MATH2621 — Higher Complex Analysis. XVI
The Cauchy–Goursat Theorem

This lecture?

In this lecture, we begin with an exercise, and then state and discuss one of the key theorems of complex analysis.

Exercise 1

Suppose that Γ is a simple closed contour in $\mathbb{C}$ and $c_0, c_1 \in \mathbb{C}$. Show that $\displaystyle\int_\Gamma (c_1 z + c_0)\, dz = 0$. Is $\displaystyle\int_\Gamma \operatorname{Re}(z)\, dz$ always 0?

Answer.

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&= c_1 \frac{\gamma^2(b)}{2} + c_0 \gamma(b) - c_1 \frac{\gamma^2(a)}{2} - c_0 \gamma(a)
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&= \left[c_1 \frac{\gamma^2(t)}{2} + c_0\gamma(t) \right]_{t=a}^{t=b} \\
&= c_1 \frac{\gamma^2(b)}{2} + c_0\gamma(b) - c_1\frac{\gamma^2(a)}{2} - c_0\gamma(a) = 0,
\end{aligned}$$

since γ is closed.

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The second part of the question needs a counterexample. Take $\gamma(t) = e^{it}$, where $0 \leq t \leq 2\pi$. By definition,

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\begin{aligned}
\int_\gamma \mathrm{Re}(z)\,dz = \int_\gamma x\,dz = \int_0^{2\pi} \cos(t)\, i\, e^{it}\,dt \\
&= i\int_0^{2\pi} \cos^2(t) + i\cos(t)\sin(t)\,dt \\
&= i\int_0^{2\pi} \cos^2(t)\,dt - \int_0^{2\pi} \cos(t)\sin(t)\,dt \\
&= i\pi.
\end{aligned}
$$

The integral $\int_\Gamma \mathrm{Re}(z)\,dz$ can be nonzero! $\triangle$

The Cauchy–Goursat theorem. I

Theorem

Suppose that Ω is a simply connected domain, that $f \in H(\Omega)$, and that Γ is a closed contour in Ω. Then

$$\int_\Gamma f(z)\,dz = 0.$$

Proof.

We prove a more general version of this result later for the case where Γ is simple. The general case follows by arguing as we did for path integrals. $\square$

Independence of contour

Corollary

Suppose that Ω is a simply connected domain in $\mathbb{C}$, that $f \in H(\Omega)$, and that Γ and Δ are contours with the same initial point p and the same final point q. Then

$$\int_\Gamma f(z)\,dz = \int_\Delta f(z)\,dz.$$

Proof. The contour $\Gamma \sqcup \Delta^*$ is closed, and so

$$0 = \int_{\Gamma \sqcup \Delta^*} f(z)\,dz = \int_\Gamma f(z)\,dz - \int_\Delta f(z)\,dz,$$

and we are done.

Existence of primitives

Corollary

Suppose that Ω is a simply connected domain in $\mathbb{C}$, and that $f \in H(\Omega)$. Then there exists a function F on Ω such that

$$\int_\Gamma f(z)\, dz = F(q) - F(p)$$

for all simple contours Γ in Ω from p to q. Further, F is differentiable, and $F' = f$. Finally, if F_1 is any other function such that $F_1' = f$, then $F_1 - F$ is a constant and

$$\int_\Gamma f(z)\, dz = F_1(q) - F_1(p),$$

where p and q are the initial and final points of Γ.

Existence of primitives. 2

Proof. Fix a "base point" b in Ω, and for $p \in \Omega$, define $F(p)$ to be $\int_\Gamma f(z)\,dz$, where Γ is any simple contour in Ω with initial point b and final point p. This definition makes sense in light of the previous corollary.

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Given p and q in Ω, and a simple contour Γ from p to q, take contours Γ_p from b to p and Γ_q from b to q.

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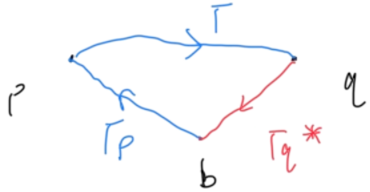
Given p and q in Ω, and a simple contour Γ from p to q, take contours Γ_p from b to p and Γ_q from b to q. Then $\Gamma \sqcup (\Gamma_q)^* \sqcup \Gamma_p$ is a closed contour for which

$$\int_{\Gamma \sqcup (\Gamma_q)^* \sqcup \Gamma_p} f(z)\,dz = 0.$$



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$$\int_{\Gamma \sqcup (\Gamma_q)^* \sqcup \Gamma_p} f(z)\,dz = 0.$$

Writing this as a combination of integrals, we see that

$\int_\Gamma - \int_{\Gamma_q} + \int_{\Gamma_p}$
$= \int_\Gamma f - F(q) + F(p) = 0$

$$\int_\Gamma f(z)\,dz = F(q) - F(p).$$

Existence of primitives. 3

Now we take $p \in \Omega$, and show that $F'(p) = f(p)$. To do so, we need to make

$$\left| \frac{F(q) - F(p)}{q - p} - f(p) \right|$$

small by taking q sufficiently close to p.

