MATH2621 — Higher Complex Analysis. XVII
Cauchy's integral formula

This lecture?

In this lecture, we

- sketch a proof of the Cauchy–Goursat Theorem,
- state and prove Cauchy's integral formula, and
- see some applications.

Proof of the Cauchy–Goursat Theorem

Theorem

Suppose that Ω is a bounded domain whose boundary $\partial\Omega$ consists of finitely many contours $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$. Suppose also that $f \in H(\Upsilon)$, where $\overline{\Omega} \subset \Upsilon$. Then

$$\sum_j \int_{\Gamma_j} f(z)\, dz = 0.$$

Proof.

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closure

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Proof. The proof of the theorem involves three steps. First, we prove it in the case where Ω is a triangle. Second, we consider the case where Ω is a domain whose boundary is made up of finitely many closed polygonal contours. Third, we treat the general case.

Proof of the Cauchy–Goursat Theorem. 2

Step one Suppose that Ω is a triangle in the complex plane. We write T_0 for Ω and $\partial \mathsf{T}_0$ for its boundary. Suppose that $f \in H(\Upsilon)$, where $\overline{\mathsf{T}}_0 \subset \Upsilon$, and let

$$\int_{\partial \mathsf{T}_0} f(z)\, dz = I.$$

We have to show that $I = 0$, and we suppose towards a contradiction that $I \neq 0$.

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We may subdivide T_0 into four congruent sub-triangles, T', T'', T''' and T'''' say, by taking the midpoint of each side, and joining these midpoints.

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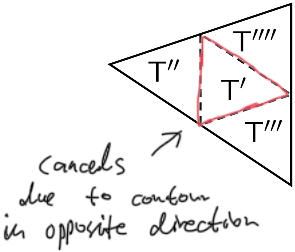


Proof of the Cauchy–Goursat Theorem. 3

Now

$$\begin{aligned} I &= \int_{\partial \mathrm{T}_0} f(z)\,dz \\ &= \int_{\partial \mathrm{T}'} f(z)\,dz + \int_{\partial \mathrm{T}''} f(z)\,dz + \int_{\partial \mathrm{T}'''} f(z)\,dz + \int_{\partial \mathrm{T}''''} f(z)\,dz. \end{aligned}$$

At least one of the triangles T', T'', T''' and T'''', which we call T_1, must satisfy

$$\left| \int_{\partial \mathrm{T}_1} f(z)\,dz \right| \geq \frac{1}{4} \left| \int_{\partial \mathrm{T}_0} f(z)\,dz \right| = \frac{|I|}{4}\,.$$

Proof of the Cauchy–Goursat Theorem. 4

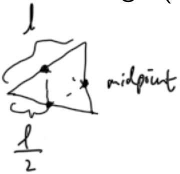
We now subdivide T_1 into 4 congruent triangles, and argue in the same way that there must be one of these, T_2 say, with the property that

$$\left|\int_{\partial \mathrm{T}_2} f(z)\,dz\right| \geq \frac{1}{4}\left|\int_{\partial \mathrm{T}_1} f(z)\,dz\right| \geq \frac{|I|}{16}\,.$$

Continuing inductively in this way, we find a sequence $(\mathrm{T}_n)_{n\in\mathbb{N}}$ of nested triangles, such that

$$\left|\int_{\partial \mathrm{T}_n} f(z)\,dz\right| \geq \frac{|I|}{4^n}\,. \tag{1}$$

Write $\mathrm{Length}(\partial \mathrm{T}_n)$ for the perimeter of T_n. Then



$$\mathrm{Length}(\partial \mathrm{T}_n) = 2^{-n}\,\mathrm{Length}(\partial \mathrm{T}_0).$$

Proof of the Cauchy–Goursat Theorem. 5

By compactness, there is a point z_0 that lies in each of the closed triangles $\overline{\mathrm{T}}_n$, and by hypothesis, f is differentiable at z_0. If $z \in \partial \mathrm{T}_n$, then $|z - z_0|$ is less than half the perimeter of T_n, that is,

$$|z - z_0| \leq \frac{1}{2} \,\mathrm{Length}(\partial \mathrm{T}_n) = 2^{-n-1} \,\mathrm{Length}(\partial \mathrm{T}_0),$$

and this tends to 0 as $n \to \infty$.

maximise at the side
but triangular inequality
$|z - z_0| \leq \frac{1}{2} \times$ perimeter.

