# MATH2621 — Higher Complex Analysis. XII
# Logarithms and roots

# This lecture?

In this lecture, we investigate inverse functions and in particular square roots and logarithms.

We just investigate a few examples.

# Definition of inverse functions

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One solution is to deal with "multi-valued functions", and another is to restrict the domain of the original function.

The archetypical example is the argument function, and we usually write $\arg(z)$ to indicate the multi-valued function and $\mathrm{Arg}(z)$ to indicate the particular choice where the values lie in the range $(-\pi, \pi]$.

# Example

Suppose that $w = z^2$. Then we may write $z = \sqrt{w}$ or $w^{1/2}$; the question is what this means. Unfortunately, different writers mean different things. Some writers mean that $z$ may be any of the two possible values; others mean that a particular choice has been made.

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We will try to use the expressions above for the multi-valued function, and add the words "principal value" (symbolically, PV) or "principal branch" to indicate that a particular choice is being made. In particular,

$$\mathrm{PV}\, w^{1/2} = \begin{cases} |w|^{1/2} e^{i\,\mathrm{Arg}(w)/2} & \text{if } w \neq 0 \\ 0 & \text{if } w = 0. \end{cases}$$

In words, we might say "the principal branch of the square root". The "branch cut" is where $\mathrm{Arg}(w)$ is not continuous, so PV $w^{1/2}$ is not continuous.

# The graph

We may think of the graph of the multi-valued function $w^{1/2}$ as being like two copies of the plane, slit along the "branch cut", and then joined together cleverly. It is not possible to do this in three dimensions, but it is possible in four dimensions.

# Example

Suppose that $w = e^z$ and $z = x + iy$. Then $w = e^x e^{iy}$, so

$$|w| = e^x \qquad \text{and} \qquad \operatorname{Arg} w = \operatorname{Arg} e^{iy}.$$

Then $x = \ln |w|$, and $x$ is single-valued, but $y = \operatorname{Arg}(w) + 2\pi k$, where $k \in \mathbb{Z}$; and $y$ is multiple-valued. When $w \neq 0$, we write $z = \log(w)$ to indicate that $z$ can be any one of the infinitely many complex numbers such that $e^z = w$ and we write $z = \operatorname{Log}(w)$ to indicate the choice that $z = \ln |w| + i \operatorname{Arg}(w)$.

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$\operatorname{Arg}(e^x e^{iy})$
$= \cancel{\operatorname{Arg} e^x} + \operatorname{Arg}(e^{iy})$
$= \tan^{-1}\left(\frac{\sin y}{\cos y}\right)$
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It is important to be very clear in what you write about multi-valued functions; always write "principal value" or "multi-valued' to avoid ambiguity.

$|w| = e^x \qquad x = \ln|w| \qquad y = \operatorname{Arg}(w) + 2\pi k \quad k \in \mathbb{Z}$

$z = \log(w)$ multi values $\qquad z = \operatorname{Log}(w) = \ln|w| + i \operatorname{Arg}(w)$ principal value

# Defining $n$th roots

Suppose that $f(z) = z^n$, where $n \geq 2$. Then $f$ is not one-to-one. However, if we restrict $f$ to a region $\Omega$ such as $\{z \in \mathbb{C} : |\operatorname{Arg}(z)| < \pi/n\}$, then $f$ becomes one-to-one, and we can define an inverse function, $z \mapsto z^{1/n}$. If we choose a different region, we get a different inverse function.

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All the possible inverse functions are constant multiples of each other (on connected sets where both are defined), where the multiplying factors are $n$th roots of unity. The different possible inverse functions are called *branches* of the $n$th root function.

# The principal branch of $w^{1/n}$

The *principal value* of the $n$th root is given by

$$\mathrm{PV}\, z^{1/n} = \exp(\mathrm{Log}(z)/n) = |z|^{1/n} e^{i\,\mathrm{Arg}(z)/n}.$$

## Proposition

*The function* $\mathrm{PV}\, z^{1/n}$ *is differentiable in* $\mathbb{C} \setminus (-\infty, 0]$*.*

## Proof.

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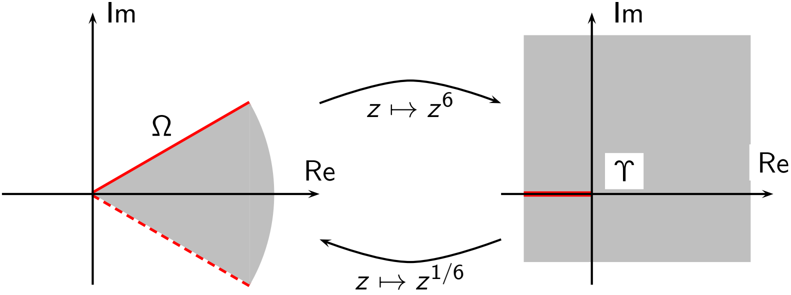
The derivative is $\frac{\mathrm{PV}\, z^{1/n}}{nz}$ where it exists.

