MATH2621 — Higher Complex Analysis. III
Sketching complex functions

This lecture?

We look at ways to represent complex functions graphically.

Just as we often write $y = f(x)$ for a real function, it is common to consider a function in the form $w = f(z)$, and write $z = x + iy$ and $w = u + iv$.

Typically, we draw "elementary" curves in the z plane, such as lines parallel to the axes, or concentric circles around and rays exiting from the origin, and then examine their images in the w plane, or we draw similar elementary curves in the w plane and then examine their preimages.

Linear mappings

Consider the map $z \mapsto az$; this mapping is 1-1.

We write $a = c + id$ and $z = x + iy$; then

$$az = (c + id)(x + iy) = (cx - dy) + i(cy + dx).$$

In Cartesian coordinates, the map may be written

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} c & -d \\ d & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

More on linear mappings

Note that, if $a \neq 0$, then

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $r = \sqrt{c^2 + d^2}$, while $\cos\theta = c/r$ and $\sin\theta = d/r$. Hence multiplying by the matrix

$$\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

is the same as rotating through the angle θ and then dilating by the real number r. This corresponds to the representation of a in the form $re^{i\theta}$, where $r = |a|$ and $\theta = \operatorname{Arg}(a)$.

Affine mappings

An affine mapping of the complex plane is a map of the form $z \mapsto az + b$; such a mapping is also 1-1. We may also represent this as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. We write $a = c + id$ and $b = e + if$. Then in Cartesian coordinates, we have

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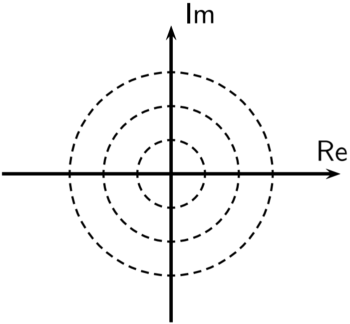
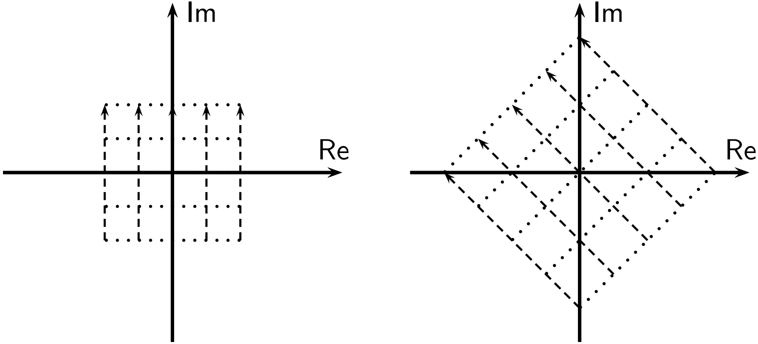
The inverse of an affine mapping is an affine mapping; hence the preimages of lines are lines and preimages of circles are circles. In particular, the preimage of a grid parallel to the axes is another grid, but not necessarily parallel to the axes.

Sketch of the translation $z \mapsto z + 1$



Sketch of the multiplication $z \mapsto (1 + i)z$



Pros and cons

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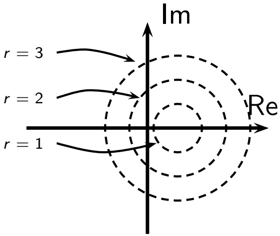

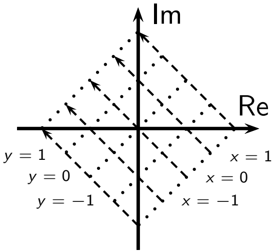

Level curves

We can look for curves in the xy plane whose images in the uv plane are the lines $u = c$ and $v = d$: we are finding the level curves of the real and imaginary parts of the function. Interpreting level curves is like reading a contour map.

Quadratic functions

Consider the function $z \mapsto z^2$, which is 2-1 in $B^\circ(0, \infty)$.

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First, we find the images in the uv plane of the lines $x = a$ and $y = b$ in the xy plane.

