MATH2621 — Higher Complex Analysis. XIV
Path Integrals

This lecture?

In this lecture, we review (or introduce) curves, paths, and line integrals.

Curves

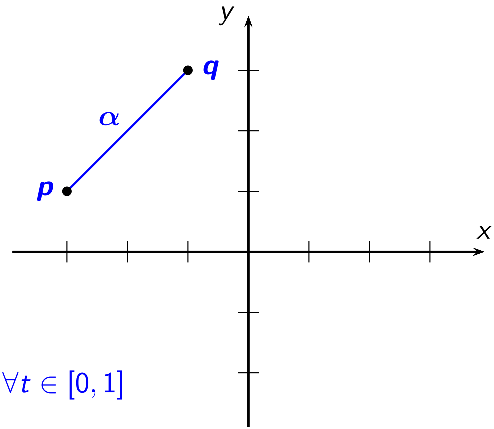
Definition

A *curve* γ in $\mathbb{R}^2$ is a continuous function from an interval $[a, b]$ of real numbers into $\mathbb{R}^2$. We may write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, where γ_1 and γ_2 are real-valued; then the continuity of γ is equivalent to the continuity of both γ_1 and γ_2.

The *initial point* of the curve is $\gamma(a)$ and the *final point* of the curve is $\gamma(b)$. The *range* of the curve is the set of points $\{\gamma(t) : t \in [a, b]\}$.

A curve $\gamma : [a, b] \to \mathbb{R}^2$ is said to be *closed* if $\gamma(a) = \gamma(b)$, and *simple* if $\gamma(s) \neq \gamma(t)$ when $s < t$, except perhaps if $s = a$ and $t = b$.

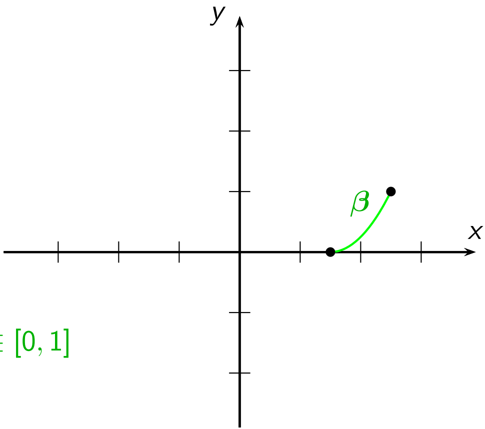
Example A



$$\alpha(t) = \boldsymbol{p} + (\boldsymbol{q} - \boldsymbol{p})t$$
$$= \boldsymbol{p}(1 - t) + \boldsymbol{q}t \quad \forall t \in [0, 1]$$

The curve α moves from $\boldsymbol{p}$ to $\boldsymbol{q}$. It is simple but not closed.

Example B

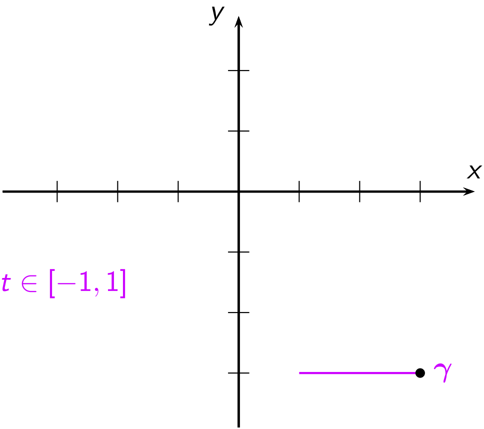


$$\beta(t) = \left(\tfrac{3}{2} + t, t^2\right) \quad \forall t \in [0, 1]$$

The curve β moves up the parabolic arc. It is simple but not closed.

