MATH2621 — Higher Complex Analysis. II
Complex functions

This lecture?

We introduce functions of a complex variable, and recall concepts such as domain and range.
We look at some examples and consider the problem of estimating the size of a complex function.

Functions

We think of a function as a machine with an input and an output.

- The *domain* of a function f, written $\text{Domain}(f)$, is the set of all the numbers you are allowed to put in. If there is no explicit restriction, you should consider the *natural domain*, that is, the largest domain possible.
- A *co-domain* is a set of numbers that includes all the numbers that you can get out, and perhaps more.
- The *range* (or *image*) of a function f, written $\text{Range}(f)$, is the set of the numbers that you can get out, and no others.
- The *image* of a subset S of the domain of a function f, sometimes written $f(S)$, is the set of all possible $f(s)$ as s varies over S.
- The *pre-image* of a subset T of the domain of a function f, sometimes written $f^{-1}(T)$, is the set of all x in $\text{Domain}(f)$ such that $f(x) \in T$.

Complex functions

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Examples of functions of a complex variable include the real part function Re, the imaginary part function Im, the modulus function $z \mapsto |z|$, and the principal value of the argument Arg; these are all real-valued. Complex conjugation $z \mapsto \bar{z}$ is an example of a complex-valued function of a complex variable.

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In this course, we will learn about a number of useful complex functions. Shortly we will define complex polynomials and rational functions. In future lectures, we will define $\log z$, $\sin z$, and $\cosh z$ for a complex number z, and there are many other functions in the menagerie of complex functions.

Exercise 1

Suppose that $f(z) = 1/z$ for all $z \in \mathbb{C} \setminus \{0\}$, and that $g(z) = z$ for all $z \in \mathbb{C}$. Show that $f \circ f(z) = z$ for all $z \in \mathbb{C} \setminus \{0\}$. Is $f \circ f = g$?

Answer.

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Answer. By definition,

$$f \circ f(z) = f(1/z) = 1/(1/z) = z = g(z).$$

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Answer. By definition,

$$f \circ f(z) = f(1/z) = 1/(1/z) = z = g(z).$$

However, the domain of $f \circ f$ is $\mathbb{C} \setminus \{0\}$ and the domain of g is $\mathbb{C}$, so these functions are different. $\triangle$

Polynomials

Definition

A *(complex) polynomial* is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_d z^d + \cdots + a_1 z + a_0,$$

where $a_d, \ldots, a_1, a_0 \in \mathbb{C}$. If $a_d \neq 0$, we say that p is of *degree* d. A *rational function* is a quotient of polynomials.

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Sums, differences, products and compositions of polynomials are polynomials. The same is true for rational functions. See the Exercise Sheet for more on this.

The fundamental theorem of algebra

Theorem

Every nonconstant complex polynomial p of degree d factorizes: there exist $\alpha_1, \alpha_2, \ldots, \alpha_d$ and c in $\mathbb{C}$ such that

$$p(z) = c \prod_{j=1}^{d} (z - \alpha_j).$$

Equivalently, every nonconstant complex polynomial has a root.

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In the factorisation above, the roots α_j may occur more than once. Thus we could also write

$$p(z) = c\prod_{j=1}^{e}(z - \beta_j)^{m_j},$$

where the β_j are distinct, and d is the sum of the *multiplicities* m_j.

Polynomial division and partial fractions

Theorem

Suppose that p and q are polynomials. Then

$$\frac{p(z)}{q(z)} = s(z) + \frac{r(z)}{q(z)},$$

where r and s are polynomials, and the degree of r is strictly less than the degree of q. Further, if

$$q(z) = c\prod_{j=1}^{e}(z-\beta_j)^{m_j},$$

then we may decompose the term r/q into partial fractions:

$$\frac{r(z)}{q(z)} = \sum_{j=1}^{e}\sum_{k=1}^{m_j}\frac{a_{jk}}{(z-\beta_j)^k}\,.$$

Comments

We will be able to prove these results later in the course.

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The natural domain of any complex polynomial is $\mathbb{C}$. Sometimes we cannot determine the range of a real polynomial exactly, because we cannot find maxima or minima exactly. However, for complex polynomials, things are easier.

