MATH1081 – Discrete Mathematics

Topic 4 – Combinatorics
Lecture 4.01 – Counting laws

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Introduction to combinatorics

Combinatorics is the study of counting/enumeration.

Counting is a natural and perhaps seemingly rudimentary topic, but features in a broad range of mathematical disciplines and can sometimes be deceptively difficult.

There are several techniques and approaches to counting that we will learn throughout this topic. The main difficulty with counting questions is usually not the application of a counting technique, but identifying which technique(s) should be used. Drawing up diagrams and specific examples can be especially useful starting approaches to questions in this topic.

Towards the end of this topic, we will study a way of evaluating enumerative sequences known as recurrence relations.

Addition law

Definition. Two events are mutually exclusive if they cannot occur together.

Theorem. (Addition law)
If n mutually exclusive events can occur in $k_1, k_2, k_3, \dots, k_n$ different ways, then the number of ways any one of the events can occur is given by $k_1 + k_2 + k_3 + \cdots + k_n$.

Using set notation, the addition law states that given finite sets $A_1, A_2, A_3, \dots, A_n$ that are pairwise disjoint ($A_i \cap A_j = \varnothing$ for all $i \neq j$), we have

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| = |A_1| + |A_2| + |A_3| + \cdots + |A_n|.$$

Example. A restaurant menu offers 4 entrée options, 5 main options, and 3 dessert options. How many possible one-course meals are there?

Solution. There are $4 + 5 + 3 = 12$ different one-course meals.

Multiplication law

Definition. Two events are independent if the outcome of one event does not affect the outcome of the other.

Theorem. (Multiplication law)
If n independent events can occur in $k_1, k_2, k_3, \dots, k_n$ different ways, then the number of ways every event can occur together is given by $k_1 k_2 k_3 \dots k_n$.

Using set notation, the multiplication law states that given finite sets $A_1, A_2, A_3, \dots, A_n$, we have

$$|A_1 \times A_2 \times A_3 \times \cdots \times A_n| = |A_1| \times |A_2| \times |A_3| \times \cdots \times |A_n|.$$

Example. A restaurant menu offers 4 entrée options, 5 main options, and 3 dessert options. How many possible three-course meals are there?

Solution. There are $4 \times 5 \times 3 = 60$ different three-course meals.

Example. A restaurant menu offers 4 entrées, 5 mains, and 3 desserts. How many possible two-course meals are there?

Solution. There are $4 \times 5 + 4 \times 3 + 5 \times 3 = 47$ different two-course meals.

Alphabets and words

A common feature of counting problems involves selections and/or arrangements of letters. The set of all allowable letters for a particular problem is called an alphabet.

- The English alphabet has 26 letters (we usually don't distinguish between upper/lower case).
- The English alphabet can be partitioned into two sets of 5 vowels ($\{\mathbf{A}, \mathbf{E}, \mathbf{I}, \mathbf{O}, \mathbf{U}\}$) and 21 consonants.

A sequence of letters is called a word or string. Unless otherwise specified, letters may be repeated, and letter order is important (so **AABB** and **BABA** are considered different words).

- Note that a word in this sense does not need to be an actual English word – any sequence of letters is considered a word, regardless of its meaning in English.

A subword or substring of a word is a sequence of contiguous letters contained inside the word. For example, **RING** is a subword of **STRING**.

The word with no letters is known as the empty word or empty string, and is sometimes denoted ε or Λ.

Example 1 – Counting words

Example. Using the English alphabet, how many three-letter words are there?

Solution. The first, second, and third letters may be any of 26 options. So there are $26 \times 26 \times 26 = 26^3$ different three-letter words.

(Notice that it is usually fine to leave your answer in unexpanded form, i.e. we did not directly calculate 26^3 here, since its actual value is less informative than the expression.)

Example. Using the English alphabet, how many three-letter words consist of only vowels or only consonants?

Solution. In this case a word can only consist of three vowels or three consonants. In the first case there are 5^3 options, while in the second case there are 21^3 options, and these two cases are mutually exclusive. So there are $5^3 + 21^3$ different three-letter words of only vowels or only consonants.

Example 2 – Alphabetising words

Example. Using the English alphabet, how many three-letter words come alphabetically before the word **DOG**?

Solution. There are three cases to consider:

Case 1. If the first letter comes from the set {**A**,**B**,**C**}, then the other two letters may take any value. So the first letter has 3 options, and the other two letters each have 26 options, meaning in this case there are 3×26^2 possible words.

Case 2. If the first letter is **D**, and the second letter comes from the set {**A**,**B**,**C**,...,**N**}, then the third letter may take any value. So the first letter has 1 option, the second letter has 14 options, and the third letter has 26 options, meaning in this case there are $1 \times 14 \times 26$ possible words.

Case 3. If the first letter is **D**, and the second letter is **O**, then the third letter must come from the set {**A**,**B**,**C**,...,**F**}. So the first letter has 1 option, the second letter has 1 option, and the third letter has 6 options, meaning in this case there are $1 \times 1 \times 6$ possible words.

Since these three cases are independent, the total number of three-letter words preceding **DOG** alphabetically is $3 \times 26^2 + 14 \times 26 + 6$.

Complement law

Theorem. (Complement law) If an event can have n different possible outcomes, and a particular outcome can occur in exactly k different ways, then the number of ways that outcome cannot occur is $n - k$.

Using set notation, the complement law states that given a finite universal set $\mathcal{U}$ and some set $A \subseteq \mathcal{U}$, we have

$$|A| = |\mathcal{U}| - |A^c|.$$

Example. A restaurant menu offers 4 entrée options, 5 main options, and 3 dessert options. How many possible two-course meals include an entrée?

Solution. There are $5 \times 3 = 15$ different two-course meals excluding an entrée, and we previously showed there are 47 total different two-course options. So there are $47 - 15 = 32$ different two-course meals including an entrée.

Of course, we could have also found the total number of two-course meals including an entrée directly by finding $4 \times 5 + 4 \times 3 = 32$.

Example 3 – More word-counting

Example. Using the English alphabet, how many three-letter words consist of at least one vowel and at least one consonant?

Solution. The complement of "at least one vowel and at least one consonant" is "no vowels or no consonants", which is the same as "all consonants or all vowels". We found in Example 1 that this number is $21^3 + 5^3$. Since the total number of three-letter words is 26^3, we therefore have that there are $26^3 - (21^3 + 5^3)$ different three-letter words with at least one vowel and at least one consonant.

Alternate solution. By way of example, the first letter might be a vowel, the second a consonant, and the third a consonant, giving $5 \times 21 \times 21$ such words. For another example, the first letter might be a vowel, the second a consonant, and the third a vowel, giving $5 \times 21 \times 5$ such words. Considering all possibilities, there is a total of

$$5 \times 21 \times 21 + 21 \times 5 \times 21 + 21 \times 21 \times 5 + 5 \times 5 \times 21 + 5 \times 21 \times 5 + 21 \times 5 \times 5,$$

which is to say $3 \times 5 \times 21^2 + 3 \times 5^2 \times 21$, different such words.

We can check that both solutions give the same result (8190).

Inclusion-exclusion law

The following theorem from Topic 1 is used extensively in combinatorics.

Theorem. (Inclusion-exclusion law)
Given a finite collection of finite sets $A_1, A_2, A_3, \dots, A_n$, we have

$$\begin{aligned} |A_1 \cup A_2| &= |A_1| + |A_2| - |A_1 \cap A_2|, \\ |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\ &\quad + |A_1 \cap A_2 \cap A_3|, \end{aligned}$$

and so on.

In general, the size of a union of sets is found by adding together the size of each set, then subtracting the size of every twofold intersection, then adding back the size of every threefold intersection, and so on.

The inclusion-exclusion principle is typically used when trying to evaluate unions of non-disjoint sets, because it is often easier to describe and count intersections of sets.

Notice that if all the sets $A_1, A_2, A_3, \dots, A_n$ are pairwise disjoint, the inclusion-exclusion law becomes the addition law.

Example 4 – Subwords condition

Example. Using the English alphabet, how many three-letter words contain the subword **AA**?

Non-Solution. If a three-letter word starts with **AA**, then the third letter may be any of 26 options. If a three-letter word ends with **AA**, then the first letter may be any of 26 options. So there are $26 + 26 = 52$ different three-letter words containing the subword **AA**.

These two cases are not mutually exclusive! The word **AAA** has been counted twice.

Solution. Let $S = \{\text{three-letter words starting with } \mathbf{AA}\}$ and $T = \{\text{three-letter words ending with } \mathbf{AA}\}$. There are 26 options for the unassigned letter in each case, meaning $|S| = |T| = 26$. Now $S \cap T = \{\text{three-letter words starting and ending with } \mathbf{AA}\} = \{\mathbf{AAA}\}$, so by the inclusion-exclusion principle, we have

$$\begin{aligned} |S \cup T| &= |S| + |T| - |S \cap T| \\ &= 26 + 26 - 1 \\ &= 51, \end{aligned}$$

meaning there are 51 different three-letter words containing **AA**.

Example 5 – Further word-counting

Example. Using the English alphabet, how many three-letter words start or end with a vowel?

Solution. Let $A = \{\text{three-letter words starting with a vowel}\}$ and $B = \{\text{three-letter words ending with a vowel}\}$. We wish to find $|A \cup B|$.

Any element of A has 5 options for its first letter, 26 options for its second letter, and 26 options for its third letter, meaning $|A| = 5 \times 26^2$. Similarly, we have $|B| = 26^2 \times 5$.

Now $A \cap B = \{\text{three-letter words starting and ending with a vowel}\}$, so any element of $A \cap B$ has 5 options for its first letter, 26 options for its second letter, and 5 options for its third letter, meaning $|A \cap B| = 5^2 \times 26$. Hence by the inclusion-exclusion principle, we have

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 5 \times 26^2 + 5 \times 26^2 - 5^2 \times 26 \\ &= 5 \times 26 \times (26 + 26 - 5), \end{aligned}$$

meaning there are $5 \times 26 \times 47$ different three-letter words that start or end with a vowel.

Example 5 – Further word-counting

Example. Using the English alphabet, how many three-letter words start or end with a vowel?

Alternate solution. The word can be one of three types: starts with a vowel and ends in a consonant ($5 \times 26 \times 21$ ways), starts with a consonant and ends in a vowel ($21 \times 26 \times 5$ ways), or starts and ends with a vowel ($5 \times 26 \times 5$ ways). These cases are mutually exclusive, so the number of three-letter words starting or ending with a vowel is

$$\begin{aligned} 5 \times 26 \times 21 + 21 \times 26 \times 5 + 5 \times 26 \times 5 &= 5 \times 26 \times (21 + 21 + 5) \\ &= 5 \times 26 \times 47. \end{aligned}$$

Alternate solution. The complement of "starts or ends with a vowel" is "starts and ends with a consonant". The number of three-letter words starting and ending with a consonant is $21 \times 26 \times 21$, and there are 26^3 total three-letter words without any further restrictions. So the total number of three-letter words starting or ending with a vowel is

$$\begin{aligned} 26^3 - 21 \times 26 \times 21 &= 26 \times (26^2 - 21^2) \\ &= 26 \times (26 - 21) \times (26 + 21) \\ &= 26 \times 5 \times 47. \end{aligned}$$

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Topic 4 – Combinatorics
Lecture 4.02 – Arrangements

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Motivating Exercise – Counting anagrams

Exercise. Find the number of ways that the letters in **DOG** can be arranged.

Solution. We can list all arrangements systematically, for example by using a tree diagram. They are **DGO**, **DOG**, **GDO**, **GOD**, **ODG**, and **OGD**. So there are 6 ways to arrange the letters in **DOG**.

