Higher Complex Analysis. XXIV
Computing Integrals. I

Today?

In this lecture, we give three examples of the use of Cauchy's residue theorem to calculate integrals.

We compute a trigonometric integral, an integral of a rational function on the real line, and an integral involving a root.

Exercise 1

Evaluate $\displaystyle\int_{-\pi}^{\pi} \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta.$

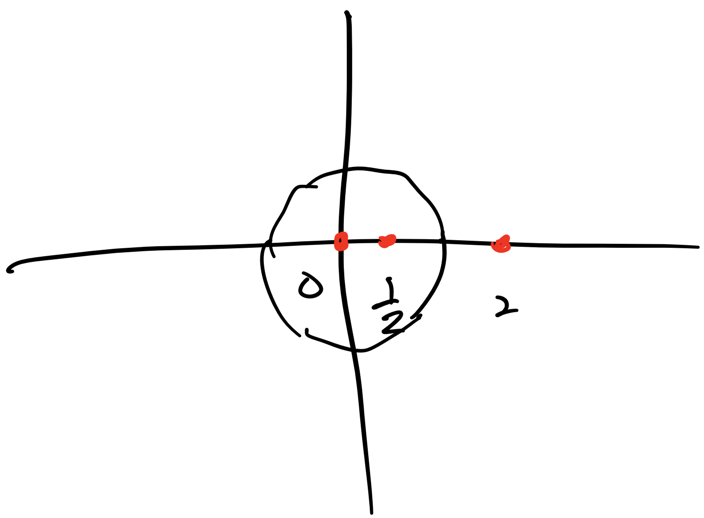
Answer. [P] The first step is to convert this to a contour integral. We take $\gamma(\theta) = e^{i\theta}$, where $-\pi \leq \theta \leq \pi$. Then $\gamma'(\theta) = ie^{i\theta} = i\gamma(\theta)$ and

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{\gamma(\theta) + \gamma(\theta)^{-1}}{2}.$$

As θ varies over its domain, $\gamma(\theta)$ travels anticlockwise around the unit circle in the complex plane. Thus . . .

Answer to Exercise 1

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta &= \int_\gamma \frac{(z + z^{-1})/2}{5 - 2(z + z^{-1})} \frac{1}{iz}\, dz \\
&= \frac{1}{2i} \int_\gamma \frac{z^2 + 1}{(-2z^2 + 5z - 2)} \frac{1}{z}\, dz \\
&= \frac{i}{2} \int_\gamma \frac{z^2 + 1}{z(2z^2 - 5z + 2)}\, dz \\
&= \frac{i}{2} \int_\gamma \frac{z^2 + 1}{z(2z - 1)(z - 2)}\, dz \\
&= \frac{i}{2} \int_\gamma f(z)\, dz,
\end{aligned}$$



where

$$f(z) = \frac{z^2 + 1}{z(2z - 1)(z - 2)}\, dz.$$

Answer to Exercise 1 (continued)

The second step is to evaluate the contour integral by evaluating the residues of f. Clearly, f has singularities at 0, $\frac{1}{2}$ and 2. Only 0 and $\frac{1}{2}$ lie inside γ. Because each of the factors in the denominator is of degree 1, and the numerator does not vanish at these points, each of the singularities is a simple pole.
[P]
Hence

$$\mathrm{Res}(f,0) = \lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{z^2+1}{(2z-1)(z-2)} = \frac{1}{2}$$

and

$$\mathrm{Res}(f,1/2) = \lim_{z\to 1/2}(z-1/2)f(z) = \lim_{z\to 1/2} \frac{z^2+1}{2z(z-2)} = \frac{5/4}{-3/2} = -\frac{5}{6}.$$

Answer to Exercise 1 (continued)

