MATH2621 — Higher Complex Analysis. V
Limits and continuity

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We define limits for complex functions much as for real functions. There are two differences from limits from Calculus: we write z instead of x, and use the word *modulus* rather than the word *absolute value*.

Recall that, given a set S, we define its *closure* $\bar{S}$ (or S^{-}) to be the set consisting of all points of S together with all its boundary points.

Definition of a limit

Suppose that f is a complex function and that z_0 is in $\overline{\text{Domain}(f)}$. We say that $f(z)$ tends to ℓ as z tends to z_0, or that ℓ is the limit of $f(z)$ as z tends to z_0, and we write $f(z) \to \ell$ as $z \to z_0$, or

$$\lim_{z \to z_0} f(z) = \ell,$$

Real axis

if, for every $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(z) - \ell| < \varepsilon$ provided that z is in $\text{Domain}(f)$ and $0 < |z - z_0| < \delta$.

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We can rewrite the conditions $0 < |z - z_0| < \delta$ and $|f(z) - \ell| < \varepsilon$ as $z \in B^{\circ}(z_0, \delta)$ and $f(z) \in B(\ell, \varepsilon)$.

Definition of a limit. 2

Suppose also S is a subset of Domain(f) and that $z_0 \in \bar{S}$. We say that $f(z)$ tends to ℓ as z tends to z_0 in S, or that ℓ is the limit of $f(z)$ as z tends to z_0 in S, and we write $f(z) \to \ell$ as $z \to z_0$ in S, or

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = \ell,$$

if, for every $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(z) - \ell| < \varepsilon$ provided that $z \in S$ and $0 < |z - z_0| < \delta$.

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Most of what follows about limits of the form $\lim_{z \to z_0} f(z)$ also applies to restricted limits, that is, limits of the form $\lim_{\substack{z \to z_0 \\ z \in S}} f(z)$.

Limits at ∞

Suppose that f is a complex function and that Domain(f) is not bounded. We say that $f(z)$ tends to ℓ as z tends to ∞, or that ℓ is the limit of $f(z)$ as z tends to ∞, and we write $f(z) \to \ell$ as $z \to \infty$, or

$$\lim_{z \to \infty} f(z) = \ell,$$

if, for every $\varepsilon \in \mathbb{R}^+$, there exists $R \in \mathbb{R}^+$ such that $|f(z) - \ell| < \varepsilon$ provided that z is in Domain(f) and $|z| > R$.

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Instead of the condition that $|z| > R$, we could put the condition that $|z| > 1/\delta$.

Limits involving ∞

We can define limits involving infinity in a uniform way by defining balls centred at infinity, and modifying our previous definition slightly.

Definition

Suppose that $\varepsilon > 0$. We define both $B(\infty, \varepsilon)$ and $B^\circ(\infty, \varepsilon)$ to be the set $\{z \in \mathbb{C} : |z| > 1/\varepsilon\}$.

Limits involving ∞. 2

Definition

Suppose that f is a complex function, that $\ell \in \mathbb{C} \cup \{\infty\}$, and that either $z_0 \in \text{Domain}(f)^{-}$ or $\text{Domain}(f)$ is not bounded and $z_0 = \infty$. We say that $f(z)$ tends to ℓ as z tends to z_0, or that ℓ is the limit of $f(z)$ as z tends to z_0, and we write $f(z) \to \ell$ as $z \to z_0$, or

$$\lim_{z \to z_0} f(z) = \ell,$$

if for all $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $f(z) \in B(\ell, \varepsilon)$ provided that $z \in B^\circ(z_0, \delta)$.

In this definition, ℓ may be ∞. We sometimes consider ∞ as a limit in this course, but this means that we have to be a bit more careful.

Standard limits

Lemma

Suppose that $\alpha, c \in \mathbb{C}$. Then

$$\lim_{z\to\alpha} c = c \qquad \lim_{z\to\infty} c = c$$

$$\lim_{z\to\alpha} z - c = \alpha - c \qquad \lim_{z\to\infty} z - \alpha = \infty$$

$$\lim_{z\to\alpha} \frac{1}{z-\alpha} = \infty \qquad \lim_{z\to\infty} \frac{1}{z-\alpha} = 0.$$

Lemmas on limits

Lemma

Suppose that f is a complex function, that $T \subseteq S \subseteq \mathrm{Domain}(f)$, and that $z_0 \in \bar{T}$. If $\lim_{\substack{z \to z_0 \\ z \in S}} f(z)$ exists, then so does $\lim_{\substack{z \to z_0 \\ z \in T}} f(z)$, and they are equal.

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Lemma

Suppose that f is a complex function, and that $z_0 \in \mathrm{Domain}(f)^{-}$. If $\lim_{z \to z_0} f(z)$ exists, then it is unique.

