MATH2621 — Higher Complex Analysis. XIX
Morera's Theorem and analytic continuation

This lecture?

In this lecture, we

- prove Morera's theorem,
- discuss *analytic continuation*, and
- compute more contour integrals.

Morera's Theorem

Theorem

Suppose that Ω is a domain, that the function $f : \Omega \to \mathbb{C}$ is continuous, and that

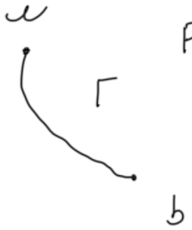
$$\int_{\mathrm{A}} f(z)\,dz = \int_{\mathrm{B}} f(z)\,dz,$$

whenever the simple contours A *and* B *have the same initial point and the same final point. Then f is holomorphic in Ω.*

Proof. This proof is in two steps.

Proof of Morera's Theorem: Step 1

We fix a base point b in Ω, and for all $w \in \Omega$, define $F(w)$ to be $\int_\Gamma f(z)\,dz$, where Γ is a contour from b to w. Then $F'(w) = f(w)$ for all $w \in \Omega$, and hence F is holomorphic. The details are in the proof of the corollary of the Cauchy–Goursat theorem on existence of primitives.



$$F(w) = \int_\Gamma f(z)\,dz$$

Proof of Morera's Theorem: Step 2

We deduce that f is holomorphic from the fact that F is holomorphic. To do this, we appeal to the corollary to Cauchy's integral formula from the preceding lecture. We take an arbitrary point $z_0 \in \Omega$, and then $B(z_0, r) \subseteq \Omega$ for some $r \in \mathbb{R}^+$ because Ω is open. Now $F \in H(B(z_0, r))$, and so by the corollary,

$$F(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

in $B(z_0, r)$. We may differentiate a power series term by term in any ball in which it converges, so

$$f(z) = F'(z) = \sum_{n=0}^{\infty} c_n n (z - z_0)^{n-1} = \sum_{m=0}^{\infty} c_{m+1}(m+1)(z - z_0)^m$$

in $B(z_0, r)$, and hence f is holomorphic in $B(z_0, r)$. Now f is holomorphic in Ω because z_0 was an arbitrary point in Ω. $\square$

A corollary

Corollary

Suppose that Λ is a (possibly infinite) line segment in an open set Ω and $\Omega \setminus \Lambda$ is open. If function $f : \Omega \to \mathbb{C}$ is continuous in Ω and is holomorphic in $\Omega \setminus \Lambda$, then f is holomorphic in Ω.

[P]

Sketch proof. By Morera's theorem, and an approximation argument, it suffices to show that $\int_\Gamma f(z)\,dz = 0$ for all closed polygonal contours Γ in Ω. We can break such an integral into a sum of integrals over closed contours in $\Omega \setminus \Lambda$ together with an error term that may be made arbitrarily small. $\square$

The logic of the theorems on contour integration

We have seen a number of results about contour integration, involving various hypotheses and conclusions, and we now summarise these results. We suppose that f is a continuous function defined in a simply connected domain Ω; and Γ denotes a contour in Ω.

The logic of the theorems on contour integration

(a) f is holomorphic in simply connected domain Ω

$\Downarrow$ Cauchy - goursat

(b) $\displaystyle\int_\Gamma f(z)\,dz = 0$ for every simple closed Γ

$\Downarrow$

(b′) $\displaystyle\int_\Gamma f(z)\,dz$ depends only on the start and end of Γ

$\Downarrow$ existence of primitive

(b″) there is a function F in Ω such that $\displaystyle\int_\Gamma f(z)\,dz = F(q) - F(p)$, where $F' = f$ and Γ goes from p to q

C.I.F

(c) $\displaystyle f(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\,dz$ for any simple closed Γ around w

$\Downarrow$ G.C.I.F

(d) $\displaystyle f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ in any open ball $B(z_0, r)$ in Ω.

Morera

power series

The logic of the theorems on contour integration

$$
\begin{array}{llll}
(a) & \implies & (b) & \text{(the Cauchy–Goursat theorem)} \\
(b) & \implies & (b') & \text{(independence of contour)} \\
(b') & \implies & (b'') & \text{(existence of primitives)} \\
(a) & \implies & (c) & \text{(Cauchy's integral formula)} \\
(c) & \implies & (d) & \text{(corollary to Cauchy's integral formula)} \\
(d) & \implies & (a) & \text{(power series)} \\
(b') & \implies & (a) & \text{(Morera's theorem} \\
(b'') & \implies & (a) & \text{(proof of Morera's theorem)}
\end{array}
$$

What is important in all this is . . .

