1. ① $\int \frac{1}{x+\sqrt{x}}dx$

(1.1) $t=\sqrt{x}$

(1.2) $\frac{1}{x+\sqrt{x}} = \frac{1}{\sqrt{x}}\frac{1}{(\sqrt{x}+1)}$

$= \frac{2(\sqrt{x})'}{\sqrt{x}+1} = \frac{2(\sqrt{x}+1)'}{\sqrt{x}+1}$

② $\int \frac{dx}{x^2+2x+5}$

$= \int \frac{d(x+1)}{(x+1)^2+4}$

$(\sin^2 x)' = 2\sin x\cdot\cos x$

$= \sin(2x)$

③ $\int \frac{\sin 2x}{4+\sin^4 x}dx$

$= \int \frac{d(\sin^2 x)}{4+(\sin^2 x)^2}$

④ $\int \frac{1}{1+\cos x}dx = \int \frac{1}{2\cos^2(\frac{x}{2})}dx \sim \tan(\frac{x}{2})$

$+C$

⑤ $\int x e^{-2x}dx$

$\int x e^{-x^2}dx$

$\frac{1}{2}\int (x^2)' e^{-x^2}dx$

⑥ $$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{x^2\cdot x}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx^2$$

$$= \frac{1}{2}\int \frac{t}{\sqrt{1-t}}\,dt$$

$\sqrt{1-x^2} = t$

$1-x^2 = t^2$

$1-t^2 = x^2$

$u=\sqrt{1-t}$

$$\frac{1}{2}\int \frac{t-1+1}{\sqrt{1-t}}\,dt$$

⑦ $$\int \frac{1-\ln x}{(x-\ln x)^2}\,dx = \int \frac{\frac{1-\ln x}{x^2}}{\frac{(x-\ln x)^2}{x^2}}\,dx$$

$$= \int \frac{\frac{1-\ln x}{x^2}}{\left(1-\frac{\ln x}{x}\right)^2}\,dx = \int \frac{\left(\frac{\ln x}{x}\right)'}{\left(1-\frac{\ln x}{x}\right)^2}\,dx$$

$$\left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

⑧ $$\int \frac{1}{(2-x)\sqrt{1-x}}\,dx$$

$$t = \sqrt{1-x}$$

$$t^2 = 1-x$$

$$x = 1-t^2$$

$$dx = -2t\,dt$$

$$= \int \frac{1}{(1+t^2)\cdot t}(-2t)\,dt$$

$$= -2\int \frac{1}{1+t^2}\,dt$$

⑨ $$\int \frac{1}{1+\sin x}\,dx$$

$$= \int \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}\,dx$$

$$= \int \frac{1-\sin x}{\cos^2 x}\,dx$$

$$= \int \frac{1}{\cos^2 x}\,dx - \int \frac{\sin x}{\cos^2 x}\,dx$$

$$\int \frac{1}{\sin x}\,dx$$

$$= \int \frac{\sin x}{\sin^2 x}\,dx$$

$$= -\int \frac{d\cos x}{1-\cos^2 x}$$

$$\int \frac{1}{\cos x}\,dx$$

$$= \int \frac{\cos x}{\cos^2 x}\,dx$$

$$= \int \frac{d\sin x}{1-\sin^2 x}$$

⑩

$$\int \frac{1}{2+\cos^2 x}\,dx$$

$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{2+\cos^2 x}{\cos^2 x}}\,dx$$

$$= \int \frac{\sec^2 x}{2\sec^2 x+1}\,dx$$

$$= \int \frac{d(\tan x)}{3+\tan^2 x}$$

$$\sec^2 x - 1$$

$$= \frac{1}{\cos^2 x} - 1$$

$$= \frac{\sin^2 x}{\cos^2 x}$$

$$= \tan^2 x$$

$\therefore\ \sec^2 x = 1+\tan^2 x$

$$\int \frac{1}{4+\sin^2 x}\,dx$$

⑪ $\int \frac{1}{\sin x \cos^4 x} dx$ $\qquad \frac{1}{u^2-1}$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^4 x} dx$$

