School of Mathematics and Statistics
MATH2111 - Higher Several Variable Calculus

5. Continuous Functions

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Elementary Functions

Definition

Let $\Omega \subseteq \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We say that f is **elementary** if it

- is a constant function; or
- maps $\mathbf{x}$ to any of x_i, $\cos x_i$, $\sin x_i$, $\exp x_i$; or
- is an inverse of an elementary function, like $\ln(x_i)$; or
- is a sum or product of elementary functions; or

finite

- is a composition of elementary functions.

Example. The following are elementary functions:

- $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = xy - z^3$;
- $g : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$, $g(x, y) = \sin(x^2 - 3y) + \ln(x^4 + y^6)$;
- $h : (0, \infty)^3 \to \mathbb{R}$, $h(x, y, z) = x^y + y^z + z^x$; $\quad x^y = e^{y \ln x}$
- $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $F(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$.

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Elementary Functions

Theorem

If $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is an elementary function, then f is continuous on Ω.

Example. Define $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ by

$$f_1(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 1 & \text{if } (x,y) = (0,0), \end{cases}$$

$$f_2(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

and define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = \big(f_1(x,y), f_2(x,y)\big)$.

Firstly, f_1 and f_2 are elementary, and hence continuous, functions on $\mathbb{R}^2 \backslash \{(0,0)\}$. We showed last lecture that $\lim\limits_{(x,y)\to(0,0)} f_2(x,y) = 0$, and so f_2 is continuous on $\mathbb{R}^2$.

Since the components of f are continuous on $\mathbb{R}^2$, we have that f is continuous on $\mathbb{R}^2$.

Images and Preimages

Recall that for a function $f : X \to Y$, $A \subseteq X$ and $B \subseteq Y$, the **image of A under f** is

$$f(A) = \{f(x) \in Y : x \in A\},$$

and the **preimage of B under f** is

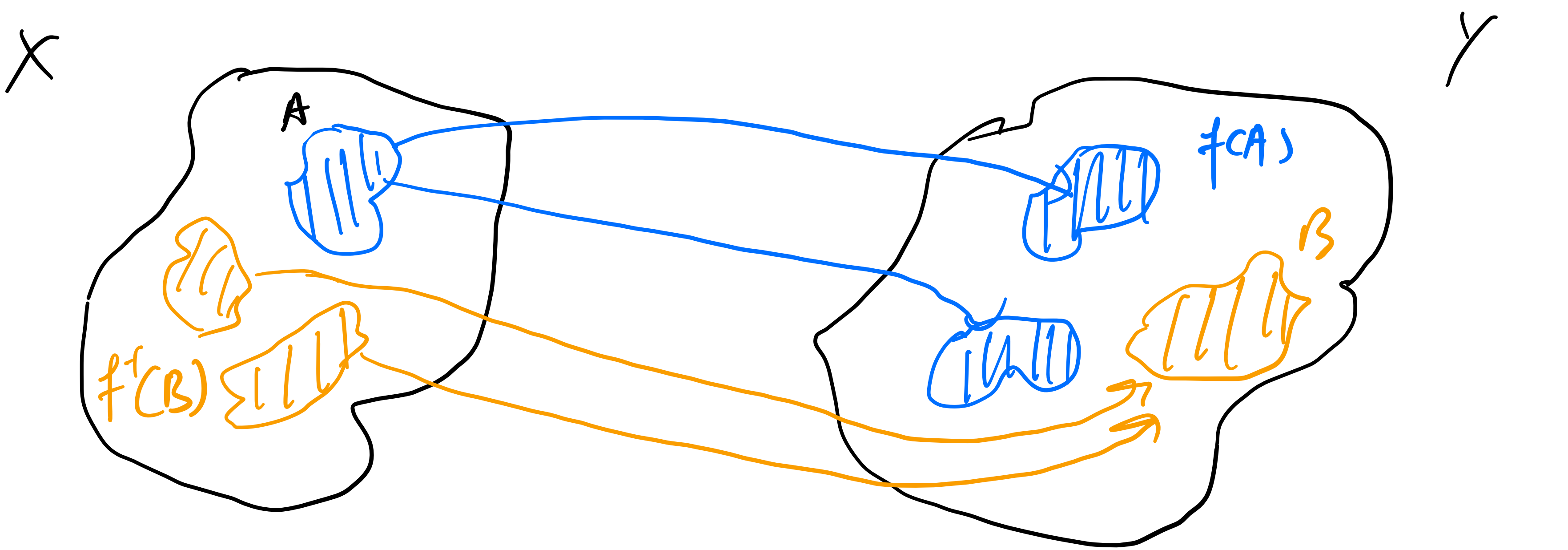
$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