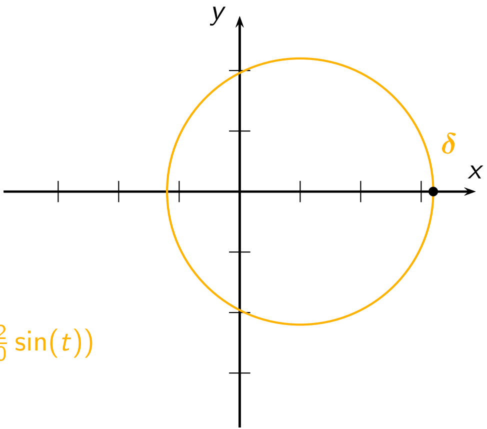
Example C

$$\gamma(t) = (1 + 2t^2, -3) \ \ \forall t \in [-1, 1]$$



The curve γ moves from the right to the left of the line segment and then back again. It is closed but not simple.

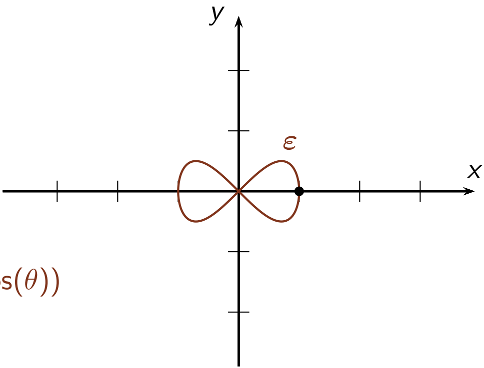
Example D



$$\delta(t) = (1 + \tfrac{22}{10}\cos(t), \tfrac{22}{10}\sin(t))$$
$$\forall t \in [0, 2\pi]$$

The curve δ moves once around the circle in the anticlockwise direction, starting at the right-most point. It is both simple and closed.

Example E

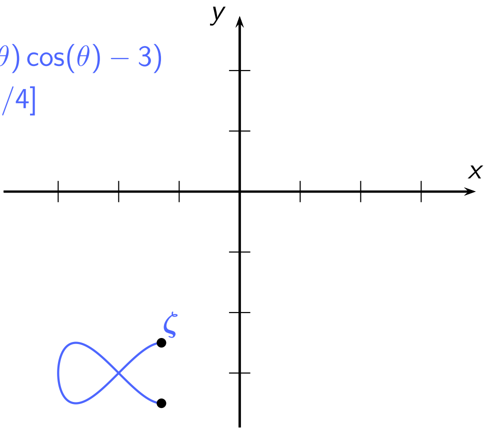


$$\varepsilon(\theta) = (\cos(\theta), \sin(\theta)\cos(\theta))$$
$$\forall \theta \in [0, 4\pi]$$

The curve ε moves *twice* around the "infinity-shaped" figure, starting at the right-most point and moving upwards. It is closed but not simple, because it intersects itself, and because it repeats itself.

Example F

$$\zeta(\theta) = (\cos(\theta) - 2, \sin(\theta)\cos(\theta) - 3)$$
$$\forall\theta \in [\pi/4, 7\pi/4]$$



The curve ζ moves along the “alpha-shaped” figure. It is neither closed not simple, because it intersects itself.

Joins of curves and reverse curves

Definition

Suppose that $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^2$ and $\boldsymbol{\beta} : [c, d] \to \mathbb{R}^2$ are curves such that the final point of $[a, b]$ is the initial point of $[c, d]$ and the final point of $\boldsymbol{\alpha}$ is the initial point of $\boldsymbol{\beta}$, that is, $b = c$ and $\boldsymbol{\alpha}(b) = \boldsymbol{\beta}(c)$. The *join* $\boldsymbol{\alpha} \sqcup \boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is the curve

$$\boldsymbol{\alpha} \sqcup \boldsymbol{\beta}(t) = \begin{cases} \boldsymbol{\alpha}(t) & \text{when } a \le t \le b \\ \boldsymbol{\beta}(t) & \text{when } c \le t \le d. \end{cases}$$

