School of Mathematics and Statistics
MATH2111 - Higher Several Variable Calculus

3. Point Set Topology on $\mathbb{R}^n$

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Balls

Recall that $d = d_2$ denotes the Euclidean distance in the lectures now. We will also write $|\mathbf{x}| = ||\mathbf{x}|| = d(\mathbf{x}, \mathbf{0})$ to denote the '**norm**' of $\mathbf{x}$.

Lots of definitions coming up!

Definition

For $\mathbf{a} \in \mathbb{R}^n$ and $\varepsilon > 0$, the **(open) ball around $\mathbf{a}$ of radius** ε is the set

$$B(\mathbf{a}, \varepsilon) = B_\varepsilon(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{a}) < \varepsilon\}.$$

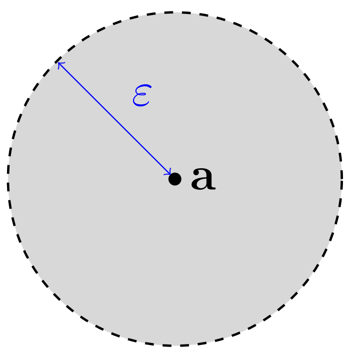


Figure: A ball centred at $\mathbf{a}$ of radius ε.

This definition can easily be adapted for arbitrary metrics by replacing d with another metric. (MATH3611)

Note. One can define the **closed ball** around $\mathbf{a} \in \mathbb{R}^n$ of radius ε as

$$\overline{B(\mathbf{a}, \varepsilon)} = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathbf{a}) \leq \varepsilon\}.$$

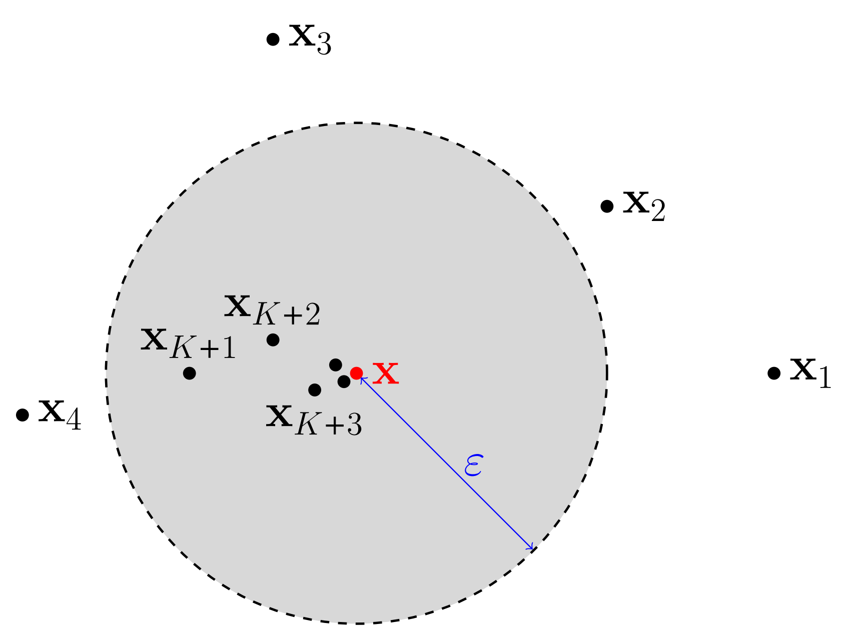
Convergence

Proposition

A sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ converges to $\mathbf{x} \in \mathbb{R}^n$ if and only if for every $\varepsilon > 0$, there exists $K \in \mathbb{Z}^+$ such that $\mathbf{x}_k \in B(\mathbf{x}, \varepsilon)$ for all $k > K$.

$$\forall \varepsilon > 0 \;\; \exists K \in \mathbb{Z}^+ \;\; \forall k > K \;\; d(\underset{\sim}{x}_k, \underset{\sim}{x}) < \varepsilon$$

$$\Leftrightarrow \forall \varepsilon > 0 \;\; \exists K \in \mathbb{Z}^+ \;\; \forall k > K \;\; \underset{\sim}{x}_k \in B(\underset{\sim}{x}, \varepsilon).$$



