Higher Complex Analysis. XXV
Computing Integrals. II

Today?

In this lecture, we compute more definite integrals, and introduce Jordan's Lemma, which can be useful in this context.

Exercise 1

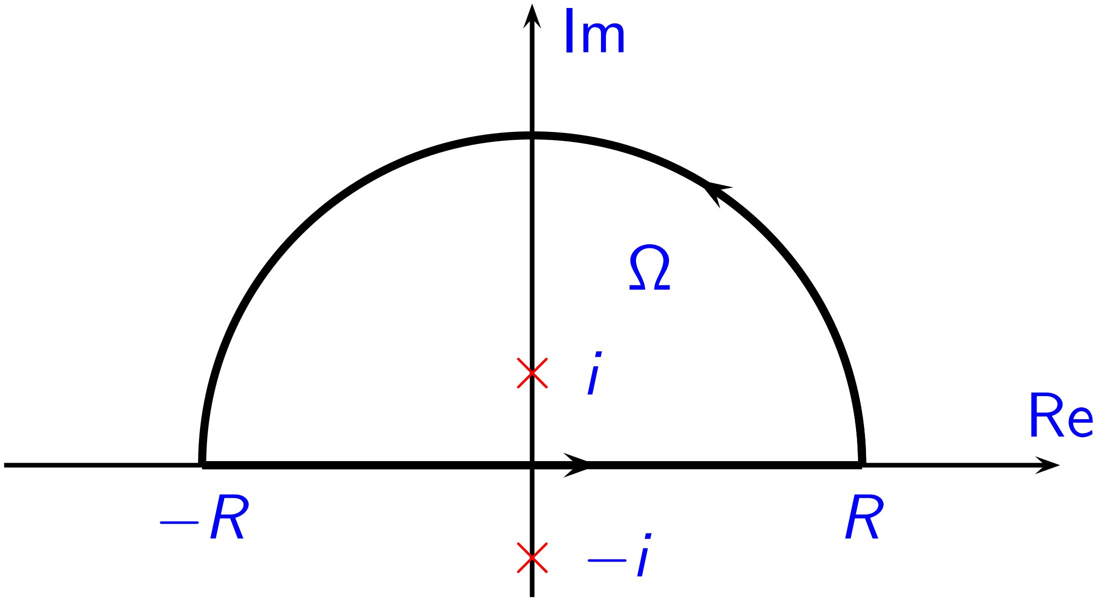
Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx$, where $\xi \in \mathbb{R}^+$.

Answer. [P] Define f by

$$f(z) = \frac{e^{i\xi z}}{z^2+1}\,.$$

The function f has two singularities, at the points $\pm i$. We integrate f around the boundary of the semicircular region Ω above the interval $[-R, R]$. We suppose that $R > 1$, so that all the singularities of f in the upper half plane lie inside the contour $\partial\Omega$.

Answer to Exercise 1



[P] Recall that

$$f(z) = \frac{e^{i\xi z}}{z^2+1} = \frac{e^{i\xi z}}{(z-i)(z+i)}\,.$$

The singularities of f are simple poles, since the power of $(z \pm i)$ in the denominator is 1.

Answer to Exercise 1 (continued)

From the p/q' formula,

$$\operatorname{Res}(f, i) = \frac{e^{-\xi}}{2i}\,.$$

[P]
We could also have done this using the formula for the residue and l'Hôpital's rule.
[P]
The contour $\partial\Omega$ has two components: the circular arc Γ and the diameter Λ.

Answer to Exercise 1 (continued)

Parametrise the diameter Λ by the function $\lambda : [-R, R] \to \mathbb{C}$, given by $\lambda(x) = x$. Then

$$\begin{aligned}\int_\Lambda f(z)\,dz &= \int_{-R}^{R} f(\lambda(x))\,\lambda'(x)\,dx \\ &= \int_{-R}^{R} f(x)\,dx \\ &= \int_{-R}^{R} \frac{e^{i\xi x}}{x^2+1}\,dx \to \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx\end{aligned}$$

as $R \to \infty$; this is exactly the integral that we were asked to compute. We have shown that, as $R \to \infty$,

$$\int_\Lambda f(z)\,dz \to \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx \qquad (1)$$

Answer to Exercise 1 (continued)

When $z \in \Gamma$, then $z = x + iy$ where $x \in \mathbb{R}$ and $y \geq 0$, so

$$|e^{i\xi z}| = |e^{i\xi x - \xi y}| = |e^{i\xi x}|\,|e^{-\xi y}| = |e^{-\xi y}| \leq 1,$$

and further, $|z^2 + 1| \geq |z|^2 - 1 = R^2 - 1$, by the triangle inequality; thus

$$\left|\frac{e^{i\xi z}}{z^2+1}\right| \leq \frac{1}{|z^2+1|} \leq \frac{1}{R^2-1} = M,$$

say.

