School of Mathematics and Statistics
MATH2111 - Higher Several Variable Calculus

10. The Inverse and Implicit Function Theorems

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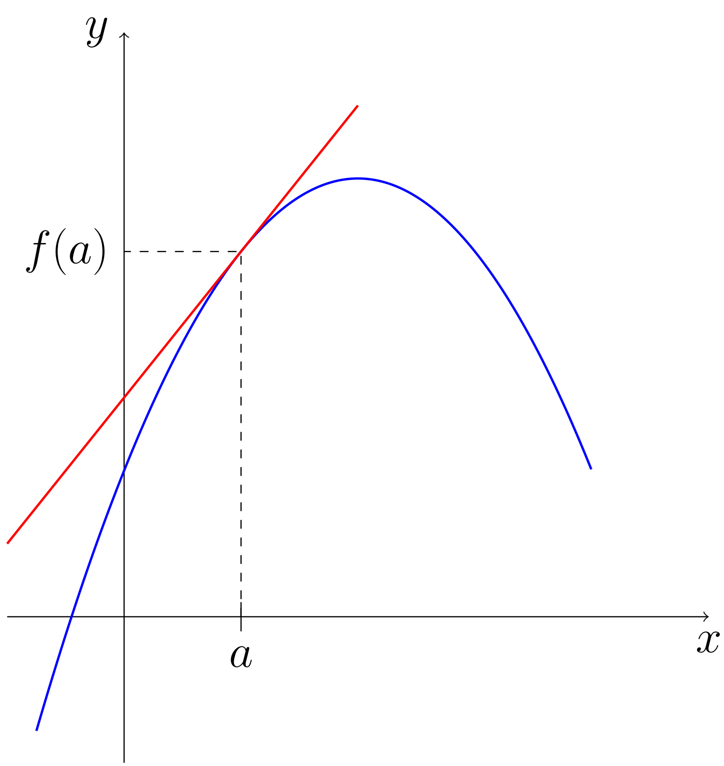
Inversion Function Theorem for $f : \mathbb{R} \to \mathbb{R}$

Recall that the mapping $x \mapsto mx + b$, where $x, m, b \in \mathbb{R}$, is invertible if and only if $m \neq 0$. That is, when the graph of $x \mapsto mx + b$ is not a horizontal line.

This morally gives us the inverse function theorem as taught in MATH1131/41: if $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies $f'(a) \neq 0$, then there is a neighbourhood U of a such that f restricted to U is invertible.

super set of an open set containing a



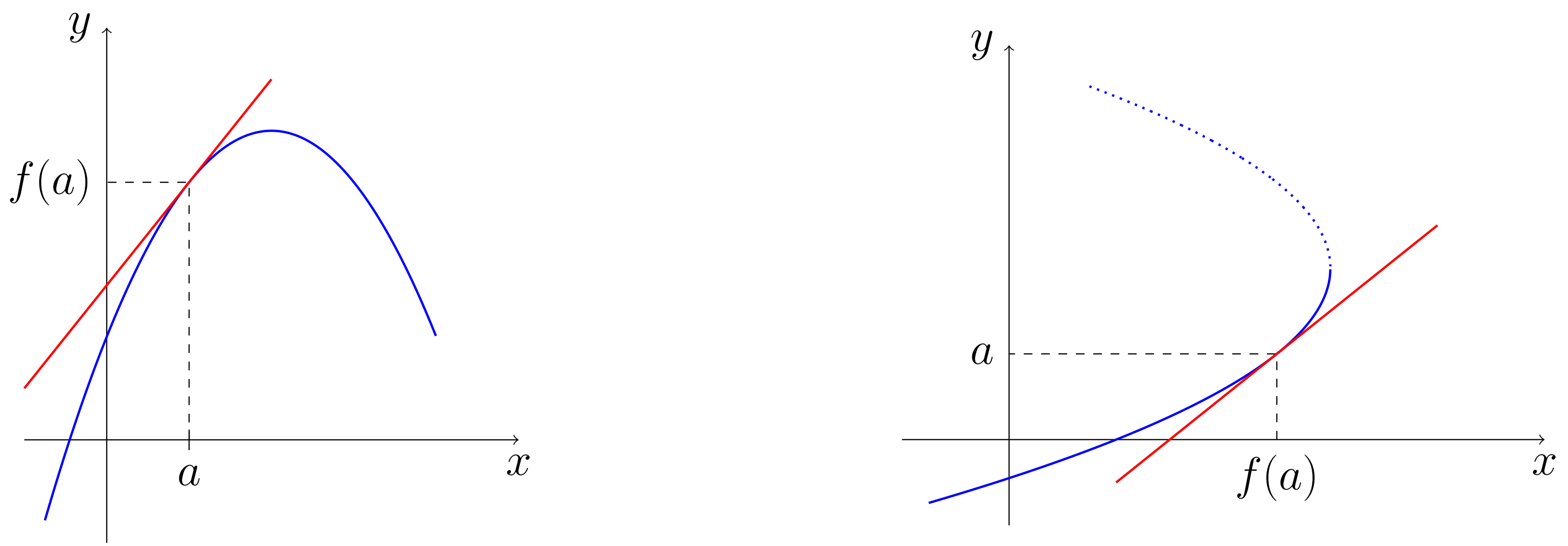
Inversion Function Theorem for $f : \mathbb{R} \to \mathbb{R}$