If $x = a$ and $y = t$, where a is fixed and t varies, then

$$w = z^2 = (a + it)^2 = a^2 - t^2 + 2iat.$$

That is, $u = a^2 - t^2$ and $v = 2at$. We eliminate t to find

$$u = a^2 - \frac{v^2}{4a^2}\,.$$

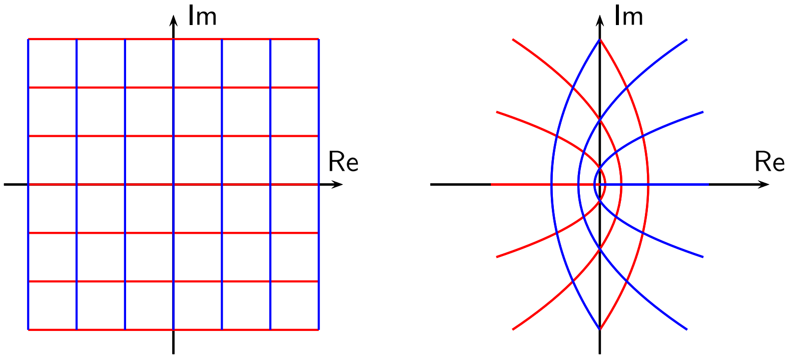
Alternatively, if $y = b$ and $x = t$, where b is fixed and t varies, then

$$w = z^2 = (t + ib)^2 = t^2 - b^2 + 2ibt.$$

That is, $u = t^2 - b^2$ and $v = 2bt$. We eliminate t to find

$$u = \frac{v^2}{4b^2} - b^2.$$

Cartesian sketch of a quadratic function



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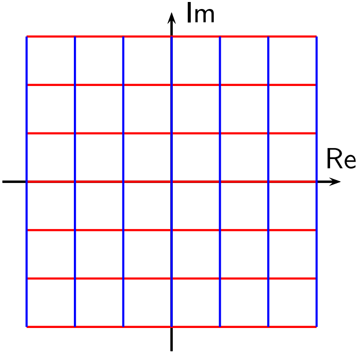
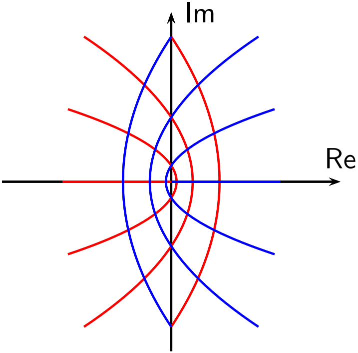




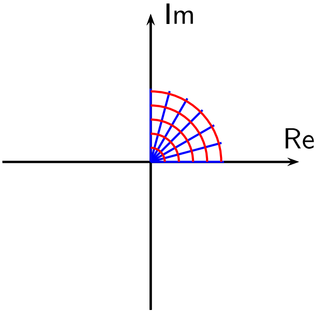
What are the foci of the parabolae?

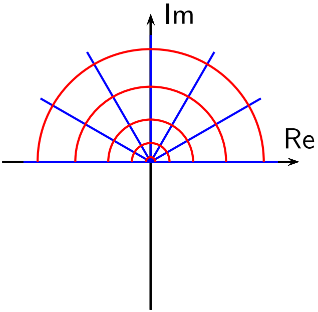
A quadratic function in polar coordinates

Again we consider the function $z \mapsto z^2$.

We find the images in the uv plane of the curves in the xy plane given by $r = a$ and $\theta = b$.

Here is the corresponding sketch using polar coordinates.





Level curves for a quadratic

On the other hand, we can look for the values of x and y so that $\mathrm{Re}(z^2)$ or $\mathrm{Im}(z^2)$ takes a fixed value. For instance, if $\mathrm{Re}(z^2) = a$, then

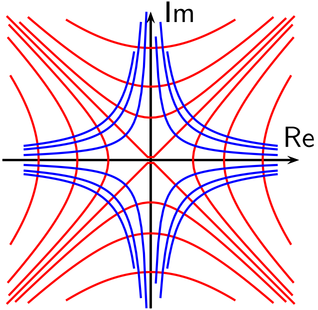
$$x^2 - y^2 = a,$$

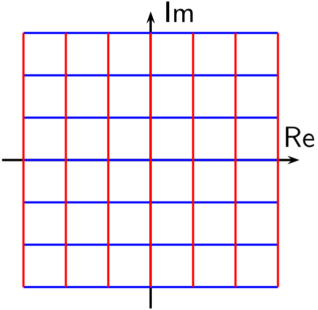
and this is a hyperbola opening to the left and right, or up and down, depending on the sign of a. Similarly, if $\mathrm{Im}(z^2) = b$, then

$$2xy = b,$$

and this is a right hyperbola in the first and third quadrants, or in the second and fourth quadrants, depending on the sign of b.