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Take $q \in \Omega$ close to p, and let Δ be the line segment from p to q. On the one hand,

$$\frac{F(q) - F(p)}{q - p} = \frac{1}{q - p} \int_{\Delta} f(z)\, dz;$$

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$$\frac{F(q) - F(p)}{q - p} = \frac{1}{q - p} \int_\Delta f(z)\, dz;$$

this requires that Δ is contained in Ω. On the other hand, by calculation, $\int_\Delta dz = q - p$, whence

$$f(p) = \frac{1}{q - p} \int_\Delta f(p)\, dz.$$

$$= \frac{f(p)}{q-p} \int_\Delta dz$$

$$= f(p)$$

Existence of primitives. 4

Thus, by the *ML* Lemma, and the fact that continuous functions attain their maximum,

$$\begin{aligned}
\left|\frac{F(q)-F(p)}{q-p}-f(p)\right| &= \left|\frac{1}{q-p}\int_\Delta f(z)\,dz - \frac{1}{q-p}\int_\Delta f(p)\,dz\right| \\
&= \left|\frac{1}{q-p}\int_\Delta \big(f(z)-f(p)\big)\,dz\right| \\
&\le \frac{1}{|q-p|}\,\max\{|f(z)-f(p)| : z\in\Delta\}\,|q-p| \\
&= \max\{|f(z)-f(p)| : z\in\Delta\} \\
&= |f(z^*)-f(p)|
\end{aligned}$$

for some z^* in Δ.

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&\leq \frac{1}{|q-p|}\,\max\{|f(z)-f(p)| : z\in\Delta\}\,|q-p| \\
&= \max\{|f(z)-f(p)| : z\in\Delta\} \\
&= |f(z^*)-f(p)|
\end{aligned}
$$

for some z^* in Δ.

We can make this small be making Δ small enough that Δ is contained in Ω and f does not vary much (f is differentiable and hence continuous).

Existence of primitives. 5

More precisely, take any small positive ε. Since Ω is open and f is continuous at p, there exists δ such that $B(p, \delta) \subset \Omega$ and $|f(z) - f(p)| < \varepsilon$ when $z \in B(p, \delta)$. Take $q \in B^\circ(p, \delta)$ and let Δ be the straight line segment from p to q. Then $\Delta \subset B(p, \delta)$ and so $|f(z) - f(p)| < \varepsilon$ for all $z \in \Delta$. In particular, $|f(z^*) - f(p)| < \varepsilon$.

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We conclude that

$$\begin{aligned}\left|\frac{F(q) - F(p)}{q - p} - f(p)\right| &= \left|\frac{1}{q - p} \int_{\Delta} \big(f(z) - f(p)\big)\, dz\right| \\ &\leq |f(z^*) - f(p)| \\ &< \varepsilon,\end{aligned}$$

so F is differentiable at p, with derivative $f(p)$, as required.

Existence of primitives. 6

If F_1 is another function such that $F_1' = f$, then $(F_1 - F)' = 0$, so $F_1 - F$ is a constant, C say. This means that

$$F_1(q) - F_1(p) = (F(q) + C) - (F(p) + C) = F(q) - F(p),$$

so that F_1 can also be used to compute $\int_\Gamma f(z)\,dz$. □

Independence of contour

We call a function F such that $F' = f$ a *primitive* or an *anti-derivative* of f. In some of our earlier computations, there are hints that it might be possible to compute contour integrals using primitives; now we have the proof of this, at least when f is holomorphic.

Multiply connected domains

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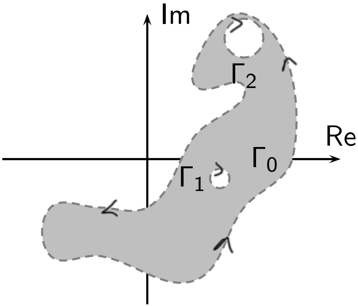

Enclosed domain on the left i.e excluded on the right

holes moves clockwise so that excluded domain on the right

Exercise 2

Determine the orientations of the contours in the figure above.