Inverse Function Theorem for $f : \mathbb{R} \to \mathbb{R}$

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable with $f'(a) \neq 0$. Then there is an open interval $I \ni a$ and a function $g : f(I) \to I$ such that

$$f \circ g = \mathrm{id}_{f(I)} \qquad \text{and} \qquad g \circ f = \mathrm{id}_I.$$

Moreover, g is continuously differentiable with $g'(y) = \dfrac{1}{f'(g(y))}$.



Inversion Function Theorem Remarks

Remarks:

- f may be still locally invertible if $f'(a) = 0$, for example $f(x) = x^3$
- A non-zero derivative does NOT imply local invertibility. Eg:

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f'(0) = \frac{1}{2}$, but $f'\left(\frac{1}{2n\pi}\right) = -\frac{1}{2}$

Inverse Function Theorem for $f : \mathbb{R}^n \to \mathbb{R}^n$

How do we generalise this theorem for a function $f : \mathbb{R}^n \to \mathbb{R}^n$?

The condition that $f'(\mathbf{a}) \neq 0$ is saying that $f'(\mathbf{a})$ is invertible in the real numbers. For $f : \mathbb{R}^n \to \mathbb{R}^n$, its derivative is represented by an $n \times n$ matrix, which could be invertible...

$\longrightarrow J_f(\underset{\sim}{a})$

Inverse Function Theorem for $f : \mathbb{R}^n \to \mathbb{R}^n$

Suppose $\Omega \subseteq \mathbb{R}^n$ is open, $\mathbf{a} \in \Omega$ and $f : \Omega \to \mathbb{R}^n$ is a C^1 function. If $J_f(\mathbf{a})$ is an invertible matrix, then f has a local inverse around $\mathbf{a}$. That is, there are open sets $U \ni \mathbf{a}$ and $V \ni f(\mathbf{a})$ and a function $g : V \to U$ such that

$$g \circ f = \mathrm{id}_U \qquad \text{and} \qquad f \circ g = \mathrm{id}_V.$$

Moreover, g is a C^1 function with $J_g(\mathbf{y}) = \big(J_f(g(\mathbf{y}))\big)^{-1}$ for all $y \in V$.

c.f. $g'(y) = f'(g(y))^{-1}$

in 1D.

Inverse Function Theorem Remarks

- A formula for the inverse is not given, and it's even more impossible in general than 1st year.

- For $\underline{a} \in \mathbb{R}^n$,
$$J_g(f(\underline{a})) = J_f(g(f(\underline{a})))^{-1} = J_f(\underline{a})^{-1}$$

- The best affine approximation for g is
$$T_{f(\underline{a})} g(\underline{y}) = g(f(\underline{a})) + J_g(f(\underline{a}))(\underline{y} - f(\underline{a})) = \underline{a} + J_f(\underline{a})^{-1}(\underline{y} - f(\underline{a}))$$

Inverse Function Theorem Example 1

Example. $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f(x, y) = (x^2 - y^2, 2xy)$

The Jacobian of f is $\begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$

which has determinant

$$(2x)^2 - (2y)(-2y) = 4(x^2 + y^2)$$

So by the Inverse Function Theorem, f has a local inverse everywhere except at the origin $(0, 0)$.

