MATH2621 — Higher Complex Analysis. XXII
Singularities

This lecture?

In this lecture, we study holomorphic functions in a punctured ball, and illustrate this study with some examples.

[P]

Recall that the punctured ball $B^\circ(z_0, R)$ is defined to be $\{z \in \mathbb{C} : 0 < |z - z_0| < R\}$; we allow R to be ∞.

[P]

From last lecture, if $f \in H(B^\circ(z_0, R))$, then f has a Laurent series that converges in the punctured ball. Vice versa, if f has a Laurent series that converges in the punctured ball, then $f \in H(B^\circ(z_0, R))$.

Laurent series at an isolated singularity

Definition

A function f has an *isolated singularity* at z_0 in $\mathbb{C}$ if f is holomorphic in the punctured ball $B^\circ(z_0, R)$ for some $R \in \mathbb{R}^+$, and f is not differentiable at z_0, perhaps because $f(z_0)$ is not defined or f is not continuous at z_0.

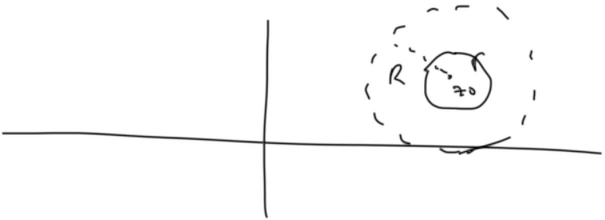
A Laurent series around an isolated singularity

Suppose that $f \in H(B^\circ(z_0, R))$. Take r such that $0 < r < R$. By Laurent's theorem,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad \forall z \in B^\circ(z_0, R),$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$



Several possibilities

If $c_n = 0$ for all $n \in \mathbb{Z}$, then f is identically zero in $B^\circ(z_0, R)$. In this case we can define $f(z_0)$ to be 0 and then $f \in H(B(z_0, R))$.

Otherwise, are three mutually exclusive and exhaustive possibilities: [P]

(i) There are infinitely many $n \in \mathbb{Z}^-$ such that $c_n \neq 0$. In this case, we say that f has an *essential singularity* at z_0. [P]

(ii) There are no $n \in \mathbb{Z}^-$ such that $c_n \neq 0$. In this case, we say that f has a *removable singularity* at z_0. If there exists $M \in \mathbb{Z}^+$ such that $c_M \neq 0$ and $c_n = 0$ for all $n < M$, then we say that f has a *zero of order* M at z_0. [P]

(iii) There are at least one and finitely many $n \in \mathbb{Z}^-$ such that $c_n \neq 0$. In this case, there exists $M \in \mathbb{Z}^-$ such that $c_M \neq 0$ and $c_n = 0$ for all $n < M$. We say that f has a *pole of order* $-M$ at z_0.

Notation

A few authors write "isolated essential singularity" rather than "essential singularity" in case (i).

[P]

Zeros of order 1 are also known as *simple zeros*, while poles of order 1 are also known as *simple poles*.

Examples

1. Suppose that $f(z) = \dfrac{\sin(z)}{z}$. The natural domain for f is $\mathbb{C} \setminus \{0\}$, because $f(0)$ is not defined. As we have seen, it is possible to extend the definition of f to 0, by setting $f(0) = 1$, and then the extended function (still written f) is entire, and the singularity has been removed.

$$g(z) = \begin{cases} f(z) & z \neq 0 \\ 1 & z = 0 \end{cases}$$

$g \in H(\mathbb{C})$

[P]

2. Suppose that $f(z) = \dfrac{(1 - \cos(z))^2}{z}$. The natural domain for f is $\mathbb{C} \setminus \{0\}$, because $f(0)$ is not defined. As we will see, it is possible to extend the definition of f to 0, by setting $f(0) = 0$, and then f becomes an entire function, and the singularity has been removed. This extended function has a zero of order three at 0.