Open Sets, Closed Sets and Boundary Points

Definition

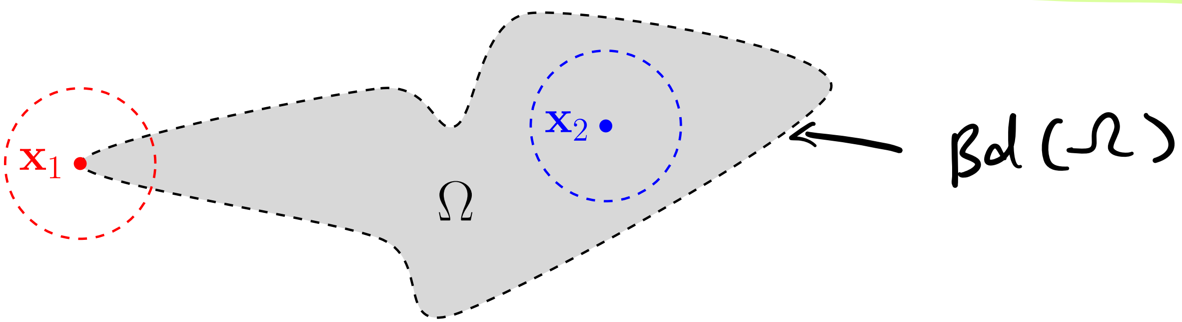
Suppose that $\Omega \subseteq \mathbb{R}^n$. We say that

- $\mathbf{x} \in \mathbb{R}^n$ is an **interior point of** Ω if there exists $\varepsilon > 0$ such that $B(\mathbf{x}, \varepsilon) \subseteq \Omega$.

  The **interior of** Ω, denoted $\text{Int}(\Omega)$ or Ω°, is the set of all interior points of Ω.
- Ω is **open** if every $\mathbf{x} \in \Omega$ is an interior point of Ω, i.e. when $\Omega = \text{Int}(\Omega)$.
- Ω is **closed** if $\Omega^c = \mathbb{R}^n \backslash \Omega$ is open.
- $\mathbf{x} \in \mathbb{R}^n$ is a **boundary point of** Ω if for every $\varepsilon > 0$, the sets $B(\mathbf{x}, \varepsilon) \cap \Omega$ and $B(\mathbf{x}, \varepsilon) \cap \Omega^c$ are non-empty.

  The **boundary of** Ω, denoted by $\text{Bd}(\Omega)$ or $\partial\Omega$, is the set of all boundary points of Ω.

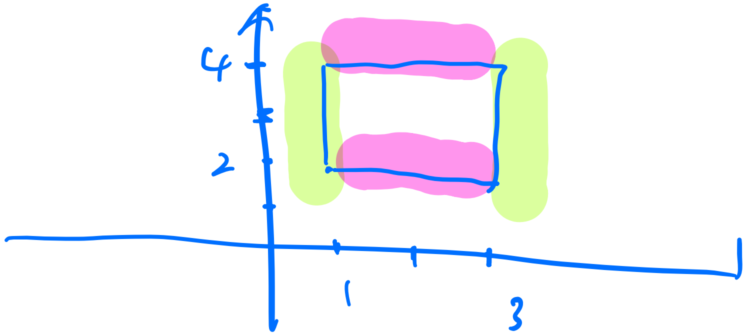
If $\mathbf{x}$ is a boundary point of Ω, then it is also a boundary point of Ω^c, so $\text{Bd}(\Omega) = \text{Bd}(\Omega^c)$.



Examples

- The set $\{ \underset{\sim}{x} \in \mathbb{R}^n : \prod_{j=1}^{n} x_j > 0 \}$ is open
- $\text{Int}([1,3] \times [2,4]) = (1,3) \times (2,4)$
- $\partial([1,3] \times [2,4]) = (\{1,3\} \times [2,4]) \cup ([1,3] \times \{2,4\})$
- $\mathbb{Z}$ is closed in $\mathbb{R}$

  $\mathbb{Z} = \left(\bigcup_{k=-\infty}^{\infty} (k, k+1) \right)^c$



- $\mathbb{Q}$ is neither open nor closed.

  $\text{Int}(\mathbb{Q}) = \emptyset$

  $\text{Bd}(\mathbb{Q}) = \mathbb{R}$

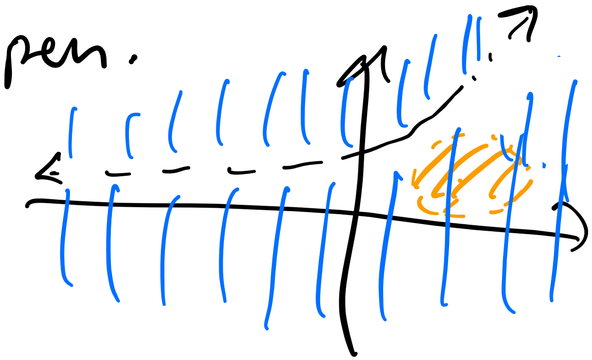
More Examples

In $\mathbb{R}$, (a,b) is open and $[a,b]$ is closed.