By the residue theorem,

$$\int_{-\pi}^{\pi} \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta = \frac{i}{2}\int_{\gamma} \frac{z^2+1}{z(2z-1)(z-2)}\, dz$$

$$= \frac{i}{2} 2\pi i \Big(\frac{1}{2} - \frac{5}{6}\Big)$$

$$= \frac{\pi}{3}\,.$$

△

$$\mathrm{Re} \int_{-\pi}^{\pi} \frac{\cos\theta + i\sin\theta}{5-4\cos\theta}\, d\theta$$

$$= \int_{-\pi}^{\pi} \frac{e^{i\theta}}{5-4\cos\theta}\, d\theta$$

$$= i\int_{\gamma} \frac{z}{2z^2-5z+2}\, dz$$

Remarks

If the integrand had contained a sin term, then we could have used the fact that

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}.$$

Similarly, expressions such as $\sin(2\theta)$ and $\tan(\theta)$ may be expressed in terms of $e^{i\theta}$ and hence of z. In summary, any integral that can be put in the form

$$\int_{-\pi}^{\pi} f(\sin(\theta), \cos(\theta))\, d\theta$$

where f is a rational function of two variables, can be tackled in this way, and becomes an integral of a different rational function around a closed contour, which can be evaluated, at least in principle, and often in practice.

Remarks on Exercise 1 (continued)

There are some other integrals that may be converted to integrals of this form. For instance, since $\cos(-\theta) = \cos(\theta)$, that is, cos is an even function, $\theta \mapsto \cos(\theta)/(5 - 4\cos(\theta))$ is also an even function,

$$\int_0^\pi \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta = \int_{-\pi}^0 \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta$$

and so

$$\int_0^\pi \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta = \frac{1}{2}\int_{-\pi}^\pi \frac{\cos(\theta)}{5 - 4\cos(\theta)}\, d\theta.$$

If the integrand involves expressions like $\cos(\theta/2)$, this method does not work: square roots appear, and these mess up the holomorphy.

Exercise 2

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx.$ $= \lim_{b\to} \lim_{a\to\infty} \displaystyle\int_a^b \frac{1}{x^4+1}\,dx$

Answer. [P] The real line is not a simple closed contour, and it is not obvious how the residue theorem can be used.
[P]
Define f by

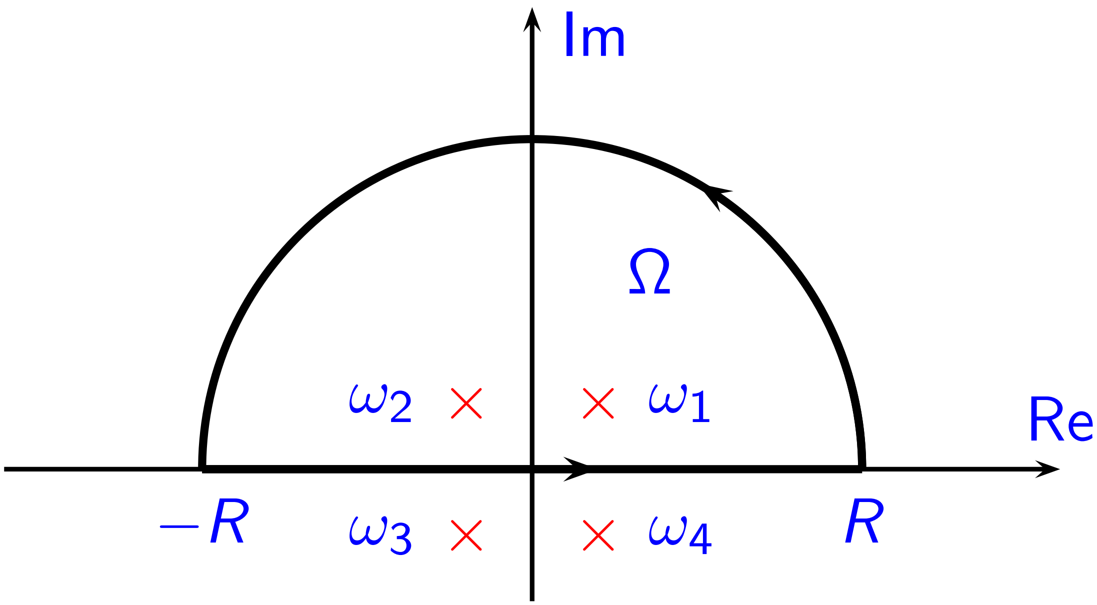
$$f(z) = \frac{1}{z^4+1} = \frac{1}{(z-\omega_1)(z-\omega_2)(z-\omega_3)(z-\omega_4)},$$

where $\omega_k = \exp(\frac{1}{2}\pi ik - \frac{1}{4}\pi i)$. The function f has four singularities, at the points ω_k.
[P]
We will integrate f, as $\int f(z)\,dz$ should be related to $\int f(x)\,dx$.

