MATH2621 — Higher Complex Analysis. XV
Contour Integrals

This lecture?

In this lecture, we define contour integrals, and compute some examples.

Curves

We define a curve in $\mathbb{C}$ much as in $\mathbb{R}^2$: a curve $t \mapsto (\gamma_1(t), \gamma_2(t))$ is replaced by a curve $t \mapsto \gamma_1(t) + i\gamma_2(t)$. The definitions of simple and closed curves are almost identical, as are joins, reverse curves, reparametrisations, and orientations.

The derivative of a curve

Definition

Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a curve and

$$\gamma(t) = \gamma_1(t) + i\gamma_2(t),$$

where $\gamma_1, \gamma_2 : [a, b] \to \mathbb{R}$. Then we define

$$\gamma'(t) = \gamma_1'(t) + i\gamma_2'(t),$$

when both $\gamma_1'(t)$ and $\gamma_2'(t)$ exist.

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when both $\gamma_1'(t)$ and $\gamma_2'(t)$ exist.

That is,

$$\mathrm{Re}(\gamma_1') = (\mathrm{Re}(\gamma))' \quad \text{and} \quad \mathrm{Im}(\gamma') = (\mathrm{Im}(\gamma))'.$$

Integrals with complex integrands

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The usual rules of differentiation hold for complex curves and functions defined thereon, and we do not always need to break things up into their real and imaginary parts in order to differentiate.

In particular, if f is a holomorphic function and α is a curve in the complex plane, then

$$\frac{d}{dt} f(\alpha(t)) = f'(\alpha(t))\, \alpha'(t).$$

We can prove this in several ways.

Smooth curves and contours

We say that $\gamma : [a, b] \to \mathbb{C}$ is continuously differentiable if γ' is defined and continuous on $[a, b]$, and smooth if γ is continuously differentiable and moreover $\gamma'(t) \neq 0$ for all $t \in [a, b]$. The curve γ is piecewise smooth if it is the join of finitely many smooth curves.

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A contour is the analogue in the complex plane of a path in $\mathbb{R}^2$. We may extend the notation for joins and reverse curves in $\mathbb{R}^2$ to curves in $\mathbb{C}$ and hence to contours.

Usually we deal with simple contours.

Examples

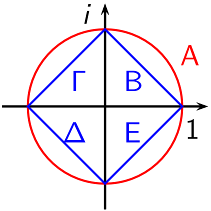
$$\alpha(t) = \cos(t) + i\sin(t) = e^{it} \quad \forall t \in [0, 2\pi]$$

$$\beta(t) = (1 - t) + it \quad \forall t \in [0, 1]$$

$$\gamma(t) = -t + i(1 - t) \quad \forall t \in [0, 1]$$

$$\delta(t) = (t - 1) - it \quad \forall t \in [0, 1]$$

$$\varepsilon(t) = t + i(t - 1) \quad \forall t \in [0, 1]$$



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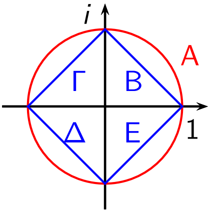

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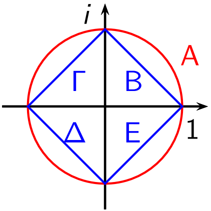

What are the derivatives of these curves?

In the absence of other information, we assume that closed contours have the standard orientation.

Integral of a complex-valued function

Definition

Suppose that $u, v : [a, b] \to \mathbb{R}$ are real-valued functions, and that $f : [a, b] \to \mathbb{C}$ is given by $f = u + iv$. We define

$$\int_a^b f(t)\,dt = \int_a^b \big(u(t) + iv(t)\big)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt,$$

provided that the two real integrals on the right hand side exist.

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provided that the two real integrals on the right hand side exist.

That is,

$$\mathrm{Re}\left(\int_a^b f(t)\,dt\right) = \int_a^b \mathrm{Re}(f(t))\,dt$$

and

$$\mathrm{Im}\left(\int_a^b f(t)\,dt\right) = \int_a^b \mathrm{Im}(f(t))\,dt.$$

Properties of integration

For $a, b, c, d \in \mathbb{R}$, $\lambda, \mu \in \mathbb{C}$, a real-valued differentiable function $h : [c, d] \to [a, b]$ such that $h(c) = a$ and $h(d) = b$, and complex-valued functions f and g,

$$\int_a^b \lambda f(t) + \mu g(t)\,dt = \lambda \int_a^b f(t)\,dt + \mu \int_a^b g(t)\,dt$$

$$\int_c^d f(h(t))\,h'(t)\,dt = \int_a^b f(s)\,ds$$

$$\int_a^b f'(t)\,g(t)\,dt = \Big[f(b)\,g(b) - f(a)\,g(a)\Big] - \int_a^b f(t)\,g'(t)\,dt$$

$$\int_a^b e^{\lambda t}\,dt = \left[\frac{e^{\lambda t}}{\lambda}\right]_{t=a}^{t=b} = \frac{e^{\lambda b} - e^{\lambda a}}{\lambda}$$

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt\,.$$

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Evaluate $\displaystyle\int_0^\pi t\, e^{it}\, dt$.

