MATH2621 — Higher Complex Analysis. XX
Taylor series

This lecture?

In this lecture, we recall some facts about power series, Cauchy's integral formula, and Taylor series. We discuss computation with and manipulation of Taylor series.

Definitions

A *power series* (with centre z_0) is an expression of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the *coefficients* a_n, the *centre* z_0, and the *variable* z are complex. A *Taylor series* (with centre z_0) for a function f is a series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

A *Maclaurin series* for a function f is a Taylor series for f with centre 0, that is, a series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

Maclaurin and Taylor series are particular kinds of power series.

Radius of convergence

Recall from the lecture on power series that a power series has a *radius of convergence*, ρ, which can often be found using the ratio test or the root test. The power series converges inside $B(z_0, \rho)$ and fails to converge outside $\overline{B}(z_0, \rho)$; if a power series converges in $B(z_0, r)$, then $r \leq \rho$, but it is possible that $r < \rho$.

More on convergence

We say (or write) that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{in } B(z_0, r)$$

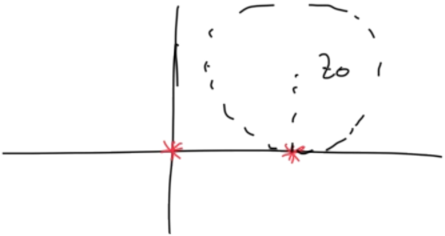
to mean that the domain of f includes $B(z_0, r)$, the power series converges in $B(z_0, r)$ and that $f(z)$ is the sum of the power series for each $z \in B(z_0, r)$. If this holds, then $f \in H(B(z_0, r))$, and moreover

$$a_n = \frac{f^{(n)}(z_0)}{n!},$$

that is, this power series is the Taylor series for f with centre z_0.
[P]
Conversely, from the lecture on Cauchy's generalised integral formula, if $f \in H(B(z_0, r))$, then f can be represented as a power series in the same ball.

Another way to find the radius of convergence

When we ask how large a ball with centre z_0 in which a function f is represented by a power series can be, we often find that the maximum value of the radius r of the ball is equal to the distance of z_0 from the set of points where f fails to be holomorphic.



Exercise 1

Show that the function f, given by $f(z) = 1/z$, can be represented as a power series in a ball $B(z_0, r)$, where $z_0 \neq 0$. Find the radius of convergence of this power series.

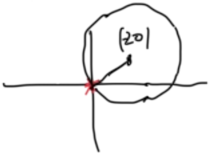
$$\frac{1}{1-\left(-\frac{z-z_0}{z_0}\right)} = \sum (-1)^n \left(\frac{z-z_0}{z_0}\right)^n$$

$$\left|\frac{z-z_0}{z_0}\right| < 1 \iff |z-z_0| < z_0$$

radius of convergence

Answer. [P]

$$\begin{aligned} f(z) &= \frac{1}{z} \\ &= \frac{1}{z_0 + z - z_0} = \frac{1}{z_0(1 + (z - z_0)/z_0)} \\ &= \frac{1}{z_0} \sum_{n=0}^{\infty} (-1)^n \frac{(z-z_0)^n}{z_0^n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}} (z-z_0)^n. \end{aligned}$$



By the ratio test, the radius of convergence is $|z_0|$. △

Comment on Exercise 1

The function f fails to be holomorphic at 0, which is on the edge of the ball $B(z_0, |z_0|)$. We say that "the *singularity* of f at 0 is a barrier to the convergence of the power series in any larger ball with centre z_0".

The algebra and calculus of power series

To determine a Taylor series, we need to be able to find *all* the derivatives of a function. There are only a few basic examples, and variations on these, for which finding all the derivatives is possible. Geometric series and exponential series are particularly important.
[P]
Putting together these basic series to find more complicated series is our next topic.
[P]
We have seen in Lecture 10 (Theorem 10.10 and 10.11 in the notes) that we can multiply by a scalar, add and multiply power series and recover easily the coefficients of the new power series. Moreover, one can differentiate a power series which corresponds to differentiate term by term. We will often use those facts for finding Taylor series.