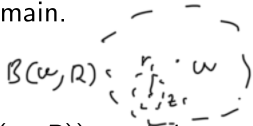
(a) for holomorphic functions in simply connected domains, integrals over contours with the same initial and final points are contour independent,

(b) to integrate a holomorphic function f along a contour from p to q, we may find a primitive F and compute $F(q) - F(p)$

(c) integrals of functions around closed contours where the function is not holomorphic at points inside the contour may be calculated in terms of the values of the function or its derivatives at these points.

Analytic continuation

We have already seen that the values of a holomorphic function f on an interval determine f in an open ball. The key to this is showing that if f is 0 on an interval, then f is 0 on the ball. Now we see that the values of a holomorphic function in an open subset of a domain determine the values in the whole domain.
[P]



Lemma

Suppose that $B(z_1, r_1) \subset B(w, R)$, that $f \in H(B(w, R))$, and that $f(z) = 0$ for all $z \in B(z_1, r_1)$. Then $f(z) = 0$ for all $z \in B(w, R)$.

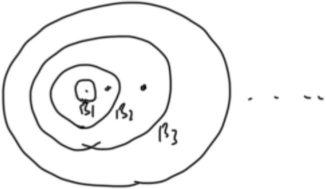
Proof. [P] We can find a finite sequence of open balls, $B(z_1, r_1)$, $B(z_2, r_2)$, ..., $B(z_J, r_J)$, say, with the properties that

$$B(z_1, r_1) \subset B(z_2, r_2) \subset \cdots \subset B(z_J, r_J) = B(w, R),$$

and the centre of $B(z_{j+1}, r_{j+1})$ is contained in $B(z_j, r_j)$.

Proof of lemma

We will show that $f(z) = 0$ for all $z \in B(z_J, r_J)$ by induction.
[P]
By hypothesis, $f(z) = 0$ for all $z \in B(z_1, r_1)$.
[P]
Suppose that $f(z) = 0$ for all $z \in B(z_j, r_j)$. Then $f(z) = 0$ for all z near the centre of $B(z_{j+1}, r_{j+1})$. Hence $f^{(n)}(z_{j+1}) = 0$ for all $n \in \mathbb{N}$. We may expand f in a power series in $B(z_{j+1}, r_{j+1})$, and the coefficients in the power series are multiples of the derivatives $f^{(n)}(z_{j+1})$. It follows that $f(z) = 0$ for all $z \in B(z_{j+1}, r_{j+1})$.
[P]
By induction, $f(z) = 0$ for all $z \in B(z_J, r_J)$. $\square$



A theorem on analytic continuation

Theorem

Suppose that Υ is a nonempty open subset of a domain Ω in $\mathbb{C}$, and that $f \in H(\Omega)$. If $f(z) = 0$ for all z in Υ, then $f(z) = 0$ for all z in Ω.

Proof[†]. Take a base point $b \in \Upsilon$. Since Ω is connected, any point $w \in \Omega$ may be joined to b by a polygonal contour Γ in Ω. Each point z on Γ is in the open set Ω, and so there is a ball $B(z, r)$ with centre z that is contained in Ω. Because Γ is compact, there are finitely many of these balls, $B(z_j, r_j)$ say, such that $B(z_j, r_j) \cap B(z_{j+1}, r_{j+1})$ is not empty, $b \in B(z_1, r_1)$ and $w \in B(z_J, r_J)$. Since $f(z) = 0$ for all z near b, the lemma implies that $f(z) = 0$ for all $z \in B(z_1, r_1)$. Now $f(z) = 0$ for all z in an open subset of $B(z_2, r_2)$, and the lemma implies that $f(z) = 0$ for all $z \in B(z_2, r_2)$. Continuing inductively, $f(z) = 0$ for all $z \in B(z_J, r_J)$ and hence $f(w) = 0$. $\square$

A corollary

Corollary

Suppose that Υ *is a nonempty open subset of a domain* Ω *in* $\mathbb{C}$*, and that* $f, g \in H(\Omega)$*. If* $f(z) = g(z)$ *for all* z *in* Υ*, then* $f(z) = g(z)$ *for all* z *in* Ω*.*

Proof. Apply the previous result to $f - g$. $\square$

[P]
Combined with results about zeroes being isolated, this means that if $f = g$ on a set containing a nonisolated point, such as an interval, then $f = g$ everywhere.