Alternate solution. We are seeking three-letter words made from the letters in $\{\mathbf{D}, \mathbf{G}, \mathbf{O}\}$ without repetition. The first letter can be any of the three available options. The second letter can be any of the two remaining options (since the first letter's value is no longer available as an option). The third letter must be the only remaining option (the value not already taken by the first and second letters). So there are $3 \times 2 \times 1 = 6$ ways to arrange the letters in **DOG**.

Notice this second approach will work in general. For an n-letter word whose letters are all distinct, there will be $n \times (n-1) \times \cdots \times 1 = n!$ ways to arrange its letters.

Arrangement of distinct objects

Definition. The factorial of any positive integer n is given by

$$n! = 1 \times 2 \times 3 \times \cdots \times n,$$

and the factorial of 0 is given by $0! = 1$.

We can think of factorials for all natural n inductively as being defined by the base case $0! = 1$, and the recursive definition $n! = n \times (n-1)!$ for all positive integers n.

Definition. An arrangement of objects is any particular ordering of all the objects. For example, an arrangement of n elements can be interpreted as an ordered tuple of those n elements.

Fact. Given n distinct objects, they can be arranged in $n!$ different ways.

Notice that it is sensible to say that 0 objects can be arranged in $0! = 1$ way, since the only arrangement of the set $\{\}$ is given by the ordered tuple $()$. Put another way, the only arrangement of the empty word is the empty word itself.

Example 1 – Arranging digits (with repetition)

Example. How many 5-digit PIN codes can be made from the digits $\{0, 1, 2, 3, 4\}$ (with repeated digits allowed)?

Solution. Since each digit in the code has a choice of 5 options, there are $5 \times 5 \times 5 \times 5 \times 5 = 5^5$ different such PIN codes.

Example. How many 5-digit numbers can be made from the digits $\{0, 1, 2, 3, 4\}$ (with repeated digits allowed)?

Solution. Since the first digit cannot be 0, it has a choice of 4 options, while each of the other digits in the number has a choice of 5 options. So there are 4×5^4 different such numbers.

Alternate solution. There are 5^5 different 5-digit strings without any restrictions. Of these, we wish to exclude those starting with 0. The number of 5-digit strings starting with 0 is found by setting the first digit to be 0, followed by any selection of the digits in $\{0, 1, 2, 3, 4\}$, of which there are 5^4 arrangements. So by the complement law, there are $5^5 - 5^4 = 4 \times 5^4$ different such numbers.

Example 2 – Arranging digits (without repetition)

Example. How many 5-digit PIN codes can be made from the digits $\{0, 1, 2, 3, 4\}$ without repeating any digit?

Solution. Since we are arranging 5 distinct objects here, there are $5! = 120$ different such PIN codes.

Example. How many 5-digit numbers can be made from the digits $\{0, 1, 2, 3, 4\}$ without repeating any digit?

Solution. Since the first digit cannot be 0, it has a choice of 4 options. The second digit could be any of the five originally available, but because it cannot repeat the value taken by the first digit, it also has a choice of 4 options. Similarly, the third digit has a choice of 3 options, the fourth digit has a choice of 2 options, and the fifth digit has a choice of just 1 option. So there are $4 \times 4!$ different such numbers.

Alternate solution. There are $5!$ different 5-digit strings of numbers using the digits $\{0, 1, 2, 3, 4\}$ without repetition. Of these, we wish to exclude those starting with 0. The number of 5-digit strings starting with 0 is found by setting the first digit to be 0, followed by any arrangement of the digits in $\{1, 2, 3, 4\}$ without repetition, of which there are $4!$ arrangements. So by the complement law, there are $5! - 4! = 4 \times 4!$ different such numbers.

Example. In how many ways can 5 boys and 4 girls be arranged in a queue?

Solution. Notice this is just an arrangement problem on 9 distinct people, so there are 9! different ways to arrange the people.

Example. In how many ways can 5 boys and 4 girls be arranged in a circle?

Solution. When arranged in a circle, we assume any rotation of a particular circle arrangement is treated as the same. So consider fixing the position of a particular person, and arranging the remaining 8 people in a circle in clockwise order. This is the same as arranging a queue of 8 people, so there are 8! different ways to arrange the people in this case.

Example. In how many ways can 5 boys and 4 girls be arranged in a queue so that the boys and girls are alternating?

Solution. There is only one way the desired arrangement can be written, in the pattern BGBGBGBGB. So we can arrange the 5 boys in 5! ways before placing them in the odd-numbered positions of the queue, and arrange the 4 girls in 4! ways before placing them in the even-numbered positions. So there are $5! \times 4!$ different ways to arrange the people in this case.

Example 4 – Queues

Example. In how many ways can 5 boys and 4 girls be arranged in a queue so that the boys are all together and the girls are all together?

Solution. Consider the boys and the girls in two separate queues. We can arrange the 5 boys in 5! ways within their queue, and arrange the 4 girls in 4! ways within their queue. Finally, we can arrange the two separate boys' and girls' queues into the complete queue in 2! different ways. So there are $5! \times 4! \times 2!$ different ways to arrange the people in this case.

Example. In how many ways can 5 boys and 4 girls be arranged in a queue so that two particular girls are next to each other?

Solution. We can treat the two particular girls as a single "block". Within this block, the girls can be arranged in 2! different ways. The total number of blocks to be arranged in the queue is 8, being comprised of the two-girl block and the 7 other individual people. These 8 blocks can be arranged in the queue in 8! different ways. So there are $2! \times 8!$ different ways to arrange the people in this case.

Motivating Exercise - Counting anagrams, revisited

Exercise. Find the number of ways that the letters in **SEE** can be arranged.

Non-Solution. Since we are arranging 3 objects here, there are $3! = 6$ different arrangements.

The three objects are not distinct in this case!

Solution. We can list all arrangements systematically, for example by using a tree diagram. They are **EES**, **ESE**, and **SEE**. So there are 3 ways to arrange the letters in **SEE**.

Alternate solution. Suppose that the three letters were in fact distinct, by labelling the letters in the set $\{\mathbf{S}, \mathbf{E}_1, \mathbf{E}_2\}$. Then there are $3!$ ways to arrange these distinct letters. Notice that each word can be paired with another word whose only difference is that $\mathbf{E}_1$ and $\mathbf{E}_2$ have swapped places (for example $\mathbf{E}_1\mathbf{S}\mathbf{E}_2$ and $\mathbf{E}_2\mathbf{S}\mathbf{E}_1$). So if the letters $\mathbf{E}_1$ and $\mathbf{E}_2$ had not in fact been distinct, we would have overcounted by a factor of 2. So there are $3! \div 2 = 3$ ways to arrange the letters in **SEE**.

Motivating Exercise – Anagrams with more repetitions

Exercise. Find the number of ways that the letters in **BANANA** can be arranged.

Solution. Suppose that the six letters were in fact distinct, by labelling the letters in the set $\{\mathbf{B}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$. Then there are $6!$ ways to arrange these distinct letters. Ignoring subscripts, any two words that differ only by swapping the positions of $\mathbf{N}_1$ and $\mathbf{N}_2$ would be considered the same, and any two words that differ only by rearranging the positions of $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ would be considered the same (for example $\mathbf{BA}_1\mathbf{N}_1\mathbf{A}_2\mathbf{N}_2\mathbf{A}_3$ and $\mathbf{BA}_2\mathbf{N}_2\mathbf{A}_3\mathbf{N}_1\mathbf{A}_1$ or $\mathbf{BA}_3\mathbf{N}_1\mathbf{A}_2\mathbf{N}_2\mathbf{A}_1$). So if the letters $\mathbf{N}_1$ and $\mathbf{N}_2$ had not in fact been distinct, we would have overcounted by a factor of $2!$, and if the letters $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ had not in fact been distinct, we would have overcounted by a factor of $3!$.

Thus there are $\dfrac{6!}{2! \times 3!} = 60$ ways to arrange the letters in **BANANA**.

Fact. Given n objects of m distinct types, where there are k_1 of one type, k_2 of another type, and so on (with $n = k_1 + k_2 + \cdots + k_m$), they can be arranged in $\dfrac{n!}{k_1!k_2!\ldots k_m!}$ different ways.

Notice that our counting argument asserts that the above formula will always give an integer. Can you explain this fact algebraically?

Notation. We sometimes write

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!k_2!\ldots k_m!},$$

which is considered to be well-defined whenever $m \in \mathbb{Z}^+$, $n \in \mathbb{N}$, $k_i \in \mathbb{N}$ for all i, and $n = k_1 + k_2 + \cdots + k_m$.

Definition. We call something of the form $\binom{n}{k_1, k_2, \ldots, k_m}$ a multinomial coefficient, for reasons explained in the next two slides.

Motivating Exercise – Multinomial coefficients

Exercise. Find the coefficient of xyz in the expression $(x+y+z)^3$.

Solution. We produce terms in the expansion of

$$(x+y+z)^3 = (x+y+z)(x+y+z)(x+y+z)$$

by selecting a choice of x, y, or z from each bracketed multinomial. For example, one way to produce a xyz term is by selecting x from the first bracketed triple, y from the second, and z from the third. Another way is to select y from the first bracketed triple, z from the second, and x from the third (which gives yzx before rearranging). In general we can see we are counting the number of ways to arrange the letters in xyz, which we know to be $3! = 6$. So the coefficient of xyz is 6.

Exercise. Find the coefficient of x^2y in the expression $(x+y+z)^3$.

Solution. Similarly to the previous exercise, we wish to count the number of ways to select two x's and one y from the three bracketed triples. In this case, we are counting the number of ways to arrange the letters in xxy, which we know to be $\frac{3!}{2!} = 3$. So the coefficient of x^2y is 3.

Multinomial coefficients

Fact. For any positive integer m and natural numbers $n, k_1, k_2, \dots, k_m$ satisfying $n = k_1 + k_2 + \cdots + k_m$, the coefficient of $x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$ in the expanded multinomial expression $(x_1 + x_2 + \cdots + x_m)^n$ is

$$\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \dots k_m!}.$$

Notice that the condition $n = k_1 + k_2 + \cdots + k_m$ is sensible, since any term in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ must have degree n.

Example. Find the coefficient of $a^3 b^2 c$ in the expression $(a + b + c + d)^6$.

Solution. The coefficient of $a^3 b^2 c$ is given by the multinomial coefficient $\binom{6}{3, 2, 1, 0} = \frac{6!}{3!2!1!0!} = 60.$

Example 5 – Grades

Example. A student studied eight different courses in a year. They received three A's, four B's, and one C altogether. In how many ways could this have happened?

Solution. Write the eight subjects down in some particular order. The grades assigned to the subjects can be written next to each subject following this order, and the grades can be read off in order as an 8-letter word, for example **AABBACBB**.

So we want to count the number of ways to arrange the letters in **AAABBBBC**, which is given by the multinomial coefficient

$$\binom{8}{3,4,1} = \frac{8!}{3!4!} = 280.$$

That is, there are 280 ways the student's grades could have been distributed.

Example 6 – Subwords

Example. How many arrangements of the word **ARRANGEMENT** include the subword **RAN**?

Non-Solution. Treat the subword **RAN** as a single "letter", so that we want all arrangements of (**A**, **R**, **RAN**, **G**, **E**, **M**, **E**, **N**, **T**). This is a collection of 9 objects, two of which are the same (**E**), so the total number of arrangements is $\frac{9!}{2!}$.

A word might contain **RAN** as a subword twice, and such a word would be counted twice using this method! (For example, the arrangements **RAN/R/A/N/G/E/M/E/T** and **R/A/N/RAN/G/E/M/E/T** would be counted twice despite representing the same word.)

