MATH2621 — Higher Complex Analysis. IX
Harmonic functions

This lecture?

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We see how to find them using holomorphic functions, and conversely, how to find holomorphic functions using harmonic functions.

Harmonic functions

Definition

Suppose that $u : \Omega \to \mathbb{R}$ is a function, where Ω is an open subset of $\mathbb{R}^2$, and that u is twice continuously differentiable, that is, all the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x\,\partial y$, $\partial^2 u/\partial y\,\partial x$ and $\partial^2 u/\partial y^2$ exist and are continuous. Then we say that u is *harmonic* in Ω if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Finding harmonic functions

Theorem

Suppose that $f \in H(\Omega)$, where Ω is an open subset of $\mathbb{C}$, that f is twice continuously differentiable in Ω, and that

$$f(x + iy) = u(x, y) + iv(x, y)$$

in Ω, where u and v are real-valued. Then u and v are harmonic functions.

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Later we will see that the hypothesis that f is twice differentiable is not needed.

Proof of theorem

Proof. Suppose that $f \in H(\Omega)$, where Ω is an open subset of $\mathbb{C}$, and that

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in Ω, where u and v are real-valued, and that all the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x\,\partial y$, $\partial^2 u/\partial y\,\partial x$ and $\partial^2 u/\partial y^2$ and $\partial v/\partial x$, $\partial v/\partial y$, $\partial^2 v/\partial x^2$, $\partial^2 v/\partial x\,\partial y$, $\partial^2 v/\partial y\,\partial x$ and $\partial^2 v/\partial y^2$ exist and are continuous.

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From vector calculus, this implies that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x} \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x}\,.$$

Proof of theorem (continued)

Now, from the Cauchy–Riemann equations,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \\
&= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \\
&= \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial^2 v}{\partial y\,\partial x} = 0,
\end{aligned}$$

so u is harmonic. Similarly, v is harmonic. $\square$

Exercise 1

Suppose that $u(x, y) = x^3 - 3xy^2$. Show that u is harmonic in $\mathbb{C}$, and find a function v such that the function f, given by $f(x + iy) = u(x, y) + iv(x, y)$, is holomorphic in $\mathbb{C}$.

Answer.

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Answer. The partial derivatives of u are given by:

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial y} = -6xy$$

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Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0,$$

and u is harmonic.

Answer to Exercise 1

If v exists such that the function f, given by $f(x+iy) = u(x,y) + iv(x,y)$, is holomorphic, then the Cauchy–Riemann equations hold, so

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 \tag{1}$$

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Fix $y \in \mathbb{R}$. Integrating (2) with respect to x shows that

$$v(x,y) = 3x^2y + C_1,$$

where C_1 does not depend on x. However, C_1 may depend on y. Thus the general solution to (2), when y can vary, is

$$v(x,y) = 3x^2y + c(y),$$

where c is an unknown function.

Answer to Exercise 1 (continued)

The function v must also satisfy (1), and this happens as long as

$$3x^2 - 3y^2 = \frac{\partial v}{\partial y} = 3x^2 + c'(y), \tag{3}$$

i.e., $c'(y) = -3y^2$, so $c(y) = -y^3 + C$, where C is a constant, and

$$v(x, y) = 3x^2 y - y^3 + C.$$

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Set $f(x + iy) = u(x, y) + iv(x, y)$. Then f is holomorphic, and

$$f(z) = f(x + iy) = x^3 - 3xy^2 + i(3x^2 y - y^3) + iC = z^3 + iC. \quad \triangle$$

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In the solution above, if we just used the constant C_1 rather than the function $c(y)$, then we wouldn't be able to satisfy (3).

Exercise 2

Suppose that $u(x, y) = x^2 - y^2$. Show that u is harmonic in $\mathbb{C}$.
Find a function v such that the function f, given by
$f(x + iy) = u(x, y) + iv(x, y)$, is holomorphic in $\mathbb{C}$.

Answer.

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Answer. The partial derivatives of u are given by:

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2.$$

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Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0,$$

and u is harmonic.

Answer to Exercise 2

If v exists such that the function f given by

$$f(x+iy) = u(x,y) + iv(x,y)$$

is holomorphic, then the Cauchy–Riemann equations must hold, so we must have

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x \qquad (1)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y. \qquad (2)$$

Answer to Exercise 2 (continued)

The general solution to equation (1) is

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For (2) to hold as well, we require that

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for all x. This tells us that $c'(x) = 0$ for all x and c has to be a constant, C say.

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Set $f(x + iy) = u(x, y) + iv(x, y)$. Then f is holomorphic and

$$f(z) = f(x + iy) = x^2 - y^2 + 2ixy + iC = z^2 + iC. \qquad \triangle$$

Existence of harmonic functions

Recall that a domain is a polygonally path-connected open set.

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Theorem

If Ω is a simply polygonally connected domain, and $u : \Omega \to \mathbb{R}$ is harmonic, then there exists a harmonic function $v : \Omega \to \mathbb{R}$ such that f, given by

$$f(x + iy) = u(x, y) + iv(x, y)$$

in Ω, is holomorphic. Any two such functions v differ by an additive constant.

Remarks

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We give a partial proof of this theorem later.

To see why it might be necessary to require that Ω be simply connected in the theorem, consider the following.

Exercise 3

Suppose that $u(x, y) = \ln(\sqrt{x^2 + y^2})$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$. Show that u is harmonic. Find a function v such that the function f in $\mathbb{C} \setminus \{0\}$, defined by $f(x + iy) = u(x, y) + iv(x, y)$, is holomorphic.

Answer.

Exercise 3

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Answer. You do this! $\triangle$

Check that having a harmonic conjugate is a local property.

The problem here is that we cannot find a global harmonic conjugate to u.

Exercise 4

Suppose that $u(x, y) = x^2 - y^2$. Show that u is harmonic in $\mathbb{C}$. Find a function v such that the function f, given by $f(x + iy) = u(x, y) + iv(x, y)$, is holomorphic in $\mathbb{C}$.

Answer.

Exercise 4

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Answer. The partial derivatives of u are given by:

$$\frac{\partial u}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -2y.$$

Hence

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2(x+iy) = 2z.$$

Answer to Exercise 4

Clearly, if $f(z) = z^2$, then $f'(z) = 2z$, so we just observe that $\operatorname{Re} f(x+iy) = u(x,y)$, so u is the real part of a holomorphic function and hence is harmonic. Further, $f = u + iv$, where $v(x,y) = 2xy$. $\triangle$

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The trouble with this method is that it may be hard to express $u_x - iu_y$ as a function of z, or hard to find the integral of this.