Examples. 2

3. Suppose that $f(z) = \dfrac{1}{z^3 - z}$. The natural domain for f is $\mathbb{C} \setminus \{0, \pm 1\}$, because $f(0)$, $f(1)$ and $f(-1)$ are not defined. As we will see, $\lim_{z \to 0} f(z) = \infty$, so we will leave $f(0)$ undefined. For this function,

$$f(z) = \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n-1} = -\frac{1}{z} + \sum_{n=1}^{\infty} (-1)^{n+1} z^{2n-1}$$

in $B^{\circ}(0,1)$, and there is a pole of order 1, i.e., a simple pole, at 0. [P]

4. Suppose that $f(z) = e^{1/z}$. The natural domain for f is $\mathbb{C} \setminus \{0\}$, because $f(0)$ is not defined. As we will see, $\lim_{z \to 0} f(z)$ does not exist, and f has an essential singularity at 0.

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} \quad \leftarrow \text{infinite singularity}$$

Examples. 3

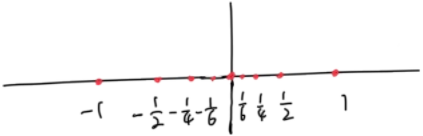
5. Suppose that $f(z) = \mathrm{Log}(z)$. The domain of f is $\mathbb{C} \setminus \{0\}$, and we say that f has a singularity at 0. However, f is not differentiable at all points on the negative real axis, due to the jump in Arg there, and so the singularity at 0 is not isolated. [P]

6. Suppose that $f(z) = \dfrac{1}{\sin(\pi/z)}$. Then

$$\mathrm{Domain}(f) = \mathbb{C} \setminus (\{0\} \cup \{1/n : n \in \mathbb{Z} \setminus \{0\}\}).$$

In this example, f has isolated singularities at the points $1/n$, where $n \in \mathbb{Z} \setminus \{0\}$, and a nonisolated singularity at 0.

$\frac{\pi}{z} = n\pi \qquad z = \frac{1}{n}$



A theorem

Theorem

Suppose that $f \in H(B^\circ(z_0, R))$*, and that*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad \forall z \in B^\circ(z_0, R).$$

Then the following are equivalent:

(i) $c_n = 0$ *for all* $n < 0$

(ii) $\lim_{z \to z_0} f(z)$ *exists in* $\mathbb{C}$*, and by defining* $f(z_0)$ *to be* $\lim_{z \to z_0} f(z)$*, we may extend* f *to a holomorphic function on* $B(z_0, R)$

(iii) $\lim_{z \to z_0} f(z)$ *exists in* $\mathbb{C}$

(iv) *there exists* $C \in \mathbb{R}^+$ *and* $r \in (0, R)$ *such that*

$$|f(z)| \leq C \qquad \forall z \in B^\circ(z_0, r).$$

Proof of theorem

Proof. Suppose that (i) holds. Then $f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ in $B^\circ(z_0, R)$, and the power series converges in $B^\circ(z_0, R)$, so its radius of convergence is at least R. Hence $\lim_{z\to z_0} f(z) = c_0$, and by defining $f(z_0)$ to be c_0, we extend f to agree with the power series in $B(z_0, R)$, which is holomorphic there, that is, (ii) holds. [P]

Suppose that (ii) holds. Then (iii) holds trivially.

Proof of theorem. 2

Suppose that (iii) holds, i.e., that $\lim_{z \to z_0} f(z) = c$ for some $c \in \mathbb{C}$. Take $\varepsilon = 1$ in the definition of the limit. Then there exists r such that $0 < r < R$ and

$$|f(z) - c| < 1$$

when $z \in B^\circ(z_0, r)$. For such z,

$$|f(z)| = |f(z) - c + c| \leq |f(z) - c| + |c| < 1 + |c|,$$

so taking C to be $1 + |c|$, we have $|f(z)| < C$, and (iv) holds.