Example 6 – Subwords

Example. How many arrangements of the word **ARRANGEMENT** include the subword **RAN**?

Solution. Treating the subword **RAN** as a single "letter", we want to count all arrangements of (**A**, **R**, **RAN**, **G**, **E**, **M**, **E**, **N**, **T**). This is a collection of 9 objects, two of which are the same (**E**), which would naively imply a total number of arrangements of $\frac{9!}{2!}$.

However, using this counting approach, any arrangement that includes the subword **RAN** twice would be counted twice when it should be counted only once. To make up for this double-counting, we need to subtract from $\frac{9!}{2!}$ the number of arrangements that contain **RAN** twice. So we also want to find the number of arrangements of (**RAN**, **RAN**, **G**, **E**, **M**, **E**, **T**), which contains 7 objects, two pairs of which are the same (**E** and **RAN**). So the number of arrangements containing **RAN** twice is $\frac{7!}{2!2!}$.

Thus the total number of arrangements containing **RAN** is $\dfrac{9!}{2!} - \dfrac{7!}{2!2!}$.

Notice this solution is an application of the inclusion-exclusion law.

Formally, we can define the set

$$S = \{\text{arrangements of } (\mathbf{A}, \mathbf{R}, \mathbf{RAN}, \mathbf{G}, \mathbf{E}, \mathbf{M}, \mathbf{E}, \mathbf{N}, \mathbf{T})\}$$

and for all $i = 1, 2, \ldots, 9$, define the sets

$$A_i = \{(w_1, w_2, \ldots, w_9) \in S \,:\, w_i = \mathbf{RAN}\}.$$

Then we want to find $|S| = |A_1 \cup A_2 \cup \cdots \cup A_9|$ by using the inclusion-exclusion law, where

- $|A_1| + |A_2| + \cdots + |A_9| = 9 \times \frac{8!}{2!} = \frac{9!}{2!}$,
- the sum of the sizes of all pairwise intersections can be found by counting all arrangements of $(\mathbf{RAN}, \mathbf{RAN}, \mathbf{G}, \mathbf{E}, \mathbf{M}, \mathbf{E}, \mathbf{T})$, so that we have $|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_8 \cap A_9| = \frac{7!}{2!2!}$, and
- the intersection of more than two sets is always empty since **RAN** cannot appear three times in an arrangement.

Thus $|S| = \dfrac{9!}{2!} - \dfrac{7!}{2!2!}$.

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Topic 4 – Combinatorics
Lecture 4.03 – Permutations and combinations

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Motivating Exercise – PIN codes revisited

Exercise. Find the number of 5-digit PIN codes that do not have any digits repeated.

Solution. We are seeking five-digit strings made from the digits in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ without repetition. The first digit can be any of the 10 available options. The second digit can be any of the 9 remaining options (since the first digit's value is no longer available as an option). Similarly the third digit has 8 remaining options, the fourth digit has 7 remaining options, and the fifth digit has 6 remaining options. So altogether there are $10 \times 9 \times 8 \times 7 \times 6 = 30240$ different 5-digit PIN codes without any repeated digits.

Notice this approach will work in general. Given an alphabet with n letters, the number of k-letter words whose letters are all distinct is $n \times (n-1) \times \cdots \times (n-k+1)$, or

$$\frac{n \times (n-1) \times \cdots \times (n-k+1) \times (n-k) \times \cdots \times 2 \times 1}{(n-k) \times \cdots \times 2 \times 1} = \frac{n!}{(n-k)!}.$$

Permutations

Definition. A permutation of objects is a selection of distinct objects whose order of selection matters.

- For example, **ABC** and **CAB** are considered different permutations of the string **ABC**.
- For example, **ABC**, **CAB**, and **DAB** are considered different permutations of size 3 from the set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$.

Notation. We denote the number of ways to choose a permutation of size k from n distinct types of object (without repetition) by $P(n, k)$ or ${}^nP_k$.

Given $k, n \in \mathbb{N}$ with $n \geq k$, we have $P(n, k) = \dfrac{n!}{(n-k)!}$.

If $n < k$, we instead have $P(n, k) = 0$.

Example. How many ways can a president, vice-president, treasurer, and secretary be selected from an executive of 9 people?

Solution. We wish to select 4 different people from a group of 9 people in a particular order (supposing the role of president is awarded first, then vice-president, etc.). So there are $P(9, 4) = \frac{9!}{5!} = 3024$ different ways to do this.

Exercise. Find the number of ways to select a subset of 3 distinct letters from the set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}\}$.

Non-Solution. We want to find the number of 3-letter words whose letters are taken from $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}\}$ without repetition. So there are $P(5,3)$ different ways to select the letters.

The order in which we choose the 3 letters doesn't matter in this case!

Solution. We are seeking all subsets of $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}\}$ that have size 3. We already know that there are $P(5,3)$ ways to select the three letters when the order of selection matters, but this method would count (for example) the set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ a total of 6 times (as **ABC**, **ACB**, **BAC**, **BCA**, **CAB**, and **CBA**). In fact, finding the number of ordered selections is overcounting by 3! in all instances. So the number of size-3 subsets is given by

$$P(5,3) \div 3! = \frac{5!}{2!3!}.$$

Notice this approach will work in general. Given a set of n distinct objects, the number of subsets with size k is $\dfrac{P(n,k)}{k!} = \dfrac{n!}{(n-k)!k!}$.

Combinations

Definition. A combination of objects is a selection of distinct objects whose order of selection does not matter.

- For example, the sets $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ and $\{\mathbf{C}, \mathbf{A}, \mathbf{B}\}$ are considered to be the same combination.
- For example, $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ and $\{\mathbf{D}, \mathbf{A}, \mathbf{B}\}$ are considered different combinations of size 3 from the set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$.

Notation. We denote the number of ways to choose a combination of size k from n distinct types of object (without repetition) by $C(n, k)$ or ${}^nC_k$ or $\binom{n}{k}$.

Given $k, n \in \mathbb{N}$ with $n \geq k$, we have $C(n, k) = \dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

If $n < k$, we instead have $C(n, k) = 0$.

Example. How many ways can an executive of 9 people be chosen from a group of 13 people?

Solution. We wish to select 9 different people from a group of 13 people, where the order of selection is unimportant. So altogether there are $C(13, 9) = \frac{13!}{9!4!} = 715$ different ways to do this.

Fact. For any natural numbers n and k with $n \geq k$, the coefficient of $x^k y^{n-k}$ in the expanded binomial expression $(x+y)^n$ is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \,.$$

We can therefore write $(x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Notice that this is just a special application of the multinomial coefficient expression, and we can interpret $\binom{n}{k}$ as $\binom{n}{k,\, n-k}$.

Example. Find the coefficient of $a^4 b^9$ in the expression $(a+b)^{13}$.

Solution. The coefficient is given by $\binom{13}{4} = \dfrac{13!}{4!9!} = 715$.

Example 1 – Committees

Example. In how many ways can 7 women and 4 men form a committee of 5 people?

Solution. Since the order of selection does not matter here, and there are 11 total people to choose from, the total number of possible committees is

$$C(11, 5) = \frac{11!}{5!6!}.$$

Example. In how many ways can 7 women and 4 men form a committee of 5 people in which there are 5 different named roles?

Solution. Since each position in the committee has a different role, the order of selection does matter here. So as there are 11 total people to choose from, the total number of possible committees in this case is

$$P(11, 5) = \frac{11!}{6!}.$$

Example 2 – Committees

Example. In how many ways can 7 women and 4 men form a committee of 5 people in which there is a president, vice-president, and treasurer?

Solution. We can break this problem into independent steps. First choose the committee as an unordered group of 5 people from the 11, which can be done in $C(11, 5)$ ways. Then choose the named roles as an ordered group of 3 people from the 5 on the committee, which can be done in $P(5, 3)$ ways. Altogether, the total number of possible committees in this case is

$$C(11, 5) \times P(5, 3) = \frac{11!}{5!6!} \times \frac{5!}{2!} = \frac{11!}{6!2!}.$$

Alternate solution. First choose the named roles as an ordered group of 3 people from the 11, which can be done in $P(11, 3)$ ways. Then choose the rest of the committee as an unordered group of 2 people from the remaining $11 - 3 = 8$ unassigned people, which can be done in $C(8, 2)$ ways. Altogether, the total number of possible committees in this case is

$$P(11, 3) \times C(8, 2) = \frac{11!}{8!} \times \frac{8!}{2!6!} = \frac{11!}{2!6!}.$$

Example 3 – Committees

Example. In how many ways can 7 women and 4 men form a committee of 3 women and 2 men?

Solution. We can break this problem into independent steps. First choose an unordered group of 3 women from the 7, which can be done in $C(7,3)$ ways. Then choose an unordered group of 2 men from the 4, which can be done in $C(4,2)$ ways. Altogether, the total number of possible committees in this case is

$$C(7,3) \times C(4,2) = \frac{7!}{3!4!} \times \frac{4!}{2!2!}.$$

Example 4 – Diverse committees

Example. In how many ways can 7 women and 4 men form a committee of 5 people in which there is at least one man?

Solution. The complement of the set of all 5-person committees with at least one man is the set of all 5-person committees with no men. The number of 5-person committees with no men is found by selecting 5 people from the 7 women, which can be done in $C(7,5)$ ways. We can subtract this from the total number of ways to select a committee, namely $C(11,5)$, to find the size of the complement. That is, the total number of possible committees in this case is

$$C(11,5) - C(7,5) = \frac{11!}{5!6!} - \frac{7!}{5!2!}.$$

Alternate solution. Break the scenario into mutually exclusive cases. If exactly one man is on the committee, this can be done in $C(4,1) \times C(7,4)$ ways. If exactly two men are on the committee, this can be done in $C(4,2) \times C(7,3)$ ways, and so on. Altogether we find the total number of possible committees in this case is

$$C(4,1) \times C(7,4) + C(4,2) \times C(7,3) + C(4,3) \times C(7,2) + C(4,4) \times C(7,1).$$

Example 5 – Contentious committees

Example. In how many ways can 7 women and 4 men form a committee of 5 people if two particular people refuse to be on the committee together?

Solution. The complement of the set of all 5-person committees without two particular people included together is the set of all 5-person committees with those two particular people included together. The number of 5-person committees with two particular people together is found by first selecting those 2 people, and then selecting another 3 from the remaining $11 - 2 = 9$ unassigned people, which can be done in $C(9, 3)$ ways. We can subtract this from the total number of ways to select a committee, namely $C(11, 5)$, to find the size of the complement. That is, the total number of possible committees in this case is

$$C(11,5) - C(9,3) = \frac{11!}{5!6!} - \frac{9!}{3!6!}.$$

MATH1081 – Discrete Mathematics

Topic 4 – Combinatorics
Lecture 4.04 – Selections with repetition

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Ordered selections with repetition allowed

Fact. The number of ways to choose an ordered selection of k objects from n distinct types of object with repetition allowed is given by n^k.

Proof. This is a direct application of the multiplication law, where we are trying to fill k spaces, each of which has n options available.

Example. How many 5-digit PIN codes are there?

Solution. We want to find the number of ordered selections of 5 digits from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. So there are 10^5 different 5-digit PIN codes.

Finding the number of unordered selections with repetition allowed is a slightly trickier task...

Motivating Exercise – Distributing objects

Exercise. Find the number of ways 3 identical computers can be distributed to 5 different offices, where an office may contain more than one computer.

Solution. Label the offices **A** through **E**. We are seeking all unordered selections of 3 letters from the set $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}\}$ with repetition allowed (for example, distributing one computer to the first office and two to the second corresponds with the selection **ABB**, which is no different from the selection **BAB**). There are 5 ways to select a single office to receive all 3 computers, $P(5, 2)$ ways to select two offices that receive 1 and 2 computers respectively, and $C(5, 3)$ ways to choose three offices that each receive 1 computer. So the total number of ways is $5 + P(5, 2) + C(5, 3)$.