$f(x) = 1$

$f([0,1]) = \{1\}$

but $f^{-1}(\{\}) = \mathbb{R}$

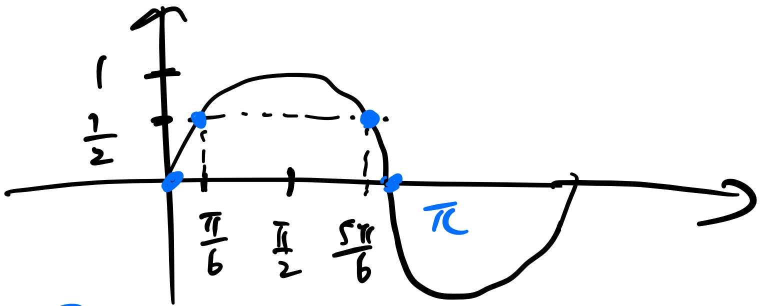
That is, $f(A)$ is the set of all outputs when the inputs are restricted to A, and $f^{-1}(B)$ is the set of all inputs that output to B. Consequently, $A \subseteq f^{-1}(f(A))$.



Images and Preimages

Example. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$. Then

- $f([0,\pi]) = [0,1]$
- $f(\{\pi, 2\pi, 3\pi\}) = \{0\}$
- $f^{-1}(\{0\}) = \{n\pi : n \in \mathbb{Z}\}$
- $f^{-1}([-2,-1]) = \emptyset$
- $f^{-1}([-1,1]) = \mathbb{R}$
- $f^{-1}([\frac{1}{2},1]) = \bigcup_{k \in \mathbb{Z}} [\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi]$



Continuity and Preimages

Theorem

Suppose that $\Omega \subseteq \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^m$. Then f is continuous if and only if $f^{-1}(U)$ is open for every open $U \subseteq \mathbb{R}^m$.

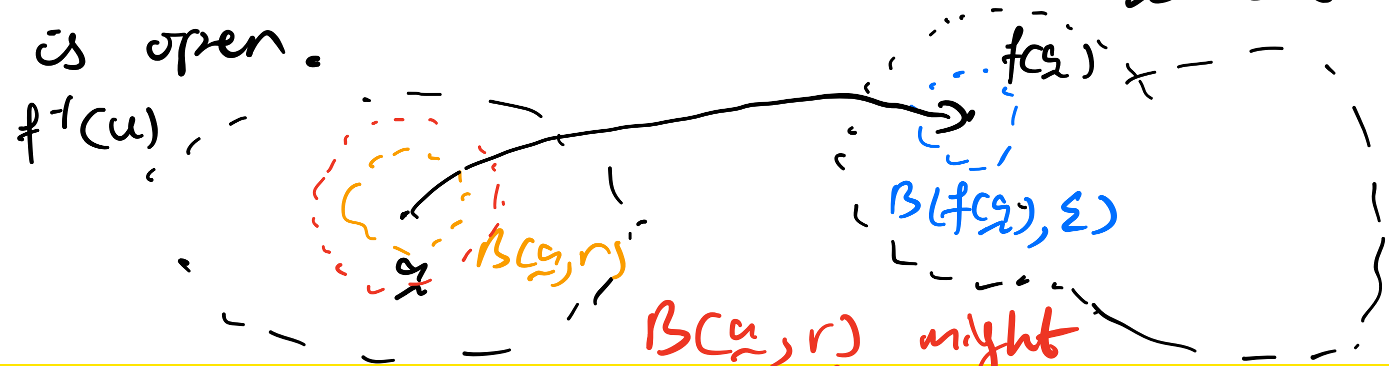
This does not work for images. Consider a constant function.