Inverse Function Theorem Example 1

If g denotes a local inverse for $F(1,2) = (-3,4)$, find the best affine approximation for g at the point

$$g(-3,4) = (1,2), \quad J_g(-3,4) = J_f(1,2)^{-1}$$

$$J_f(1,2) = \begin{pmatrix} 2 & -4 \\ 4 & 2 \end{pmatrix} \longrightarrow J_f(1,2)^{-1} = \frac{1}{20}\begin{pmatrix} 2 & 4 \\ -4 & 2 \end{pmatrix}$$

So the required best affine approx is

$$T_{(-3,4)}\, g(u,v) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{20}\begin{pmatrix} 2 & 4 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} u+3 \\ v-4 \end{pmatrix}$$

Inverse Function Theorem Example 2

Exercise. Find when the polar map $p : \mathbb{R}^2 \to \mathbb{R}^2$, $p(r,\theta) = (r\cos\theta, r\sin\theta)$, has a local inverse and, when it exists, find its best affine approximation at

$$(x_0, y_0) = (r_0\cos\theta_0, r_0\sin\theta_0).$$

$$J_p(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \quad |J_p(r,\theta)| = r$$

So p has a local inverse whenever $r \neq 0$

$$J_p(r_0,\theta_0)^{-1} = \frac{1}{r}\begin{pmatrix} r\cos\theta_0 & r_0\sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{pmatrix}$$

Best affine approximation for the local inverse is

$$T_{p^{-1}}(r_0\cos\theta_0, r_0\sin\theta_0) = \begin{pmatrix} r_0 \\ \theta_0 \end{pmatrix} + J_p(r_0,\theta_0)^{-1}\begin{pmatrix} x - r_0\cos\theta_0 \\ y - r_0\sin\theta_0 \end{pmatrix}$$

$p^{-1}(r_0\cos\theta_0, r_0\sin\theta_0)$

Implicit Function Theorem

$\phi \in C^1$

Given a surface $\mathcal{S}$ in $\mathbb{R}^3$ defined by $\phi(x, y, z) = 0$ and $\mathbf{x}_0 \in \mathcal{S}$, can we write $z = f(x, y)$ for all (x, y, z) in the vicinity of $\mathbf{x}_0$?

If so, we would say that the surface is **implicitly defined by** f at the point $\mathbf{x}_0$.

For example, consider the sphere $x^2 + y^2 + z^2 = 1$. For any point (x_0, y_0, z_0) in the upper hemisphere, we can express the surface 'locally' via $z = \sqrt{1 - x^2 - y^2}$ for all (x, y) in an open set $U \ni (x_0, y_0)$. Likewise, we can write $z = -\sqrt{1 - x^2 - y^2}$ for any point in the lower hemisphere.

However, for any point on the equator of the sphere ($z = 0$), we cannot find find a function f that implicitly defines the surface via $z = f(x, y)$.

More generally, it is possible to implicitly define a surface with a function around $\mathbf{x}_0$ when

$$\phi \in C^1 \qquad \text{and} \qquad \frac{\partial \phi}{\partial z}(\mathbf{x}_0) \neq 0.$$

$\frac{\partial \phi}{\partial z} = 2z = 0$ at $z = 0$

$\Rightarrow \quad x^2 + y^2 = 1$ and $z = 0$

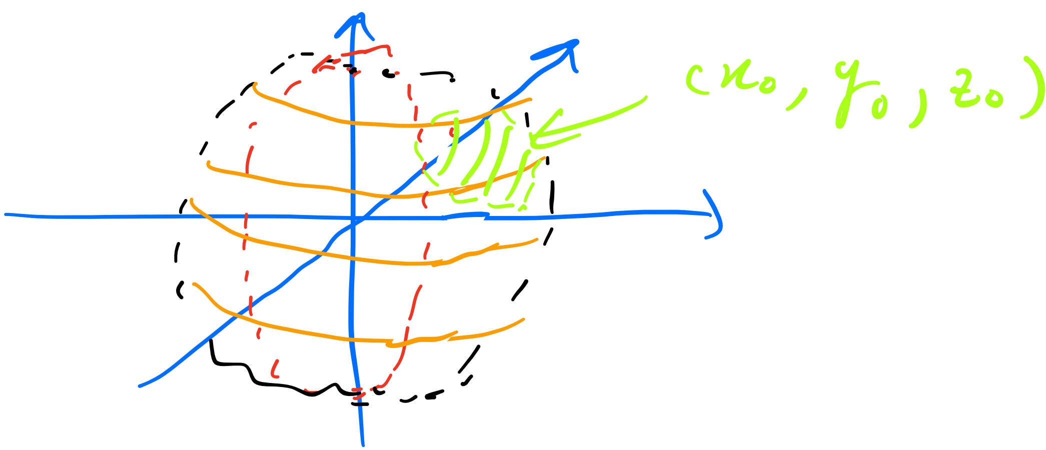
so z cannot be defined in terms of x and y.