Notice that this approach will not scale well – how many cases would we need to consider if there were 4 computers to distribute instead?

Motivating Exercise – Distributing objects

Exercise. Find the number of ways 3 identical computers can be distributed to 5 different offices, where an office may contain more than one computer.

Alternate solution. Represent the offices diagrammatically by 4 lines representing the walls between them:

$$\mathbf{A} \mid \mathbf{B} \mid \mathbf{C} \mid \mathbf{D} \mid \mathbf{E}$$

The computers can then be represented as stars drawn between these dividing bars. For example:

$\star \mid \star \mid \star \mid \mid$	$\star\star \mid \mid \mid \star \mid$	$\mid \mid \mid \mid \star\star\star$
represents	represents	represents
ABC,	**AAD**,	**EEE**.

So we want to find the number of ways to arrange the characters in the string $\mid\mid\mid\mid\star\star\star$, which contains 7 characters total with multiplicities 4 and 3. Thus the number of ways to distribute the computers is given by

$$\binom{7}{4,3} = \binom{7}{3} = \frac{7!}{4!3!}.$$

Notice that this approach will work in general.

Unordered selections with repetition allowed

Fact. The number of ways to choose an unordered selection of k objects from n distinct types of object with repetition allowed is given by

$$\binom{k+n-1}{k} = \binom{k+n-1}{n-1} = \frac{(k+n-1)!}{k!(n-1)!}.$$

Proof. This comes from the "stars and bars" approach, where the number of stars is k and the number of bars is one less than the total number of available positions/options, so $n-1$.

Example. How many ways can 4 fruits be selected from a grocer selling apples, bananas, cherries, dates, elderberries, and figs?

Solution. We want to find the number of unordered selections of 4 fruits from the set of 6 fruit types with repetition allowed. This can be represented by drawing 4 stars between $6-1=5$ dividing bars, where stars left of the first bar represent the number of apples bought, stars between the first and second bars represent the number of bananas bought, etc. So there are $C(9,4)$ ways to do this.

Often the difficulty with these sorts of problems is identifying which part of the problem maps to bars and which to stars.

Example 1 – Multinomial terms

Example. How many different terms are there in the expanded multinomial $(a+b+c+d+e+f)^4$?

Solution. Each term in the expansion is an unordered selection of 4 objects from the set $\{a, b, c, d, e, f\}$, with repetition allowed. We can think of any term in the expansion as a distribution of 4 objects amongst 6 distinct boxes. Diagrammatically, this can be represented by drawing 4 stars amongst $6-1=5$ dividing bars. For example:

$\star\star|\star|||\star|$ represents a^2be, $|||||\star\star\star\star$ represents f^4.

So altogether, we are finding the number of ways to arrange 9 objects with multiplicities 4 and 5. Thus the number of solutions is given by

$$\binom{9}{5} = \frac{9!}{5!4!}.$$

Example 2 – Integer sums

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 10$, where $x_i \in \mathbb{N}$ for all i?

Solution. We can think of any solution to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 10$ as a distribution of 10 objects amongst five distinct boxes. Diagrammatically, this can be represented by drawing 10 stars amongst $5 - 1 = 4$ dividing bars. For example:

$$\star\star\,|\,\star\star\,|\,\star\star\,|\,\star\star\,|\,\star\star$$

represents

$$x_1 = x_2 = x_3 = x_4 = x_5 = 2,$$

$$\star\star\star\star\star\,|\,\star\star\star\,|\,\star\star\,|\,|$$

represents

$$x_1 = 5, x_2 = 3, x_3 = 2, x_4 = x_5 = 0.$$

So altogether, we are finding the number of ways to arrange 14 objects with multiplicities 10 and 4. Thus the number of solutions is given by

$$\binom{14}{4} = \frac{14!}{4!10!}.$$

Example 3 – Integer sums with greater-than conditions

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 18$, where $x_i \in \mathbb{N}$ and $x_i \geq 2$ for all i?

Solution. We can think of any solution to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 18$ as a distribution of 18 objects amongst 5 boxes. However, we only want to count solutions for which there are at least 2 objects in each box. So first place 2 objects in each box, and then distribute the remaining $18 - 2 \times 5 = 8$ objects amongst the 5 boxes without any further restrictions. Diagrammatically, this can be represented by drawing 2 grey stars in each position, and then distributing another 8 stars amongst $5 - 1 = 4$ bars. For example:

$$\begin{array}{c|c|c|c|c} \star\star & \star\star & \star\star & \star\star & \star\star \\ \star\star & \star\star & \star\star & \star\star & \end{array}$$

represents

$$x_1 = x_2 = x_3 = x_4 = 4, x_5 = 2,$$

$$\begin{array}{c|c|c|c|c} \star\star & \star\star & \star\star & \star\star & \star\star \\ \star\star\star\star & \star\star & \star\star & & \end{array}$$

represents

$$x_1 = 6, x_2 = x_3 = 4, x_4 = x_5 = 2.$$

So altogether, we are finding the number of ways to arrange 12 (blue) objects with multiplicities 8 and 4, which is

$$\binom{12}{4} = \frac{12!}{4!8!}.$$

Example 3 – Integer sums with greater-than conditions

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 18$, where $x_i \in \mathbb{N}$ and $x_i \geq 2$ for all i?

Alternate solution. Choose y_i such that $x_i = 2 + y_i$ for each i. Then the equation and conditions

$$x_1 + x_2 + x_3 + x_4 + x_5 = 18 \quad \text{where} \quad x_i \in \mathbb{N} \quad \text{and} \quad x_i \geq 2 \quad \text{for all } i$$

become the equation and conditions

$$y_1 + y_2 + y_3 + y_4 + y_5 = 8 \quad \text{where} \quad y_i \in \mathbb{N} \quad \text{and} \quad y_i \geq 0 \quad \text{for all } i.$$

We have already shown that the number of solutions to $y_1 + y_2 + \cdots + y_n = k$ where $y_i \in \mathbb{N}$ for all i (without any further restrictions) is the number of ways to distribute k objects amongst n boxes, so the number of solutions in this case is

$$\binom{8+5-1}{4} = \frac{12!}{4!8!}.$$

Example 4 – Integer sums with parity conditions

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 12$, where $x_i \in \mathbb{N}$ and x_i is even for all i?

Solution. We can think of any solution to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 12$ as a distribution of 12 objects amongst 5 boxes. However, we only want to count solutions for which there are an even number of objects in each box. So consider distributing the 12 objects amongst the 5 boxes two at a time. Diagrammatically, this can be represented by drawing 6 "double-stars" amongst $5 - 1 = 4$ bars. For example:

$\star\star\ \star\star \mid \star\star \mid \star\star \mid \star\star \mid \star\star$
represents
$x_1 = 4, x_2 = x_3 = x_4 = x_5 = 2,$

$\star\star\ \star\star \mid \mid \star\star\ \star\star\ \star\star \mid \star\star \mid$
represents
$x_1 = 4, x_3 = 6, x_4 = 2, x_2 = x_5 = 0.$

So altogether, we are finding the number of ways to arrange 10 objects with multiplicities 6 and 4, which is

$$\binom{10}{4} = \frac{10!}{4!6!}.$$

Example 4 – Integer sums with parity conditions

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 12$, where $x_i \in \mathbb{N}$ and x_i is even for all i?

Alternate solution. Choose y_i such that $x_i = 2y_i$ for each i. Then the equation and conditions

$$x_1 + x_2 + x_3 + x_4 + x_5 = 12 \quad \text{where} \quad x_i \in \mathbb{N} \quad \text{and} \quad x_i \text{ is even} \quad \text{for all } i$$

become the equation and conditions

$$y_1 + y_2 + y_3 + y_4 + y_5 = 6 \quad \text{where} \quad y_i \in \mathbb{N} \quad \text{and} \quad y_i \text{ is an integer} \quad \text{for all } i.$$

We have already shown that the number of solutions to $y_1 + y_2 + \cdots + y_n = k$ where $y_i \in \mathbb{N}$ for all i (without any further restrictions) is the number of ways to distribute k objects amongst n boxes, so the number of solutions in this case is

$$\binom{6+5-1}{4} = \frac{10!}{4!6!}.$$

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 17$, where $x_i \in \mathbb{N}$ and $x_i \leq 4$ for all i?

Solution. We can think of any solution to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 17$ as a distribution of 17 objects amongst 5 boxes. However, we only want to count solutions for which there are no more than 4 objects in each box. So consider first placing 4 objects into each box, and then removing $5 \times 4 - 17 = 3$ of these objects. Diagrammatically, this can be represented by drawing 4 grey stars in each position, and then distributing 3 crosses (deletion indicators) amongst $5 - 1 = 4$ bars. For example:

$$\begin{array}{c|c|c|c|c} \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star \\ & \times & \times & \times & \end{array}$$

represents

$x_1 = x_5 = 4, x_2 = x_3 = x_4 = 3,$

$$\begin{array}{c|c|c|c|c} \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star \\ \times\times\times & & & & \end{array}$$

represents

$x_1 = 1, x_2 = x_3 = x_4 = x_5 = 4.$

So altogether, we are finding the number of ways to arrange 7 (blue) objects with multiplicities 3 and 4, which is

$$\binom{7}{4} = \frac{7!}{4!3!}.$$

Example 5 – Integer sums with less-than conditions

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 17$, where $x_i \in \mathbb{N}$ and $x_i \leq 4$ for all i?

Alternate solution. Choose y_i such that $x_i = 4 - y_i$ for each i. Then the equation and conditions

$$x_1 + x_2 + x_3 + x_4 + x_5 = 17 \text{ where } x_i \in \mathbb{N} \text{ and } 0 \leq x_i \leq 4 \text{ for all } i$$

become the equation and conditions

$$y_1 + y_2 + y_3 + y_4 + y_5 = 3 \text{ where } y_i \in \mathbb{N} \text{ and } 0 \leq y_i \leq 4 \text{ for all } i.$$

However, the restriction $y_i \leq 4$ is trivially satisfied here, since the total sum being 3 already implies no y_i can exceed 3. So we can omit the restriction $y_i \leq 4$.

We have already shown that the number of solutions to $y_1 + y_2 + \cdots + y_n = k$ where $y_i \in \mathbb{N}$ for all i (without any further restrictions) is the number of ways to distribute k objects amongst n boxes, so the number of solutions here is

$$\binom{3+5-1}{4} = \frac{7!}{4!3!}.$$

Example 6 – Integer sums with less-than conditions

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 15$, where $x_i \in \mathbb{N}$ and $x_i \le 4$ for all i?

Non-Solution. We can think of any solution to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 15$ as a distribution of 15 objects amongst 5 boxes. However, we only want to count solutions for which there are no more than 4 objects in each box. So consider first placing 4 objects into each box, and then removing $5 \times 4 - 15 = 5$ of these objects. Diagrammatically, this can be represented by drawing 4 grey stars in each position, and then distributing 5 crosses (deletion indicators) amongst $5 - 1 = 4$ bars. For example:

$$\begin{array}{c|c|c|c|c} \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star \\ \times & \times & \times & \times & \times \end{array}$$

represents

$x_1 = x_2 = x_3 = x_4 = x_5 = 3,$

$$\begin{array}{c|c|c|c|c} \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star \\ \times\times\times & \times & \times & & \end{array}$$

represents

$x_1 = 1, x_2 = x_3 = 3, x_4 = x_5 = 4.$

This method will count arrangements where there are more than 4 crosses assigned to a single section. For example, if 5 crosses are assigned to the first section, this would represent $x_1 = -1 \notin \mathbb{N}$.