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Answer. Take a nonconstant complex polynomial p, and a complex number w. We need to show that there is $z \in \mathbb{C}$ such that $p(z) = w$. Define $q(z) = p(z) - w$. Then q is also a nonconstant complex polynomial, so has a root by the fundamental theorem of algebra. That is, there exists $z \in \mathbb{C}$ such that $q(z) = 0$. It follows that $p(z) = w$. $\triangle$

Real and imaginary parts

To a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$, we associate two *real-valued* functions u and v of *two real variables*:

$$f(x + iy) = u(x, y) + iv(x, y).$$

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Then $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$.

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Then $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$.

It is *very useful and very important* to be able to view a complex-valued function of a complex variable in this way.

Exercise 3

Suppose that $f(z) = z$ and that $g(z) = z^2$. Find the real and imaginary parts of f and g.

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Answer. Clearly $\operatorname{Re} f(x + iy) = x$ and $\operatorname{Im} f(x + iy) = y$.

Exercise 3

Suppose that $f(z) = z$ and that $g(z) = z^2$. Find the real and imaginary parts of f and g.

Answer. Clearly $\operatorname{Re} f(x+iy) = x$ and $\operatorname{Im} f(x+iy) = y$. If $z = x + iy$, then $z^2 = x^2 + 2ixy - y^2$. Thus $\operatorname{Re} g(x+iy) = x^2 - y^2$ and $\operatorname{Im} g(x+iy) = 2xy$. $\triangle$

Exercise 4

Suppose that $f(z) = z^3 + \overline{z} - 2$. Write the real and imaginary parts of this function as functions u and v of (x, y), where $z = x + iy$.

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Answer. Observe that

$$\begin{aligned} f(x+iy) &= (x+iy)^3 + \overline{x+iy} - 2 \\ &= x^3 + 3ix^2y - 3xy^2 - iy^3 + x - iy - 2 \\ &= (x^3 - 3xy^2 + x - 2) + i(3x^2y - y^3 - y). \end{aligned}$$

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Thus $u(x, y) = x^3 - 3xy^2 + x - 2$ and
$v(x, y) = 3x^2y - y^3 - y$. $\triangle$

Exercise 5

Suppose that $f(z) = 1/z$. Write the real and imaginary parts of this function as functions of x and y, where $z = x + iy$.

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Answer. Note that

$$f(x+iy) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right).$$

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Thus $\operatorname{Re} f(x+iy) = \dfrac{x}{x^2+y^2}$ and $\operatorname{Im} f(x+iy) = \dfrac{-y}{x^2+y^2}$.
$\triangle$

Exercise 6

Write e^z in the form $u(x, y) + iv(x, y)$, where $z = x + iy$.

Answer.

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Write e^z in the form $u(x, y) + iv(x, y)$, where $z = x + iy$.

Answer. Observe that

$$\begin{aligned} e^z &= e^x(\cos(y) + i\sin(y)) \\ &= e^x\cos(y) + ie^x\sin(y) \\ &= u(x, y) + iv(x, y), \end{aligned}$$

where $u(x, y) = e^x\cos(y)$ and $v(x, y) = e^x\sin(y)$. $\triangle$

Polar coordinates

Sometimes we view the complex number z in polar coordinates, that is, we write $z = re^{i\theta}$. In this case, we consider the real functions u and v as functions of r and θ:

$$f(z) = u(r, \theta) + iv(r, \theta),$$

where $z = re^{i\theta}$.

Exercise 7

Write e^z in the form $u(r, \theta) + iv(r\theta)$, where $z = re^{i\theta}$.

Answer.

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Write e^z in the form $u(r,\theta) + iv(r\theta)$, where $z = re^{i\theta}$.

Answer. Observe that

$$
\begin{aligned}
e^z &= e^{r\cos\theta + ir\sin\theta} \\
&= e^{r\cos\theta}(\cos(r\sin\theta) + i\sin(r\sin\theta)) \\
&= e^{r\cos\theta}\cos(r\sin\theta) + i\,e^{r\cos\theta}\sin(r\sin\theta).
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Thus $u(r,\theta) = e^{r\cos\theta}\cos(r\sin\theta)$ and $v(r,\theta) = e^{r\cos\theta}\sin(r\sin\theta)$. $\triangle$

The function $z \mapsto 1/z$

It is obvious that if $w = 1/z$, then $z = 1/w$, and the function $z \mapsto 1/z$ is one-to-one (injective). Further, the domain and the range of the function are both equal to $\mathbb{C} \setminus \{0\}$.

Exercise 8

Suppose that z varies on the line $x = 1$, and let $w = 1/z$. Show that w varies on the circle $|w - \frac{1}{2}| = \frac{1}{2}$.

Answer.

