack{m\in\mathbb{N}\\ m \text{ even}}} \frac{z^m}{2^m}.$$

Answer. The centre is 0. Take $a_m = 0$ when m is odd while $a_m = 2^{-m}$ when m is even. We cannot use the ratio test, as a_m/a_{m+1} is not defined when m is odd. The root test also fails because $|a_m|^{-1/m}$ is equal to 2 when m is even and 0 when m is odd, so $\lim_{m\to\infty} |a_m|^{-1/m}$ does not exist.

Answer to Exercise 6

One way to deal with this series is to take $w = z^2$, and rewrite the series:

$$\sum_{\substack{m\in\mathbb{N}\\ m \text{ even}}} \frac{z^m}{2^m} = \sum_{n\in\mathbb{N}} \frac{z^{2n}}{2^{2n}} = \sum_{n\in\mathbb{N}} \frac{w^n}{4^n}\,.$$

This is a geometric series with ratio $z^2/4$, whose sum is $1/(1-z^2/4)$. The series converges when $\left|z^2/4\right| < 1$, that is, when $|z| < 2$, so the centre is 0 and the radius of convergence is 2. $\triangle$

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This is a geometric series with ratio $z^2/4$, whose sum is $1/(1 - z^2/4)$. The series converges when $\left|z^2/4\right| < 1$, that is, when $|z| < 2$, so the centre is 0 and the radius of convergence is 2. $\triangle$

Note that $|a_m|^{-1/m}$ is equal to 0 if m is odd and to 2 if m is even, and

$$\limsup_{m\to\infty} |a_m|^{-1/m} = 2.$$

The algebra of power series

Theorem

Suppose that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ converge in $B(z_0,\rho)$ to $f(z)$ and $g(z)$, and that $c \in \mathbb{C}$. Then the following series also converge in $B(z_0,\rho)$:

(a) $\displaystyle\sum_{n=0}^{\infty} c\, a_n(z-z_0)^n$, *and its sum is* $c\,f(z)$;

(b) $\displaystyle\sum_{n=0}^{\infty} (a_n+b_n)(z-z_0)^n$, *and its sum is* $f(z)+g(z)$;

(c) $\displaystyle\sum_{n=0}^{\infty} c_n(z-z_0)^n$, *where* $\displaystyle c_n=\sum_{j=0}^{n} a_j\, b_{n-j}$, *and its sum is* $f(z)\,g(z)$.

Proof. We omit the proof. $\square$

Power series are differentiable

Theorem
Suppose that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ in $B(z_0, \rho)$ and $\rho > 0$. Then f is differentiable in $B(z_0, \rho)$, and

$$f'(z) = \sum_{n=1}^{\infty} a_n n (z - z_0)^{n-1} = \sum_{m=0}^{\infty} a_{m+1}(m+1)(z - z_0)^m$$

in $B(z_0, \rho)$.

Proof. We omit the proof. □

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Theorem

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ *in* $B(z_0, \rho)$ *and* $\rho > 0$. *Then* f *is differentiable in* $B(z_0, \rho)$, *and*

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in $B(z_0, \rho)$.

Proof. We omit the proof. □

This theorem allows us to differentiate power series term by term.

Repeatedly differentiating power series

Corollary

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ in $B(z_0, \rho)$. Then f may be differentiated as many times as desired, and

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)a_n(z-z_0)^{n-k}.$$

In particular,

$$f^{(k)}(z_0) = k!\, a_k.$$

Further, the real-valued functions u and v, such that $f(x+iy) = u(x,y) + iv(x,y)$, may be differentiated as many times as desired, and all their partial derivatives are continuous.

Proof. We omit the proof. ∎

Power series that vanish on an interval

Corollary

Suppose that $g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ *in* $B(z_0, \rho)$*, and that* $\varepsilon > 0$*. If* $g(z_0 + t) = 0$ *for all real* t *in* $(-\varepsilon, \varepsilon)$*, then* $g(z) = 0$ *for all* z *in* $B(z_0, \rho)$*.*

Proof.

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Proof. First, we prove by induction that $g^{(n)}(z_0 + t) = 0$ for all real t in $(-\varepsilon, \varepsilon)$ and all natural numbers n.