Exercise 1

Compute $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{5+3\cos\theta}\,d\theta.$

Answer. [P] Then

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos\theta}{5+3\cos\theta}\,d\theta &= \int_0^{2\pi} \frac{2\cos\theta}{10+6\cos\theta}\,d\theta \\
&= \int_0^{2\pi} \frac{e^{i\theta}+e^{-i\theta}}{10+3(e^{i\theta}+e^{-i\theta})}\,d\theta \\
&= \int_0^{2\pi} \frac{e^{2i\theta}+1}{3e^{2i\theta}+10e^{i\theta}+3}\,d\theta \\
&= \frac{1}{i}\int_0^{2\pi} \frac{e^{2i\theta}+1}{e^{i\theta}(3e^{2i\theta}+10e^{i\theta}+3)}\,ie^{i\theta}\,d\theta.
\end{aligned}$$

Answer to Exercise 1

Let $\gamma(\theta) = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Then

$$\begin{aligned}
&\int_0^{2\pi} \frac{\cos\theta}{5 + 3\cos\theta}\,d\theta \\
&\quad = \frac{1}{i}\int_\gamma \frac{z^2+1}{z(3z^2+10z+3)}\,dz \\
&\quad = \frac{1}{i}\int_\gamma \frac{1}{3z} - \frac{5}{4(3z+1)} + \frac{5}{12(z+3)}\,dz \\
&\quad = \frac{1}{3i}\int_\gamma \frac{1}{z}\,dz - \frac{5}{12i}\int_\gamma \frac{1}{z+1/3}\,dz + \frac{5}{12i}\int_\gamma \frac{1}{z+3}\,dz \\
&\quad = \frac{2\pi i}{3i} - \frac{10\pi i}{12i} + 0 = \frac{-\pi}{6},
\end{aligned}$$

by the Cauchy integral formula (first two integrals) and the Cauchy–Goursat theorem (last integral). $\triangle$

Exercise 2

Compute $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{\cos\theta}\, d\theta$.

Answer. [P] This is a trick question: the integral is not defined. $\triangle$

[P]

In all the following exercises, Γ is the circle of radius 3 and centre 0, traversed in the usual anticlockwise direction, and $f \in H(B(0,\pi))$.

Exercise 3

Compute $\displaystyle\int_\Gamma \frac{f(z)}{z^2-1}\,dz$.

Answer. [P] We may use partial fractions:

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{(1/2)}{z-1} + \frac{(-1/2)}{z+1},$$

[P] so

$$\begin{aligned}\int_\Gamma \frac{f(z)}{z^2-1}\,dz &= \frac{1}{2}\int_\Gamma \frac{f(z)}{z-1}\,dz - \frac{1}{2}\int_\Gamma \frac{f(z)}{z+1}\,dz \\ &= \frac{2\pi i}{2} f(1) - \frac{2\pi i}{2} f(-1) \\ &= \pi i(f(1) - f(-1)),\end{aligned}$$

by the Cauchy integral formula. △

Exercise 4

Compute $\displaystyle\int_\Gamma \frac{f(z)}{(z-1)^2}\,dz$.

Answer. [P] By the Cauchy integral formula for higher derivatives,

$$\int_\Gamma \frac{f(z)}{(z-1)^2}\,dz = 2\pi i[f'(1)]. \qquad \triangle$$

Exercise 5

Compute $\displaystyle\int_\Gamma \frac{f(z)}{(z^2-1)^2}\,dz$.

Answer. [P] We may use partial fractions again:

$$\frac{1}{(z-1)^2(z+1)^2} = \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{(z+1)^2} + \frac{D}{z+1},$$

[P] from which it follows that

$$A(z+1)^2 + B(z-1)(z+1)^2 + C(z-1)^2 + D(z-1)^2(z+1) = 1.$$

[P] Looking at the coefficients of z^3 tells us that $B + D = 0$. Putting $z = 1$ and then putting $z = -1$ tells us first that $4A = 1$ and then that $4C = 1$. Putting $z = 0$ tells us that $A - B + C + D = 1$. [P] It follows that

$$A = -B = C = D = \frac{1}{4}.$$

Answer to Exercise 5

We conclude that

$$\begin{aligned}\int_\Gamma \frac{f(z)}{(z^2-1)^2}\,dz &= \frac{1}{4}\Bigg[\int_\Gamma \frac{f(z)}{(z-1)^2}\,dz - \int_\Gamma \frac{f(z)}{z-1}\,dz \\ &\qquad + \int_\Gamma \frac{f(z)}{(z+1)^2}\,dz + \int_\Gamma \frac{f(z)}{z+1}\,dz\Bigg] \\ &= \frac{2\pi i}{4}\big[f'(1) - f(1) + f'(-1) + f(-1)\big] \\ &= \frac{\pi i}{2}\big[f'(1) - f(1) + f'(-1) + f(-1)\big],\end{aligned}$$

by Cauchy's (generalised) integral formula. △

Remarks

The exercises that we have done suggest that we will be able to compute "arbitrary" integrals around simple closed contours of functions of the form $f(z)/p(z)$, where p is a polynomial. To do this, we will need to be able to factorise p, and then expand $1/p$ into partial fractions, that is, a sum of terms of the form $1/(z-\alpha)^k$. We will need to be able to compute with partial fractions!