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Answer. We set $w = u + iv$. If z is on the line $x = 1$, then $\operatorname{Re} z = 1$, whence $\operatorname{Re}(1/w) = 1$ (and $w \neq 0$). Now

$$1 = \operatorname{Re}\left(\frac{\bar{w}}{|w|^2}\right) = \frac{u}{u^2 + v^2}\,.$$

It follows that $u^2 + v^2 = u$, whence $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$, and the result follows. $\triangle$

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We can reverse the argument and show that every point on the circle except 0 arises in this way. Thus the image of the line is the circle with the point 0 removed.

Exercise 9

Suppose that z varies on the line $ax + by = c$, where $a, b, c \in \mathbb{R}$, and let $w = 1/z$. Show that w varies on a line when $c = 0$ and on a circle otherwise.

Answer.

Exercise 9

Suppose that z varies on the line $ax + by = c$, where $a, b, c \in \mathbb{R}$, and let $w = 1/z$. Show that w varies on a line when $c = 0$ and on a circle otherwise.

Answer. We leave this exercise to the reader. $\triangle$

Fractional linear transformations

The fractional linear transformations form an important family of complex functions. These are the functions of the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. We will study these functions in more detail later, but at the moment we just point out that if f is a fractional linear transformation and z varies on a line or a circle, then $f(z)$ varies on a line or on a circle.

Fractional linear transformations (continued)

Note that when $z \to -d/c$, then $cz = d \to 0$ and $f(z) \to \infty$ (we will define limits formally later). We can tell whether $f(z)$ varies on a line or on a circle as follows: if the points where z varies include $-d/c$, then the points where $f(z)$ varies will include ∞, and this means that $f(z)$ must vary on a line. Conversely, if the points where z varies do not include $-d/c$, then $f(z)$ will stay bounded, and this means that $f(z)$ must vary on a circle. Once we know whether $f(z)$ varies on a line or on a circle, we can find the equation of the line or the circle quite easily by finding a few values of $f(z)$.

Example

Let $f(z) = 1/z$. As z varies on the line $x = 1$, its image $f(z)$ varies on a circle, because z stays away from 0 and so $1/z$ stays away from ∞. This circle passes through the points 1 and 0 (since $f(z) \to 0$ as $z \to \infty$), and is symmetric about the real axis, since $1/(1 - it) = (1/(1 + it))^{-}$. This must be the circle that we found above.

Estimating the size of the values of a function

We will need to use what we know about inequalities to estimate how large the values of a complex function are.

Exercise 10

Suppose that $f(z) = \dfrac{1}{z^4 - 1}$ for all $z \in \mathbb{C} \setminus \{\pm 1, \pm i\}$. Show that

$$|f(z)| \leq \frac{1}{15}$$

if $|z| \geq 2$.

Answer.

Exercise 10

Suppose that $f(z) = \dfrac{1}{z^4 - 1}$ for all $z \in \mathbb{C} \setminus \{\pm 1, \pm i\}$. Show that

$$|f(z)| \leq \frac{1}{15}$$

if $|z| \geq 2$.

Answer. If $|z| \geq 2$, then by the circle inequality from Lecture 1,

$$|z^4 - 1| \geq |z^4| - |1| \geq 16 - 1 = 15.$$

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Answer. If $|z| \geq 2$, then by the circle inequality from Lecture 1,

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Hence

$$|f(z)| = \frac{1}{|z^4 - 1|} \leq \frac{1}{15},$$

as required. △

Exercise 11

Suppose that $p(z) = 10z^4 - 3z^3 + z - 10$. Show that $|p(z)| \leq 11|z|^4$ when $|z|$ is large enough.

Answer.

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Answer. Write $p(z) = z^4(10 - 3z^{-1} + z^{-3} - 10z^{-4})$. If $|z| > 10$, then, by the triangle inequality, used several times,

$$|10 - 3z^{-1} + z^{-3} - 10z^{-4}| \leq 10 + \frac{3}{10} + \frac{1}{1000} + \frac{10}{10000} \leq 11,$$

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Answer. Write $p(z) = z^4(10 - 3z^{-1} + z^{-3} - 10z^{-4})$. If $|z| > 10$, then, by the triangle inequality, used several times,

$$|10 - 3z^{-1} + z^{-3} - 10z^{-4}| \leq 10 + \frac{3}{10} + \frac{1}{1000} + \frac{10}{10000} \leq 11,$$

and so $|p(z)| \leq 11|z|^4$ when $|z| > 10$. $\triangle$