MATH2621 — Higher Complex Analysis. XVIII
Cauchy's generalised integral formula

This lecture?

In this lecture, we establish various consequences of Cauchy's integral formula. These include both theoretical results and explicit computations.

Cauchy's integral formula

$$f(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-w}\, dz,$$

where w lies inside a simple closed contour Γ in a simply connected domain Ω, and $f \in H(\Omega)$,

[P]

This formula is perhaps the most important formula in complex analysis.

Cauchy's generalised integral formula

Corollary

Suppose that $f \in H(B(z_0, R))$, and that Γ is a simple closed contour in $B(z_0, R)$ such that $z_0 \in \mathrm{Int}(\Gamma)$. Then

$$f(w) = \sum_{n=0}^{\infty} c_n (w - z_0)^n \qquad \forall w \in B(z_0, R),$$

where

$$c_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz.$$

The radius of convergence of the power series is at least R.

Cauchy's generalised integral formula

Note that this corollary, combined with the fact that $f^{(n)}(z_0) = n!\, c_n$, implies that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

This is Cauchy's generalised integral formula.

[P]

Notice that we just assumed that f is differentiable once; the corollary implies that f is actually infinitely differentiable.

let $g(z) = \frac{f(z)}{(z-z_0)^{n+1}}$ by residue theorem

$$\mathrm{Res}(g, z_0) = \frac{1}{n!} \lim_{z \to z_0} \frac{d^n}{dz^n} \underbrace{(z-z_0)^{n+1} g(z)}_{f(z)}$$

$$\Rightarrow \frac{1}{2\pi i} \int_\Gamma g(z)\, dz = \mathrm{Res}(g, z_0) = \frac{1}{n!} f^{(n)}(z_0)$$

$$\Leftrightarrow f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

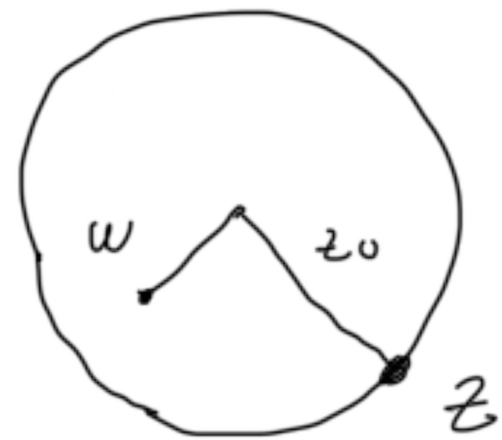
Proof of Cauchy's integral formula

Proof. Write Γ_r for the circle with centre z_0 and radius r, where $r < R$. By independence of contour, encloses same singularity z_0

$$\int_{\Gamma} \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \int_{\Gamma_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz,$$

so we may assume that $\Gamma = \Gamma_r$. By the Cauchy integral formula, if $w \in B(z_0, r)$, then

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(z)}{z-w}\,dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(z)}{(z-z_0)-(w-z_0)}\,dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(z)}{(z-z_0)}\,\frac{1}{(1-(w-z_0)/(z-z_0))}\,dz. \end{aligned}$$



Proof of Cauchy's integral formula

Observe that $|w - z_0| < |z - z_0| = r$ for all $z \in \Gamma_r$, so

$$\frac{1}{1 - (w - z_0)/(z - z_0)} = \sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^n},$$

and, for fixed w, this series converges uniformly for $z \in \Gamma_r$.
[P]
Now look at the last formula on the previous page . . .

Let $S = \sum_{k=0}^{n} q^k$ $\qquad qS = \sum_{k=0}^{n} q^{k+1}$

$S(q-1) = q^{n+1} - 1$ $\qquad S = \frac{q^{n+1} - 1}{q - 1}$

$|q| < 1 \implies \lim_{n \to \infty} S = \frac{1}{1 - q}$

Proof of Cauchy's integral formula

This means that

$$
\begin{aligned}
f(w) &= \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{(z - z_0)} \sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^n}\, dz \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\Gamma_r} \frac{f(z)}{(z - z_0)} \frac{(w - z_0)^n}{(z - z_0)^n}\, dz \\
&= \sum_{n=0}^{\infty} c_n (w - z_0)^n,
\end{aligned}
$$

[P] where

$$
c_n = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}}\, dz,
$$

by independence of contour. Here we have exchanged the order of summation and integration.

Proof of Cauchy's integral formula

Once we know that f has this power series representation, it follows that $c_n = n!\, f^{(n)}(z_0)$ from results on power series.
[P]
We chose r and w such that $|w - z_0| < r < R$. If we take any $w \in B(z_0, R)$, then we may choose r such that these inequalities hold, so the series converges at w. Since w is an arbitrary element of $B(z_0, R)$, the series converges in $B(z_0, R)$, and the radius of convergence is at least R. $\square$

Exercise 1

Compute $\displaystyle\int_\Gamma \frac{e^z}{z^{n+1}}\,dz$, where Γ is the circle with centre 0 and radius 1.