Algebra of limits

Theorem

Suppose that f and g are complex functions and that $c \in \mathbb{C}$. Then

$$\begin{aligned}
\lim_{z \to z_0} cf(z) &= c \lim_{z \to z_0} f(z) \\
\lim_{z \to z_0} f(z) + g(z) &= \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) \\
\lim_{z \to z_0} f(z)\, g(z) &= \lim_{z \to z_0} f(z) \lim_{z \to z_0} g(z) \\
\lim_{z \to z_0} \frac{f(z)}{g(z)} &= \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)},
\end{aligned}$$

in the sense that if the right hand side exists, then so does the left hand side, and they are equal. In particular, for the quotient, we require that $\lim_{z \to z_0} g(z) \neq 0$.

Limits and complex conjugation

Theorem

Suppose that f is a complex function and that either $\mathrm{Domain}(f)$ *is unbounded and $z_0 = \infty$ or $z_0 \in \mathrm{Domain}(f)^{-}$. Then*

$$\begin{aligned}
\lim_{z\to z_0} \overline{f(z)} &= \overline{\lim_{z\to z_0} f(z)}\\
\lim_{z\to z_0} \mathrm{Re}(f(z)) &= \mathrm{Re} \lim_{z\to z_0} f(z)\\
\lim_{z\to z_0} \mathrm{Im}(f(z)) &= \mathrm{Im} \lim_{z\to z_0} f(z)\\
\lim_{z\to z_0} f(z) &= \lim_{z\to z_0} \mathrm{Re}(f(z)) + i \lim_{z\to z_0} \mathrm{Im}(f(z)),
\end{aligned}$$

in the sense that if the right hand side exists, then so does the left hand side, and they are equal. In particular, $f(z)$ tends to ℓ as z tends to z_0 if and only if $\mathrm{Re}(f(z))$ *tends to* $\mathrm{Re}(\ell)$ *and* $\mathrm{Im}(f(z))$ *tends to* $\mathrm{Im}(\ell)$ *as z tends to z_0.*

Sketch proof of theorem

Proof. The proof of part (1) uses that fact that

$$\left|\overline{f(z)} - \bar{\ell}\right| = |f(z) - \ell|\,.$$

The rest follows from the first part and the algebra of limits. $\square$

Exercise 1

Show from first principles that $\lim_{z \to z_0} z = z_0$.

Answer.

$\forall \varepsilon > 0$, $\exists \delta > 0$ st

if $0 < |z - z_0| < \delta$ then

$|z - z_0| < \varepsilon$

choose $\delta = \varepsilon$

Exercise 1

Show from first principles that $\lim_{z \to z_0} z = z_0$.

Answer. Given $\varepsilon \in \mathbb{R}^+$, take $\delta = \varepsilon$. Then if $0 < |z - z_0| < \delta$, it follows immediately that $|z - z_0| < \varepsilon$; the result is proved.

$\triangle$

Exercise 2

Suppose that $f(z) = z^2 - \bar{z} + i$. Does $\lim_{z \to 2i} f(z)$ exist: if so find it, and if not, why not.

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Answer. By the algebra of complex limits,

$$\lim_{z\to 2i} z^2 - \bar{z} + i = \left(\lim_{z\to 2i} z\right)^2 - \overline{\left(\lim_{z\to 2i} z\right)} + i = -4 + 2i + i = -4 + 3i.$$

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We now consider an important example.

Exercise 3

Suppose that $a > 0$. Show that

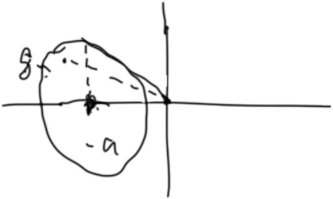
$$\lim_{\substack{z \to -a \\ \mathrm{Im}(z) \geq 0}} \mathrm{Arg}(z) = \pi \qquad \text{and} \qquad \lim_{\substack{z \to -a \\ \mathrm{Im}(z) < 0}} \mathrm{Arg}(z) = -\pi.$$

Does $\lim_{z \to -a} \mathrm{Arg}(z)$ exist, and if so, what is it?

Answer. $|z + a| < \delta \implies$

$|\mathrm{Arg}(z) - \mathrm{Arg}(-a)| < \varepsilon$ if $\mathrm{Im}(z) \geq 0$

$\mathrm{Arg}(z) < \pi - \sin^{-1}\left(\frac{\delta}{a}\right)$



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Does $\lim_{z \to -a} \mathrm{Arg}(z)$ exist, and if so, what is it?