The Cauchy–Goursat Theorem. II

Theorem (Cauchy–Goursat)

Suppose that Ω is a bounded domain whose boundary $\partial\Omega$ consists of finitely many contours $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$. Suppose also that $f \in H(\Upsilon)$, where $\overline{\Omega} \subset \Upsilon$. Then

$$\int_{\partial\Omega} f(z)\,dz = \sum_{j=0}^{n} \int_{\Gamma_j} f(z)\,dz = 0.$$

A corollary

Corollary

Suppose that Υ is a simply connected domain, that Γ is a simple closed contour in Υ, and that f is a differentiable function in Υ. Then

$$\int_{\Gamma} f(z)\, dz = 0.$$

Proof. We let Ω be the interior of Γ and apply the previous result. $\square$

Existence of primitives in multiply connected domains

Corollary

Suppose that Ω is a bounded domain whose boundary $\partial\Omega$ consists of finitely many contours $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$, that $\overline{\Omega} \subset \Upsilon$, and that f is a differentiable function in Υ. If $\int_{\Gamma_j} f(z)\,dz = 0$ when $j = 1, \ldots, n$, then $\int_\Gamma f(z)\,dz = 0$ for any closed contour in Ω, and further, there is a differentiable function F in Ω such that $F' = f$ and

$$\int_\Delta f(z)\,dz = F(q) - F(p)$$

for all simple contours Δ in Ω from p to q.

Significance

The Cauchy–Goursat theorem is mathematically important because it will lead to the Cauchy integral formula, one of the most useful formulae in complex analysis.

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One of the uses of this formula is to compute integrals. Another is to show that a holomorphic function in a domain Ω has continuous partial derivatives, and indeed is infinitely differentiable.

We stated several theorems earlier about holomorphic functions which include the hypotheses that f is holomorphic and that f' is continuous, and it is useful to know that the continuity hypothesis is automatically true. At least in principle, we should check that the hypotheses of a theorem are satisfied before we apply the theorem, and so it is good to make these hypotheses unnecessary.

History

Augustin Cauchy was one of the finest French mathematicians of the first half of the 1800s, and he developed much of what is in a course on complex analysis today, as well as making precise the idea of limit that had been worrying mathematicians and philosophers of mathematics since the time of Newton and Leibniz.

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Some of Cauchy's ideas were being developed simultaneously by George Green, an "uneducated miller" from Nottingham, who gave us Green's theorem in 1828.

Connection with vector fields

Take a simple piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$, and write $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ and $f(z) = u(x, y) + iv(x, y)$. Let $\boldsymbol{\gamma}(t)$ be the analogue of γ in $\mathbb{R}^2$, that is, $\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$.

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Define the vector fields U and V by

$$U(x, y) = (u(x, y), -v(x, y)) \quad \text{and} \quad V(x, y) = (v(x, y), u(x, y)).$$

Then ...

Connection with vector fields. 2

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

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$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \\ &= \int_a^b (u(\boldsymbol{\gamma}(t)) + iv(\boldsymbol{\gamma}(t)))\,(\gamma_1'(t) + i\gamma_2'(t))\,dt \\ &= \int_a^b (u(\boldsymbol{\gamma}(t))\,\gamma_1'(t) - v(\boldsymbol{\gamma}(t)))\,\gamma_2'(t))\,dt\end{aligned}$$

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&= \int_a^b (u(\boldsymbol{\gamma}(t))\,\gamma_1'(t) - v(\boldsymbol{\gamma}(t)))\,\gamma_2'(t))\,dt \\
&\qquad\qquad + i\int_a^b (u(\boldsymbol{\gamma}(t))\,\gamma_2'(t) + v(\boldsymbol{\gamma}(t)\,\gamma_1'(t))\,dt
\end{aligned}
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&\qquad\qquad + i\int_a^b (u(\boldsymbol{\gamma}(t))\,\gamma_2'(t) + v(\boldsymbol{\gamma}(t)\,\gamma_1'(t))\,dt \\
&= \int_a^b U(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt + i\int_a^b V(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt
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&\qquad + i\int_a^b (u(\boldsymbol{\gamma}(t))\,\gamma_2'(t) + v(\boldsymbol{\gamma}(t)\,\gamma_1'(t))\,dt \\
&= \int_a^b U(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt + i\int_a^b V(\boldsymbol{\gamma}(t))\cdot\boldsymbol{\gamma}'(t)\,dt \\
&= \int_{\boldsymbol{\gamma}} U(\boldsymbol{s})\,d\boldsymbol{s} + i\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\,d\boldsymbol{s}.
\end{aligned}
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Connection with vector fields. 3

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The condition that U and V be closed is the condition

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}.$$

These are exactly the Cauchy–Riemann equations.

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So in a simply connected domain in $\mathbb{C}$, the complex line integrals depend only on the initial point and the final point when the Cauchy–Riemann equations hold.