Proof. $(\Leftarrow)$ Suppose $f^{-1}(U)$ is open for every open $U \in \mathbb{R}^m$. Let $\varepsilon > 0$ and $\underset{\sim}{a} \in \Omega$. Since $B(f(\underset{\sim}{a}), \varepsilon)$ is open, then $V = f^{-1}(B(f(\underset{\sim}{a}), \varepsilon))$ is open in $\mathbb{R}^n$. As $\underset{\sim}{a} \in V$ and V is open there is $\delta > 0$ s.t. $B(\underset{\sim}{a}, \delta) \subseteq V$. Then for any $\underset{\sim}{x} \in B(\underset{\sim}{a}, \delta)$, we have that $f(\underset{\sim}{x}) \in B(f(\underset{\sim}{a}), \varepsilon)$. ie, $d(f(\underset{\sim}{x}), f(\underset{\sim}{a})) < \varepsilon$, and so f

Continuity and Preimages

Proof cont.

Conversely suppose that f is continuous on Ω and let $U \subseteq \mathbb{R}^n$ be open. If $U = \emptyset$, then $f^{-1}(U) = \emptyset$ which is open. Otherwise, let $\underset{\sim}{a} \in f^{-1}(U)$. As U and Ω are open, we can find $r, \varepsilon > 0$ s.t. $B(a, r) \subseteq \Omega$ and $B(f(a), \varepsilon) \subseteq U$. As f is continuous, there is $\delta > 0$ s.t. $f(x) \in B(f(a), \varepsilon) \subseteq U$ for all $\underset{\sim}{x} \in B(\underset{\sim}{a}, \delta)$. That is, $B(\underset{\sim}{a}, \delta) \subseteq f^{-1}(U)$, so $\underset{\sim}{a}$ is an interior point of $f^{-1}(U)$. As $\underset{\sim}{a}$ was arbitrary, $f^{-1}(U)$ is open.



$B(\underset{\sim}{a}, r)$ might go outside $f^{-1}(U)$ hence picking $\delta > r$.

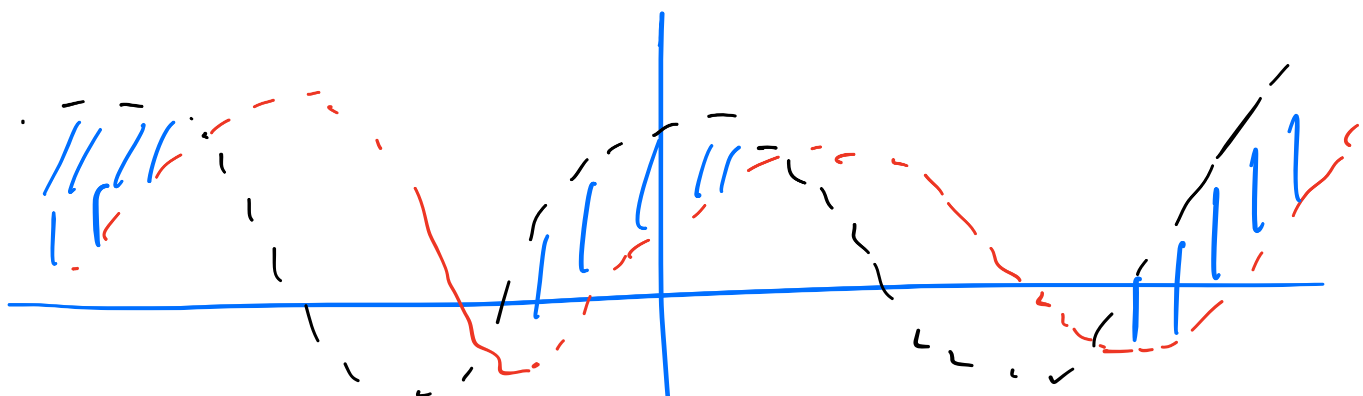
Continuity and Preimages

Corollary

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$. Then f is continuous if and only if $f^{-1}(\Omega)$ is a closed subset of $\mathbb{R}^n$ for every closed $\Omega \subseteq \mathbb{R}^m$.