Answer to Exercise 2

We will take Ω to be the semicircular region Ω above the interval $[-R, R]$, and integrate f around $\partial\Omega$. We suppose that $R > 1$, so that all the singularities of the integrand in the upper half plane lie in Ω.

[P]



Answer to Exercise 2 (continued)

We defined f by

$$f(z) = \frac{1}{z^4+1} = \frac{1}{(z-\omega_1)(z-\omega_2)(z-\omega_3)(z-\omega_4)},$$

where $\omega_k = \exp(\frac{1}{2}\pi ik - \frac{1}{4}\pi i)$. The only singularities of f that lie in Ω are ω_1 and ω_2. Since the power of $(z-\omega_k)$ in the denominator is 1, these are simple poles. Further, $\omega_k^4 = -1$, and so $\omega_k^3 = -1/\omega_k$. We deduce that

$$\begin{aligned} \mathrm{Res}(f, \omega_k) &= \lim_{z\to\omega_k} (z-\omega_k) f(z) = \lim_{z\to\omega_k} \frac{z-\omega_k}{z^4+1} \\ &= \lim_{z\to\omega_k} \frac{1}{4z^3} = \frac{1}{4\omega_k^3} = -\frac{1}{4}\omega_k \,. \end{aligned}$$

[P] The p/q' formula would have given this somewhat more quickly.

Answer to Exercise 2 (continued)

Hence $2\pi i$ multiplied by the sum of the residues at the singularities in Ω is given by

$$\begin{aligned} 2\pi i(\operatorname{Res}(f,\omega_1)+\operatorname{Res}(f,\omega_2)) &= -\frac{2\pi i}{4}\Big(\frac{1+i}{\sqrt{2}}+\frac{-1+i}{\sqrt{2}}\Big) \\ &= -\frac{\pi i}{2}\Big(\frac{2i}{\sqrt{2}}\Big)=\frac{\pi\sqrt{2}}{2}\,. \end{aligned}$$

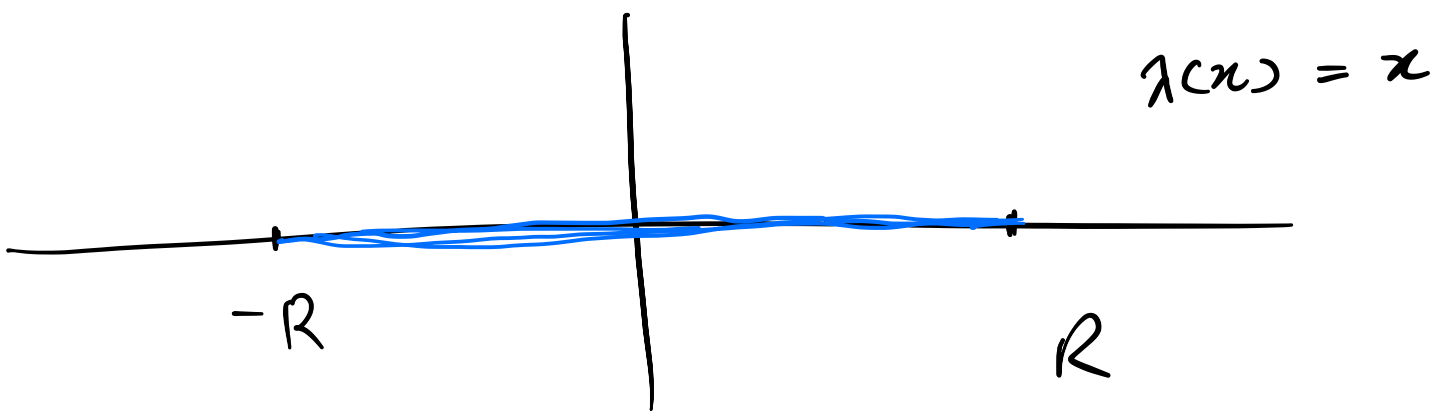
[P]
The contour $\partial\Omega$ has two components: the circular arc and the diameter.

