School of Mathematics and Statistics
MATH2111 - Higher Several Variable Calculus

4. Limits and Functions

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Limit of a function at a point

Definition

$\Omega \cup \partial\Omega$

Suppose that $\mathbf{b} \in \mathbb{R}^m$, $\Omega \subseteq \mathbb{R}^n$, $\mathbf{a} \in \overline{\Omega}$ and $f : \Omega \to \mathbb{R}^m$. We say that $f(\mathbf{x})$ **converges to $\mathbf{b}$ as $\mathbf{x}$ approaches $\mathbf{a}$**, written $f(\mathbf{x}) \to \mathbf{b}$ **as** $\mathbf{x} \to \mathbf{a}$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\mathbf{x} \in \Omega$, $0 < d(\mathbf{x}, \mathbf{a}) < \delta$ implies $d(f(\mathbf{x}), \mathbf{b}) < \varepsilon$.
In quantifiers and in terms of balls,

$$\forall \varepsilon > 0 \qquad \exists \delta > 0 \qquad \forall x \in \Omega \cap B(\mathbf{a}, \delta)\backslash\{\mathbf{a}\} \qquad f(\mathbf{x}) \in B(\mathbf{b}, \varepsilon).$$

We often write $\lim\limits_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = \mathbf{b}$.

As before, when the limit of a function exists, it is unique:

suppose that $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = \underset{\sim}{b_1}$ and $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = \underset{\sim}{b_2}$

fix $\varepsilon > 0$. Then there is $\delta > 0$ st. for all $\underset{\sim}{x} \in B(\underset{\sim}{a}, \delta)$ we have that $d(f(\underset{\sim}{x}), \underset{\sim}{b_1})$, $d(f(\underset{\sim}{x}), \underset{\sim}{b_2}) < \frac{\varepsilon}{2}$. So

$0 \leq d(\underset{\sim}{b_1}, \underset{\sim}{b_2}) \leq d(f(\underset{\sim}{x}), \underset{\sim}{b_1}) + d(f(\underset{\sim}{x}), \underset{\sim}{b_2}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Since ε was arbitrary, it must be that $d(\underset{\sim}{b_1}, \underset{\sim}{b_2}) = 0$.

Exercise: Limit of a function at a point

Let $\Omega = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $f : \Omega \to \mathbb{R}$ be defined by $f(x, y) = \dfrac{x^4 + x^2 + y^2 + y^4}{x^2 + y^2}$.

Note that Ω is open, f is not defined at $\mathbf{0}$ but $\mathbf{0} \in \overline{\Omega}$.

Claim. $f(x, y) \to 1$ as $(x, y) \to \mathbf{0}$.

Let $\varepsilon > 0$, choose $\delta = \sqrt{\varepsilon}$. Then if $d((x,y),(0,0)) < \delta$, we have $d(f(x,y),1)$

$$= \left| \frac{x^4 + x^2 + y^2 + y^4}{x^2 + y^2} - 1 \right| = \frac{x^4 + y^4}{x^2 + y^2}$$

$$\leq \frac{x^4 + y^4 + 2x^2y^2}{x^2 + y^2}$$

$$= x^2 + y^2 = d((x,y),(0,0))^2$$

$$< \delta^2 = \varepsilon.$$

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Limits of functions via components

Theorem

Suppose that $\mathbf{b} = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$, $\Omega \overset{\subseteq}{\not\subseteq} \mathbb{R}^n$, $\mathbf{a} \in \overline{\Omega}$ and $f : \Omega \to \mathbb{R}^m$ with components $f_i : \Omega \to \mathbb{R}$, $1 \leqslant i \leqslant m$ satisfying

$$f(\mathbf{x}) = \big(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x})\big).$$

Then $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{b}$ if and only if $\lim\limits_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = b_i$ for all $1 \leqslant i \leqslant m$.

This theorem is useful for showing limits exist.

Example. Consider $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}^2$, $f(t) = \left(\dfrac{e^t - 1}{t}, \dfrac{\sin t}{t}\right)$.