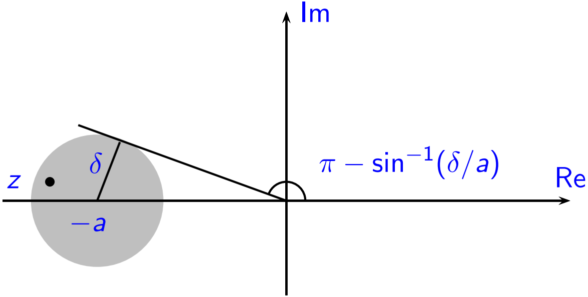
Answer. We consider first the limit where $\mathrm{Im}\, z \geq 0$. Suppose that $\varepsilon > 0$. Let $\delta = a \sin(\min\{\varepsilon, \frac{\pi}{6}\})$; then

$$\delta \leq \frac{a}{2},$$

and so $0 \notin B(-a, \delta)$, and moreover

$$\sin^{-1}(\delta/a) = \min\{\varepsilon, \frac{\pi}{6}\} \leq \varepsilon.$$

Answer to Exercise 3



If $z \in B(-a, \delta)$, then z lies below the (upper) tangent to the circle with centre $-a$ and radius δ from 0. If z also lies in the upper half plane, then

$$\pi - \varepsilon \leq \pi - \sin^{-1}(\delta/a) < \operatorname{Arg}(z) \leq \pi.$$

Hence $|\operatorname{Arg}(z) - \pi| < \varepsilon$.

Answer to Exercise 3 (continued)

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The function Arg is one of many important discontinuous complex functions. The graph of the argument function is like a spiral staircase.

Exercise 4

Suppose that $f(z) = \bar{z}/z$ and $g(z) = z^2/\bar{z}$. Does $\lim_{z\to 0} f(z)$ or $\lim_{z\to 0} g(z)$ exist? If so, find the limit; otherwise, explain why it does not exist.

Answer.

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Answer. On the one hand, if $z = re^{i\theta}$, then $f(z) = e^{-2i\theta}$. As we approach 0 along a ray from the origin, θ is fixed and $r \to 0+$, so the limit along the ray is $e^{-2i\theta}$; different rays give different limits. By the uniqueness of the limit, $\lim_{z\to 0} f(z)$ does not exist.

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On the other hand, $|g(z) - 0| = |z|$, and as $z \to 0$, $|z| \to 0$. Hence $g(z) \to 0$ as $z \to 0$. $\triangle$

Continuity

Definition

Suppose that the complex function f is defined in a set $S \subseteq \mathbb{C}$, and that $z_0 \in S$. We say that f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0);$$

that is, the limit exists, $f(z_0)$ exists, and they are equal.

We say that f is continuous in S if it is continuous at all points of S, and continuous if it is continuous in its domain.

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The functions $z \mapsto z$, $z \mapsto \overline{z}$, $z \mapsto |z|$, $z \mapsto \mathrm{Re}(z)$, and $z \mapsto \mathrm{Im}(z)$ are all continuous. The function Arg is continuous in the set $\{z \in \mathbb{C} \setminus \{0\} : \mathrm{Arg}(z) \neq \pi\}$.

Properties of continuous functions

Theorem

Suppose that $c \in \mathbb{C}$, and that $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$ are continuous complex functions in $S \subseteq \mathbb{C}$. Then cf, $f + g$, $|f|$, $\overline{f}$, $\operatorname{Re} f$, $\operatorname{Im} f$ and fg are continuous in S, as is f/g provided that $g(z) \neq 0$ for any z in S.

Theorem

Suppose that $f : S \to \mathbb{C}$ and $g : T \to \mathbb{C}$ are continuous complex functions in $S \subseteq \mathbb{C}$ and $T \subseteq \mathbb{C}$. Then $f \circ g$ is continuous where it is defined, that is, in $\{z \in T : g(z) \in S\}$.

Applications of continuity

By using the theorems above, it follows that functions that are composed of the standard functions (except Arg), such as

$$z \mapsto \frac{\mathrm{Re}(z^2) + i\,\mathrm{Im}(z^3)}{|z| + 1 + \overline{z}}\,,$$

are also continuous where they are defined.

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Generally speaking, any function that can be written down using the standard functions, and without choices in the definition, is continuous in its domain of definition, except when Arg is involved. Where there are choices in the definition, the difficulties usually lie where the different definitions meet.

Continuity and boundedness

Theorem

Suppose that the set $S \subseteq \mathbb{C}$ is compact (i.e., closed and bounded) and that f is a continuous complex function defined on S. Then there exists a point z_0 in S such that

$$|f(z_0)| = \max\{|f(z)| : z \in S\}.$$

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One says that the modulus of a continuous function *attains its maximum* in a compact set.

Consequently, if f is a continuous complex function defined in a compact set $S \subseteq \mathbb{C}$, then f is *bounded in S*, that is, there is a number R such that

$$|f(z)| \leq R \qquad \forall z \in S.$$

More on continuity

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Continuous functions in compact sets are *uniformly continuous*. We will not explain this now.