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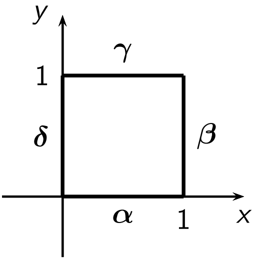
Suppose that $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$ is a curve. The *reverse curve* $\boldsymbol{\gamma}^* : [-b, -a] \to \mathbb{R}^2$ of $\boldsymbol{\gamma}$ is given by

$$\boldsymbol{\gamma}^*(t) = \boldsymbol{\gamma}(-t) \qquad \text{where } -b \leq t \leq -a.$$

Example of a join

$$\boldsymbol{\alpha}(t) = (t, 0) \quad \forall t \in [0, 1]$$
$$\boldsymbol{\beta}(t) = (1, t-1) \quad \forall t \in [1, 2]$$
$$\boldsymbol{\gamma}(t) = (3-t, 1) \quad \forall t \in [2, 3]$$
$$\boldsymbol{\delta}(t) = (0, 4-t) \quad \forall t \in [3, 4]$$



The join $\boldsymbol{\alpha} \sqcup \boldsymbol{\beta} \sqcup \boldsymbol{\gamma} \sqcup \boldsymbol{\delta}$ moves anticlockwise around the perimeter of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$, starting and ending at $(0, 0)$, as t varies from 0 to 4.

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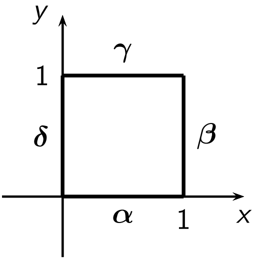

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What does $(\alpha \sqcup \beta \sqcup \gamma \sqcup \delta)^*$ do?

Reparametrising curves

Definition

Suppose that $\gamma : [a, b] \to \mathbb{R}^2$ is a curve, and that h is a continuous bijection from $[c, d]$ to $[a, b]$ such that $h(c) = a$ and $h(d) = b$. Then $\gamma \circ h : [c, d] \to \mathbb{R}^2$ is also a curve, called a *reparametrisation* of γ.

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A reparametrised curve $\gamma \circ h$ has the same initial and final point as the original curve, and the same range; further it is traversed in the same direction and it goes through each point the same number of times. What may change is the interval of definition, and the speed.

Smooth curves

Some very strange curves, such as space-filling curves and snowflake curves, violate some of our intuitions about curves. It is helpful to restrict to certain "nice" curves, that are more tractable.

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$$\boldsymbol{\gamma}'(t) = (\gamma_1'(t), \gamma_2'(t)),$$

provided that both $\gamma_1'(t)$ and $\gamma_2'(t)$ exist. We say that $\boldsymbol{\gamma}$ is *continuously differentiable* if the derivative $\boldsymbol{\gamma}'$ exists and is continuous in $[a, b]$, and $\boldsymbol{\gamma}$ is *smooth* if it is continuously differentiable and moreover $\boldsymbol{\gamma}'(t) \neq 0$ for all $t \in [a, b]$.

More definitions

The vector $\gamma'(t)$ is the tangent vector to γ at time t, or at the point $\gamma(t)$. It represents the velocity of a point moving so as to be at $\gamma(t)$ at time t.

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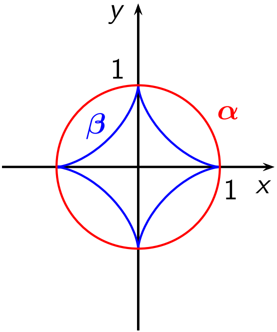
If two smooth curves pass through the same point, we may define the angle between them to be the angle between their tangent vectors.