Since $\dfrac{e^t - 1}{t}, \dfrac{\sin t}{t} \to 1$ as $t \to 0$, we have that $f(t) \to (1, 1)$ as $t \to 0$.

Limits of functions via sequences

Theorem

Suppose that $\mathbf{b} \in \mathbb{R}^m$, $\Omega \subseteq \mathbb{R}^n$, $\mathbf{a} \in \overline{\Omega}$ and $f : \Omega \to \mathbb{R}^m$. Then $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \mathbf{b}$ if and only if for every sequence $\{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq \Omega$ with $\mathbf{x}_k \to \mathbf{a}$, we have that $\lim\limits_{k\to\infty} f(\mathbf{x}_k) = \mathbf{b}$.

The above theorem is particularly useful for showing that a limit does NOT exist.

The proof is a problem sheet exercise.

Exercise. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist, where $f(x,y) = \dfrac{xy}{x^2+y^2}$.

$k \in \mathbb{Z}^+$, $\underset{\sim}{x}_k = (\frac{1}{k}, 0)$, $f(\underset{\sim}{x}_k) = \dfrac{0}{\frac{1}{k^2}+0} = 0 \to 0$ as $k \to \infty$

But if $\underset{\sim}{x}_k = (\frac{1}{k}, \frac{1}{k})$, then $f(\underset{\sim}{x}_k) = \dfrac{\frac{1}{k^2}}{\frac{2}{k^2}} = \frac{1}{2}$

Since limits are unique $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE.

Limits of functions via sequences

TIP!

Often to show that a limit does not exist, we will use one of the axes and a diagonal, as in the previous example.

But sometimes, we need to be a bit more creative!

Exercise. Show that $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist, where $g(x,y) = \dfrac{x^2y}{x^4+y^2}$.

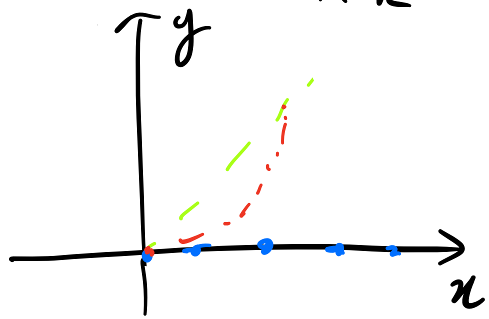
if $\underset{\sim}{x}_k = \left(\frac{1}{k}, 0\right)$, $k \in \mathbb{Z}^+$, then $g\left(\frac{1}{k}, 0\right) = 0 \to 0$

if $\underset{\sim}{x}_k = \left(\frac{1}{k}, \frac{1}{k}\right)$, then $g\left(\frac{1}{k}, \frac{1}{k}\right) = \dfrac{\frac{1}{k^3}}{\frac{1}{k^4}+\frac{1}{k^2}} \times \dfrac{k^4}{k^4}$

$= \dfrac{k}{1+k^2} \to 0$ as $k \to \infty$

if $\underset{\sim}{x}_k = \left(\frac{1}{k}, \frac{1}{k^2}\right)$,

$g\left(\frac{1}{k}, \frac{1}{k^2}\right) = \dfrac{\frac{1}{k^4}}{\frac{1}{k^4}+\frac{1}{k^4}} = \dfrac{1}{2}$



Algebra of Limits

Theorem

Suppose that $\Omega \subseteq \mathbb{R}^n$, $\mathbf{x}_0 \in \overline{\Omega}$, $a, b \in \mathbb{R}$, and $f, g : \Omega \to \mathbb{R}$ are such that $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = a$ and $\lim_{\mathbf{x} \to \mathbf{x}_0} g(\mathbf{x}) = b$. Then

- $\lim_{\mathbf{x} \to \mathbf{x}_0} (f + g)(\mathbf{x}) = a + b$;
- $\lim_{\mathbf{x} \to \mathbf{x}_0} (fg)(\mathbf{x}) = ab$;
- $\lim_{\mathbf{x} \to \mathbf{x}_0} (f/g)(\mathbf{x}) = \dfrac{a}{b}$, provided $b \neq 0$.