To correct this error, we should subtract the number of cases where more than 4 crosses are assigned to a single section.

Example 6 – Integer sums with less-than conditions

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 15$, where $x_i \in \mathbb{N}$ and $x_i \leq 4$ for all i?

Solution. We can think of any solution to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 15$ as a distribution of 15 objects amongst 5 boxes. However, we only want to count solutions for which there are no more than 4 objects in each box. So consider first placing 4 objects into each box, and then removing $5 \times 4 - 15 = 5$ of these objects. Diagrammatically, this can be represented by drawing 4 grey stars in each position, and then distributing 5 crosses (deletion indicators) amongst $5 - 1 = 4$ bars. For example:

$$\begin{array}{c|c|c|c|c} \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star \\ \times & \times & \times & \times & \times \end{array}$$

represents

$$x_1 = x_2 = x_3 = x_4 = x_5 = 3,$$

$$\begin{array}{c|c|c|c|c} \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star & \star\star\star\star \\ \times\times\times & \times & \times & & \end{array}$$

represents

$$x_1 = 1, x_2 = x_3 = 3, x_4 = x_5 = 4.$$

Amongst these cases, we must exclude all cases where more than 4 crosses (that is, exactly 5 crosses) are assigned to a single section, to prevent any x_i taking on negative values. There are exactly 5 ways that this can happen, so the number of solutions here is

$$\binom{9}{4} - 5 = \frac{9}{4!5!} - 5.$$

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 15$, where $x_i \in \mathbb{N}$ and $x_i \le 4$ for all i?

Alternate non-solution. Choose y_i such that $x_i = 4 - y_i$ for each i. Then the equation and conditions

$$x_1 + x_2 + x_3 + x_4 + x_5 = 15 \;\text{ where }\; x_i \in \mathbb{N} \;\text{ and }\; 0 \le x_i \le 4 \text{ for all } i$$

become the equation and conditions

$$y_1 + y_2 + y_3 + y_4 + y_5 = 5 \;\text{ where }\; y_i \in \mathbb{N} \;\text{ and }\; 0 \le y_i \le 4 \text{ for all } i.$$

In this case, the restriction $y_i \le 4$ is not trivially satisfied, so it must still be taken into consideration. This means this substitution has not converted the problem into one we are familiar with.

To adapt this method, we should proceed as usual but then subtract the number of cases where $y_i > 4$.

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 15$, where $x_i \in \mathbb{N}$ and $x_i \leq 4$ for all i?

Alternate solution. Choose y_i such that $x_i = 4 - y_i$ for each i. Then the equation and conditions

$$x_1 + x_2 + x_3 + x_4 + x_5 = 15 \;\text{ where }\; x_i \in \mathbb{N} \;\text{ and }\; 0 \leq x_i \leq 4 \text{ for all } i$$

become the equation and conditions

$$y_1 + y_2 + y_3 + y_4 + y_5 = 5 \;\text{ where }\; y_i \in \mathbb{N} \;\text{ and }\; 0 \leq y_i \leq 4 \text{ for all } i.$$

We cannot omit the restriction that $0 \leq y_i \leq 4$ since it is not trivially satisfied in this case. Instead, we could ignore the restriction and then exclude all cases where $y_i > 4$. Since the total sum must equal 5, this can only occur in 5 different ways (when one of the y_i terms is 5, and the rest are 0). So the number of solutions here is

$$\binom{9}{4} - 5 = \frac{9}{4!5!} - 5.$$

Example 7 – Integer sums with less-than conditions

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 28$, where $x_i \in \mathbb{N}$ and $x_i \leq 8$ for all i?

Motivation: The methods in Example 6 become a little more unwieldy when applied here. It was easier to solve this sort of problem when the less-than conditions were greater-than conditions, so let's try to use the complement law.

Solution. Set $\mathcal{U}$ to be the universal set of solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 28$ where $x_i \in \mathbb{N}$. Then the complement of the set of solutions where $x_i \leq 8$ for all i is the set of solutions where $x_i > 8$ for some i, that is, the set of solutions where $x_i \geq 9$ for some i.

Let $A_i = \{\text{solutions to } x_1 + x_2 + x_3 + x_4 + x_5 = 28 \text{ where } x_i \geq 9\}$ for each i. Then the set of solutions where $x_i \geq 9$ for some i is $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$, and the number of solutions to the original set of conditions is given by

$$|\mathcal{U}| - |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|.$$

In order to find this value, we can use the inclusion-exclusion principle and determine each set size using techniques established in earlier examples...

Example 7 – Integer sums with less-than conditions

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 28$, where $x_i \in \mathbb{N}$ and $x_i \leq 8$ for all i?

Solution (continued). Using the "stars and bars" approach explained earlier, the size of $\mathcal{U}$ can be found by counting the number of ways to distribute 28 stars amongst $5 - 1 = 4$ bars, so $|\mathcal{U}| = \binom{32}{4}$.

There are $\binom{5}{1}$ sets of the form A_i. The size of each A_i is given by assigning 9 stars to the ith position, and then distributing the remaining $28 - 9 = 19$ stars amongst 4 bars, so $|A_i| = \binom{23}{4}$ for each i.

There are $\binom{5}{2}$ intersections of the form $A_i \cap A_j$ for any distinct i, j. The size of each $A_i \cap A_j$ is found by assigning 9 stars each to the ith and jth positions, and then distributing the remaining $28 - 18 = 10$ stars amongst 4 bars, so $|A_i \cap A_j| = \binom{14}{4}$ for each pair of distinct i, j.

There are $\binom{5}{3}$ intersections of the form $A_i \cap A_j \cap A_k$ for any distinct i, j, k, and by a similar argument we find the size of each three-fold intersection is given by $|A_i \cap A_j \cap A_k| = \binom{5}{4}$.

Any four- or five-fold intersection will be empty, since if more than three x_i terms are at least 9, their sum will be at least 36 which exceeds 28...

Example 7 – Integer sums with less-than conditions

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 28$, where $x_i \in \mathbb{N}$ and $x_i \leq 8$ for all i?

Solution (continued). Putting this all together, we have by the inclusion-exclusion principle that

$$|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = \binom{5}{1}\binom{23}{4} - \binom{5}{2}\binom{14}{4} + \binom{5}{3}\binom{5}{4},$$

and so the number of solutions to the original question is

$$|\mathcal{U}| - |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = \binom{32}{4} - \binom{5}{1}\binom{23}{4} + \binom{5}{2}\binom{14}{4} - \binom{5}{3}\binom{5}{4}.$$

Notice the patterns followed by the numbers in each combination, especially if we premultiply the first term by $\binom{5}{0}$. The bottom numbers in the first term of each pair increase by 1, while the top numbers in the second term of each pair decrease by $8 + 1 = 9$. This provides a useful way to check that an answer looks correct.

Challenge: Note that an alternative approach could be to combine this argument with the substitution $x_i = 8 - y_i$ as seen in Example 6. Show that this leads to the simpler solution $\binom{16}{4} - \binom{5}{1}\binom{7}{4}$.

MATH1081 – Discrete Mathematics

Topic 4 – Combinatorics
Lecture 4.05 – Counting arguments and the pigeonhole principle

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Summary of counting rules

The number of ways to arrange n distinct objects is $n!$.

Given n objects of m distinct types, where there are k_1 of one type, k_2 of another type, and so on (with $n = k_1 + k_2 + \cdots + k_m$), they can be arranged in $\binom{n}{k_1, k_2, \ldots, k_m} = \dfrac{n!}{k_1!k_2! \ldots k_m!}$ different ways.

The number of ways to select k objects from n distinct types of object is given by:

	ordered selection	unordered selection
without repetition	$P(n,k) = \dfrac{n!}{(n-k)!}$	$\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$
with repetition	n^k	$\dbinom{k+n-1}{k} = \dfrac{(k+n-1)!}{k!(n-1)!}$

We have so far used these rules to enumerate answers to various word problems, but it is also sometimes possible to prove certain mathematical identities by using counting (or "word") arguments.

Example 1 – Symmetry of combinations

Example. Prove that for any $k, n \in \mathbb{N}$ with $k \leq n$, we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

Algebraic solution. We have in general that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}.$$

Counting solution. Suppose we wish to select k different objects from n distinct objects to be placed into a box. This can be done in $\binom{n}{k}$ different ways.

Equivalently, we could select $n-k$ different objects from the n distinct objects to be left out of the box, and move the remaining objects into the box. This can be done in $\binom{n}{n-k}$ different ways.

Since these are two different ways of enumerating the same result, we have shown that

$$\binom{n}{k} = \binom{n}{n-k}.$$

Example 2 – Pascal's formula

Example. Prove that for any $k, n \in \mathbb{N}$ with $k \leq n$, we have

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Algebraic solution. We have in general that

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} \\
&= \frac{(k+1)n! + (n-k)n!}{(k+1)!(n-k)!} \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} \\
&= \binom{n+1}{k+1}.
\end{aligned}$$

Example. Prove that for any $k, n \in \mathbb{N}$ with $k \leq n$, we have

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

Counting solution. Suppose a committee of $k+1$ people must be chosen from a group of $n+1$ people. This can be done in $\binom{n+1}{k+1}$ different ways.

Let X be one of the people in the group. Then we can select the committee in two ways:

- Select X for the committee. Then the remaining k committee members must be chosen from the other n people in the group, which can be done in $\binom{n}{k}$ ways.
- Do not select X for the committee. Then $k+1$ committee members must be chosen from the other n people in the group, which can be done in $\binom{n}{k+1}$ ways.

Since these two cases are exhaustive and mutually exclusive, we must have that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

Example 3 – Sum of rows in Pascal's triangle

Example. Prove that for any $n \in \mathbb{N}$, we have

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Algebraic solution. Recall that for any natural number n and real numbers x and y, we have by the binomial theorem that

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n}x^ny^0.$$

Substituting $x = y = 1$ into both sides of the above equation, we then find that

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}.$$

Example 3 – Sum of rows in Pascal's triangle

Example. Prove that for any $n \in \mathbb{N}$, we have

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Counting solution. Suppose we wish to count the number of subsets of the set $A = \{1, 2, 3, \ldots, n\}$.

To form a subset of size k, we must choose k elements from A (with unordered selection and without repetition), which can be done in $\binom{n}{k}$ ways. So to enumerate all subsets of A, we can count all subsets of size $0, 1, 2, \ldots, n$, which is done in $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ ways.

From another perspective, to form any subset of A, we can go through each of its elements in turn, deciding whether that element is or is not included in the subset. Since each choice is between 2 options and each choice is independent, this can be done in 2^n ways.

Since these are two different ways of enumerating the same result, we have shown that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

Pigeonhole Principle

Theorem. (Pigeonhole principle)
Given n objects to distribute amongst k boxes, if $n > k$ then at least one box must contain at least 2 objects.

Notice this rule is only an existential statement, and doesn't allow us to directly calculate particular values in general.

Example. Suppose 12 pigeons come to roost amongst 10 pigeonholes. Then at least one pigeonhole contains more than one pigeon (that is, at least two pigeons).

Notice that in the above example:

- It is possible more than one pigeonhole contains more than one pigeon. For example, two pigeonholes could each house 6 pigeons.
- It is possible some pigeonhole contains less than one pigeon, like in the previous example.
- No pigeonhole is guaranteed to have more than two pigeons. For example, six pigeonholes could each house 2 pigeons.

Example 4 – Pigeonhole principle

Example. Show that amongst any collection of 7 integers, at least two must leave the same remainder upon division by 6.