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Corollary

Suppose that $g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ in $B(z_0, \rho)$, and that $\varepsilon > 0$. If $g(z_0 + t) = 0$ for all real t in $(-\varepsilon, \varepsilon)$, then $g(z) = 0$ for all z in $B(z_0, \rho)$.

Proof. First, we prove by induction that $g^{(n)}(z_0 + t) = 0$ for all real t in $(-\varepsilon, \varepsilon)$ and all natural numbers n.

This is true when $n = 0$, by hypothesis.

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Suppose that $g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ *in* $B(z_0, \rho)$*, and that* $\varepsilon > 0$*. If* $g(z_0 + t) = 0$ *for all real* t *in* $(-\varepsilon, \varepsilon)$*, then* $g(z) = 0$ *for all* z *in* $B(z_0, \rho)$*.*

Proof. First, we prove by induction that $g^{(n)}(z_0 + t) = 0$ for all real t in $(-\varepsilon, \varepsilon)$ and all natural numbers n.

This is true when $n = 0$, by hypothesis.

Suppose now that $g^{(k)}(z_0 + t) = 0$ for all real t in $(-\varepsilon, \varepsilon)$ for some natural number k, and write $f(z) = g^{(k)}(z)$ for all $z \in B(z_0, \rho)$, so that $f(z_0 + t) = 0$ for all real t in $(-\varepsilon, \varepsilon)$. By the theorem on differentiating power series, applied as many times as necessary, f is holomorphic in $B(z_0, \rho)$.

Power series that vanish near the centre. 2

As usual, set $z_0 = x_0 + iy_0$, and $f(x + iy) = u(x, y) + iv(x, y)$.

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As usual, set $z_0 = x_0 + iy_0$, and $f(x + iy) = u(x, y) + iv(x, y)$. Then $u(x_0 + t, y_0) = 0$ and $v(x_0 + t, y_0) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Hence $\partial u/\partial x(x_0 + t, y_0) = 0$ and $\partial v/\partial x(x_0 + t, y_0) = 0$ for all $t \in (-\varepsilon, \varepsilon)$.

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$$f'(z_0+t) = \frac{\partial u}{\partial x}(x_0+t, y_0) + i\frac{\partial v}{\partial x}(x_0+t, y_0),$$

$f'(z_0+t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Therefore $g^{(k+1)}(z_0+t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$, and the inductive step is established.

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$f'(z_0+t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Therefore $g^{(k+1)}(z_0+t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$, and the inductive step is established.

To conclude, we recall that the power series for $g^{(k)}$ is of the form

$$\sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(z-z_0)^{n-k},$$

whence $g^{(k)}(z_0) = k!\,a_k$, and so $a_k = 0$. This is true for all k, and hence $g(z) = \sum_{n=0}^{\infty} 0(z-z_0)^n = 0$ for all $z \in B(z_0, \rho)$. $\square$

Power series that are equal near the centre

Corollary

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ *and moreover that* $g(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$ *in* $B(z_0, \rho)$*. If* $f(z_0 + t) = g(z_0 + t)$ *for all* $t \in (-\varepsilon, \varepsilon)$*, then* $f(z) = g(z)$ *for all* $z \in B(z_0, \rho)$*.*

Proof. We apply the previous corollary to $f - g$. $\square$

Generalisation

There is a stronger version of the second last corollary that says that if f is holomorphic in a domain Ω, and $f(z_n) = 0$ for distinct points $z_n \in \Omega$ such that $z_n \to z_0 \in \Omega$ as $n \to \infty$, then $f = 0$. This leads to a stronger version of the last corollary.

Analytic continuation

These last corollaries lead to the concept of *analytic continuation*: if a function is defined in a domain Ω, then it is determined by its values in a "infinite set with a finite limit point". In particular, it is determined by its values on $\mathbb{R}$.

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These last corollaries lead to the concept of *analytic continuation*: if a function is defined in a domain Ω, then it is determined by its values in a "infinite set with a finite limit point". In particular, it is determined by its values on $\mathbb{R}$.

This explains why, in finding a holomorphic function with certain properties, it suffices to find it on $\mathbb{R}$. Once you know it on $\mathbb{R}$, you know it everywhere. This is useful when dealing with harmonic functions.