$\mathrm{Int}((a,b)) = \mathrm{Int}([a,b]) = (a,b)$.

$\mathrm{Bd}((a,b)) = \mathrm{Bd}([a,b]) = \{a,b\}$

- $\{x\}$ are closed.
- ϕ and $\mathbb{R}^n$ are both open and closed.
- $S^2 = \{\underset{\sim}{x} \in \mathbb{R}^3 : x^2+y^2+z^2=1\}$ is closed.
- $\{(x,y) \in \mathbb{R}^2 : y \neq e^x\}$ is open.



Open balls are open sets...who would have thought?

Proposition

For $\mathbf{x}_0 \in \mathbb{R}^n$ and $r > 0$, the set $B(\mathbf{x}_0, r)$ is open.

To prove this, we need to show that any point in $B(\mathbf{x}_0, r)$ has an open ball around it contained within $B(\mathbf{x}_0, r)$.

Proof.

let $\underset{\sim}{x} \in B(\underset{\sim}{x_0}, r)$ and

$\varepsilon = r - d(\underset{\sim}{x}, \underset{\sim}{x_0})$. Then

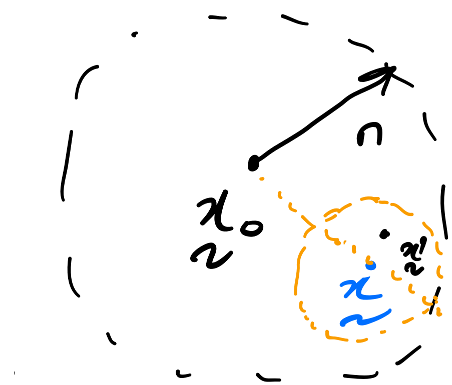
if $\underset{\sim}{x}' \in B(\underset{\sim}{x}, \varepsilon)$, we have

$d(\underset{\sim}{x}', \underset{\sim}{x_0}) \leq d(\underset{\sim}{x}', \underset{\sim}{x}) + d(\underset{\sim}{x}, \underset{\sim}{x_0})$

$< \varepsilon + d(\underset{\sim}{x} + \underset{\sim}{x_0})$

$= r$, so $\underset{\sim}{x}' \in B(\underset{\sim}{x_0}, r)$. This works for all $\underset{\sim}{x}$ and x',

so $B(\underset{\sim}{x_0}, r)$ is open.



Some Facts

Proposition

1. Every non-empty open set can be written as a union of open balls.
2. The intersection of two open sets is itself an open set.

Proof. 1. Let Ω be a non-empty open set. For each $\underset{\sim}{x} \in \Omega$, there is $\varepsilon_{\underset{\sim}{x}} > 0$ such that $B(\underset{\sim}{x}, \varepsilon_{\underset{\sim}{x}}) \subseteq \Omega$. Then $\Omega = \bigcup_{\underset{\sim}{x} \in \Omega} B(\underset{\sim}{x}, \varepsilon_{\underset{\sim}{x}})$.

2. Let Ω_1, Ω_2 be open subsets of $\mathbb{R}^n$, if $\Omega_1 \cap \Omega_2 = \emptyset$, we're done so let $\underset{\sim}{x} \in \Omega_1 \cap \Omega_2$. We have $\varepsilon_1, \varepsilon_2 > 0$ such that $B(\underset{\sim}{x}, \varepsilon_i) \subseteq \Omega_i$, set $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.
Then $B(\underset{\sim}{x}, \varepsilon) \subseteq \Omega_1 \cap \Omega_2$,
so $\Omega_1 \cap \Omega_2$ is open.

Exercises

respectively

Ex.

1. If $\{A_i\}_{i \in I}$ is a collection of open (resp. closed) sets, then $\bigcup_{i \in I} A_i$ (resp. $\bigcap_{i \in I} A_i$) is open (resp. closed).
2. If $\{A_i\}_{i=1}^{n}$ is a finite collection of open (resp. closed) sets, then $\bigcap_{i=1}^{n} A_i$ (resp. $\bigcup_{i=1}^{n} A_i$) is open (resp. closed).
3. $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are open if and only if $A \times B$ is open in $\mathbb{R}^{n+m}$.

Lemma

$\Omega \subseteq \mathbb{R}^n$ is closed if and only if $\partial\Omega \subseteq \Omega$.