Solution. The set of possible remainders upon division by 6 is $\{0, 1, 2, 3, 4, 5\}$. We wish to assign 7 integers (pigeons) to these 6 possible remainders (pigeonholes), so by the pigeonhole principle at least one remainder will have at least two integers assigned to it. That is, at least two integers in any collection of 7 will have the same remainder upon division by 6.

Example 5 – Injectivity

Example. Show that any function from $f : X \to Y$ with $|X| > |Y|$ cannot be injective.

Solution. A function is not injective if two different input values give the same output value. Any function f must map the set of $|X|$ possible inputs (pigeons) to the set of $|Y|$ possible outputs (pigeonholes), so by the pigeonhole principle, since $|X| > |Y|$, we have that at least one output will have at least two inputs assigned to it. That is, no function $f : X \to Y$ with $|X| > |Y|$ can be injective.

Example 6 – Reverse pigeonhole principle

Example. How many different people are needed to guarantee that at least two share the same birthday?

Solution. The set of possible birthdays has 366 total elements. We wish to assign a group of n people (pigeons) to these 366 possible dates (pigeonholes) so that at least one date is assigned at least twice. By the pigeonhole principle, we want $n > 366$, so at least 367 different people are needed to guarantee that at least two share the same birthday.

Example 7 – Harder pigeonhole principle

Example. Show that at a meeting of 9 people, there are two people who shook the same number of hands.

Solution. The set of possible number of hands shaken is $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. We wish to assign 9 people (pigeons) to these 9 possible values (pigeonholes). The pigeonhole principle doesn't apply here, since the number of objects being distributed is not greater than the number of boxes.

However, notice that it is not possible for both 0 and 8 hands to be shaken at the same meeting, since this would imply one person shook nobody's hand while another person shook everybody's hand. So in fact, there are two sets of possible number of hands shaken: either $\{0, 1, 2, 3, 4, 5, 6, 7\}$ or $\{1, 2, 3, 4, 5, 6, 7, 8\}$, each of which has size 8.

We wish to assign 9 people (pigeons) amongst one of these sets of 8 possible values (pigeonholes), so by the pigeonhole principle at least one "handshake number" will have at least two people assigned to it. That is, at least two people in any collection of 9 will have shaken the same number of hands.

Theorem. (Generalised pigeonhole principle)
Given n objects to distribute amongst k boxes, at least one box must contain at least $\left\lceil \frac{n}{k} \right\rceil$ objects.

Notice this is consistent with the basic pigeonhole principle, since if $n > k$ then $\frac{n}{k} > 1$ and so $\left\lceil \frac{n}{k} \right\rceil \geq 2$.

Example. Suppose 24 pigeons come to roost amongst 10 pigeonholes. Then at least one pigeonhole contains at least $\left\lceil \frac{24}{10} \right\rceil = 3$ pigeons.

Notice that in the above example:

- It is possible more than one pigeonhole contains more than 3 pigeons. For example, two pigeonholes could each house 12 pigeons.
- It is possible some pigeonhole contains less than 3 pigeons, like in the previous example.
- No pigeonhole is guaranteed to have more than 3 pigeons. For example, eight pigeonholes could each house 3 pigeons.

Example 8 – Generalised pigeonhole principle

Example. Show that amongst a cohort of 1500 students, there are at least 5 who share the same birthday.

Solution. The set of possible birthdays has 366 total elements. We wish to assign a group of 1500 students (pigeons) to these 366 possible dates (pigeonholes), so by the (generalised) pigeonhole principle at least one date will have at least $\left\lceil \dfrac{1500}{366} \right\rceil = 5$ students assigned to it. That is, at least 5 students share the same birthday.

Example 9 – Reverse generalised pigeonhole principle

Example. A collection of integers must be at least how large for it to be certain that at least five of them end in the same digit?

Solution. The set of possible final digits is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We wish to find n such that when assigning any set of n integers (pigeons) to these 10 possible digits (pigeonholes), at least 5 are assigned to the same digit. By the (generalised) pigeonhole principle, we wish to solve $\left\lceil \frac{n}{10} \right\rceil \geq 5$, which is equivalent to solving $\frac{n}{10} > 4$.

The smallest integer solution to $n > 40$ is $n = 41$, so at least 41 integers are needed for it to be certain that at least five of them end in the same digit.

MATH1081 – Discrete Mathematics

Topic 4 – Combinatorics
Lecture 4.06 – Recurrence relations (homogeneous)

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Notation. We typically refer to a sequence of numbers $a_0, a_1, a_2, \ldots$ as (a_n) or $(a_n)_{n\in\mathbb{N}}$. The ith term in the sequence is given by a_i, so in particular, a_0 is usually referred to as the "zeroth term".

Definition. A recurrence relation is any relation which determines a sequence of numbers by defining successive terms in relation to previous terms. Some examples of recurrence relations are:

- $a_n = 2a_{n-1}$.
- $b_n = b_{n-1} + b_{n-2}$.
- $c_{n+1} = c_0c_n + c_1c_{n-1} + \cdots + c_nc_0$.

A sequence defined by a recurrence relation is not explicitly described unless it also includes initial conditions, which usually give the first value(s) of the sequence. For example:

- $a_n = 2a_{n-1}$ for all $n \in \mathbb{Z}^+$, and $a_0 = 1$.
- $b_n = b_{n-1} + b_{n-2}$ for all integers $n \geq 2$, and $b_0 = 0, b_1 = 1$.
- $c_{n+1} = c_0c_n + c_1c_{n-1} + \cdots + c_nc_0$ for all $n \in \mathbb{N}$, and $c_0 = 1$.

Example 1 – Amoebas

Example. An amoeba splits into two amoebas every hour. A Petri dish contains a single amoeba. Let a_n be the number of amoebas present in the Petri dish n hours later. Find a recurrence relation and initial conditions defining the sequence (a_n).

Solution. The first term is given by $a_0 = 1$, since the dish begins with just one amoeba.

At n hours past the initial time, there are twice as many amoebas as there were at $n-1$ hours past the initial time. That is, there are $2a_{n-1}$ amoebas at n hours past the initial time.

So we can conclude that $a_n = 2a_{n-1}$ for all integers $n \geq 1$, with initial condition $a_0 = 1$.

Example 2 – Binary words

Example. Let a_n be the number of words of length n using the alphabet $\{\mathbf{A}, \mathbf{B}\}$ that do not include **AA** as a subword. Find a recurrence relation and initial conditions defining the sequence (a_n).

Solution. The first few terms are given by $a_0 = 1$ and $a_1 = 2$, corresponding to the sets $\{\varepsilon\}$ (the empty word) and $\{\mathbf{A}, \mathbf{B}\}$ respectively.

An allowed word of length $n \geq 2$ can end in two different ways: **B** or else **BA**. (It cannot end in **AA** since that subword is not allowed).

<u>Case 1.</u> Every allowed word ending in **B** can be made by appending **B** to the end of any allowed word of length $n-1$. So there are a_{n-1} such words in this case.

<u>Case 2.</u> Every allowed word ending in **BA** can be made by appending **BA** to the end of any allowed word of length $n-2$. So there are a_{n-2} such words in this case.

Since the above cases are mutually exclusive and no other possible cases exist, we can conclude that $a_n = a_{n-1} + a_{n-2}$ for all integers $n \geq 2$, with initial conditions $a_0 = 1$ and $a_1 = 2$.

Example 2 – Notes

- The first step for these sorts of problems is to try listing the first few examples. In the solution we only gave the ones required for the initial conditions, but in your working you'll likely want to list more, e.g. $1, 2, 3, 5, 8, \ldots$ in this case. This can help with spotting patterns and guessing at the recurrence relation.
- Often the $n = 0$ case for these sorts of problems can be tricky to explain. Sometimes it can be easier to deduce the recurrence relation and the first few terms for positive n, and then work backwards to find a_0. Other times it might make sense to not even define a_0.
- It's important when splitting the enumeration into cases that you check that all possible cases have been accounted for (no undercounting) and that all of the cases are mutually exclusive (no overcounting).
- Because the recurrence relation defined a_n in relation to the two preceding terms a_{n-1} and a_{n-2}, we needed to provide two initial conditions. (This might remind you of proof by strong induction!)

Recurrence relations terminology

Definition. A linear recurrence relation for a sequence (a_n) recursively defines a_n in relation to a linear combination of previous terms. For example:

- $a_n = 2a_{n-1}$ is linear.
- $b_n = b_{n-1} + b_{n-2}$ is linear.
- $c_{n+1} = c_0c_n + c_1c_{n-1} + \cdots + c_nc_0$ is not linear.

Definition. A homogeneous linear recurrence relation is one in which all terms are constant multiples of terms from the sequence (a_n). If all terms in a_i are moved to the left-hand side, the right-hand side is 0. For example:

- $a_n = 2a_{n-1}$ is homogeneous (since $a_n - 2a_{n-1} = 0$).
- $b_n = b_{n-1} + b_{n-2}$ is homogeneous (since $b_n - b_{n-1} - b_{n-2} = 0$).
- $t_n = t_{n-1} + n$ is not homogeneous (since $t_n - t_{n-1} = n \neq 0$).

Definition. A linear recurrence relation of order k (or kth-order linear recurrence) is one in which a_n is defined recursively as far back as k earlier terms in the sequence (a_n). That is, the smallest subscript to appear in the relation is $n - k$. For example:

- $a_n = 2a_{n-1}$ is a first-order recurrence.
- $b_n = b_{n-1} + b_{n-2}$ is a second-order recurrence.
- $t_n = t_{n-1} + n$ is a first-order recurrence.

Recurrence relations terminology – solutions

Definition. The closed form solution for a sequence (a_n) is an equation that gives a_n as a function of n only. When we are asked to solve a recurrence relation, we are being asked to provide a closed form solution.

For example:

- The closed form solution for the sequence defined by $a_0 = 1$ and $a_n = 2a_{n-1}$ for all $n \in \mathbb{Z}^+$ is the equation $a_n = 2^n$.
- The general solution for the sequence defined by just $a_n = 2a_{n-1}$ for all $n \in \mathbb{Z}^+$ is the equation $a_n = A2^n$ for some constant A.

For this course we will only investigate how to solve linear recurrence relations, since they can be solved in general using a systematic approach. Non-linear recurrence relations are much more difficult to solve in general.

For this lecture we are interested in finding a solution for any kth-order homogeneous linear recurrence relation, which can be expressed as

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0$$

for given constants c_i.

Exercise. Find the closed form solution for the sequence defined by $a_0 = 1$ and $a_n = 2a_{n-1}$ for all $n \in \mathbb{Z}^+$.

Solution. Notice that the sequence begins $1, 2, 4, 8, 16, \ldots$, implying $a_n = 2^n$ is the closed form solution here.

We can confirm this via induction. Let $P(n)$ be the statement " $a_n = 2^n$ ".

Base case. When $n = 0$, we have $a_0 = 1 = 2^0$, so $P(0)$ is true.

Induction. Suppose $P(k)$ is true for some $k \in \mathbb{N}$. That is, suppose $a_k = 2^k$. Then we wish to show $P(k+1)$ is true, that is, we want to show a_{k+1} equals 2^{k+1}.

Notice $a_{k+1} = 2a_k = 2 \times 2^k$ by the inductive hypothesis, so $a_{k+1} = 2^{k+1}$ and thus $P(k+1)$ is true whenever $P(k)$ is true.

So $a_n = 2^n$ for all $n \in \mathbb{N}$ by induction.

We can use this example to establish a general formula, so that we don't have to guess a solution and give a proof by induction.

Solving first-order homogeneous linear recurrences

The solution for any first-order homogeneous linear recurrence relation $a_n = ra_{n-1}$ is given by $a_n = Ar^n$, where $A = a_0$.