Answer to Exercise 2 (continued)

Parametrise the diameter by the function $\lambda : [-R, R] \to \mathbb{C}$, defined by $\lambda(x) = x$. Then one component of the integral around the contour is

$$\begin{aligned} \int_\lambda f(z)\,dz &= \int_{-R}^{R} f(\lambda(x))\,\lambda'(x)\,dx = \int_{-R}^{R} f(x)\,dx \\ &= \int_{-R}^{R} \frac{1}{x^4+1}\,dx \to \int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx \end{aligned} \tag{1}$$

as $R \to \infty$; this limit is exactly the integral that we were asked to compute.



Answer to Exercise 2 (continued)

Parametrise the semicircular arc by the function $\gamma : [0, \pi] \to \mathbb{C}$, given by $\gamma(t) = Re^{it}$. When $z \in \text{Range}(\gamma)$, then, by the triangle inequality, $|z^4 + 1| \geq |z|^4 - 1 = R^4 - 1$; thus

$$\frac{1}{|z^4 + 1|} \leq \frac{1}{R^4 - 1} .$$

By the *ML* lemma,

$$\left| \int_\gamma f(z)\, dz \right| \leq \frac{1}{R^4 - 1} \text{Length}(\gamma) = \frac{\pi R}{R^4 - 1} \to 0$$

as $R \to \infty$, that is,

$$\int_\gamma f(z)\, dz \to 0. \tag{2}$$

Answer to Exercise 2 (continued)

By the residue theorem, $= \int_\gamma f + \int_\lambda f$

$$\int_{\partial\Omega} f(z)\,dz = 2\pi i\big(\mathrm{Res}(f,\omega_1) + \mathrm{Res}(f,\omega_2)\big) = \frac{\pi\sqrt{2}}{2}.$$

That is,

$$\int_\lambda f(z)\,dz = \frac{\pi\sqrt{2}}{2} - \int_\gamma f(z)\,dz. \tag{3}$$

From (1), (3) and (2),

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx &= \lim_{R\to\infty} \int_\lambda f(z)\,dz \\ &= \lim_{R\to\infty} \left(\frac{\pi\sqrt{2}}{2} - \int_\gamma f(z)\,dz\right) \\ &= \frac{\pi\sqrt{2}}{2}.\end{aligned}$$

△

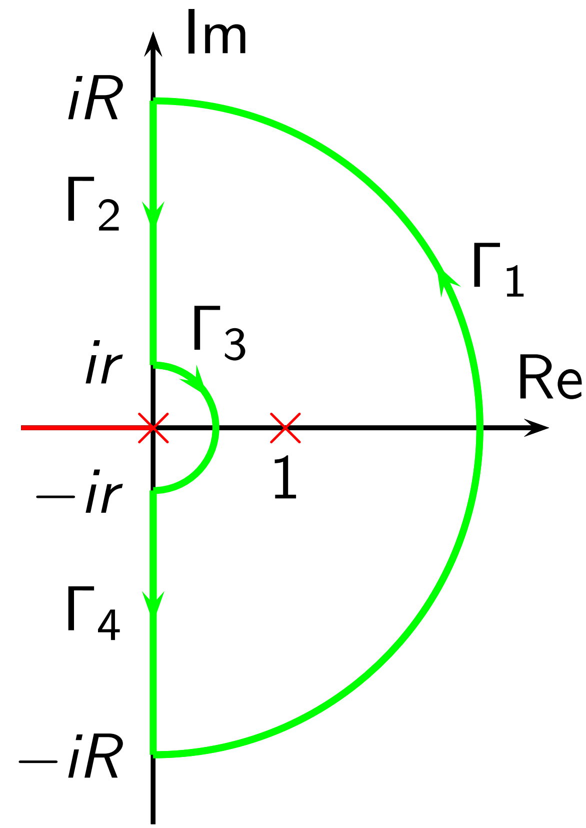
Exercise 3

Compute the contour integral

$$\int_\Gamma \frac{z^{1/2}}{z^2-1}\,dz,$$

where Γ is the join of the contours Γ_1, Γ_2, Γ_3 and Γ_4 shown in the figure, and $z^{1/2}$ denotes the principal branch of the square root. What happens to the component integrals when $r \to 0$ and $R \to \infty$? Find

$$\int_0^{+\infty} \frac{x^{1/2}}{x^2+1}\,dx.$$



Here $r < 1 < R$.