Proof of above lemma will be a Board Tutorial exercise, but it is similar to the proof that will be presented on slide 13.

Limit Points and Closure

Definitions

Suppose that $\Omega \subseteq \mathbb{R}^n$.

- $\mathbf{x} \in \mathbb{R}^n$ is a **limit point of** Ω if there is a sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ in $\Omega \backslash \{\mathbf{x}\}$ such that $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$. (Note that $\mathbf{x}_k \neq \mathbf{x}$ for all $k \in \mathbb{Z}^+$.)

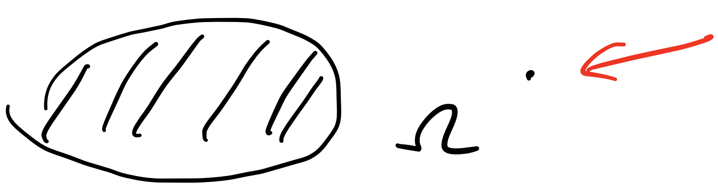
  Limit points are also known as **accumulation** points or **cluster** points.

- The **closure of** Ω, denoted by $\mathrm{cl}(\Omega)$ or $\overline{\Omega}$, is the union of Ω and all of its limit points.

we would say that x is a limit point of Ω, if $(B(x, \varepsilon) \backslash \{x\}) \cap \Omega \neq \emptyset$ for all $\varepsilon > 0$.

Proposition

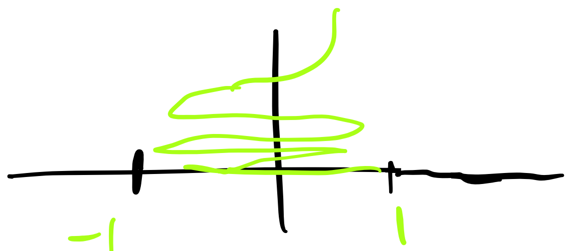
If $\Omega \subseteq \mathbb{R}^n$, then $\overline{\Omega} = \Omega \cup \partial\Omega$.



not a limit point

Examples

- 0 is the only limit point of $\Omega = \{\frac{1}{k}\}_{k=1}^{\infty} \cup \{0\}$
- $\pm\frac{\pi}{2}$ are the only limit point of $\{\arctan(k) : k \in \mathbb{Z}\}$
- $\overline{(a,b)} = [a,b] = \{\text{limit points of } \overset{(a,b]}{[a,b]}\}$
- $\overline{\mathbb{Q}} = \mathbb{R} = \mathbb{Q} \cup \mathrm{Bd}(\mathbb{Q})$



- $\mathrm{cl}(\underbrace{\{(\cos\frac{1}{y}, y) : y > 0\}}_{\Omega}) = \Omega \cup ([-1,1] \times \{0\})$
- $\overline{\{(\cos k, \sin k) : k \in \mathbb{Z}^+\}} = S^1 = \mathbb{T} \leftarrow$ \mathbb{T}

Remarks

Proposition

Suppose that $\Omega \subseteq \mathbb{R}^n$. Then

- $\mathrm{Int}(\Omega)$ is an open set;
- $\overline{\Omega}$ is a closed set;
- $\mathrm{Int}(\Omega) = \Omega \backslash \partial\Omega$.

In a sense, $\mathrm{Int}(\Omega)$ is the 'largest' open subset of Ω, and the closure of Ω is the 'smallest' closed superset of Ω.

Exercise. Show that for $\Omega \subseteq \mathbb{R}^n$,

- $\mathrm{Int}(\Omega) = \bigcup \{U \subseteq \Omega : U \text{ is open}\}$;
- $\overline{\Omega} = \bigcap \{V \supseteq \Omega : V \text{ is closed}\}$.

Some More Facts

Proposition

Let $\mathbf{x} \in \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$.

1. If $\mathbf{x}$ is an interior point of Ω, then it is a limit point of Ω;
2. If $\mathbf{x}$ is a limit point of Ω, then it is either a boundary point or an interior point of Ω;
3. If $\mathbf{x} \in \overline{\Omega}$, then there is a sequence $\{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq \Omega$ with $\mathbf{x}_k \to \mathbf{x}$ as $k \to \infty$;
   closure
4. If $\mathbf{x} \in \Omega$ is not an interior point of Ω, then it is a boundary point of Ω;
5. If $\mathbf{x} \in \partial\Omega \backslash \Omega$, then $\mathbf{x}$ is a limit point of Ω;
6. $\mathbb{R}^n = \text{Int}(\Omega) \cup \partial\Omega \cup \text{Int}(\Omega^c)$ as a disjoint union.