Answer. [P] Take $f(z) = e^z$ and $w = 0$, and apply Cauchy's generalised integral formula:

$$\int_\Gamma \frac{e^z}{z^{n+1}}\,dz = \int_\Gamma \frac{f(z)}{(z-w)^{n+1}}\,dz = \frac{2\pi i}{n!}\, f^{(n)}(0) = \frac{2\pi i}{n!},$$

since the derivative, and hence also the nth derivative, of the exponential function is the exponential function itself, and $e^0 = 1$. $\triangle$

Liouville's theorem

Theorem

Suppose that f is a bounded entire function. Then f is constant.

Proof. Since f is bounded, we may choose a positive constant C such that $|f(z)| \leq C$ for all $z \in \mathbb{C}$.
[P]
Since f is entire, we may take Γ_R to be the circle centre 0 and radius R, and use Cauchy's generalised integral formula to find the power series for f inside Γ_R:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \tag{1}$$

where

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{\Gamma_R} \frac{f(z)}{z^{n+1}} \, dz.$$

Proof of Liouville's theorem

The power series (1) converges inside Γ_R, so its radius of convergence ρ is at least R; as R is arbitrarily large, $\rho = \infty$.
[P]
Further, when $|z| = R$,

$$\left| \frac{f(z)}{z^{n+1}} \right| \leq \frac{C}{R^{n+1}},$$

[P] and so, by the ML Lemma,

$$\left| f^{(n)}(0) \right| = \frac{n!}{2\pi} \left| \int_{\Gamma_R} \frac{f(z)}{z^{n+1}}\, dz \right| \leq \frac{n!}{2\pi} \frac{C}{R^{n+1}}\, 2\pi R = \frac{n!\, C}{R^n}\,.$$

[P] If $n \geq 1$, then the left hand side of the formula above is 0, since R may be made arbitrarily large, and so $f^{(n)}(0) = 0$. Thus the power series (1) simplifies to show that $f(z) = f(0)$. $\square$

The fundamental theorem of algebra

Corollary

Suppose that f is a nonconstant complex polynomial. Then f has at least one root, and hence f may be factorised as a product of a constant and finitely many linear factors.

Sketch. Suppose that f has no root.

[P]

First, $f(z) \to \infty$ as $z \to \infty$, and so there exists R such that $|f(z)| \geq 1$ when $|z| \geq R$. Next, in the compact set $\overline{B}(0, R)$, the function $|f|$ is continuous and takes positive values, so it has a minimum value, m say, which cannot be 0 as f has no root. Thus $|f(z)| \geq m$ when $|z| \leq R$. by EVT

Proof of the fundamental theorem of algebra

It follows that the function $1/f$ is bounded and entire, so $1/f$ is constant, and f is constant. Since f is not constant by hypothesis, f must have a root.

[P]

The complete factorisation of a polynomial follows by dividing out a factor of $z - r$ for each root r. If the quotient is a nonconstant polynomial, we can find another root, and keep on dividing out and finding more roots until the quotient is constant and we have a complete factorisation. $\square$

A holomorphic function near a zero

A zero of a function f that is holomorphic in an open set Ω is a point $w \in \Omega$ such that $f(w) = 0$.
[P]
For such a point w, Cauchy's generalised integral formula implies that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-w)^n \qquad \forall z \in B(w,r),$$

for some open ball $B(w,r)$. [P] Note that $a_0 = f(w) = 0$.
[P]
If all a_n are 0, then $f(z) = 0$ for all $z \in B(w,r)$. Otherwise, we define $N = \min\{n \in \mathbb{N} : a_n \neq 0\}$; then $a_n = 0$ when $n < N$ and $a_N \neq 0$.
[P]
One idea that we will use in several ways in this course is that $f(z)$ behaves like $a_N(z-w)^N$ near to w.

A holomorphic function near a zero. 2

Proposition

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z-w)^n$ *for all* $z \in B(w,r)$, *and that* $a_n \neq 0$ *for some* $n \in \mathbb{N}$. *Let* $N = \min\{n \in \mathbb{N} : a_n \neq 0\}$. *Then*

$$\lim_{z \to w} \frac{f(z)}{a_N(z-w)^N} = 1.$$

Proof[†]. We show that, given $\varepsilon \in \mathbb{R}^+$, there exists $r_\varepsilon \in \mathbb{R}^+$ such that

$$\left| \sum_{n=N+1}^{\infty} a_n(z-w)^n \right| < \varepsilon \left| a_N(z-w)^N \right| \qquad \forall z \in B(w, r_\varepsilon). \quad (2)$$

A proof †

Since $a_n = 0$ when $n < N$, it follows that

$$\frac{f(z)}{a_N(z-w)^N} = \frac{\sum_{n=0}^{\infty} a_n(z-w)^n}{a_N(z-w)^N} = 1 + \frac{\sum_{n=N+1}^{\infty} a_n(z-w)^n}{a_N(z-w)^N},$$

and the lemma follows.