Other operations on power series

It is sometimes possible to divide power series, since the quotient of holomorphic functions is homomorphic, at least when the denominator does not vanish, and holomorphic functions may be represented as power series. [P] It is also sometimes possible to compose power series, since the composition of holomorphic functions is homomorphic. [P] Rather than try to state theorems about these operations, we will look at some examples.

Exercise 2

Consider the function Log. Determine its Taylor series with centre $i-1$. What is the radius of convergence ρ of this series. Does the series represent Log in *all* the ball $B(i-1,\rho)$?

Answer. [P] We compute the derivatives of Log: as long as z is not a nonpositive real number, then

$$\mathrm{Log}^{(0)}(z) = \mathrm{Log}(z), \qquad \mathrm{Log}^{(1)}(z) = \frac{1}{z}\,, \qquad \mathrm{Log}^{(2)}(z) = \frac{-1}{z^2}\,,$$
$$\mathrm{Log}^{(3)}(z) = \frac{2}{z^3}\,, \qquad \mathrm{Log}^{(4)}(z) = \frac{-6}{z^4}\,, \qquad \mathrm{Log}^{(5)}(z) = \frac{24}{z^5}\,,$$

and we may guess (and verify by induction if necessary) that

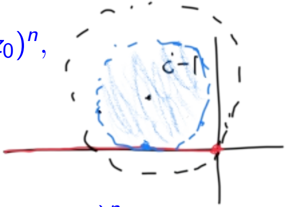
$$\mathrm{Log}^{(n)}(z) = \frac{(-1)^{n-1}(n-1)!}{z^n},$$

when $n > 0$.

Answer to Exercise 2

We deduce that the Taylor series for Log around z_0 is

$$\begin{aligned}\mathrm{Log}(z) &= \mathrm{Log}(z_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!\, z_0^n}(z-z_0)^n \\ &= \mathrm{Log}(z_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n z_0^n}(z-z_0)^n,\end{aligned}$$



[P] and hence, when $z_0 = i-1$, we get

$$\begin{aligned}\mathrm{Log}(z) &= \mathrm{Log}(i-1) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(i-1)^n}(z+1-i)^n \\ &= \frac{\ln 2}{2} + \frac{3}{4}i\pi + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(i-1)^n}(z+1-i)^n.\end{aligned}$$

Using the ratio test we obtain that the radius of convergence is equal to $|z_0| = |i-1| = \sqrt{2}$.

Answer to Exercise 2 (continued)

The Taylor series does not represent Log on the other side of the branch cut along the negative real axis. However, it does represent the branch of log where the imaginary part of $\log(w)$ lies in $[0, 2\pi)$. $\triangle$

Exercise 3

Define the function f by $f(z) = \dfrac{\sin(z)}{z}$ if $z \neq 0$ and $f(0) = 1$.

Does this function have a Maclaurin series? If so, then what is its radius of convergence?

$$f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

Answer. [P] By definition, for all $z \in \mathbb{C}$,

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.$$

[P] The series converges for all $z \in \mathbb{C}$. Thus, when $z \neq 0$,

$$\frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \underset{z \neq 0}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}.$$

[P] When $z = 0$, the power series on the right hand side of this last expression is equal to 1.

Answer to Exercise 3

It follows that

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}$$

for *all* $z \in \mathbb{C}$.

[P]

This power series converges for all z in $\mathbb{C}$. Thus the radius of convergence of the Maclaurin series for $f(z)$ is infinite. $\triangle$

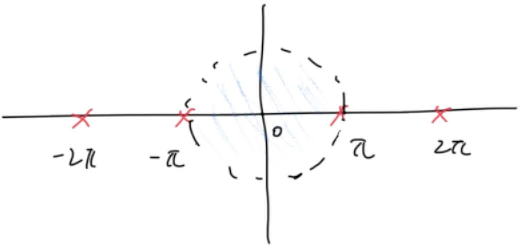
Remarks

It follows that $f^{(2n)}(0) = \dfrac{(-1)^n}{(2n+1)}$ and $f^{(2n+1)}(0) = 0$ for all $n \in \mathbb{N}$.
[P]
It is also possible to compute the derivatives of the function $z \mapsto \sin(z)/z$ "by hand", and to find a formula for these using induction, but this is much longer.
[P]
When we discuss singularities in more detail, we will say that "the function $z \mapsto \sin(z)/z$ has a removable singularity at 0".