It would be a good exercise for you to prove some of these results.

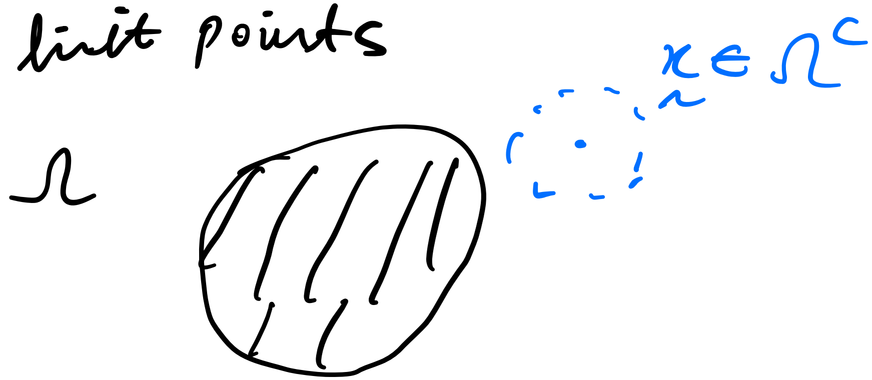
Some of the above results, such as the first one, are not true if we use other metrics, like the discrete metric, instead of the Euclidean distance d_2 in all of our definitions.

Closed Sets and Limit Points

Lemma

A set $\Omega \subseteq \mathbb{R}^n$ is closed if and only if it contains all of its limit points.

Proof. Suppose that Ω is closed, so Ω^c is open. If $\underset{\sim}{x} \in \Omega^c$, which is an interior points of Ω^c, so there is $\varepsilon > 0$ such that $B(\underset{\sim}{x}, \varepsilon) \subseteq \Omega^c$. That is $B(\underset{\sim}{x}, \varepsilon) \cap \Omega = \emptyset$, and $\underset{\sim}{x}$ cannot be a limit point of Ω so Ω contains all of it's limit points

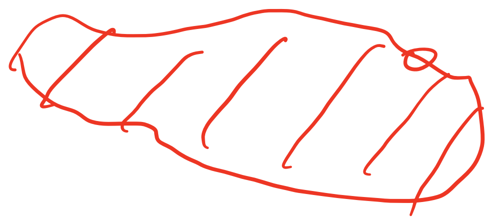


Conversely, suppose Ω is not closed.

Then Ω^c is not open, so there is $x \in \Omega^c$ that is not an interior point of Ω^c. So for every $\varepsilon > 0$, $B(x, \varepsilon) \cap (\Omega^c)^c \neq \emptyset$. In particular, let $x_k \in B(x, \frac{1}{k}) \cap \Omega$. Then $d(x, x_n) < \frac{1}{k}$ for

$\neq x$ all $n \geq k$, so $\{x_k\}_{k=1}^{\infty}$ converges to x.

So x is a limit point of Ω but it is not in Ω.



Closed Sets and Limit Points

Remark. From the preceding lemma, we get that $\Omega \subseteq \mathbb{R}^n$ is closed if only if every convergent sequence in Ω has its limit also in Ω.

Corollary

Suppose that $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. Then A and B are closed if and only if $A \times B$ is closed in $\mathbb{R}^{n+m}$. $\{a_1, \dots a_n, b_1 \dots b_m\}$

Proof. Suppose that $A \in \mathbb{R}^n$, $B \in \mathbb{R}^m$ are closed and that $\{(\underset{\sim}{a}_k, \underset{\sim}{b}_k)\}_{k=1}^{\infty} \subseteq A \times B$ converges to $(\underset{\sim}{x}, \underset{\sim}{y})$. By convergence of components, $\underset{\sim}{a}_k \longrightarrow \underset{\sim}{x}$ and $\underset{\sim}{b}_k \longrightarrow y$ as $k \longrightarrow \infty$. As A and B are closed, $x \in A$ and $y \in B$, so $(\underset{\sim}{a}, \underset{\sim}{b}) \in A \times B$. So $A \times B$ is closed.

Conversely is similar.

$x > y \in \mathbb{R} \setminus \mathbb{Q}$ distinct

$x - y > 0$

$\Rightarrow \exists\, n \in \mathbb{Z}^+$ s.t.

$n(x-y) > 1$

$nx - ny > 1$

$\Rightarrow \exists\, m \in \mathbb{Z}^+$ s.t.

$nx > m > ny$

$\Rightarrow x > \frac{m}{n} > y$