Corollary

Suppose that $\Omega \subseteq \mathbb{R}^n$, $\mathbf{x}_0 \in \overline{\Omega}$, and $f, g : \Omega \to \mathbb{R}^m$ are such that $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{a}$ and $\lim_{\mathbf{x} \to \mathbf{x}_0} g(\mathbf{x}) = \mathbf{b}$. Then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} (f + g)(\mathbf{x}) = \mathbf{a} + \mathbf{b}.$$

Pinching Theorem

Theorem

Suppose that $\Omega \subseteq \mathbb{R}^n$, $\mathbf{a} \in \overline{\Omega}$, $\varepsilon > 0$, and $f, g, h : \Omega \to \mathbb{R}$ satisfy $g(\mathbf{x}) \leqslant f(\mathbf{x}) \leqslant h(\mathbf{x})$ for all $\mathbf{x} \in \Omega \cap B(\mathbf{a}, \varepsilon) \setminus \{\mathbf{a}\}$. If

$$\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = b = \lim_{\mathbf{x}\to\mathbf{a}} h(\mathbf{x}),$$

then $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = b$.

Exercise. Consider $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined by $f(x,y) = \dfrac{x^2 y}{x^2 + y^2}$. Show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

$2|xy| \leq x^2 + y^2$

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First bound $|f(x,y)|$ as follows:

$$|f(x,y)| = \left| \frac{x^2 y}{x^2+y^2} \right| = |x| \left| \frac{xy}{x^2+y^2} \right| \leq |x| \frac{1}{2}$$

Pinching Theorem

Ex. cont. Consider $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ defined by $f(x,y) = \dfrac{x^2 y}{x^2+y^2}$. Show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

So if $g(x,y) = -\dfrac{|x|}{2}$ and $h(x,y) = \dfrac{|x|}{2}$

then $g(x,y) \le f(x,y) \le h(x,y)$ and

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0 = \lim_{(x,y)\to(0,0)} h(x,y)$$

By the pinching theorem

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Definition of Continuity

Definition

Suppose that $\Omega \subseteq \mathbb{R}^n$, $\mathbf{a} \in \overset{\Omega}{\cancel{\mathbb{R}^n}}$, and $f : \Omega \to \mathbb{R}^m$. Then f **is continuous at** $\mathbf{a}$ if $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$. That is,

$$\forall \varepsilon > 0 \qquad \exists \delta > 0 \qquad \forall \mathbf{x} \in \Omega \cap B(\mathbf{a}, \delta) \qquad f(\mathbf{x}) \in B(f(\mathbf{a}), \varepsilon).$$

If f is continuous at $\mathbf{a}$ for all $\mathbf{a} \in \Omega$, we say that f **is continuous on** Ω.