Answer to Exercise 3

Answer. [P] First of all, this is a simple closed contour, and the integrand is holomorphic on and inside the contour, except for a simple pole at 1, so

$$\begin{aligned}
\int_\Gamma \frac{z^{1/2}}{z^2-1}\,dz &= 2\pi i\,\mathrm{Res}\left(\frac{z^{1/2}}{z^2-1}, z=1\right) \\
&= 2\pi i \lim_{z\to 1} \frac{z^{1/2}(z-1)}{z^2-1} \\
&= \pi i.
\end{aligned}$$

Answer to Exercise 3

$$\left|\frac{z^{\frac{1}{2}}}{z^2-1}\right| \leq \frac{|z|^{\frac{1}{2}}}{\left||z|^2-1\right|} = \frac{r^{\frac{1}{2}}}{|r^2-1|}$$

$$(r<1) = \frac{\sqrt{r}}{1-r^2}$$

Now, by the ML lemma,

$$\left|\int_{\Gamma_3} \frac{z^{1/2}}{z^2-1}\,dz\right| \leq \max\left\{\left|\frac{z^{1/2}}{z^2-1}\right| : |z| = r,\ \mathrm{Re}(z) \geq 0\right\}\pi r$$

$$\leq \frac{r^{1/2}}{1-r^2}\,\pi r \to 0 \qquad \text{as } r \to 0.$$

Similarly, by the ML lemma,

$$\left|\int_{\Gamma_1} \frac{z^{1/2}}{z^2-1}\,dz\right| \leq \max\left\{\left|\frac{z^{1/2}}{z^2-1}\right| : |z| = R,\ \mathrm{Re}(z) \geq 0\right\}\pi R$$

$$\leq \frac{R^{1/2}}{R^2-1}\,\pi R \to 0 \qquad \text{as } R \to \infty.$$

Answer to Exercise 3

Parametrise Γ_2 by $\gamma_2(y) = iy$, where y goes from R to r, and Γ_4 by $\gamma_4(y) = -iy$, where where y goes from r to R. [P] Then,

$$\int_{\Gamma_2} \frac{z^{1/2}}{z^2-1}\,dz = \int_R^r \frac{(iy)^{1/2}}{-y^2-1}\,i\,dy = i(+i)^{1/2}\int_r^R \frac{y^{1/2}}{y^2+1}\,dy,$$

and

$$\int_{\Gamma_4} \frac{z^{1/2}}{z^2-1}\,dz = \int_r^R \frac{(-iy)^{1/2}}{-y^2-1}\,(-i)\,dy = i(-i)^{1/2}\int_r^R \frac{y^{1/2}}{y^2+1}\,dy.$$

As $r \to 0$ and $R \to \infty$,

$$\int_r^R \frac{y^{1/2}}{y^2+1}\,dy \to \int_0^\infty \frac{y^{1/2}}{y^2+1}\,dy.$$

Answer to Exercise 3 (continued)

Now

$$\int_{\Gamma_1} \frac{z^{1/2}}{z^2-1}\,dz + \int_{\Gamma_2} \frac{z^{1/2}}{z^2-1}\,dz + \int_{\Gamma_3} \frac{z^{1/2}}{z^2-1}\,dz + \int_{\Gamma_4} \frac{z^{1/2}}{z^2-1}\,dz$$
$$= \int_{\Gamma} \frac{z^{1/2}}{z^2-1}\,dz = \pi i\,,$$

whence

$$\lim_{\substack{r\to 0\\ R\to\infty}} \left(\int_{\Gamma_1} \frac{z^{1/2}}{z^2-1}\,dz + \int_{\Gamma_2} \frac{z^{1/2}}{z^2-1}\,dz + \int_{\Gamma_3} \frac{z^{1/2}}{z^2-1}\,dz + \int_{\Gamma_4} \frac{z^{1/2}}{z^2-1}\,dz\right)$$
$$= \pi i,$$

so

$$\left(i(i^{1/2}) + i(-i)^{1/2}\right) \int_0^\infty \frac{y^{1/2}}{y^2+1}\,dy = \pi i,$$

and hence ...

Answer to Exercise 3 (continued)

$$\int_0^\infty \frac{y^{1/2}}{y^2+1}\,dy = \frac{\pi}{\sqrt{2}}\,. \qquad \triangle$$

Remarks on Exercise 3

We can compute integrals involving functions with branches as long as the branch cut is outside the contour. The process of avoiding a singularity, such as 0 here, by adding a small circular arc to the contour, is called *indenting*.