Using our general form provided earlier, we can also write that the solution for $a_n + c_1 a_{n-1} = 0$ is given by $a_n = A(-c_1)^n$, where $A = a_0$.

Example. Solve the recurrence relation $b_n = \frac{2}{5} b_{n-1}$ with initial condition $b_0 = 3$.

Solution. The general formula gives $b_n = A\left(\frac{2}{5}\right)^n$, and substituting $n = 0$ shows that $A = 3$ here, so the closed form solution is $b_n = 3\left(\frac{2}{5}\right)^n$.

We can now try to adapt this technique for higher-order linear recurrences...

Motivating Exercise – Second-order homogeneous

Exercise. Find the general solution for the sequence defined by $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$.

Solution. Based on the previous exercise for a first-order linear recurrence, we might expect an answer for a_n of the form Ar^n.

Substituting $a_n = Ar^n$ yields $Ar^n - 5Ar^{n-1} + 6Ar^{n-2} = 0$. Supposing $A, r \neq 0$ (else our solution is trivial), we can divide through by Ar^{n-2} and factorise to find $(r-2)(r-3) = 0$. So some solutions to the recurrence relation are $a_n = A2^n$ and $a_n = B3^n$ for any constants A and B. However, depending on the initial conditions for a_0 and a_1, it is possible that neither of these solutions will work for any values A or B.

Notice that any linear combination of 2^n and 3^n will also provide a solution for the recurrence, since if $a_n = A2^n + B3^n$ for constants A and B, then

$$\begin{aligned}
5a_{n-1} - 6a_{n-2} &= 5(A2^{n-1} + B3^{n-1}) - 6(A2^{n-2} + B3^{n-2}) \\
&= (10A - 6A)2^{n-2} + (15B - 6B)3^{n-2} \\
&= A2^n + B3^n \\
&= a_n.
\end{aligned}$$

In fact, $a_n = A2^n + B3^n$ for constants A and B is the complete general solution for the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$.

Motivating Exercise – Second-order homogeneous

Exercise. Find the closed form solution for the sequence defined by $a_0 = 3, a_1 = 7$ and $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$.

Solution. In the previous exercise, we saw that $a_n = A2^n + B3^n$ for constants A and B is the complete general solution for the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$.

From here, we can find the values of A and B by substituting in the initial conditions.

When $n = 0$ we have $a_0 = A + B = 3$, and when $n = 1$ we have $a_1 = 2A + 3B = 7$. Solving simultaneously yields $A = 2$ and $B = 1$, so altogether we have that the closed form solution is $a_n = 2 \times 2^n + 1 \times 3^n$, or just $a_n = 2^{n+1} + 3^n$.

This method of solution will work for almost all second-order linear recurrences (even if the roots of the quadratic in r are distinct complex conjugates). There is just one case where the method must be adjusted, which we'll address in the next exercise.

Exercise. Find the closed form solution for the sequence defined by $a_0 = 1, a_1 = 9$ and $a_n = 6a_{n-1} - 9a_{n-2}$ for all integers $n \geq 2$.

Solution. As before, we can first rearrange the recurrence to $a_n - 6a_{n-1} + 9a_{n-2} = 0$ and substitute $a_n = Ar^n$, giving $Ar^n - 6Ar^{n-1} + 9Ar^{n-2} = 0$. Supposing $A, r \neq 0$ and dividing through by Ar^{n-2} gives $(r-3)^2 = 0$. Adapting the general formula in the previous exercise, we might write $a_n = A3^n + B3^n$, though note this is just a solution of the form $a_n = C3^n$ where $C = A + B$. However, this solution cannot satisfy the given initial conditions for any value of C.

This means our solution is not yet completely general. This problem arises because the two terms $A3^n$ and $B3^n$ are not independent of each other, but the solution should contain two arbitrary constants to be determined by the two initial conditions. To rectify this, it turns out that we can instead multiply one of the terms by n, giving $a_n = A3^n + Bn3^n$ for some constants A and B...

Exercise. Find the closed form solution for the sequence defined by $a_0 = 1, a_1 = 9$ and $a_n = 6a_{n-1} - 9a_{n-2}$ for all integers $n \geq 2$.

Solution (cont'd). We can check that the solution $a_n = A3^n + Bn3^n$ works in general:

$$\begin{aligned} 6a_{n-1} - 9a_{n-2} &= 6(A3^{n-1} + B(n-1)3^{n-1}) - 9(A3^{n-2} + B(n-2)3^{n-2}) \\ &= (18A + 18Bn - 18B - 9A - 9Bn + 18B)3^{n-2} \\ &= A3^n + Bn3^n \\ &= a_n. \end{aligned}$$

In fact, $a_n = A3^n + Bn3^n$ for constants A and B is the complete general solution for the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$.

From here, we can find the values of A and B by substituting in the initial conditions.

When $n = 0$ we have $a_0 = A = 1$, and when $n = 1$ we have $a_1 = 3A + 3B = 9$, implying $B = 2$. So altogether we have that the closed form solution is $a_n = 1 \times 3^n + 2n \times 3^n$, or just $a_n = 3^n(2n + 1)$.

Solving second-order homogeneous linear recurrences

The solution for any second-order homogeneous linear recurrence relation $a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0$ is found by following these steps:

- Write the characteristic equation $r^2 + c_1 r + c_2 = 0$ (by substituting $a_n = r^n$ (or $a_n = Ar^n$) and dividing through by r^{n-2}).
- Find the solutions r_1 and r_2 to the characteristic equation.
  - If $r_1 \neq r_2$, the general solution for the recurrence relation is given by $a_n = Ar_1^n + Br_2^n$ for arbitrary constants A and B.
  - If $r_1 = r_2$, the general solution for the recurrence relation is given by $a_n = Ar_1^n + Bnr_1^n$ for arbitrary constants A and B.
- Substitute the initial conditions into the general solution to find the values of A and B.

Example 3 – Practice with general homogeneous solutions

Example. Find general solutions for the following recurrence relations (without initial conditions).

- $a_n = 7a_{n-1}$.
  Sol'n. $a_n = A7^n$ for arbitrary constant A.
- $b_n = 7b_{n-1} - 10b_{n-2}$.
  Sol'n. $b_n = A2^n + B5^n$ for arbitrary constants A and B.
- $c_n = 4c_{n-1} - 3c_{n-2}$.
  Sol'n. $c_n = A + B3^n$ for arbitrary constants A and B.
- $d_n = 4d_{n-1} + 5d_{n-2}$.
  Sol'n. $d_n = A(-1)^n + B5^n$ for arbitrary constants A and B.
- $e_n = 4e_{n-1} - 4e_{n-2}$.
  Sol'n. $e_n = A2^n + Bn2^n$ for arbitrary constants A and B.
- $f_n = 7f_{n-2}$.
  Sol'n. $f_n = A(\sqrt{7})^n + B(-\sqrt{7})^n$ for arbitrary constants A and B.
- $g_n = -g_{n-2}$.
  Sol'n. $g_n = Ai^n + B(-i)^n$ for arbitrary constants A and B.

Example 4 – Second-order homogeneous recurrence

Example. Find the closed form solution for the sequence defined by $a_0 = 1, a_1 = 3$ and $a_n = 4a_{n-2}$ for all integers $n \geq 2$.

Solution. The characteristic equation for the recurrence relation is $r^2 - 4 = 0$, which has roots 2 and -2. The general solution will thus be of the form $a_n = A2^n + B(-2)^n$. Substituting in the initial conditions gives $a_0 = A + B = 1$ and $a_1 = 2A - 2B = 3$, which together imply $A = \frac{5}{4}$ and $B = -\frac{1}{4}$. So the closed form solution is $a_n = \frac{5}{4}2^n - \frac{1}{4}(-2)^n$, or $a_n = 5 \times 2^{n-2} - (-2)^{n-2}$.

Example. Find the closed form solution for the sequence defined by $a_0 = 1, a_1 = 2$ and $a_n + 2a_{n-1} + a_{n-2} = 0$ for all integers $n \geq 2$.

Solution. The characteristic equation for the recurrence relation is $r^2 + 2r + 1 = 0$, which has roots -1 and -1. Since there is a repeated root, the general solution will be of the form $a_n = A(-1)^n + Bn(-1)^n$. Substituting in the initial conditions gives $a_0 = A = 1$ and $a_1 = -A - B = 2$, so $A = 1$ and $B = -3$. Hence the closed form solution is $a_n = (-1)^n - 3n(-1)^n = (-1)^n(1 - 3n)$.

Example 5 – Fibonacci

Example. Find the closed form solution for the Fibonacci sequence (F_n), which is defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$.

Solution. The characteristic equation for the recurrence relation is $r^2 - r - 1 = 0$, which has roots $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. The general solution will thus be of the form

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Substituting in the initial conditions gives $F_0 = A + B = 0$ and $F_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$, which together imply $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$. So the closed form solution is

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Solving general homogeneous linear recurrences

To solve a kth-order homogeneous linear recurrence relation

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0$$

for given constants c_i:

- Find the solutions $r_1, r_2, \ldots, r_k$ of the characteristic equation $r^k + c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k = 0$.
- Write the general solution for a_n:
  - If there are no repeated roots, write $a_n = A_1 r_1^n + A_2 r_2^n + \cdots + A_k r_k^n$ for arbitrary constants A_i.
  - If any root is repeated, multiply each repeated instance by n (possibly multiple times) to preserve independence of terms. For example, if $r_1 = r_2 = r_3$, write $a_n = A_1 r_1^n + A_2 n r_1^n + A_3 n^2 r_1^n + \cdots + A_k r_k^n$ for arbitrary constants A_i.
- Substitute the initial conditions into the general solution to find the values of each A_i.

Example 6 – More practice with homogeneous solutions

Example. Find general solutions for the following recurrence relations (without initial conditions).

- $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$. (Hint: consider $(r-1)(r-2)(r-3)$.)
  Sol'n. $a_n = A + B2^n + C3^n$ for arbitrary constants A, B, C.
- $b_n + b_{n-1} = 2b_{n-2} + 2b_{n-3}$. (Hint: consider $(r+1)(r^2-2)$.)
  Sol'n. $b_n = A(-1)^n + B\left(\sqrt{2}\right)^n + C\left(-\sqrt{2}\right)^n$ for constants A, B, C.
- $c_n = 8c_{n-1} - 21c_{n-2} + 18c_{n-3}$. (Hint: consider $(r-2)(r-3)^2$.)
  Sol'n. $c_n = A2^n + B3^n + Cn3^n$ for constants A, B, C.
- $d_n = 6d_{n-1} - 12d_{n-2} + 8d_{n-3}$. (Hint: consider $(r-2)^3$.)
  Sol'n. $d_n = A2^n + Bn2^n + Cn^2 2^n$ for constants A, B, C.
- $e_n = 2e_{n-1} + 3e_{n-2} - 4e_{n-3} - 4e_{n-4}$. (Hint: consider $(r+1)^2(r-2)^2$.)
  Sol'n. $a_n = A2^n + Bn2^n + C(-1)^n + Dn(-1)^n$ for constants A, B, C, D.
- $f_n = 5f_{n-2} - 4f_{n-4}$. (Hint: consider $(r^2-1)(r^2-4)$.)
  Sol'n. $f_n = A + B(-1)^n + C2^n + D(-2)^n$ for constants A, B, C, D.
- $g_n = 4g_{n-1} - 6g_{n-2} + 4g_{n-3} - g_{n-4}$. (Hint: consider $(r-1)^4$.)
  Sol'n. $a_n = A + Bn + Cn^2 + Dn^3$ for constants A, B, C, D.