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Taking real and imaginary parts shows that $\int_0^\pi t\cos(t)\,dt = -2$ and $\int_0^\pi t\sin(t)\,dt = \pi$. It is much easier to find an integral like $\int e^{(1+i)t}\,dt$ in this way than to evaluate $\int e^t\cos(t)\,dt$.

Complex line integrals

Definition

Given a (not necessarily simple) piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$ and a continuous (not necessarily differentiable) function f defined on the range of γ, we define the *complex line integral* $\int_\gamma f(z)\,dz$ by

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt,$$

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provided that the integral on the right hand side exists.

In general,

$$\mathrm{Re}\left(\int_\gamma f(z)\, dz\right) \neq \int_\gamma \mathrm{Re}(f(z))\, dz,$$

because $\gamma'(t)$ may not be purely real.

How to remember this formula

To remember this formula, write $z(t)$ rather than $\gamma(t)$: then $dz = \dfrac{dz}{dt}\,dt$, and

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,\frac{dz}{dt}\,dt.$$

If we now convert back to writing $\gamma(t)$, which is a good idea, because we already use z to mean too many different things, then we get the formula in the definition.

Properties of complex line integrals

Theorem

Suppose that $\lambda, \mu \in \mathbb{C}$, *that* $\gamma : [a, b] \to \mathbb{C}$ *is a piecewise smooth curve, and that* f *and* g *are complex functions defined on* Range(γ). *Then the following hold.*

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(a) *The integral is linear:*

$$\int_\gamma \lambda f(z) + \mu g(z)\, dz = \lambda \int_\gamma f(z)\, dz + \mu \int_\gamma g(z)\, dz.$$

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(a) *The integral is linear:*

$$\int_\gamma \lambda f(z) + \mu g(z)\, dz = \lambda \int_\gamma f(z)\, dz + \mu \int_\gamma g(z)\, dz.$$

(b) *The integral is independent of parametrisation: if δ is a reparametrisation of γ that is also a piecewise smooth curve, then*

$$\int_\gamma f(z)\, dz = \int_\delta f(z)\, dz.$$

Properties of complex line integrals (continued)

(c) *The integral is additive for joins: if* $\gamma = \alpha \sqcup \beta$, *then*

$$\int_\gamma f(z)\,dz = \int_\alpha f(z)\,dz + \int_\beta f(z)\,dz.$$

Properties of complex line integrals (continued)

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(e) *We may estimate the size of the integral:*

$$\left|\int_\gamma f(z)\,dz\right| \leq ML,$$

where L is the length of γ and M is a number such that $|f(z)| \leq M$ for all $z \in \mathrm{Range}(\gamma)$.

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$$\begin{aligned}\left|\int_{\gamma} f(z)\,dz\right| = \left|\int_{\gamma} f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right| \\ \leq \int_a^b \left|f(\gamma(t))\right| \left|\gamma'(t)\right| dt \leq \int_a^b M \left|\gamma'(t)\right| dt = ML,\end{aligned}$$

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Part (e) of the theorem is often called the *ML* Lemma. Note that M is a *maximiser* but need not be the *maximum*.

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Definition

We define

$$\int_\Gamma f(z)\,dz = \int_\gamma f(z)\,dz,$$

where γ is any parametrisation of Γ.

Exercise 2

Suppose that $p, q \in \mathbb{C}$, that Γ is the line segment from p to q, and that $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$. Find $\displaystyle\int_\Gamma \lambda_1 + \lambda_2 z + \lambda_3 \bar{z} + \lambda_4 e^z \, dz$.

Answer.

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$$\gamma(t) = (1-t)p + tq = p + t(q-p),$$

where $0 \le t \le 1$. Then $\gamma'(t) = q - p$.

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First

$$\int_\gamma dz = \int_0^1 \gamma'(t)\, dt = \int_0^1 (q-p)\, dt = q - p.$$

Answer to Exercise 2

Next

$$\int_\gamma z\,dz = \int_0^1 \gamma(t)\,\gamma'(t)\,dt = \int_0^1 [p + t(q-p)]\,(q-p)\,dt$$
$$= p(q-p) + \tfrac{1}{2}(q-p)^2 = \frac{q^2 - p^2}{2}.$$

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Further,

$$\int_\gamma \overline{z}\,dz = \int_0^1 \overline{\gamma(t)}\,\gamma'(t)\,dt = \int_0^1 \left(\overline{p} + t\overline{(q-p)}\right)(q-p)\,dt$$
$$= \overline{p}(q-p) + \tfrac{1}{2}\overline{(q-p)}(q-p) = \frac{(\overline{q}+\overline{p})(q-p)}{2}.$$

Answer to Exercise 2 (continued)