A proof †

Now the argument involves power series. Take $z_0 \in B^\circ(w, r)$ such that $|z_0 - r|$ is close to r. Since the series $\sum_{n=0}^{\infty} a_n(z_0 - w)^n$ converges, there is a constant C such that $|a_n(z_0 - w)^n| \leq C$ for all $n \in \mathbb{N}$, so $|a_n| \leq C|z_0 - w|^{-n}$. It follows that, when $|z - w| < |z_0 - w|$,

$$\begin{aligned}\sum_{n=N+1}^{\infty} |a_n| \, |z - w|^n &\leq \sum_{n=N+1} C \frac{|z - w|^n}{|z_0 - w|^n} \\ &= C \frac{|z - w|^N}{|z_0 - w|^N} \frac{|z - w|}{|z_0 - w| - |z - w|}.\end{aligned}$$

A proof †

By taking r_ε small enough, we can ensure that

$$C\,\frac{1}{\left|z_0 - w\right|^N}\,\frac{r_\varepsilon}{\left|z_0 - w\right| - r_\varepsilon} < \varepsilon\left|a_N\right|,$$

and then when $z \in B(w, r_\varepsilon)$, it follows that

$$C\,\frac{\left|z - w\right|^N}{\left|z_0 - w\right|^N}\,\frac{\left|z - w\right|}{\left|z_0 - w\right| - \left|z - w\right|} < \varepsilon\left|a_N(z - w)^N\right|,$$

and then (2) holds. $\square$

[P]
Later we will use this fact to prove l'Hôpital's rule.

A corollary

Corollary

Suppose that Ω is an open set, that $f \in H(\Omega)$, and that $f(w) = 0$ for some $w \in \Omega$. Then there exists $r \in \mathbb{R}^+$ such that either $f(z) = 0$ for all $z \in B(w, r)$ or $f(z) \neq 0$ for all $z \in B^\circ(w, r)$.

Proof. [P] Write $f(z) = \sum_{n=0}^{\infty} a_n(z - w)^n$ for all $z \in B(w, r)$, and suppose that $a_n \neq 0$ for some $n \in \mathbb{N}$. Then $f^{(n)}(w) \neq 0$, and so f is not identically equal to 0 near w. Take N as in the proposition. [P] Then there exists $r \in \mathbb{R}^+$ such that

$$\left| \frac{f(z)}{a_N(z - w)^N} - 1 \right| < \frac{1}{2} \qquad \forall z \in B^\circ(w, r),$$

and for these z it follows that $f(z) \neq 0$. □

[P] In summary, if a holomorphic function f is not constant, then the zeroes of f are *isolated*.

Exercise 2

Compute $\displaystyle\int_0^{2\pi} \frac{4}{5+3\cos(\theta)}\,d\theta.$

Answer. [P] Take $\gamma(\theta) = e^{i\theta}$, where $0 \le \theta \le 2\pi$. [P] Then

$$\begin{aligned}
\int_0^{2\pi} \frac{4}{5+3\cos(\theta)}\,d\theta &= \int_0^{2\pi} \frac{8}{10+3e^{i\theta}+3e^{-i\theta}}\,d\theta \\
&= \int_0^{2\pi} \frac{8e^{i\theta}}{3(e^{i\theta})^2+10e^{i\theta}+3}\,d\theta \\
&= \frac{1}{i}\int_0^{2\pi} \frac{8}{3(e^{i\theta})^2+10e^{i\theta}+3}\,ie^{i\theta}\,d\theta \\
&= \frac{1}{i}\int_0^{2\pi} \frac{8}{3(\gamma(\theta))^2+10\gamma(\theta)+3}\,\gamma'(\theta)\,d\theta \\
&= \frac{1}{i}\int_\gamma \frac{8}{3z^2+10z+3}\,dz.
\end{aligned}$$

Answer to Exercise 2

Clearly $3z^2+10z+3=(3z+1)(z+3)$, and it is easy to check that

$$\frac{8}{3z^2+10z+3}=\frac{3}{3z+1}-\frac{1}{z+3}.$$

[P] Thus

$$\begin{aligned}\int_0^{2\pi}\frac{4}{5+3\cos(\theta)}\,d\theta &= \frac{1}{i}\int_\gamma \frac{3}{3z+1}-\frac{1}{z+3}\,dz\\ &= \frac{1}{i}\int_\gamma \frac{1}{z+1/3}\,dz-\frac{1}{i}\int_\gamma\frac{1}{z+3}\,dz\\ &= \frac{1}{i}\int_\gamma \frac{1}{z+1/3}\,dz,\end{aligned}$$

because $z\mapsto 1/(z+3)$ is holomorphic in $B(0,3)$ and $\mathrm{Range}(\gamma)\cup\mathrm{Int}(\gamma)\subset B(0,3)$, so the second integral is zero by the Cauchy–Goursat theorem.

Answer to Exercise 2 (continued)

Finally, we apply Cauchy's integral formula, where $f(z) = 1$ for all $z \in \mathbb{C}$ and $w = -1/3$, which lies inside the curve γ. It follows that

$$\int_0^{2\pi} \frac{4}{5 + 3\cos(\theta)}\, d\theta = \frac{2\pi i}{i} f(-1/3)$$
$$= 2\pi. \qquad \triangle$$

[P]
This integral may also be computed using the substitution $t = \tan(\theta/2)$.