MATH2621 — Higher Complex Analysis. VIII
Properties of differentiable functions

This lecture?

In this lecture, we look at more consequences of the Cauchy–Riemann equations.

An example

The Cauchy Riemann equations enable us to define new complex functions and show that they are complex differentiable. For instance, recall the definition of the hyperbolic functions:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \forall x \in \mathbb{R}.$$

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Recall also that $\cosh' = \sinh$ and $\sinh' = \cosh$, and that

$$\begin{aligned} \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y) \\ \sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y). \end{aligned}$$

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We define two new functions of a complex variable as follows:

$$\begin{aligned} \mathrm{ch}(x+iy) &= \cosh(x)\cos(y) + i\sinh(x)\sin(y) \\ \mathrm{sh}(x+iy) &= \sinh(x)\cos(y) + i\cosh(x)\sin(y). \end{aligned}$$

Exercise 1

Is ch differentiable? What is its derivative? What about sh?

Answer.

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Answer. Write $\operatorname{ch}(x+iy) = u(x,y) + iv(x,y)$. Then

$$
\begin{aligned}
u(x,y) &= \cosh(x)\cos(y) & v(x,y) &= \sinh(x)\sin(y)\\
u_x(x,y) &= \sinh(x)\cos(y) & v_x(x,y) &= \cosh(x)\sin(y)\\
u_y(x,y) &= -\cosh(x)\sin(y) & v_y(x,y) &= \sinh(x)\cos(y).
\end{aligned}
$$

The partial derivatives are continuous and the Cauchy–Riemann equations hold in the open set $\mathbb{C}$, so ch is differentiable in $\mathbb{C}$. Moreover,

$$
\begin{aligned}
\operatorname{ch}'(x+iy) &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)\\
&= \sinh(x)\cos(y) + i\cosh(x)\sin(y),
\end{aligned}
$$

that is, $\operatorname{ch}' = \operatorname{sh}$. $\triangle$

Exercise 2

Is sh differentiable? What is its derivative?

Answer.

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Answer. You do this! $\triangle$

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Answer. You do this! $\triangle$

Recall that we defined $\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$ for all $z \in \mathbb{C} \setminus \{0\}$.

Exercise 3

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Answer. As usual, we may write

$$\operatorname{Log}(x+iy) = u(x,y) + iv(x,y),$$

where $u(x,y) = \ln(x^2+y^2)^{1/2}$ and $v(x,y) = \operatorname{Arg}(x+iy)$.

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Note that Log cannot be differentiable at 0 since it is not defined there, nor on the negative real axis $(-\infty, 0)$, as it is not continuous there.

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In $\mathbb{C} \setminus (-\infty, 0]$, which is open, we may apply the Cauchy–Riemann equations.

Answer to Exercise 3

Observe that

$$\begin{aligned}\frac{\partial u(x,y)}{\partial x} &= \frac{1}{(x^2+y^2)^{1/2}}\,\frac{1}{2(x^2+y^2)^{1/2}}\,2x = \frac{x}{x^2+y^2}\,,\\ \frac{\partial u(x,y)}{\partial y} &= \frac{1}{(x^2+y^2)^{1/2}}\,\frac{1}{2(x^2+y^2)^{1/2}}\,2y = \frac{y}{x^2+y^2}\,.\end{aligned}$$

Answer to Exercise 3 (continued)

In the right half-plane where $x > 0$, we may write $\operatorname{Arg}(x + iy) = \tan^{-1}(y/x)$, and so

$$\begin{aligned}\frac{\partial v(x,y)}{\partial x} &= \frac{1}{1+(y/x)^2}\,\frac{-y}{x^2} = \frac{-y}{x^2+y^2},\\ \frac{\partial v(x,y)}{\partial y} &= \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{x}{x^2+y^2}.\end{aligned}$$

In the upper half-plane where $y > 0$, we may write $\operatorname{Arg}(x + iy) = -\tan^{-1}(x/y) + \pi/2$; in the lower half plane where $y < 0$, we may write $\operatorname{Arg}(x + iy) = -\tan(x/y) - \pi/2$. Two very similar calculations show that the same formulae hold for the partial derivatives in these cases too.