Example 7 – Third-order homogeneous recurrence

Example. Find the closed form solution for the sequence (a_n) defined by $a_0 = 2, a_1 = 8, a_2 = 31$ and $a_n = 8a_{n-1} - 21a_{n-2} + 18a_{n-3}$ for all integers $n \geq 3$. (Hint: consider $(r-2)(r-3)^2$.)

Solution. The characteristic equation for the recurrence relation is $r^3 - 8r^2 + 21r - 18 = 0$, which has roots 2, 3, and 3. Since 3 is a repeated root, the general solution will be of the form $a_n = A2^n + B3^n + Cn3^n$ for some constants A, B, C. Substituting in the initial conditions gives $a_0 = A + B = 2$, $a_1 = 2A + 3B + 3C = 8$, and $a_2 = 4A + 9B + 18C = 31$, which together imply $A = B = C = 1$. So the closed form solution is $a_n = 2^n + 3^n + n3^n$.

In general, finding roots of polynomials with degree higher than 2 and solving simultaneous equations in more than 2 variables is the sort of content covered by Mathematics 1A Algebra. You are welcome to use those techniques here, though you can expect that most recurrence relation questions will only go up to second-order for this course.

UNSW SYDNEY

MATH1081 – Discrete Mathematics

Topic 4 – Combinatorics
Lecture 4.07 – Recurrence relations (inhomogeneous)

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Recurrence relations terminology

Recall that a homogeneous linear recurrence relation is one in which all terms are constant multiples of terms from the sequence (a_n). If all terms in a_i are moved to the left-hand side, the right-hand side is 0.

Definition. An inhomogeneous linear recurrence relation is one in which not all terms are constant multiples of terms from the sequence (a_n). If all terms in a_i are moved to the left-hand side, the right-hand side is non-zero. For example:

- $a_n = a_{n-1} + a_{n-2}$ is homogeneous (since $a_n - a_{n-1} - a_{n-2} = 0$).
- $b_n = b_{n-1} + n$ is inhomogeneous (since $b_n - b_{n-1} = n$).
- $c_n = 3c_{n-1} + 2^n - 1$ is inhomogeneous (since $c_n - 3c_{n-1} = 2^n - 1$).

For this lecture, we are interested in finding a general solution for any kth-order inhomogeneous linear recurrence relation, which can be expressed as

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$$

for given constants c_i and nonzero function f of n.

Motivating Exercise – First-order inhomogeneous

Exercise. Find the closed form solution for the sequence defined by $a_0 = 0$ and $a_n - 2a_{n-1} = 1$ for all $n \in \mathbb{Z}^+$.

Solution. Notice that the sequence begins $0, 1, 3, 7, 15, \dots$, implying $a_n = 2^n - 1$ might be the closed form solution here.

We know that if the right-hand side were 0, the simplified recurrence relation $h_n - 2h_{n-1} = 0$ is homogeneous and thus has solution $h_n = A2^n$ for some constant A.

The difference between the relations $h_n - 2h_{n-1} = 0$ and $a_n - 2a_{n-1} = 1$ is the addition of a constant term. Notice that if $a_n = C$ were a solution for some constant C, we would have $C = -1$. So this is another solution to the recurrence relation, which is independent of the homogeneous solution.

In fact, we can add -1 to the homogeneous solution h_n to obtain a general solution $a_n = h_n - 1$. This is because for all $n \geq 1$, we would have

$$a_n - 2a_{n-1} = (h_n - 1) - 2(h_{n-1} - 1) = h_n - 2h_{n-1} + 1 = 0 + 1 = 1.$$

So here we find $a_n = A2^n - 1$ as a general solution, and substituting in the initial condition yields $A = 1$ in this case. That is, $a_n = 2^n - 1$ is the closed form solution.

- From this example, we saw that to find a general solution for a_n, we needed to find both the homogeneous solution h_n to a simplified recurrence relation (found by setting all inhomogeneous terms in the original recurrence to be 0), as well as a particular solution to the original recurrence relation, which is often labelled p_n (so here we had $p_n = -1$).
- So long as we can find a particular solution p_n that works, $a_n = h_n + p_n$ will always represent the general solution for a_n. To see why this is true, suppose p_n is a particular solution to the recurrence relation

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n),$$

that is,

$$p_n + c_1 p_{n-1} + c_2 p_{n-2} + \cdots + c_k p_{n-k} = f(n),$$

while h_n is the general solution to the homogeneous recurrence relation

$$h_n + c_1 h_{n-1} + c_2 h_{n-2} + \cdots + c_k h_{n-k} = 0.$$

Making the substitution $a_n = h_n + p_n$ into the original recurrence yields

$$(h_n + p_n) + c_1(h_{n-1} + p_{n-1}) + \cdots + c_k(h_{n-k} + p_{n-k}) = 0 + f(n)$$

as required, and since h_n is a general solution, so is a_n.

Finding a particular solution

The general rule for finding an appropriate particular solution is to set p_n to be a function in n resembling the inhomogeneous component $f(n)$ as generally as possible, including unknown constants as coefficients throughout, and substituting this guess for p_n into the recurrence relation for a_n to determine the values of the unknown constants. For example:

- If $f(n)$ is a constant, set $p_n = C$ for unknown constant C.
- If $f(n)$ is a polynomial in n with degree d, set $p_n = C_d n^d + C_{d-1} n^{d-1} + \cdots + C_1 n + C_0$ for unknown constants C_i.
- If $f(n)$ is an exponential form in n like m^n for some constant m, set $p_n = Cm^n$ for unknown constant C.
- If $f(n)$ is some combination of the above (like a sum or product of polynomials and exponentials), set p_n to be the same combination of the components' corresponding guesses.

Remember that all the terms in the general solution for a_n must be independent of each other. So if any term in your guess for p_n already appears in the general homogeneous solution h_n, multiply the offending term(s) in p_n by n until they are independent (as we saw before).

Example 1 – Practice with particular solutions

Example. Give the expected form of the particular solution p_n (without evaluating the constants) for each of the following inhomogeneous recurrence relations.

- $a_n = 7a_{n-1} + 5n + 3$. (Hint: $h_n = A7^n$.)
  Sol'n. $p_n = C_1 n + C_0$ for constants C_0, C_1.

- $b_n = 7b_{n-1} - n^2$. (Hint: $h_n = A7^n$.)
  Sol'n. $p_n = C_2 n^2 + C_1 n + C_0$ for constants C_0, C_1, C_2.

- $c_n = 7c_{n-1} - 3 \times 2^n$. (Hint: $h_n = A7^n$.)
  Sol'n. $p_n = C2^n$ for some constant C.

- $d_n = 7d_{n-1} - 7^n$. (Hint: $h_n = A7^n$.)
  Sol'n. $p_n \neq C7^n$ so $p_n = Cn7^n$ for some constant C.

- $e_n = 4e_{n-1} - 4e_{n-2} + 2^n$. (Hint: $h_n = A2^n + Bn2^n$.)
  Sol'n. $p_n \neq C2^n$ and $p_n \neq Cn2^n$ so $p_n = Cn^2 2^n$ for some constant C.

- $f_n = 4f_{n-1} - 3f_{n-2} + 2n$. (Hint: $h_n = A + B3^n$.)
  Sol'n. $p_n \neq C_1 n + C_0$ so $p_n = n(C_1 n + C_0)$ for constants C_0, C_1.

- $g_n = 7g_{n-1} - n7^n$. (Hint: $h_n = A7^n$.)
  Sol'n. $p_n \neq (C_1 n + C_0)7^n$ so $p_n = n(C_1 n + C_0)7^n$ for constants C_0, C_1.

Solving general inhomogeneous linear recurrences

To solve a k-th order inhomogeneous recurrence relation

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$$

for given constants c_i and function f of n:

- Find the homogeneous solution h_n by solving $h_n + c_1 h_{n-1} + c_2 h_{n-2} + \cdots + c_k h_{n-k} = 0$ as before, but omitting the final step (do not yet find the values of each constant A_i).
- Guess the particular solution p_n by setting p_n to be a generalised form of $f(n)$ (for example a general polynomial of same degree as $f(n)$, or a general exponential with same base as $f(n)$).
  - If any term in the guess for p_n already appears as a term in the homogeneous solution h_n, multiply it by n (possibly multiple times) to preserve independence of terms.
- Substitute p_n into the recurrence to find the specific form of p_n (by resolving its arbitrary constants).
- Write the general solution $a_n = h_n + p_n$.
- Substitute the initial conditions into the general solution to find the values of each constant A_i found in the first step.

Example 2 – First-order inhomogeneous recurrence

Example. Find the closed form solution for the sequence defined by $a_0 = 0$ and $a_n - 2a_{n-1} = n^2$ for all integers $n \geq 1$.

Solution.

Find h_n: First we find the homogeneous solution h_n to the modified recurrence relation $h_n - 2h_{n-1} = 0$. This has characteristic equation $r - 2 = 0$, which has the root 2. The general homogeneous solution will thus be of the form $h_n = A2^n$ for some constant A.

Find p_n: Next we find a particular solution p_n to the original recurrence relation. Since the inhomogeneous component is a quadratic in n, we set $p_n = C_2 n^2 + C_1 n + C_0$ for some unknown constants C_0, C_1, and C_2. Substituting p_n as a_n into the original recurrence relation gives

$$\begin{aligned} n^2 &= (C_2 n^2 + C_1 n + C_0) - 2(C_2(n-1)^2 + C_1(n-1) + C_0) \\ &= -C_2 n^2 + (4C_2 - C_1)n + (-2C_2 + 2C_1 - C_0), \end{aligned}$$

yielding that $C_2 = -1, C_1 = -4, C_0 = -6$, and hence $p_n = -(n^2 + 4n + 6)$.

Find a_n: Finally we may write $a_n = h_n + p_n = A2^n - (n^2 + 4n + 6)$. Substituting in the initial conditions gives $a_0 = A - 6 = 0$, so $A = 6$. Thus the closed form solution is $a_n = 6 \times 2^n - (n^2 + 4n + 6)$.

Example 3 – Second-order inhomogeneous recurrence

Example. Find the closed form solution for the sequence defined by $a_0 = 1, a_1 = 9$ and $a_n = 5a_{n-1} - 6a_{n-2} + 3^n$ for all integers $n \geq 2$.

Solution. Rearranging the recurrence gives $a_n - 5a_{n-1} + 6a_{n-2} = 3^n$.

Find h_n: First we find the homogeneous solution h_n to the modified recurrence relation $h_n - 5h_{n-1} + 6h_{n-2} = 0$. This has characteristic equation $r^2 - 5r + 6 = 0$, which has roots 2 and 3. The general homogeneous solution will thus be of the form $h_n = A2^n + B3^n$ for some constants A and B.

Find p_n: Next we find a particular solution p_n to the original recurrence relation. Since the inhomogeneous component 3^n already appears as a term in the homogeneous solution, we set $p_n = Cn3^n$ for some unknown constant C. Substituting p_n as a_n into the original recurrence relation gives

$$\begin{aligned} 3^n &= Cn3^n - 5C(n-1)3^{n-1} + 6C(n-2)3^{n-2} \\ &= 3^{n-2}(9Cn - 15C(n-1) + 6C(n-2)) = C3^{n-1}, \end{aligned}$$

yielding that $C = 3$ and hence $p_n = n3^{n+1}$.

Find a_n: Finally we may write $a_n = h_n + p_n = A2^n + B3^n + n3^{n+1}$. Substituting in the initial conditions gives $a_0 = A + B = 1$ and $a_1 = 2A + 3B + 9 = 9$, which together imply $A = 3$ and $B = -2$. So the closed form solution is $a_n = 3 \times 2^n - 2 \times 3^n + n3^{n+1}$.