TIP! The previous theorem and it's corollary are useful for proving that sets are open or closed.

Exercise. Show that the set $\Omega = \{(x, y) \in \mathbb{R}^2 : \sin x < y < \cos x\}$ is open.



$f(x,y) = \sin x - y$

$\Omega_1 = f^{-1}((-\infty, 0))$

open

$g(x,y) = \cos x - y$

$\Omega_2 = g^{-1}((0, \infty))$

open

$\Omega = \Omega_1 \cup \Omega_2$ is open

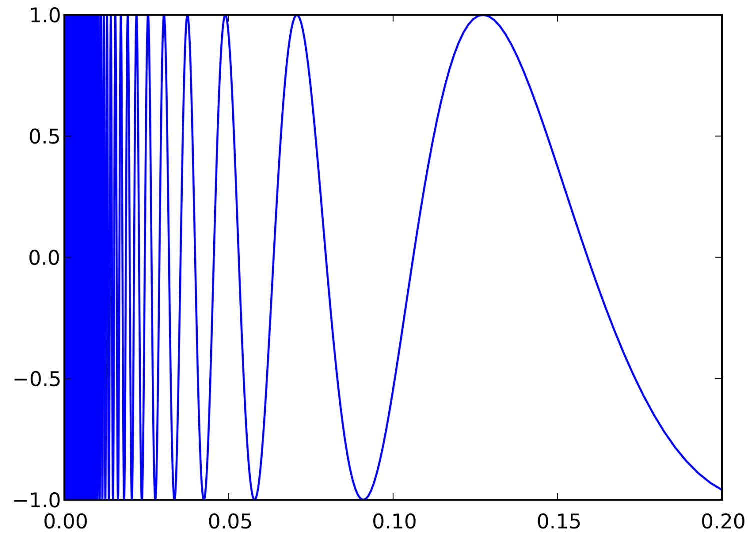
Path-connected Sets

Definition

A set $\Omega \subseteq \mathbb{R}^n$ is **path-connected** if for every $\mathbf{x}, \mathbf{y} \in \Omega$, there is a continuous function $\varphi : [0,1] \to \Omega$ such that $\varphi(0) = \mathbf{x}$ and $\varphi(1) = \mathbf{y}$.

Examples.

- The path-connected subsets of $\mathbb{R}$ are the empty set, singleton sets, intervals, and $\mathbb{R}$.
- The unit circle $\mathbb{T}$ and $\{(x, y) : xy \geq 0\}$ are path-connected subsets of $\mathbb{R}^2$.
- The Topologist's Sine Curve $\{(x, \sin \frac{1}{x}) : x \in (0, \infty)\} \cup \{(0,0)\}$ is NOT path-connected.



Path-connected Sets

Proposition

If Ω_1, Ω_2 are path-connected with $\Omega_1 \cap \Omega_2 \neq \varnothing$, then $\Omega_1 \cup \Omega_2$ is path-connected.

Proof.

Generalising the IVT

Intermediate Value Theorem (IVT)

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then for any z between $f(a)$ and $f(b)$, there is a $c \in [a, b]$ such that $f(c) = z$.

The following theorem generalises the IVT.

Theorem

Suppose that $\Omega \subseteq \mathbb{R}^n$ is path-connected and $f : \Omega \to \mathbb{R}^m$. Then $f(\Omega)$ is path-connected.

Continuous functions preserve path-connectedness.

Generalising the IVT

Proof.

Path-Connected Example

Example. Show that $\Omega = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$ is path-connected.