Example. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

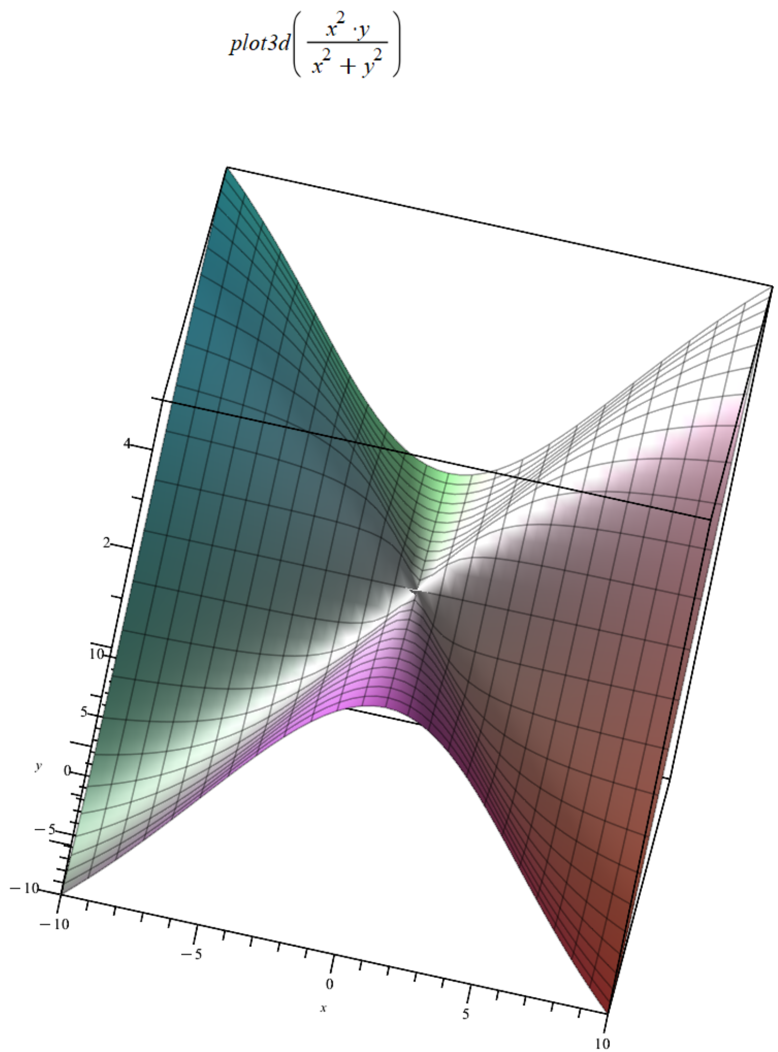
is continuous on $\mathbb{R}^2$, whereas the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

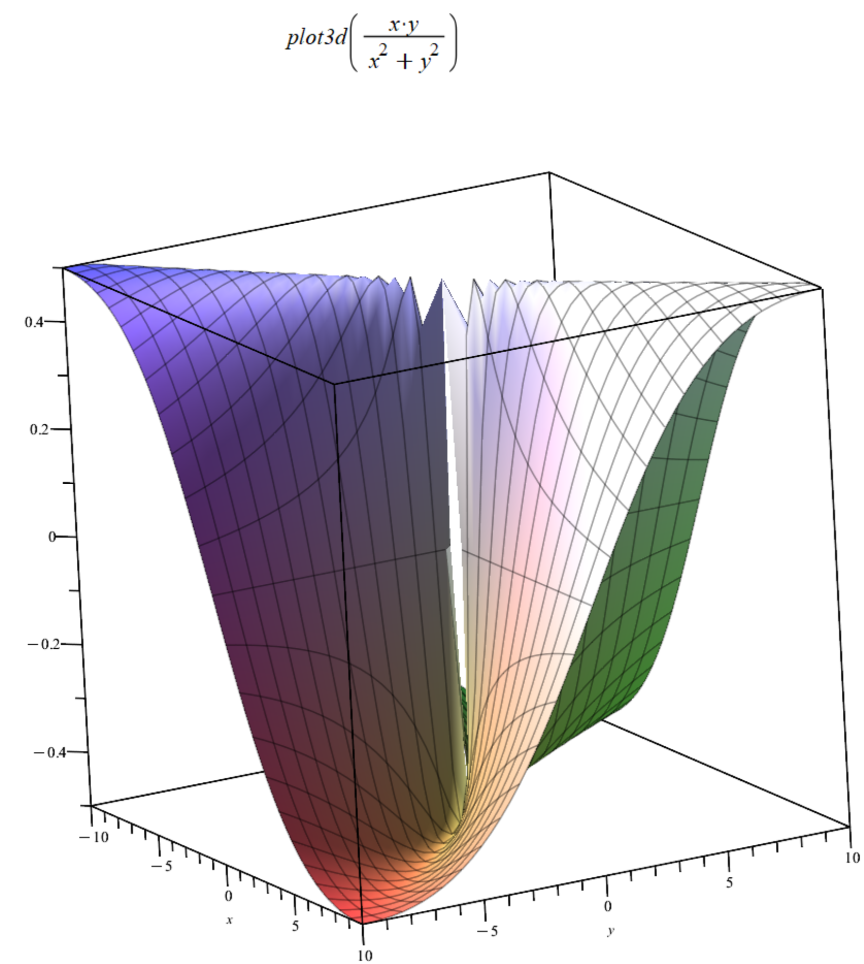
$$g(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$ since $\lim_{(x,y)\to(0,0)} g(x,y)$ was shown earlier not to exist.