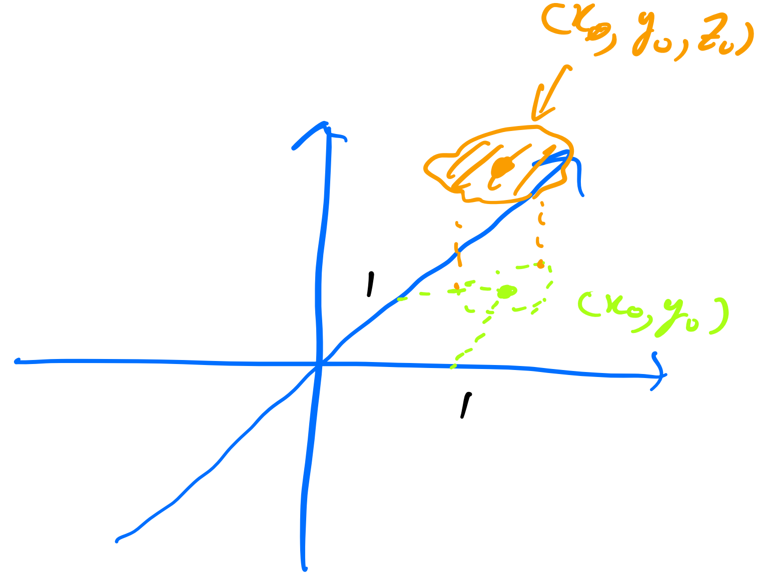
Implicit Function Theorem v1

Implicit Function Theorem (Special Case)

Let $\phi : \mathbb{R}^{m+1} \to \mathbb{R}$ be a C^1 function and express an element of $\mathbb{R}^{m+1}$ as $(\mathbf{x}, z) = (x_1, \ldots, x_m, z)$. Let $(\mathbf{a}, b) \in \mathbb{R}^{m+1}$ satisfy $\phi(\mathbf{a}, b) = 0$. If $\dfrac{\partial \phi}{\partial z}(\mathbf{a}, b) \neq 0$, then there are open sets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}$ and a surjective C^1 function $f : U \to V$ such that $\mathbf{a} \in U$ and $b \in V$ and $\phi(\mathbf{x}, f(\mathbf{x})) = 0$ for all $\mathbf{x} \in U$.

$z = f(x) \quad \forall \underline{x} \in U$





Implicit Function Theorem Example

$\swarrow$ $\phi(1,0,0) = 0$

Example. Consider the surface defined by $\phi(x,y,z) = xy^2 - x^2 + z^3 + e^{xz}\cos y = 0$. Can we write $z = f(x,y)$ in the vicinity of $(1,0,0)$?

Clearly ϕ is C^1. Now,

$$\frac{\partial \phi}{\partial z} = 3z^2 + xe^{xz}\cos y$$

$$\frac{\partial \phi}{\partial z}(1,0,0) = 0 + 1e^0\cos(0) = 1 \neq 0$$

By the Implicit Function Theorem, there is a C^1 function f such that $z = f(x,y)$ if and only if $\phi(x,y,z) = 0$ for all (x,y,z) in an open neighbourhood of $(1,0,0)$. $\swarrow$ $f(1,0) = 0$

Best affine approx: $f(x,y) \approx f(1,0) + \nabla f(1,0) \cdot \begin{pmatrix} x-1 \\ y-0 \end{pmatrix}$

Implicit Function Theorem Example

or chain Rule

To get ∇f, we use Implicit Differentiation

$$0 = \frac{\partial}{\partial x}\left(xy^2 - x^2 + z^3 + e^{xz}\cos y\right) = y^2 - 2x + 3z^2\frac{\partial z}{\partial x} + e^{x}\cos y\,\frac{\partial}{\partial x}(xz)$$

$$= y^2 - 2x + 3z^2\frac{\partial z}{\partial x} + e^{xz}\cos y\left(z + x\frac{\partial z}{\partial x}\right)$$