Example of smooth and nonsmooth curves

$$\boldsymbol{\alpha}(t) = (\cos(t), \sin(t)) \quad \forall t \in [0, 2\pi]$$



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$$\boldsymbol{\alpha}(t) = (\cos(t), \sin(t)) \quad \forall t \in [0, 2\pi]$$
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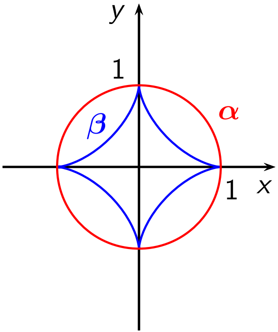

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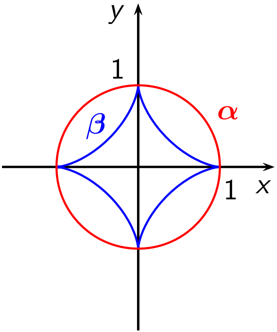

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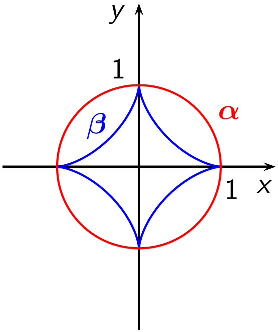
$$\boldsymbol{\beta}(t) = (\cos^3(t), \sin^3(t)) \quad \forall t \in [0, 2\pi]$$
$$\boldsymbol{\beta}'(t) = (-3\sin(t)\cos^2(t), 3\sin^2(t)\cos(t)) \quad \forall t \in [0, 2\pi]$$



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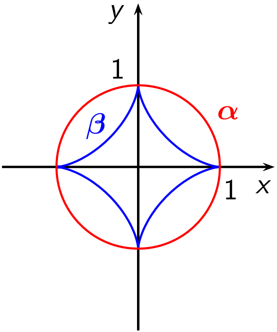

Both curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are continuously differentiable. Only $\boldsymbol{\alpha}$ is smooth.

The "best" curves and their length

Definition

A curve is *piecewise smooth* if it is a join of finitely many smooth curves. The length of a piecewise smooth curve $\gamma : [a, b] \to \mathbb{R}^2$ is given by

$$\text{Length}(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

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Note that $\gamma'(t)$ may not be defined at finitely many points, where different smooth curves are joined. This is just a formal difficulty; we take $\gamma'(t)$ to be 0 wherever it is not defined, and then the integral exists as a Riemann integral. For this, it is important that γ be bounded.

Orientation

If γ is a simple curve that is not closed, then it has an initial point and an endpoint and a "direction of motion", and even if we reparametrise it, these do not change. Similarly, simple closed curves have a "direction of motion", called *orientation*.

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For simple closed curves, the next theorem lets us find an orientation that does not depend on the parametrisation.

The Jordan curve theorem

Theorem

If $\gamma : [a, b] \to \mathbb{R}^2$ is a simple closed curve, then the complement of the range of γ is the union of two disjoint domains. One of these is bounded and the other is not. The bounded component is called the interior of γ and written $\mathrm{Int}(\gamma)$, and the unbounded component is called the exterior of γ and written $\mathrm{Ext}(\gamma)$.

Proof. We do not prove this. $\square$

Remarks

This theorem seems obvious but is actually quite difficult, particularly when curves such as snowflakes are concerned. It relies on approximating a curve by a polygonal curve.

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We will consider only curves for which it is easy to identify the interior and exterior.

More on orientation

If γ is a simple closed curve, then as we travel around γ, the interior of γ will always lie on our left, or always on our right. The *standard orientation* of γ is the direction of motion such that the interior is always on our left. With the standard orientation, we move around the perimeter of $\mathrm{Int}(\gamma)$ in an anti-clockwise direction.

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What is the standard orientation of each of the simple closed curves described above? Can you define orientation for closed curves that are not simple?

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We will use the expression *oriented range* of a curve to describe the image of a simple curve in $\mathbb{R}^2$ and a "direction of motion" along the curve.

Vector fields and line integrals

A *vector field* V on an open subset Ω of $\mathbb{R}^2$ is a function from Ω to $\mathbb{R}^2$. A vector field may be viewed as a pair of functions (v, w), where $v, w : \Omega \to \mathbb{R}$.