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Hence the partial derivatives are continuous and the Cauchy–Riemann equations hold in the open set $\mathbb{C} \setminus (-\infty, 0]$, and so Log is differentiable in this set.

Answer to Exercise 3 (continued)

The derivative of Log is given by

$$\text{Log}'(x+iy) = \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$$

in $\mathbb{C} \setminus (-\infty, 0]$.

$\triangle$

A topological question

If f is defined in an open set Ω, and f is constant along all horizontal and vertical line segments contained in Ω, must f be constant? It is easy to see that if Ω is not polygonally path-connected, then f need not be constant. But what if Ω is polygonally path-connected? Recall that a polygonally path-connected open subset of $\mathbb{C}$ is called a domain.

An answer

Proposition

Suppose that f is a function defined on a domain Ω in $\mathbb{C}$, and f is constant along all horizontal and vertical line segments contained in Ω. Then f is constant in Ω.

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Proof. Two distinct points of Ω may always be connected by a polygonal path, involving finitely many line segments, so it will suffice to know that f is constant along line segments.

Take a line segment ℓ, with endpoints z_0 and z_1, and write $z_t = (1-t)z_0 + tz_1$, where $t \in [0,1]$. Then z_t moves along the line segment ℓ from z_0 to z_1 as t varies between 0 and 1.

Proof

Suppose that $z \in \Omega$ and z lies on ℓ. Since Ω is open, there is a ball $B(z, \varepsilon)$ contained in Ω. We claim that f is constant on $B(z, \varepsilon) \cap \ell$.

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Indeed, by elementary geometry, if $z' \in B(z, \varepsilon) \cap \ell$, then the line segments joining $x + iy$ and $x' + iy$ and joining $x' + iy$ and $x' + iy'$ both lie inside $B(z, \varepsilon)$ and hence in Ω. By hypothesis, f is constant along both these segments, and so $f(z) = f(z')$, and our claim is proved.

Proof (continued)

To show that f is constant along the whole line segment ℓ, we argue by contradiction.

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If f is not constant on ℓ, then we define

$$t^* = \inf\{t \in [0,1] : f(z_t) \neq f(z_0)\};$$

Since f is not constant, the set is not empty, and $0 \leq t^* \leq 1$.

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Since f is not constant, the set is not empty, and $0 \leq t^* \leq 1$. But f is constant on $B(z_{t^*}, \varepsilon_{t^*}) \cap \ell$ for some positive ε_{t^*}, so f is constant near t^*, contradicting the definition of t^*, and showing that f is constant on ℓ. $\square$

Consequences of the Cauchy–Riemann equations

Theorem

Suppose that f is differentiable in a domain Ω in $\mathbb{C}$. Then

(a) *if $f' = 0$ in Ω, then f is constant on Ω;*

(b) *if $|f|$ is constant, then f is constant on Ω;*

(c) *if $\mathrm{Re}(f)$ or $\mathrm{Im}(f)$ is constant, then f is constant on Ω.*

Proof. As usual, write $f(x + iy) = u(x, y) + iv(x, y)$.
First, suppose that $f' = 0$. Then $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ are all 0. By the proposition, f is constant.

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Next, suppose that $|f|$ is a constant, C say, in Ω. If $C = 0$, then $f = 0$, a constant, so without loss of generality we may suppose that $C \neq 0$.

Proof of theorem

Then

$$u^2 + v^2 = C^2 > 0. \tag{1}$$

Differentiating (1) with respect to x and with respect to y gives

$$2u\,\frac{\partial u}{\partial x} + 2v\,\frac{\partial v}{\partial x} = 0 \tag{2}$$

$$2u\,\frac{\partial u}{\partial y} + 2v\,\frac{\partial v}{\partial y} = 0. \tag{3}$$

Using the Cauchy–Riemann equations with (3), we get

$$2v\,\frac{\partial u}{\partial x} - 2u\,\frac{\partial v}{\partial x} = 0. \tag{4}$$

Eliminating $\frac{\partial v}{\partial x}$ from (2) and (4) shows that $2(u^2 + v^2)\,\frac{\partial u}{\partial x} = 0$.
From (1), $\frac{\partial u}{\partial x} = 0$. Similarly, $\frac{\partial v}{\partial x} = 0$, so $f' = 0$, and f is constant.