Proof of theorem. 3

Suppose that (iv) holds. According to Laurent's theorem, if $0 < r < R$, then

$$c_n = \frac{1}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(z)}{(z-z_0)^{n+1}} \, dz \, .$$

Clearly when z lies on $\partial B(z_0, r)$,

$$\left| \frac{f(z)}{(z-z_0)^{n+1}} \right| = \frac{|f(z)|}{r^{n+1}} \leq \frac{C}{r^{n+1}}.$$

The length of $\partial B(z_0, r)$ is $2\pi r$, and so, by the *ML* Lemma,

$$|c_n| \leq \frac{1}{2\pi} \left| \int_{\partial B(z_0,r)} \frac{f(z)}{(z-z_0)^{n+1}} \, dz \right| \leq \frac{1}{2\pi} \frac{C}{r^{n+1}} 2\pi r = Cr^{-n}.$$

If $n < 0$, then $-n > 0$, and $r^{-n} \to 0$ as $r \to 0$. In this case, we can make r^{-n} arbitrarily small, so $c_n = 0$, i.e., (i) holds. $\square$

Removing singularities

The reason for the terminology "removable singularity" should now be clear: by defining (or perhaps redefining) $f(z_0)$ appropriately, we may extend f to be holomorphic in $B(z_0, R)$, that is, we *remove the singularity* of f at z_0.

[P]

The proof of the theorem has similarities to that of Liouville's theorem.

[P]

It is possible to generalize the proof of the theorem above to prove the following result.

A more general theorem

$$F(z) = \frac{f(z)}{(z-z_0)^M} = \sum_{n=M}^{\infty} c_n \frac{(z-z_0)}{(z-z_0)^M} = c_M + \cdots\cdot$$

Theorem

Suppose that $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ *in* $B^\circ(z_0, R)$*. Then the following are equivalent:*

(i) $c_n = 0$ *for all* $n < M$ *and* $c_M \neq 0$

(ii) *there exists* $F \in H(B(z_0, R))$ *such that* $f(z) = (z-z_0)^M F(z)$ *in* $B^\circ(z_0, R)$ *and* $F(z_0) \neq 0$

(iii) $\lim_{z \to z_0} (z-z_0)^{-M} f(z)$ *exists and is in* $\mathbb{C} \setminus \{0\}$

(iv) *there exists* $C \in \mathbb{R}^+$ *and* $r \in (0, R)$ *such that*

$$|f(z)| \leq C|z-z_0|^M \qquad \forall z \in B^\circ(z_0, r),$$

but $\lim_{z \to z_0} (z-z_0)^{-M} f(z) \neq 0$ *(either because the limit does not exist or because it exists and is not* 0*).*

Notation

We write $f(z) \sim c(z - z_0)^N$ as $z \to z_0$, where $c \in \mathbb{C} \setminus \{0\}$ and $N \in \mathbb{Z}$, to mean that

$$\lim_{z \to z_0} \frac{f(z)}{c(z - z_0)^N} = 1.$$

Note that if $f(z) \sim c(z - z_0)^N$ as $z \to z_0$, then there is a punctured ball $B(z_0, r)$ in which f does not vanish.

Summary

If f has a zero of order N at z_0, then $f(z) \sim c(z - z_0)^N$ as $z \to z_0$, where $c \neq 0$, and f vanishes at z_0; when the order of the zero is higher, $f(z)$ vanishes more rapidly.

[P]

If f has a pole of order N at z_0, then $f(z) \sim c(z - z_0)^{-N}$ as $z \to z_0$, where $c \neq 0$, and diverges to ∞; when the order of the pole is higher, the divergence is more rapid.

[P]

If f has a removable singularity at z_0, then f is bounded near z_0, and vice versa.

Manipulating functions at singularities

Corollary

Suppose that $f, g, h \in H(B^\circ(z_0, R))$ and that f has a zero of order M at z_0 while g has a zero of order N at z_0, while h has a pole of order P at z_0. Then:

$f \sim a(z-z_0)^M \quad M \geq 0$

$g \sim b(z-z_0)^N \quad N \geq 0$

$h \sim c\frac{1}{(z-z_0)^P} \quad P \geq 0$

(i) $1/h$ *has a zero of order P at z_0;*

(ii) $1/f$ *has a pole of order M at z_0;*

(iii) fg *has a zero of order $M+N$ at z_0;*

(iv) *if $M \geq N$, then f/g has a removable singularity at z_0, and if $M > N$, then f/g has a zero of order $M-N$;*

(v) *if $M < N$, then f/g has a pole of order $N-M$.*

$$(z-z_0)^a (z-z_0)^b \sim (z-z_0)^{a+b}$$

$$a, b \in \mathbb{Z}$$

Proof

We prove only (v).
Observe that $f(z) \sim a(z-z_0)^M$ and $g(z) \sim b(z-z_0)^N$ as $z \to z_0$, where $ab \neq 0$. Now

$$\frac{f(z)}{g(z)} \sim \frac{a(z-z_0)^M}{b(z-z_0)^N} = \frac{a}{b}(z-z_0)^{M-N}$$

as $z \to z_0$, and the result follows. $\square$

[P]
If f has an essential singularity at z_0, what about $1/f$?