Finally,

$$\int_\gamma e^z\,dz = \int_0^1 e^{\gamma(t)}\,\gamma'(t)\,dt = \int_0^1 e^p e^{t(q-p)}\,(q-p)\,dt$$
$$= e^p\left[e^{t(q-p)}\right]_{t=0}^{t=1} = e^p\left(e^{(q-p)} - 1\right) = e^q - e^p.$$

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Hence

$$\int_\gamma \lambda_1 + \lambda_2 z + \lambda_3\overline{z} + \lambda_4 e^z\,dz$$
$$= \lambda_1(q-p) + \lambda_2\frac{q^2-p^2}{2} + \lambda_3\frac{(\overline{q}+\overline{p})(q-p)}{2} + \lambda_4(e^q-e^p).$$

△

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&= \lambda_1 (q - p) + \lambda_2 \frac{q^2 - p^2}{2} + \lambda_4 (e^q - e^p).
\end{aligned}$$

This approach fails when $\lambda_3 \neq 0$, because we cannot find a function of t whose derivative is $\overline{\gamma(t)}\, \gamma'(t)$, or at least not easily. At this point, we do not know whether such a function might exist, but later we will see that this is problematic.

Exercise 3

Suppose that Γ is the circle $\{z \in \mathbb{C} : |z| = r\}$ with the standard orientation, traversed k times, where $k \in \mathbb{Z}^+$, and that $n \in \mathbb{Z}$. Compute $\displaystyle\int_\Gamma z^n \, dz$.

Answer.

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Answer. First, we parametrise Γ: define $\gamma(t) = re^{it}$ where $0 \leq t \leq 2k\pi$, so $\gamma'(t) = ire^{it}$. Now $\int_\Gamma z^n \, dz = \int_\gamma z^n \, dz$, and ...

Answer to Exercise 3

$$\int_\gamma z^n \, dz$$

Answer to Exercise 3

$$\int_\gamma z^n \, dz = \int_0^{2k\pi} \gamma(t)^n \, \gamma'(t) \, dt$$

Answer to Exercise 3

$$\begin{aligned}\int_\gamma z^n \, dz &= \int_0^{2k\pi} \gamma(t)^n \, \gamma'(t) \, dt \\ &= \int_0^{2k\pi} r^n e^{int} \, ire^{it} \, dt\end{aligned}$$

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\begin{aligned}
\int_\gamma z^n \, dz &= \int_0^{2k\pi} \gamma(t)^n \, \gamma'(t) \, dt \\
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&= ir^{n+1} \int_0^{2k\pi} e^{i(n+1)t} \, dt
\end{aligned}
$$

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&= ir^{n+1} \left[\frac{e^{i(n+1)t}}{i(n+1)} \right]_{t=0}^{t=2k\pi}
\end{aligned}$$

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\int_\gamma z^n\,dz &= \int_0^{2k\pi} \gamma(t)^n\,\gamma'(t)\,dt \\
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&= ir^{n+1}\int_0^{2k\pi} e^{i(n+1)t}\,dt \\
&= ir^{n+1}\left[\frac{e^{i(n+1)t}}{i(n+1)}\right]_{t=0}^{t=2k\pi} \\
&= ir^{n+1}\left(\frac{e^{2k\pi i(n+1)} - e^0}{i(n+1)}\right)
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&= ir^{n+1}\left[\frac{e^{i(n+1)t}}{i(n+1)}\right]_{t=0}^{t=2k\pi} \\
&= ir^{n+1}\left(\frac{e^{2k\pi i(n+1)} - e^0}{i(n+1)}\right) \\
&= 0,
\end{aligned}$$

unless $n = -1$, in which case . . .

Answer to Exercise 3 (continued)

$$\int_\gamma z^{-1}\,dz = \int_0^{2k\pi} r^{-1}e^{-it}\,ire^{it}\,dt = \int_0^{2k\pi} i\,dt = 2k\pi i. \qquad \triangle$$

Alternatively . . .

We may also write, when $n \neq 0$,

$$\int_\gamma z^n \, dz = \int_0^{2k\pi} \gamma^n(t)\,\gamma'(t)\,dt = \int_0^{2k\pi} \frac{1}{(n+1)} \frac{d\gamma^{n+1}(t)}{dt}\,dt$$

$$= \left[\frac{\gamma^{n+1}(t)}{n+1}\right]_0^{2k\pi} = 1 - 1 = 0.$$

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When $n = -1$,

$$\int_\gamma \frac{1}{z}\,dz = \int_0^{2k\pi} \frac{\gamma'(t)}{\gamma(t)}\,dt = \int_0^{2k\pi} \frac{d\log\gamma(t)}{dt}\,dt$$

$$= \Big[\log\gamma(t)\Big]_0^{2k\pi} = 2k\pi i,$$

where we take a branch of log *that varies continuously along the curve.*

Alternatively ...

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where we take a branch of log *that varies continuously along the curve.*

We would get the same answer if the parameter t varied from $-k\pi$ to $k\pi$.