$$= \int \frac{\sin x}{\cos^4 x} dx + \int \frac{1}{\sin x \cos^2 x} dx$$

$$= -\int \frac{d\cos x}{\cos^4 x} + \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^2 x} dx$$

$$= -\left(\frac{\cos^{-3} x}{-3}\right) + \int \frac{\sin x}{\cos^2 x} dx$$

$$+ \int \frac{1}{\sin x} dx$$

$$\int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx \quad ---$$

(12) $$\int \frac{1+\cos x}{1+\sin x}\,dx$$

$$= \int \frac{1}{1+\sin x}\,dx + \int \frac{\cos x}{1+\sin x}\,dx$$

$$= \int \frac{1-\sin x}{1-\sin^2 x}\,dx + \int \frac{d\sin x}{1+\sin x}$$

(13) $$\int \frac{\cos\sqrt{x}-1}{\sqrt{x}\,\sin^2\sqrt{x}}\,dx$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$$= 2\int (\sqrt{x})' \frac{\cos\sqrt{x}-1}{\sin^2\sqrt{x}}\,dx$$

$$= 2\int \frac{\cos t-1}{\sin^2 t}\,dt$$

⑭ $$\int \frac{(x+1)\arcsin x}{\sqrt{1-x^2}}dx$$

$$= \int \frac{x\arcsin x}{\sqrt{1-x^2}}dx - \int \frac{\arcsin x}{\sqrt{1-x^2}}dx$$

$$= -\frac{1}{2}\int \frac{\arcsin x}{\sqrt{1-x^2}}\, d(1-x^2)$$

$$- \int \arcsin x\, (\arcsin x)' dx$$

$$\int \arcsin x\, d(\sqrt{1-x^2})$$

$$= \arcsin x \cdot \sqrt{1-x^2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2}\, dx$$

$$= \arcsin x\sqrt{1-x^2} - \int 1\, dx$$

$$(\sqrt{1-x^2})' = \frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$(\sqrt{x^2-1})' = \frac{1}{2\sqrt{x^2-1}} \cdot 2x = \frac{x}{\sqrt{x^2-1}}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

Ex. ODE

$$(x+1)y'(x) - ny(x) = e^x (x+1)^{n+1}, \quad n>0.$$

- if $x=-1$, then $-n \cdot y(-1) = 0$

$$\Rightarrow y(-1)=0.$$

$\therefore$ solution goes through $(-1,0)$

- if $x \neq -1$

$$y'(x) - \frac{n}{x+1} y(x) = e^x (x+1)^n$$

step 1. $y'(x) - \frac{n}{x+1} y(x) = 0$

⊙ $y \equiv 0$. ✓ solution

⊙ $y \neq 0$. $\dfrac{y'(x)}{y(x)} = \dfrac{n}{x+1}$

$$\Rightarrow \int \frac{dy}{y} = n \int \frac{1}{x+1} dx$$

$$\Rightarrow \ln|y| = n\ln|x+1| + C$$

$$= \ln|x+1|^n + C$$

$$\therefore |y| = e^{\ln|x+1|^n} e^{c}$$

$$\therefore y = \pm e^{c}(x+1)^n$$

$$\therefore y = A(x+1)^n. \qquad A \in \mathbb{R}.$$

step 2. let $y = A(x)(x+1)^n$

substitu to

$$y' - \frac{n}{x+1}y = e^x(x+1)^n$$

$$\Rightarrow A'(x)(x+1)^n + A(x)\,n(x+1)^{n-1} - \frac{n}{x+1}\cdot A(x)(x+1)^n$$

$$= e^x(x+1)^n$$

$$\Rightarrow A'(x)(x+1)^n = e^x(x+1)^n$$

$$\Rightarrow A'(x) = e^x \quad \Rightarrow A(x) = e^x + C$$

$\therefore \quad y(x) = (e^x + C)(x+1)^n, \quad C \in \mathbb{R},$

$x \neq -1$

check all these curves satisfies

$y(-1) = 0.$

$\therefore$ solution

$$y(x) = (e^x + C)(x+1)^n,$$

$\forall x \in \mathbb{R}.$