L'Hôpital's rule

Theorem

Suppose that $f, g \in H(\Omega)$ and $z_0 \in \Omega$. Suppose also that $\lim_{z \to z_0} f(z)/g(z)$ is of the form $0/0$. If $\lim_{z \to z_0} f'(z)/g'(z)$ exists, then so does $\lim_{z \to z_0} f(z)/g(z)$, and the limits are equal.

Proof. By hypothesis, f and g have Taylor series in some ball centred at z_0, and $f(z) \sim a(z - z_0)^M$ while $g(z) \sim b(z - z_0)^N$ as $z \to z_0$. Further, $f'(z) \sim aM(z - z_0)^{M-1}$ and $g'(z) \sim bN(z - z_0)^{N-1}$. If $\lim_{z \to z_0} f'(z)/g'(z)$ exists, then

$$\begin{aligned} \lim_{z \to z_0} \frac{f(z)}{g(z)} &= \lim_{z \to z_0} \frac{a(z - z_0)^M}{b(z - z_0)^N} \\ &= \lim_{z \to z_0} \frac{N}{M} \frac{aM(z - z_0)^{M-1}}{bN(z - z_0)^{N-1}} = \frac{N}{M} \lim_{z \to z_0} \frac{f'(z)}{g'(z)} . \end{aligned}$$

[P]

The result follows by considering the cases $M < N$, $M = N$ and $M > N$.

Exercise 1

Suppose that $f(z) = (1 - \cos(z))^2/z$. What kind of singularity does f have at 0?

Answer. [P] Clearly

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{(1 - \cos(z))^2}{z} = \lim_{z \to 0} \frac{2(1 - \cos(z))\sin(z)}{1} = 0$$

by l'Hôpital's rule. So f has a zero of some currently unknown order at 0.

Answer to Exercise 1

There are various ways to proceed. Here is one:

$$\begin{aligned}
(1-\cos(z))^2 &= \left(1-(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\dots)\right)^2 \\
&= \left(\frac{z^2}{2!}-\frac{z^4}{4!}+\dots\right)^2 \\
&= \frac{z^4}{4}-\frac{z^6}{24}+\dots
\end{aligned}$$

whence $(1-\cos(z))^2$ has a zero of order 4; by dividing by z, we obtain the series for f, which has a zero of order 3.

Answer to Exercise 1 (continued)

Here is another: by L'Hôpital's rule, $\lim_{z\to 0} \dfrac{1-\cos(z)}{z^2} = \dfrac{1}{2}$, whence

$$\lim_{z\to 0} \frac{(1-\cos(z))^2}{z^4} = \frac{1}{4},$$

and so $(1-\cos(z))^2$ has a zero of order 4 at 0, whence f has a zero of order 3. $\triangle$

Exercise 2

Suppose that $f(z) = e^{1/z}$. How does f behave near 0?

Answer. [P] From Taylor series,

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \dots.$$

Hence

$$e^{1/z} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots.$$

Since there are infinitely many nonzero negative powers of z, the singularity at 0 is essential. $\triangle$

Picard's theorem†

Theorem

If f has an essential singularity at z_0, then for all $\delta > 0$, the set $\mathbb{C} \setminus f(B^\circ(z_0, \delta))$ has at most one element.

[P]

We shall not prove this result. But consider the example $f(z) = e^{1/z}$.

[P]

Given any c in $\mathbb{C} \setminus \{0\}$, we can find w such that $e^w = c$. Let w_n be $w + 2\pi in$, for all $n \in \mathbb{N}$. Then $e^{w_n} = c$. Now take $z_n = 1/w_n$; this sequence tends to 0. Thus there is a z_n as close to 0 as we like for which $e^{1/z_n} = c$. This illustrates Picard's theorem.