Exercise 5

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Answer.

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Answer. Take z_0 in $\mathbb{C}$. We claim that $\lim_{z \to z_0} |z| = |z_0|$. Given $\varepsilon \in \mathbb{R}^+$, take $\delta = \varepsilon$. If $0 < |z - z_0| < \delta$, then

$$\big||z| - |z_0|\big| \leq |z - z_0| < \varepsilon,$$

proving our claim. Since $\lim_{z \to z_0} |z| = |z_0|$ for all $z_0 \in \mathbb{C}$, $z \mapsto |z|$ is continuous in $\mathbb{C}$. $\triangle$

Exercise 6

Show that $\mathrm{Arg}(z)$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$.

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Answer. Fix z_0 in $\mathbb{C} \setminus (-\infty, 0]$, and suppose that $\varepsilon \in \mathbb{R}^+$.
Choose $\delta = |z_0| \sin \min\{\varepsilon, \pi - |\operatorname{Arg}(z_0)|\}$.

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Answer. Fix z_0 in $\mathbb{C} \setminus (-\infty, 0]$, and suppose that $\varepsilon \in \mathbb{R}^+$. Choose $\delta = |z_0| \sin \min\{\varepsilon, \pi - |\operatorname{Arg}(z_0)|\}$.

By trigonometry, if $0 < |z - z_0| < \delta$, then $|\operatorname{Arg}(z) - \operatorname{Arg}(z_0)| < \varepsilon$, as required. $\triangle$

Another complex function

Definition

The function $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is defined by

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z).$$

Exercise 7

Show that $\mathrm{Log}(z)$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$, and is not continuous elsewhere.

Answer.

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Show that $\operatorname{Log}(z)$ is continuous in $\mathbb{C} \setminus (-\infty, 0]$, and is not continuous elsewhere.

Answer. We know that $z \mapsto |z|$ is continuous, and from Calculus, ln is also continuous. Hence $z \mapsto \ln |z|$ is continuous.

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We claim that Log is continuous where Arg is continuous.

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Answer. We know that $z \mapsto |z|$ is continuous, and from Calculus, ln is also continuous. Hence $z \mapsto \ln |z|$ is continuous.

We claim that Log is continuous where Arg is continuous. Indeed, if Arg is continuous. then so is Log, as it is the sum of continuous functions; conversely, if Log is continuous, then so is $z \mapsto \operatorname{Log}(z) - \ln |z|$, for the same reason, whence Arg is continuous. △

The Riemann sphere

To review the role of ∞, we review the Riemann sphere.

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The balls $B(0, \varepsilon)$ correspond to spherical caps in the Riemann sphere, centred at $\sigma(0)$, while balls $B(z_0, \varepsilon)$ correspond to spherical caps in the Riemann sphere containing the point $\sigma(z_0)$, and the "punctured balls" $B^\circ(\infty, \varepsilon)$ correspond to spherical caps in the Riemann sphere centred at $\sigma(\infty)$. These spherical caps shrink down to $\sigma(0)$, to $\sigma(z)$ and to $\sigma(\infty)$ as ε tends to 0.

The Riemann sphere

To review the role of ∞, we review the Riemann sphere.

The balls $B(0, \varepsilon)$ correspond to spherical caps in the Riemann sphere, centred at $\sigma(0)$, while balls $B(z_0, \varepsilon)$ correspond to spherical caps in the Riemann sphere containing the point $\sigma(z_0)$, and the "punctured balls" $B^\circ(\infty, \varepsilon)$ correspond to spherical caps in the Riemann sphere centred at $\sigma(\infty)$. These spherical caps shrink down to $\sigma(0)$, to $\sigma(z)$ and to $\sigma(\infty)$ as ε tends to 0. So $\lim_{z \to z_0} f(z) = \ell$ when $f(z)$ is close to ℓ in the Riemann sphere provided that z is close to z_0 in the Riemann sphere.

End Notes

1. Standard real-valued functions of a real variable may be considered as complex functions; the definitions from complex analysis then coincide with those from calculus.
2. Unless problems about limits use expressions like "from first principles", then you can use standard limits and the algebra of limits to solve them.
3. Potentially there are many different kinds of limits: in $\lim_{z \to z_0} f(z) = \ell$, neither, either or both of z_0 or ℓ might be ∞. Rather than spend lots of time discussing all these possibilities, we discuss two and then give a general definition that encompasses all four possibilities.
4. Many properties of limits and continuity mimic those in calculus of one variable. It is not worth spending much time on these aspects of the theory; rather, just point out that things are as in calculus and multivariable calculus.
5. Arg and Log are both very important.