The level curves for Re(z^2) and Im(z^2)





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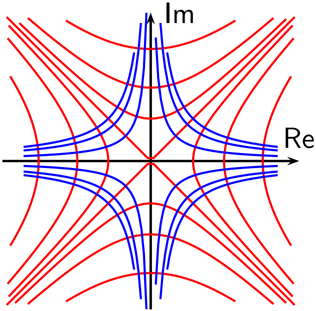

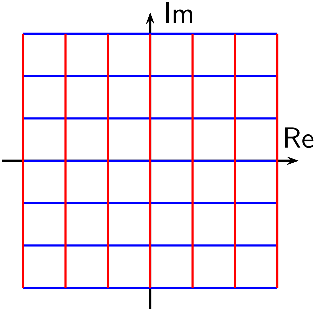

Often we just present the left hand picture, labelling the curves to make it clear what they represent.

Exercise 1

How would you understand the function $w = (z-1)^2 - 1$?

Answer.

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Answer. Taking first $z \mapsto z - 1$, then composing with $w = z^2$, and finally shifting to the left by 1.

The function $w = 1/z$

The function $w = 1/z$ is 1-1. It sends lines through the origin to lines through the origin. This is easy to see using polar coordinates.

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As r varies in $\mathbb{R}^+$, the point $z = re^{i\theta}$ varies along a ray from the origin. Now $w = (1/r)e^{-i\theta}$, and $1/r$ also varies in $\mathbb{R}^+$ and this point too varies along a ray. However the new ray is the reflection of the old ray in the real axis.

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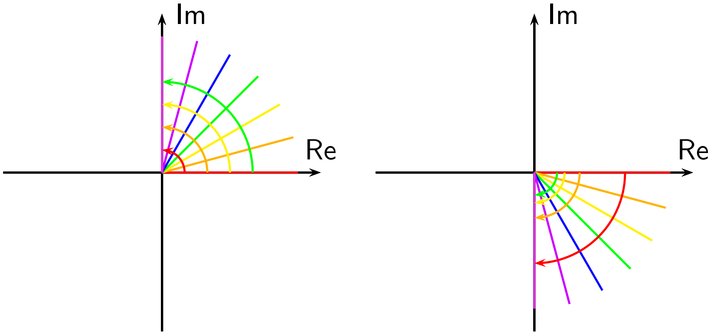
Similarly, as θ varies, the point $re^{i\theta}$ moves around a circle centred at the origin. Now $w = (1/r)e^{-i\theta}$, and this point moves around the cricle centred at the origin of radius $1/r$, but in the opposite direction.

Images of curves $r = c$ and $\theta = d$ for $w = 1/z$



More on $w = 1/z$

Lemma

1. *The image of a line through* 0 *(with the origin removed) is a line through* 0 *(with the origin removed).*
2. *The image of a line that does not pass though* 0 *is a circle (with the origin removed). If* p *is the closest point on the line to* 0*, then the line between* 0 *and* $1/p$ *is a diameter of the circle.*
3. *The image of a circle that passes through* 0 *is a line. If* q *is the furthest point on the circle from* 0*, then the closest point on the line to* 0 *is* $1/q$*.*
4. *The image of a circle that does not pass through* 0 *is a circle. If* p *and* q *are the closest and the furthest point on the circle from* 0*, then the closest and furthest point on the image circle to* 0 *are* $1/q$ *and* $1/p$*.*

The exponential function

The exponential function $w = e^z$ is ∞ to 1; that is, infinitely many different points in the xy plane are sent to the same point in the uv plane.

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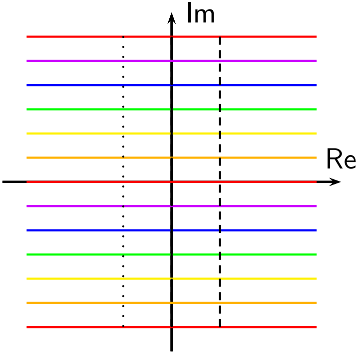

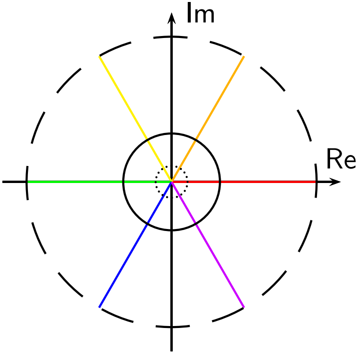

More on graphical representations of complex functions

There are many good web-sites that explore different ways to represent complex functions.