Exercise 4

Define the function g by $g(z) = \dfrac{z}{\sin(z)}$ when $z \neq 0$ and $g(0) = 1$. Does this function have a Maclaurin series? If so, then what is its radius of convergence?

Answer. [P] The function g is $1/f$, where f is as in the previous exercise. Since f is entire, g is holomorphic where f does not vanish, that is, in $\mathbb{C} \setminus \{\pm\pi, \pm 2\pi, \pm 3\pi, \dots\}$. So g has a Maclaurin series in $B(0, \pi)$, with radius of convergence π. △



Answer to Exercise 4

To determine the series, we may argue in different ways. Note that f is even, so g is even. Assuming that $g(z) = \sum_{n=0}^{\infty} b_{2n} z^{2n}$, we may write

$\frac{\sin z}{z}$ $\qquad$ $\frac{z}{\sin z}$

$$\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots\right)\left(b_0 + b_2 z^2 + b_4 z^4 + b_6 z^6 + \dots\right) = 1,$$

whence, equating the coefficient of 1 to 0, and the coefficients of z^2, z^4, and so on, to 0, we get

$$b_0 = 1, \qquad b_2 = \frac{1}{6}, \qquad b_4 = \frac{7}{360}, \qquad b_6 = \frac{31}{15120},$$

and we could continue if we wished. In practice, this kind of computation would be done using a computer. $\triangle$

Exercise 5

Find the Maclaurin series for e^{z^2}.

Answer. [P] We know that $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$, so $e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$.

$\triangle$

Composing power series

If $f(z) = \sum_{m=0}^{\infty} a_m(z - z_0)^m$ and $g(w) = \sum_{n=0}^{\infty} b_n(w - w_0)^n$, then

$$g(f(z)) = \sum_{n=0}^{\infty} b_n \left(\sum_{m=0}^{\infty} a_m(z - z_0)^m - w_0 \right)^n ;$$

in principle we can expand the inner power and then gather terms. Usually this is only done if $a_0 = w_0$; then the first nonzero term of the expanded series is $[a_1(z - z_0)]^n$, and only finitely many terms are involved in the coefficient of each power $(z - z_0)^k$.

Lagrange inversion theorem[†]

Theorem

Suppose that $f \in H(\Omega)$, and that $f(a) = b$ and $f'(a) \neq 0$. Then there is a holomorphic function g, defined in an open set Υ that contains b, such that $g \circ f(z) = z$ for all z near to a, and $f \circ g(w) = w$ for all w near to b. Further, $g(w) = a + \sum_{n=1}^{\infty} c_n (w-b)^n$, where

$$c_n = \frac{1}{n!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z-a}{f(z)-f(a)} \right)^n$$

when $n \geq 1$.

Proof. Omitted. $\square$

Exercise 6

Define the function f by $f(z) = e^{1/z}$ when $z \neq 0$. Find a series that represents f in $B(1,1)$.

Answer. [P] The function is holomorphic in $B(1,1)$, and so there is a convergent power series in powers of $(z-1)$ that represents the function in the ball. To find the coefficients, we have to compute the derivatives of f: this is tricky.
[P]
But we can also observe that

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$

so

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}.$$

This answers the question! $\triangle$

End notes

Power series are still a topic for research.

The “Bieberbach conjecture”, formulated in 1916, and proved in 1985, states that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and f is holomorphic and *injective* in $B(0, 1)$, then $|a_n| \leq n$.