Proof of the Cauchy–Goursat Theorem. 6

Since f is differentiable at z_0, we may write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z),$$

from taylor approximation

where the error term $E(z)$ satisfies

$$\frac{|E(z)|}{|z - z_0|} \to 0 \quad \text{as } z \to z_0.$$

In particular, we can ensure that

$$\frac{|E(z)|}{|z - z_0|} \le \frac{|I|}{\mathrm{Length}(\partial \mathrm{T}_0)^2} \qquad \forall z \in \partial \mathrm{T}_n \tag{2}$$

by taking n large enough. In what follows, we take such an n.

Proof of the Cauchy–Goursat Theorem. 8

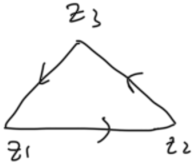
Recall that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z).$$

This means that

$$\int_{\partial T_n} f(z)\,dz = \int_{\partial T_n} f(z_0)\,dz + \int_{\partial T_n} f'(z_0)(z - z_0)\,dz + \int_{\partial T_n} E(z)\,dz.$$

The first two integrals on the right hand side are 0, by calculation, and hence



$$\int_{\partial T_n} f(z)\,dz = \int_{\partial T_n} E(z)\,dz.$$

$$f(z_0)\left[\int_{z_1 z_2} + \int_{z_2 z_1} + \int_{z_3 z_1}\right] = 0$$

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The first two integrals on the right hand side are 0, by calculation, and hence

$$\int_{\partial \mathrm{T}_n} f(z)\,dz = \int_{\partial \mathrm{T}_n} E(z)\,dz.$$

Thus. . .

Proof of the Cauchy–Goursat Theorem. 9

by (1), the ML Lemma, properties of maxima, and (2),

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&\le 4^n \max\{|E(z)| : z \in \partial \mathrm{T}_n\} \,\mathrm{Length}(\partial \mathrm{T}_n) \\
&= 4^n \max\left\{ \frac{|E(z)|}{|z - z_0|} \, |z - z_0| : z \in \partial \mathrm{T}_n \right\} \mathrm{Length}(\partial \mathrm{T}_n)
\end{aligned}
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&\qquad \times \mathrm{Length}(\partial \mathrm{T}_n)
\end{aligned}$$

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&\qquad \times \operatorname{Length}(\partial \mathrm{T}_n) \\
&\le 4^n \frac{|I|}{\operatorname{Length}(\partial \mathrm{T}_0)^2}\, \frac{\operatorname{Length}(\partial \mathrm{T}_n)^2}{2}
\end{aligned}
$$

$\le \frac{1}{2^{n+1}}$ Length (∂T_0)

$\le \frac{|I|}{\text{Length}(\partial T_0)^2}$

Proof of the Cauchy–Goursat Theorem. 9

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&\qquad \times \mathrm{Length}(\partial \mathrm{T}_n) \\
&\le 4^n \frac{|I|}{\mathrm{Length}(\partial \mathrm{T}_0)^2} \, \frac{\mathrm{Length}(\partial \mathrm{T}_n)^2}{2} \\
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\end{aligned}
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&\le 4^n \frac{|I|}{\mathrm{Length}(\partial \mathrm{T}_0)^2} \, \frac{\mathrm{Length}(\partial \mathrm{T}_n)^2}{2} \\
&= \frac{|I|}{2},
\end{aligned}
$$

contradiction

which is absurd. Hence $I = 0$.

Proof of the Cauchy–Goursat Theorem. 10

Step 2 The next step is to deal with a domain Ω with a polygonal boundary. Any such domain may be subdivided into triangles T_n, in such a way that

$$\int_{\partial\Omega} f(z)\,dz = \sum_n \int_{\partial \mathrm{T}_n} f(z)\,dz;$$

by the result of the previous step,

$$\int_{\partial\Omega} f(z)\,dz = 0.$$

Proof of the Cauchy–Goursat Theorem. 11

Step 3 Finally, we have to deal with a domain whose boundary is the union of finitely many disjoint closed contours. This can be done by approximating unions of general contours by unions of polygonal contours; the integral is 0 for all the unions of approximating polygonal contours, and so the integral around the union of general contours that we want is also 0. $\square$