At $(1,0,0)$: $0 = 0 - 2 + 0 + (1)\cos(0)\left(0 + \frac{\partial z}{\partial x}\right)$

$\Rightarrow \frac{\partial z}{\partial x}(1,0) = 2$

$$0 = \frac{\partial}{\partial y}\left(xy^2 - x^2 + z^3 + e^{xz}\cos y\right) = 2xy + 3z^2\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial y}e^{xz}\cos y - e^{xz}\sin y$$

At $(1,0,0)$: $0 = 0 + 0 + 1\frac{\partial z}{\partial y}e^0\cos(0) - e^0\sin(0)$

$\frac{\partial z}{\partial y} = 0$. So $f(x,y) \approx 2(x-1)$ in the vicinity of $(1,0)$

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Implicit Function Theorem v1 Proof

Proof sketch. Define $G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $G(\underset{\sim}{x}, z) = (\underset{\sim}{x}, \phi(\underset{\sim}{x}, z))$

suppose that $\phi(\underset{\sim}{a}, b) = 0$ with $\frac{\partial \phi}{\partial z}(\underset{\sim}{a}, b) \neq 0$ Then

$$J_G(\underset{\sim}{a}, b) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \vdots \\ 0 & & 1 & & \vdots \\ \vdots & & & \ddots & \\ \phi_{x_1}(\underset{\sim}{a}, b) & \cdots & & & \phi_z(\underset{\sim}{a}, b) \end{pmatrix}$$

This is invertible since $\frac{\partial \phi}{\partial z}(\underset{\sim}{a}, b) \neq 0$

By the Inverse Function Theorem, There are open sets $W_1 \times (-\delta, \delta)$, W_2 and a bijection $F : W_1 \times (-\delta, \delta) \to W_2$ such that $\underset{\sim}{a} \in W_1$, $(\underset{\sim}{a}, b) \in W_2$ and

$$(\underset{\sim}{x}, t) = G(F(\underset{\sim}{x}, z)) = (F_1(\underset{\sim}{x}, z), F_2(\underset{\sim}{x}, z) \cdots F_n(\underset{\sim}{x}, z), \phi(F_1(\underset{\sim}{x}, z), \ldots, F_{n+1}(\underset{\sim}{x}, z)))$$

$\forall (\underset{\sim}{x}, z) \in W_1 \times (-\delta, \delta)$

Implicit Function Theorem v1 Proof

Proof sketch cont. In particular, $x_i = f_i(\underset{\sim}{x}, z)$ $\forall$ $1 \leq i \leq m$, and so

$$z = \phi(x_1, x_2, \ldots, x_m, f_{m+1}(\underset{\sim}{x}, z))$$
$$= \phi(\underset{\sim}{x}, f_{m+1}(\underset{\sim}{x}, z)) \quad \forall\ (\underset{\sim}{x}, z) \in W_1 \times (-\delta, \delta)$$

In particular, setting $z = 0$ gives $\underbrace{\phi(\underset{\sim}{x}, f_{m+1}(\underset{\sim}{x}, 0))}_{\text{This means } (\underset{\sim}{x},\, f_{m+1}(\underset{\sim}{x}, 0)) \in S.} = 0$ for all $\underset{\sim}{x} \in W_1$.

As $(\underset{\sim}{a}, b) \in W_2$, which is open, there are open $U' \subseteq \mathbb{R}$ and $V \in \mathbb{R}$ such that $(\underset{\sim}{a}, b) \in U' \times V \subseteq W_2$

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Implicit Function Theorem v1 Proof

$$G(\underset{\sim}{a}, b) = (\underset{\sim}{a}, \phi(\underset{\sim}{a}, b)) = (\underset{\sim}{a}, 0)$$

Proof sketch cont.

Let $U = \{ \underset{\sim}{x} \in U' : f_{n+1}(\underset{\sim}{x}, 0) \in V \}$, which is open since it is the preimage of an open set under a continuous function.