Answer to Exercise 1 (continued)

From the *ML* lemma,

$$\left|\int_\Gamma f(z)\,dz\right| \leq M\ \text{Length}(\Gamma) = \frac{1}{R^2-1}\,\pi R = \frac{\pi R}{R^2-1} \to 0$$

as $R \to \infty$, that is,

$$\int_\Gamma f(z)\,dz \to 0. \tag{2}$$

By the Residue Theorem,

$$\int_\Lambda f(z)\,dz + \int_\Gamma f(z)\,dz = 2\pi i\,\mathrm{Res}(f, i) = 2\pi i\,\frac{e^{-\xi}}{2i} = \pi e^{-\xi},$$

that is,

$$\int_\Lambda f(z)\,dz = \pi e^{-\xi} - \int_\Gamma f(z)\,dz. \tag{3}$$

Answer to Exercise 1 (continued)

We conclude from (1), (3) and (2) that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx &= \lim_{R\to\infty} \int_{\Lambda} f(z)\,dz \\ &= \lim_{R\to\infty} \left(\pi e^{-\xi} - \int_{\Gamma} f(z)\,dz\right) \\ &= \pi e^{-\xi}.\end{aligned}$$

$\triangle$

[P]
This gives us formulae for two real integrals.

Remarks on this calculation

There are many integrals on the whole line $\mathbb{R}$ that may be evaluated in this way. Integrals between 0 and ∞ may also be tackled if the integrand is even. When the integrand involves $\cos(\xi x)$ or $\sin(\xi x)$, then we write the trigonometric function in terms of exponentials, and if necessary change variables to get an integral involving $e^{i\xi x}$, where $\xi \geq 0$. Then $|e^{i\xi z}| \leq 1$ when $z \in \Gamma$. Sometimes this inequality is not enough and we have to use Jordan's Lemma (see below).

More remarks on this calculation

For example, by the changes of variable $x' = -x$ and then $x = x'$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2+1}\,dx &= \frac{1}{2}\Big(\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx + \int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{x^2+1}\,dx\Big) \\
&= \frac{1}{2}\Big(\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx + \int_{-\infty}^{\infty} \frac{e^{i\xi x'}}{(x')^2+1}\,dx'\Big) \\
&= \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx,
\end{aligned}$$

which we have just computed. Since $\cos(x)$ and x^2 are even functions of x,

$$\int_{0}^{\infty} \frac{\cos(\xi x)}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(\xi x)}{x^2+1}\,dx = \frac{\pi e^{-\xi}}{2}\,.$$

Jordan's Lemma

Lemma

Suppose that f is continuous in $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0, |z| \geq S\}$, for some positive S, that Γ_R is the upper half of the circle with centre 0 and radius R, and that $|f(z)| \leq M_R$ for all $z \in \Gamma_R$ where $\lim_{R\to\infty} M_R = 0$.
Then for any $\xi > 0$,

$$\lim_{R\to\infty} \left| \int_{\Gamma_R} e^{i\xi z} f(z)\, dz \right| = 0.$$

Proof. We give the proof at the end of this lecture. $\square$

Exercise 2

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x\, e^{i\xi x}}{x^2+1}\, dx$, where $\xi \in \mathbb{R}^+$.

Answer. [P] Define f by

$$f(z) = \frac{z\, e^{i\xi z}}{z^2+1} = \frac{z\, e^{i\xi z}}{(z-i)(z+i)}\,.$$

The function f has two singularities, at the points $\pm i$.
[P]
As before, we integrate around the boundary of the semicircular region Ω above the interval $[-R, R]$. The only singularity of f in the region Ω is at i. Since the power of $(z-i)$ in the denominator of the expression defining f is 1, this is a simple pole.