Proof of theorem (continued)

Suppose now that $\mathrm{Re}(f)$ is constant. Then $\partial u/\partial x = \partial u/\partial y = 0$, so from the Cauchy–Riemann equations, $\partial v/\partial x = \partial v/\partial y = 0$, whence $f' = 0$ and f is constant.

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In fact, it is possible to show that if f is complex differentiable in a polygonally path-connected open set Ω, then either f is constant or $\mathrm{Range}(f)$ is open, which means that f cannot satisfy any equations that restrict its range to lie in a one-dimensional set, such as a curve.

The Cauchy–Riemann equations in polar coordinates

Theorem

Suppose that the complex function f is differentiable at the point $z_0 \in \mathbb{C} \setminus \{0\}$, and that $z_0 = r_0 e^{i\theta_0}$. Then

$$\frac{\partial u}{\partial \theta}(r_0, \theta_0) = -r_0 \frac{\partial v}{\partial r}(r_0, \theta_0) \quad \textit{and} \quad \frac{\partial v}{\partial \theta}(r_0, \theta_0) = r_0 \frac{\partial u}{\partial r}(r_0, \theta_0).$$

Further,

$$\begin{aligned} f'(z_0) &= e^{-i\theta_0} \left(\frac{\partial u}{\partial r}(r_0, \theta_0) + i \frac{\partial v}{\partial r}(r_0, \theta_0) \right) \\ &= \frac{-ie^{-i\theta_0}}{r} \left(\frac{\partial u}{\partial \theta}(r_0, \theta_0) + i \frac{\partial v}{\partial \theta}(r_0, \theta_0) \right). \end{aligned}$$

Conversely, if f is defined on a ball $B = B(z_0, \epsilon)$ with $z_0 \in \mathbb{C} \setminus \{0\}$ such that the the four partial derivatives of above exist on B, are continuous at z_0 and satisfy the two equation of above at z_0, then f is differentiable at z_0.

A corollary

Corollary

The function Log *is differentiable in* $\mathbb{C} \setminus (-\infty, 0]$.

Proof. Write $\text{Log}(re^{i\theta})$ in the form $u(r,\theta) + iv(r,\theta)$. Then $u(r,\theta) = \ln(r)$ and $v(r,\theta) = \theta$. Hence

$$\begin{aligned} \frac{\partial u}{\partial r}(r_0,\theta_0) &= \frac{1}{r_0} & \frac{\partial u}{\partial \theta}(r_0,\theta_0) &= 0 \\ \frac{\partial v}{\partial r}(r_0,\theta_0) &= 0 & \frac{\partial v}{\partial \theta}(r_0,\theta_0) &= 1. \end{aligned}$$

Clearly, the polar form of the Cauchy–Riemann equations is satisfied. Further, the partial derivatives are continuous in the open set $\mathbb{C} \setminus (-\infty, 0]$, so Log is differentiable in this set. $\square$

Inverse functions

Suppose that Ω and Υ are open subsets of $\mathbb{C}$, and that f is one-to-one from Ω onto Υ. Then f has an inverse function, usually written f^{-1}, from Υ to Ω: we define $f^{-1}(w) = z$ if $f(z) = w$.

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Theorem

Suppose that Ω and Υ are open subsets of $\mathbb{C}$, that $f : \Omega \to \Upsilon$ is one-to-one, and that $f(z_0) = w_0$. If f is differentiable at z_0 and f^{-1} is differentiable at w_0, then $(f^{-1})'(w_0) = 1/f'(z_0)$.

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$$1 = \frac{df}{dw}^{-1}(w_0)\, f'(z_0),$$

and the desired result follows. $\square$

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Later, we will investigate whether the inverse function is differentiable.

Definition

The examples and theorems above show that functions that are complex differentiable in an open set have special properties; we are going to study them in much greater detail. This justifies giving them a name.

Definition

Suppose that Ω is an open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is a function. If f is differentiable in Ω, that is, if it is differentiable at every point of Ω, then we say that f is *holomorphic* or *(complex) analytic* or *regular* in Ω, and we write $f \in H(\Omega)$.

If $\Omega = \mathbb{C}$ and f is differentiable in Ω, then we say that f is *entire*.