Define $f : U \longrightarrow V$ by $f(\underset{\sim}{x}) = f_{n+1}(\underset{\sim}{x}, 0)$

Firstly $\underline{a} \in U$ since $f(\underset{\sim}{a}) = f_{n+1}(\underset{\sim}{a}, 0) = f_{n+1}(G(\underset{\sim}{a}, b))$

identity $= (F \circ G)_{n+1}(\underset{\sim}{a}, b) = b$

Earlier, we observed that

$$\phi(\underset{\sim}{x}, f_{n+1}(\underset{\sim}{x}, 0)) = 0 \quad \forall x \in W,$$

$$\iff \phi(\underset{\sim}{x}, f(x)) = 0 \quad \forall x \in W,$$

But wait, there's more!

The slides from here are not examinable!

So given a C^1 function $\phi : \mathbb{R}^{m+1} \to \mathbb{R}$ and the surface defined by $\phi(x_1, \ldots, x_{m+1}) = 0$, we have conditions for when the surface can be locally expressed as the graph of a function.

What if we want to express more variables as functions of the others? For example, given

$$\begin{aligned} e^{xyu} + yuv + x - 3 &= 0 \\ \ln(yv) + xu^3v - x^3u &= 0, \end{aligned}$$

can we express u and v (locally) as functions of x and y?

Implicit Function Theorem v2

Implicit Function Theorem (Marsden & Tromba Ch. 3.5)

Let $\phi : \mathbb{R}^{m+n} \to \mathbb{R}^n$ be a C^1 function and write $(\mathbf{x}, \mathbf{y}) = (x_1, \dots, x_m, y_1, \dots y_m)$ for an element of $\mathbb{R}^{m+n}$. Let $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{m+n}$ satisfy $\phi(\mathbf{a}, \mathbf{b}) = 0$ and consider

$$J_\phi(\mathbf{a}, \mathbf{b}) = \left(\frac{\partial(\phi_1, \dots, \phi_n)}{\partial(x_1, \dots, x_m)} \middle| \frac{\partial(\phi_1, \dots, \phi_n)}{\partial(y_1, \dots, y_n)} \right)$$

$$= \left(\begin{array}{ccc|ccc} \frac{\partial \phi_1}{\partial x_1}(\mathbf{a}, \mathbf{b}) & \cdots & \frac{\partial \phi_1}{\partial x_m}(\mathbf{a}, \mathbf{b}) & \frac{\partial \phi_1}{\partial y_1}(\mathbf{a}, \mathbf{b}) & \cdots & \frac{\partial \phi_1}{\partial y_n}(\mathbf{a}, \mathbf{b}) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial x_1}(\mathbf{a}, \mathbf{b}) & \cdots & \frac{\partial \phi_n}{\partial x_m}(\mathbf{a}, \mathbf{b}) & \frac{\partial \phi_n}{\partial y_1}(\mathbf{a}, \mathbf{b}) & \cdots & \frac{\partial \phi_n}{\partial y_n}(\mathbf{a}, \mathbf{b}) \end{array} \right).$$

If $\frac{\partial(\phi_1, \dots, \phi_n)}{\partial(y_1, \dots, y_n)}$ is invertible, then there are open sets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ and a surjective C^1 function $f : U \to V$ such that $\mathbf{a} \in U$, $\mathbf{b} \in V$, $f(\mathbf{a}) = \mathbf{b}$ and for all $\mathbf{x} \in U$, $\mathbf{y} \in V$, we have that $\phi(\mathbf{x}, \mathbf{y}) = 0$ if and only if $f(\mathbf{x}) = \mathbf{y}$.

Implicit Function Theorem v2

Example. Given the equations

$$
\begin{aligned}
e^{xyu} + yuv + x - 3 = 0 \\
\ln(yv) + xu^3v - x^3u = 0,
\end{aligned}
$$

can u and v be expressed as functions of x and y in the vicinity of $(x_0, y_0, u_0, v_0) = (2, 1, 0, 1)$?

Consider the C^1 function $\phi : \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2$ defined by

$$
\phi(x, y, u, v) = \begin{pmatrix} e^{xyu} + yuv + x - 3 \\ \ln(yv) + xu^3v - x^3u \end{pmatrix},
$$

which satisfies $\phi(2, 1, 0, 1) = 0$. The Jacobian of ϕ at $(2, 1, 0, 1)$ is

$$
J_\phi(2, 1, 0, 1) = \left(\begin{array}{cc|cc} 1 & 0 & 3 & 0 \\ 0 & 1 & -8 & 1 \end{array}\right).
$$

Since $\dfrac{\partial(\phi_1, \phi_2)}{\partial(u, v)} = \begin{pmatrix} 3 & 0 \\ -8 & 1 \end{pmatrix}$ is invertible, the Implicit Function Theorem gives us open sets U and V and a function $f : U \to V$ such that $(2, 1) \in U$, $(0, 1) \in V$, $f(2, 1) = (0, 1)$ and $\phi\big(x, y, f(x, y)\big) = 0$ for all $(x, y) \in U$.