Answer to Exercise 2

We deduce that

$$\operatorname{Res}(f, i) = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} z\, e^{i\xi z} \frac{z - i}{z^2 + 1}$$
$$= i e^{-\xi} \lim_{z \to i} \frac{1}{2z} = \frac{1}{2}\, e^{-\xi} \ .$$

The contour has two components: the circular arc Γ and the diameter Λ.

Answer to Exercise 2 (continued)

Parametrise the diameter Λ by the function $\lambda : [-R, R] \to \mathbb{C}$, given by $\lambda(x) = x$. Then

$$\begin{aligned}\int_\Lambda f(z)\,dz &= \int_{-R}^{R} f(\lambda(x))\,\lambda'(x)\,dx \\ &= \int_{-R}^{R} f(x)\,dx \qquad (4)\\ &= \int_{-R}^{R} \frac{x\,e^{i\xi x}}{x^2+1}\,dx.\end{aligned}$$

It is not obvious what happens to this integral when $R \to \infty$.

Answer to Exercise 2 (continued)

When $z \in \Gamma$, then $z = x + iy$ where $x \in \mathbb{R}$ and $y \geq 0$, so, as before,

$$|e^{i\xi z}| \leq 1;$$

further, $|z^2 + 1| \geq |z|^2 - 1 = R^2 - 1$, by the triangle inequality; thus

$$\left|\frac{z\, e^{i\xi z}}{z^2+1}\right| \leq \frac{|z|}{|z^2+1|} \leq \frac{R}{R^2-1}.$$

From the ML Lemma,

$$\left|\int_\Gamma f(z)\, dz\right| \leq \frac{R}{R^2-1}\,\mathrm{Length}(\Gamma) = \frac{\pi R^2}{R^2-1} \to \pi$$

as $R \to \infty$. This is not helpful. However, since $M \to 0$ as $R \to \infty$, Jordan's Lemma tells us that

$$\int_\Gamma f(z)\, dz \to 0. \tag{5}$$

Answer to Exercise 2 (continued)

By the Residue Theorem,

$$\int_{\Lambda} f(z)\,dz + \int_{\Gamma} f(z)\,dz = 2\pi i\,\mathrm{Res}(f,i) = 2\pi i\,\frac{e^{-\xi}}{2} = i\pi e^{-\xi},$$

that is,

$$\int_{\Lambda} f(z)\,dz = i\pi e^{-\xi} - \int_{\Gamma} f(z)\,dz. \tag{6}$$

We conclude from (4), (6) and (5) that, as $R \to \infty$,

$$\int_{-R}^{R} \frac{x\,e^{i\xi x}}{x^2+1}\,dx \to i\pi e^{-\xi}. \qquad \triangle$$

Comments

We may write the integral in the previous exercise as

$$\int_{\mathbb{R}} \frac{x\cos(\xi x)}{x^2+1}\,dx + i\int_{\mathbb{R}} \frac{x\sin(\xi x)}{x^2+1}\,dx.$$

The result implies that the first integral is 0, as it should be since the integrand is odd, while the second integral is $\pi e^{-\xi}$.

Proof of Jordan's Lemma[†]

For convenience, we restate the result.

Lemma

Suppose that f is continuous in $\{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0, |z| \geq S\}$, for some positive S, that Γ_R is the upper half of the circle with centre 0 and radius R, and that $|f(z)| \leq M_R$ for all $z \in \Gamma_R$, where $\lim_{R\to\infty} M_R = 0$.
Then for any $\xi > 0$,

$$\lim_{R\to\infty} \left| \int_{\Gamma_R} e^{i\xi z} f(z)\, dz \right| = 0.$$

Proof. We parametrise Γ_R by setting $\gamma_R(\theta) = Re^{i\theta}$.

Proof of Jordan's Lemma (continued)