Cauchy's integral formula

Theorem

Suppose that Ω is a simply connected domain in $\mathbb{C}$, that $f \in H(\Omega)$, that Γ is a simple closed contour in Ω and that $w \in \mathrm{Int}(\Gamma)$. Then

$$f(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - w}\, dz. \tag{3}$$

Cauchy's integral formula

z is on the contour Γ
w has to be inside Γ

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Proof. Let Γ_ε be the circle with centre w and radius ε, *traversed clockwise*, and take ε small enough that $\Gamma_\varepsilon \subset \mathrm{Int}(\Gamma)$. We consider the domain Υ consisting of $\mathrm{Int}(\Gamma) \cap \mathrm{Ext}(\Gamma_\varepsilon)$, the domain between Γ and Γ_ε, whose boundary consists of Γ, traversed anti-clockwise, and Γ_ε, traversed clockwise. The quotient function $z \mapsto f(z)/(z-w)$ is holomorphic in $\Omega \setminus \{w\}$, a domain that contains $\Upsilon \cup \partial\Upsilon$.

Proof of Cauchy's Integral Formula

By the Cauchy–Goursat theorem,

$$\int_{\partial\Upsilon} \frac{f(z)}{z-w}\,dz = \int_{\Gamma} \frac{f(z)}{z-w}\,dz + \int_{\Gamma_\varepsilon} \frac{f(z)}{z-w}\,dz = 0;$$

that is,

$$\int_{\Gamma} \frac{f(z)}{z-w}\,dz = \int_{\Gamma_\varepsilon^*} \frac{f(z)}{z-w}\,dz.$$

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The left hand side of this equality does not depend on ε, so the limit as ε tends to 0 of the right hand side exists.

We can chose any ε st. $w \in \operatorname{int}(T_\varepsilon)$

Proof of Cauchy's Integral Formula. 2

To compute the limit, we parametrise Γ_ε^*. Define $\gamma_\varepsilon^*(\theta) = w + \varepsilon e^{i\theta}$, where $0 \le \theta \le 2\pi$, and observe that

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&= i\int_0^{2\pi} f(w)\,d\theta \\
&= 2\pi i f(w).
\end{aligned}$$

Proof of Cauchy's Integral Formula. 3

We can move the limit inside the integral because

$$\lim_{\varepsilon \to 0} f(w + \varepsilon e^{i\theta}) = f(w)$$

uniformly in θ, since $\lim_{z \to w} f(z) = f(w)$. Formula (3) follows.

$\square$

Independence of contour

Corollary

Suppose that w lies in a simply connected domain Ω, and that $f \in H(\Omega)$. If Γ and Δ are simple closed contours such that $w \in \mathrm{Int}(\Gamma)$ and $w \in \mathrm{Int}(\Delta)$, then

$$\int_\Gamma \frac{f(z)}{z-w}\,dz = \int_\Delta \frac{f(z)}{z-w}\,dz.$$

Proof. This follows from Cauchy's integral formula; both are equal to $2\pi i f(w)$. $\square$

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This means that if we need to compute the integral $\displaystyle\int_\Gamma \frac{f(z)}{z-w}\,dz$, we may change the contour to make the calculation easier.

Mean Value Formula

Corollary

Suppose that Ω is a simply connected domain in $\mathbb{C}$, that $f \in H(\Omega)$, and that $w \in \Omega$. If $\overline{B}(w, r) \subset \Omega$, then

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{i\theta})\, d\theta. \tag{4}$$

Proof of the Mean Value Formula

Proof.

This formula is virtually proved in the course of the proof of the Cauchy integral formula; let $\gamma(\theta) = w + re^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Then

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\,dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(w+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}\,d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(w+re^{i\theta})\,d\theta, \end{aligned}$$

as required. □

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as required. □

The Cauchy integral formula expresses $f(w)$ as a weighted average of the values of $f(z)$ around any contour surrounding w.

Exercise 1

Compute $\displaystyle\int_{\Gamma} \frac{\sin z}{z}\, dz$, where Γ is the circle with centre 0 and radius R.

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$$\int_\Gamma \frac{\sin z}{z}\,dz = \int_\Gamma \frac{f(z)}{z-w}\,dz = 2\pi i f(w) = 0. \qquad \triangle$$

singularity at $z = 0$ which is included contour. Hence Cauchy - Goursat does not apply

End notes

Precise statements of the Cauchy–Goursat theorem and of Cauchy's integral formula may be examined.

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The proofs of the results might be examined.