Continuity on Maple

"with(plots);"





Continuity Theorems

Theorem (Continuity by components)

Let $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$. Then f is continuous at $\mathbf{a}$ if and only if every component of f is continuous at $\mathbf{a}$.

Theorem (Continuity via sequences)

Let $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$. Then f is continuous at $\mathbf{a}$ if and only if for every sequence $\{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq \Omega$ converging to $\mathbf{a}$, we have that $\{f(\mathbf{x}_k)\}_{k=1}^{\infty}$ converges to $f(\mathbf{a})$.

As with sequences, the first theorem is useful for claiming that a function is continuous, and the second theorem is useful for claiming that a function is not continuous.

Algebra of Continuous Functions

Theorem

Let $\Omega \subseteq \mathbb{R}^n$, $c \in \mathbb{R}$, and $f, g : \Omega \to \mathbb{R}$ be continuous on Ω. Then cf, $f + g$ and fg are continuous on Ω, and f/g is continuous on $\Omega \backslash \{\mathbf{x} \in \Omega \, : \, g(\mathbf{x}) = 0\}$.

Via continuity of components, we get the following.

Corollary

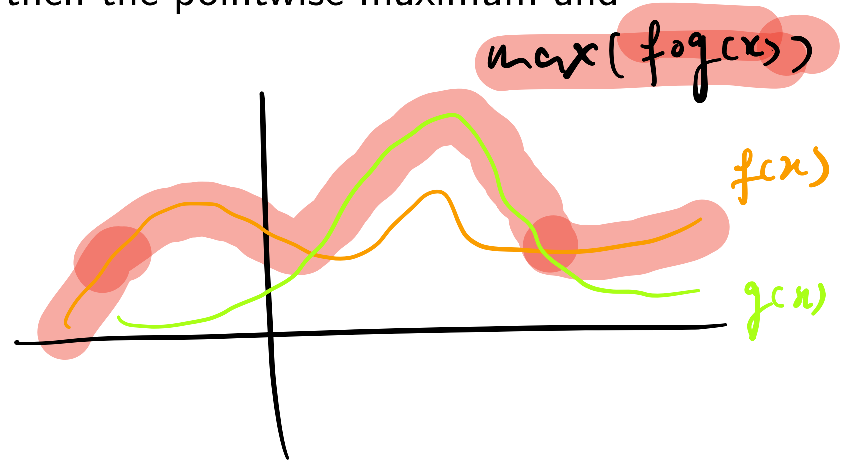
Let $\Omega \subseteq \mathbb{R}^n$, $c \in \mathbb{R}$, and $f, g : \Omega \to \mathbb{R}^m$ be continuous on Ω. Then cf and $f + g$ are continuous on Ω.

Theorem

Suppose that $f : \Omega \to \mathbb{R}^n$ and $g : f(\Omega) \to \mathbb{R}^m$ are continuous on their respective domains. Then $g \circ f : \Omega \to \mathbb{R}^m$ defined by $g \circ f(\mathbf{x}) = g(f(\mathbf{x}))$ is continuous on Ω.

Algebra of Continuous Functions

Exercise. Show that if $f, g : \Omega \to \mathbb{R}$ are continuous, then the pointwise maximum and minimum $\max(f,g)$ and $\min(f,g)$ are continuous.



$$\max(a,b) = \frac{a+b+|a-b|}{2}$$

So $\max(f,g(x))$

$$= \frac{f(x)+g(x)+|f(x)-g(x)|}{2}$$

which is a combinations of continuous functions and hence is continuous.

similarly $\min(a,b) = \dfrac{f(x)+g(x)-|f(x)-g(x)|}{2}$

hence is continuous.