Option 1: for every $\mathbf{a}, \mathbf{b} \in \Omega$, find a continuous function $\varphi : [0, 1] \to \Omega$ with $\varphi(0) = \mathbf{a}$ and $\varphi(1) = \mathbf{b}$.

Option 2: find a continuous function $f : D \to \mathbb{R}^2$ where D is path-connected and $f(D) = \Omega$.

Consider the **polar map** $P : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$P(r, \theta) = (r \cos\theta, r \sin\theta).$$

We observe that P is continuous since its components are elementary functions and, if we set $D = (1, 2) \times [0, 2\pi)$, then $P(D) = \Omega$. Clearly D is path-connected, and so Ω is also path-connected.

Note that the polar map is not invertible. Even if we restrict the domain to $\mathbb{R}_{>0} \times [0, 2\pi)$ so that it is invertible, the inverse is not continuous!

Bounded Sets and Compact Sets

Definition

Let Ω be a subset of $\mathbb{R}^n$. We say that Ω **is bounded** if there is an $M \in \mathbb{R}$ such that $d(\mathbf{x}, \mathbf{0}) \leqslant M$ for all $\mathbf{x} \in \Omega$.

We also say that Ω is **compact** if it is closed and bounded.

Examples.

- $\varnothing$ is compact but $\mathbb{R}^n$ is not;
- $(0, 1)$, $(0, 1]$ and $[0, 1)$ are bounded but not compact, but $[0, 1]$ is compact;
- The union of two bounded sets is bounded, hence the union of two compact sets is compact;
- The unit circle, sphere and (closed) disc are compact;
- A convergent sequence is bounded. The set $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is bounded, but $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\}$ is compact.

Monotone Subsequences of Real Numbers

Recall that a sequence of real numbers $\{a_k\}_{k=1}^{\infty}$ is said to be **monotonic** if it is non-decreasing ($a_{k+1} \geqslant a_k$ for all $k \in \mathbb{Z}^+$) or if it is non-increasing ($a_{k+1} \leqslant a_k$ for all $k \in \mathbb{Z}^+$).

Lemma

Suppose that $\{a_k\}_{k=1}^{\infty}$ is a sequence of real numbers. Then there is a subsequence $\{a_{k_j}\}_{j=1}^{\infty}$ that is monotonic.

Monotone Subsequences of Real Numbers

Proof.

Bolzano-Weierstrass Theorem

- A monotone bounded sequence in $\mathbb{R}$ converges to its supremum (least upper bound) if it is non-decreasing or to its infimum (greatest lower bound) if it is non-increasing.
- Every bounded sequence in $\mathbb{R}$ has a convergent subsequence by the previous lemma.
- By taking subsequences of subsequences (finitely many times), every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Bolzano-Weierstrass Theorem

A set $\Omega \subseteq \mathbb{R}^n$ is compact if and only if every sequence in Ω has a convergent subsequence whose limit is in Ω.

The above reasoning and Ω being closed proves the forward direction. The backward direction follows since if Ω is unbounded, we can construct a sequence that is not Cauchy (and hence not convergent), or if Ω is not closed, then it does not contain one of its limit points, say $\mathbf{x}$, but there will be a sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ in Ω converging to $\mathbf{x}$.

Continuous Functions Preserve Compactness

Theorem

Suppose that $K \subseteq \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}^m$ is continuous. Then $f(K)$ is compact.

Proof.

Applications

Corollary (Extreme Value Theorem)

Suppose that $K \subseteq \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}$ is continuous. Then f attains a maximum and a minimum on K. That is, there are $\mathbf{a}, \mathbf{b} \in K$ such that for all $\mathbf{x} \in K$,

$$f(\mathbf{a}) \leqslant f(\mathbf{x}) \leqslant f(\mathbf{b}).$$

Let $S_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$ (compact) and $S_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ (not compact). Is there a continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

1. $f(S_1) = S_2$?
2. $f(S_2) = S_1$?

Answers:

1.
2.