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Definition

Given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{R}^2$ and a continuous vector field V defined on Range(γ), we define the *line integral* $\int_\gamma V(\boldsymbol{s})\, d\boldsymbol{s}$ by

$$\int_\gamma V(\boldsymbol{s})\, d\boldsymbol{s} = \int_a^b V(\gamma(t)) \cdot \gamma'(t)\, dt.$$

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Again, $\gamma'(t)$ may not be defined for finitely many points, where different continuously differentiable curves are joined. We take $\gamma'(t)$ to be 0 wherever it is not defined, and then the integral exists as a Riemann integral.

Approximating curves by polygonal curves

If $\gamma : [a, b] \to \mathbb{R}^2$ is a piecewise smooth curve, we may define "approximating curves" γ_N (where $N \in \mathbb{Z}^+$) as follows. Fix N, and subdivide the interval $[a, b]$ into N equal subintervals of equal length, $[a_{n-1}, a_n]$ say, where $n = 1, \ldots, N$ (we do this by choosing $a_n = (N - n)a/N + nb/N$). Now let γ_N be the polygonal curve composed of the N line segments from $\gamma(a_{n-1})$ to $\gamma(a_n)$, in order.

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If we could approximate a simple closed curve by a simple polygonal closed curve then the Jordan curve theorem would follow.

Approximation lemma

Lemma

Suppose also V is a continuous vector field in a domain Ω in $\mathbb{R}^2$. Suppose that γ is a piecewise smooth curve in Ω, and that the polygonal curves γ_N defined above also lie in Ω. Then the curves γ_N approximate γ, in the sense that $\gamma_N(t) \to \gamma(t)$ for all $t \in [a, b]$ and $\mathrm{Length}(\gamma_N) \to \mathrm{Length}(\gamma)$ as $N \to \infty$; further,

$$\int_{\gamma_N} V(\boldsymbol{s})\, d\boldsymbol{s} \to \int_{\gamma} V(\boldsymbol{s})\, d\boldsymbol{s}.$$

Proof.

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Proof. Omitted. □

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Proof. Omitted. $\square$

The point of this lemma is that we may prove many results for integration along general curves by proving easier results for integration along polygonal curves.

Properties of line integrals

Theorem

Suppose that $\lambda, \mu \in \mathbb{R}$, *that* $\gamma : [a, b] \to \mathbb{R}^2$ *is a piecewise smooth curve, and that* V *and* W *are vector fields defined on* $\mathrm{Range}(\gamma)$. *Then the following hold.*

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Suppose that $\lambda, \mu \in \mathbb{R}$, that $\boldsymbol{\gamma} : [a, b] \to \mathbb{R}^2$ is a piecewise smooth curve, and that V and W are vector fields defined on $\mathrm{Range}(\boldsymbol{\gamma})$. Then the following hold.

(a) *The integral is linear:*

$$\int_{\boldsymbol{\gamma}} \lambda V(\boldsymbol{s}) + \mu W(\boldsymbol{s})\, d\boldsymbol{s} = \lambda \int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s} + \mu \int_{\boldsymbol{\gamma}} W(\boldsymbol{s})\, d\boldsymbol{s}.$$

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(b) *The integral is independent of parametrisation: if* $\boldsymbol{\delta}$ *is a reparametrisation of* $\boldsymbol{\gamma}$ *that is also a piecewise smooth curve, then*

$$\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s} = \int_{\boldsymbol{\delta}} V(\boldsymbol{s})\, d\boldsymbol{s}.$$

Properties of line integrals (continued)

(c) The integral is additive for joins: if $\boldsymbol{\gamma} = \boldsymbol{\alpha} \sqcup \boldsymbol{\beta}$, then

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(d) The integral depends on the orientation:

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$$\int_{\boldsymbol{\gamma}^*} V(\boldsymbol{s})\, d\boldsymbol{s} = -\int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s}.$$

(e) The size of the integral is controlled by the size of the vector field V and the length of the curve $\boldsymbol{\gamma}$:

$$\left| \int_{\boldsymbol{\gamma}} V(\boldsymbol{s})\, d\boldsymbol{s} \right| \le ML,$$

where L is the length of $\boldsymbol{\gamma}$ and M is a number such that $|V(\boldsymbol{s})| \le M$ for all $\boldsymbol{s} \in \text{Range}(\boldsymbol{\gamma})$.