Then

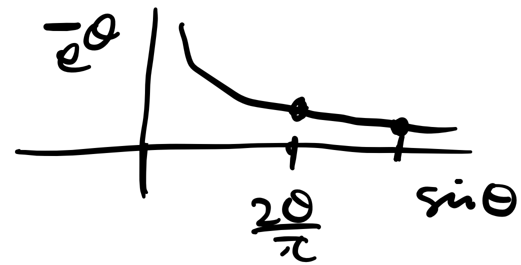
$$\begin{aligned}
\left|\int_{\Gamma_R} e^{i\xi z} f(z)\, dz\right| &= \left|\int_0^\pi e^{i\xi Re^{i\theta}} f(Re^{i\theta})\, iRe^{i\theta}\, d\theta\right| \\
&\le \int_0^\pi \left|e^{i\xi R(\cos(\theta)+i\sin(\theta))} f(Re^{i\theta})\, iRe^{i\theta}\right| d\theta \\
&= \int_0^\pi e^{-\xi R\sin(\theta)} \left|f(Re^{i\theta})\right| R\, d\theta \qquad (7) \\
&\le \int_0^\pi e^{-\xi R\sin(\theta)} M_R\, R\, d\theta \\
&= 2\int_0^{\pi/2} e^{-\xi R\sin(\theta)} M_R\, R\, d\theta,
\end{aligned}$$

$$|e^{i\xi R\cos\theta}| = 1$$

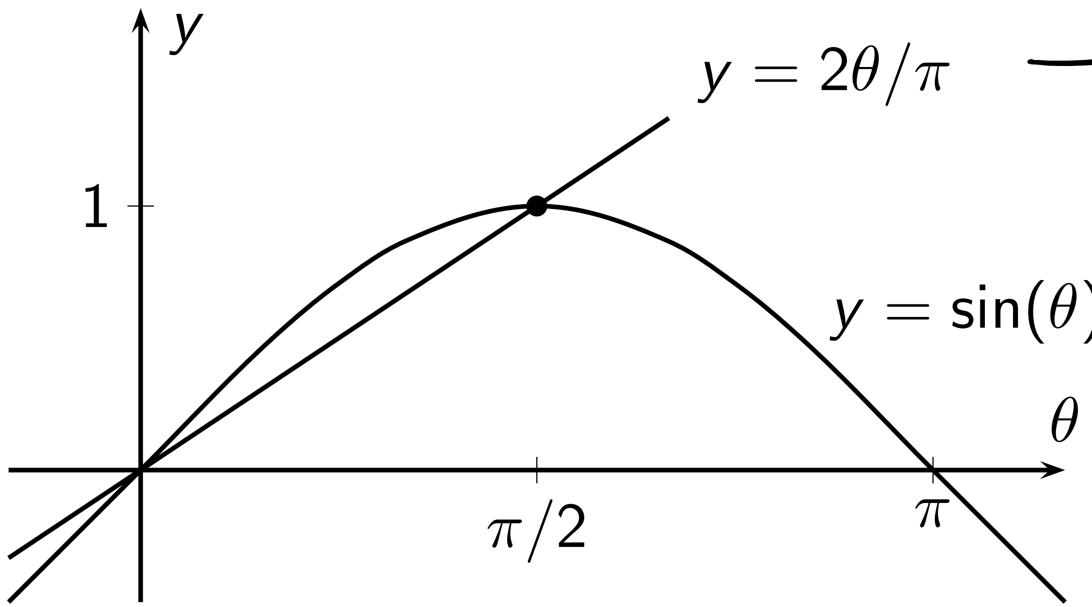
because sin is symmetric about $\pi/2$.

Proof of Jordan's Lemma (continued)

$\frac{2\theta}{\pi} \leq \sin(\theta)$



$e^{-\frac{2\theta}{\pi}} \geq e^{-\sin\theta}$



[P]
It is clear from the graph, and could be proved by using the mean value theorem or convexity, that $\sin(\theta) \geq 2\theta/\pi$ when $0 \leq \theta \leq \pi/2$, so $-\xi R \sin(\theta) \leq -2\xi R\theta/\pi$ and

$$e^{-\xi R \sin(\theta)} \leq e^{-2\xi R\theta/\pi}.$$

Proof of Jordan's Lemma (continued)

Thus, substituting $t = 2\xi R\theta/\pi$, we see from (7) that

$$
\begin{aligned}
\left| \int_{\Gamma_R} e^{i\xi z} f(z)\, dz \right| &\le 2M_R \int_0^{\pi/2} e^{-2\xi R\theta/\pi}\; R\, d\theta \\
&= \frac{\pi}{\xi} M_R \int_0^{\xi R} e^{-t}\; dt \\
&\le \frac{\pi}{\xi} M_R \int_0^{\infty} e^{-t}\; dt \\
&= \frac{\pi}{\xi} M_R \\
&\to 0
\end{aligned}
$$

as $R \to \infty$, as required. $\square$