Implict Function Theorem v2

Example cont. Find the best affine approximation to f at $(2,1)$.

$$\begin{aligned}
0 &= \frac{\partial}{\partial x}\left(e^{xyu} + yuv + x - 3\right) \\
&= ye^{xyu}\left(u + x\frac{\partial u}{\partial x}\right) + y\left(v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}\right) + 1 \\
\frac{\partial u}{\partial x}(2,1) &= -\frac{1}{3} \\
0 &= \frac{\partial}{\partial y}\left(e^{xyu} + yuv + x - 3\right) \\
&= xe^{xyu}\left(u + y\frac{\partial u}{\partial y}\right) + uv + yv\frac{\partial u}{\partial y} + yu\frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y}(2,1) &= 0 \\
0 &= \frac{\partial}{\partial x}\left(\ln(yv) + xu^3v - x^3u\right) \\
&= \frac{1}{v}\frac{\partial v}{\partial x} + u^3v + 3xu^2v\frac{\partial u}{\partial x} + xu^3\frac{\partial v}{\partial x} - 3x^2u - x^3\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial x}(2,1) &= -\frac{8}{3}
\end{aligned}$$

Implict Function Theorem v2

Example cont.

$$\begin{aligned} 0 &= \frac{\partial}{\partial y}\left(\ln(yv) + xu^3v - x^3u\right) \\ &= \frac{1}{y} + \frac{1}{v}\frac{\partial v}{\partial y} + x\left(3u^2v\frac{\partial u}{\partial y} + u^3\frac{\partial v}{\partial y}\right) - x^3\frac{\partial u}{\partial y} \end{aligned}$$

$$\frac{\partial v}{\partial y}(1,2) = -1$$

So the best affine approximation to f at $(1,2)$ is

$$f(x,y) \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/3 & 0 \\ -8/3 & -1 \end{pmatrix}\begin{pmatrix} x-2 \\ y-1 \end{pmatrix}$$

Inverse $\Longleftrightarrow$ Implict

We saw earlier how the Inverse Function Theorem can give us the Implicit Function Theorem. Suppose that $\mathbf{c} \in \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function such that $J_g(\mathbf{c})$ is invertible. Define a C^1 function $\phi : \mathbb{R}^{2n} \to \mathbb{R}^n$ by

$$\phi(\mathbf{x}, \mathbf{y}) = \mathbf{x} - g(\mathbf{y}).$$

Then $\phi(g(\mathbf{c}), \mathbf{c}) = 0$ and $J_\phi(\mathbf{c}) = \big(I| - J_g(\mathbf{c})\big)$, and so the Implicit Function Theorem yields open sets $U, V \subseteq \mathbb{R}^n$ and a surjective function $f : U \to V$ such that $g(\mathbf{c}) \in U$, $\mathbf{c} \in V$, $f(g(\mathbf{c})) = \mathbf{c}$ and for all $\mathbf{x} \in U$ and $\mathbf{y} \in V$, we have that $\phi(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ if and only if $f(\mathbf{x}) = \mathbf{y}$. That is,

$$\mathbf{0} = \phi(\mathbf{x}, f(\mathbf{x})) = \mathbf{x} - g(f(\mathbf{x}))$$

and so $g \circ f = \mathrm{id}_U$. Moreover, U and V can be chosen so that the continuity of g implies that $g(\mathbf{y}) \in U$ for all $\mathbf{y} \in V$. So for any $\mathbf{y} \in V$,

$$\phi\big(g(\mathbf{y}), \mathbf{y})\big) = g(\mathbf{y}) - g(\mathbf{y}) = \mathbf{0},$$

and so $f(g(\mathbf{y})) = \mathbf{y}$.

Proving one of the Inverse or Implicit Function Theorems is very challenging. One method uses the Contraction Mapping Theorem (MATH3611).