Proof

Omitted.

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Omitted.
Note that M is a *maximiser* for $|V|$ on γ, not necessarily the *maximum* of $|V|$ on γ.

Paths

Definition

We define a path $\boldsymbol{\Gamma}$ to be the oriented range of a piecewise smooth curve γ. The theorem above implies that the line integral depends only on γ, and hence we define

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We may extend the notation for joins and reverse curves to paths.

Closed paths start where?

Note that for paths that are not closed, different parametrisations differ by a change of variable, and part (d) of the theorem applies. But for closed paths, parametrisations might have different initial points and endpoints.

Closed paths start where?

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To show that the integral around a closed path does not depend on where we start, we break up the closed path $\boldsymbol{\Gamma}$ into two parts: $\boldsymbol{\Gamma} = \mathbf{A} \sqcup \mathbf{B}$, and observe that

$$\begin{aligned} \int_{\mathbf{A}\sqcup\mathbf{B}} V(\boldsymbol{s})\, d\boldsymbol{s} &= \int_{\mathbf{A}} V(\boldsymbol{s})\, d\boldsymbol{s} + \int_{\mathbf{B}} V(\boldsymbol{s})\, d\boldsymbol{s} \\ &= \int_{\mathbf{B}} V(\boldsymbol{s})\, d\boldsymbol{s} + \int_{\mathbf{A}} V(\boldsymbol{s})\, d\boldsymbol{s} = \int_{\mathbf{B}\sqcup\mathbf{A}} V(\boldsymbol{s})\, d\boldsymbol{s}. \end{aligned}$$

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Hence the integral around $\mathbf{\Gamma}$ starting at the initial point of $\mathbf{A}$ is the same as the integral around $\mathbf{\Gamma}$ starting at the initial point of $\mathbf{B}$.

Closed and exact vector fields

Suppose that the vector field $V : \Omega \to \mathbb{R}^2$ is equal to (v, w). Then V is said to be *curl-free* or *closed* if $\partial w/\partial x = \partial v/\partial y$. Further, V is said to be *conservative* or *exact* if there is a function $u : \Omega \to \mathbb{R}$ such that $\partial u/\partial x = v$ and $\partial u/\partial y = w$; this is often written as $V = \nabla u$.

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If the function u is twice continuously differentiable, then

$$\frac{\partial w}{\partial x} = \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial v}{\partial y}.$$

This leads to the conclusion that a continuously differentiable exact vector field is closed. The converse of this is very useful.

Sometimes closed = exact

Theorem

Suppose that Ω is a simply connected domain, and that V is a closed continuously differentiable vector field in Ω. Then V is exact.

Proof.

We omit this proof, but mention that Green's theorem is often used when γ is simple. The approximation lemma and some thiniing about polygonal curves then completes the proof. We will prove a more general result about contour integrals, whose proof may be modified to prove this result. $\square$

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This theorem has variants; in particular, it is equivalent to the statement that a line integral depends only on the initial point and the end point.

End Notes

Different authors use the words “curve” and “path” in different ways. The word “arc” is also often used.

We have seen that not all vector fields can be integrated “nicely”. However, the integral of a derivative is the function that we started with.

We will prove similar results in the context of complex analysis, where line integrals become contour integrals, vector fields become complex-valued functions, and the condition for a vector field to be closed becomes the Cauchy–Riemann equations.

The material is this lecture is not examinable, except that we re-use without further comment definitions such as closed and simple curves in future lectures.