# Sketch of the principal branch of $w^{1/n}$

$$\operatorname{PV} z^{1/n} = |z|^{1/n} e^{i \operatorname{Arg}(z)/n}.$$

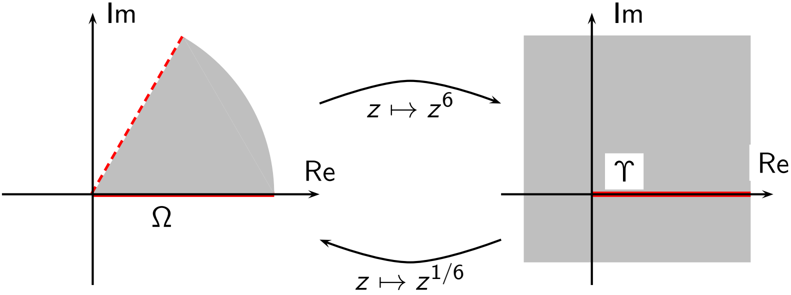


# Another branch of $w^{1/n}$

Another choice of the inverse function is given by

$$f^{-1}(z) = |z|^{1/6} e^{i\,\mathrm{Arg}^0(z)/6},$$

where $\mathrm{Arg}^0$ denotes the choice of argument in the range $[0, 2\pi)$.



# Comment

The regions $\Omega$ that we chose are relatively simple, but we could have chosen more complicated versions, such as a region between two curves coming out of the origin. This would have led to a curved branch cut.

# Branch cuts and branch points

In general, when we restrict a function $f$ to a smaller domain $\Omega$ in order to define an inverse function for $f$, we try to make $\Omega$ as large as possible, so that the domain $\Upsilon$ of the inverse function is as large as possible. The boundary of $\Upsilon$ is called the *branch cut* (or cuts, as it may have a number of connected pieces). By choosing $\Omega$ and $\Upsilon$ carefully, we can usually avoid having an inverse function with discontinuities where we want to work. Although we can vary these sets by varying the branch cut, there are some points, called *branch points*, which must appear in any branch cut. For the case of the function $f(z) = z^n$, the only branch point is 0.

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First, we consider the problem of defining $(z \pm 1)^{1/2}$. The principal branch of $(z-1)^{1/2}$ is given by

$$\mathrm{PV}(z-1)^{1/2} = \begin{cases} |z-1|^{1/2} e^{i\,\mathrm{Arg}(z-1)/2} & \text{if } z \neq 1 \\ 0 & \text{if } z = 1. \end{cases}$$

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This function is continuous in $\mathbb{C} \setminus (-\infty, 1)$; the branch cut is the infinite interval $(-\infty, 1]$ and the branch point is 1.

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This function is continuous in $\mathbb{C} \setminus (-\infty, 1)$; the branch cut is the infinite interval $(-\infty, 1]$ and the branch point is 1.

Similarly, the principal branch of $(z+1)^{1/2}$ is given by

$$\mathrm{PV}(z+1)^{1/2} = \begin{cases} |z+1|^{1/2} e^{i\,\mathrm{Arg}(z+1)/2} & \text{if } z \neq -1 \\ 0 & \text{if } z = -1. \end{cases}$$

# $(z^2-1)^{1/2}$

We could define

$$(z^2-1)^{1/2} = \mathrm{PV}(z^2-1)^{1/2} = \begin{cases} |z^2-1|^{1/2} e^{i\,\mathrm{Arg}(z^2-1)/2} & \text{when } z \neq \pm 1 \\ 0 & \text{when } z = \pm 1. \end{cases}$$

Alternatively, since $z^2-1 = (z-1)(z+1)$, we could define

$$\begin{aligned} (z^2-1)^{1/2} &= \mathrm{PV}(z-1)^{1/2}\,\mathrm{PV}(z+1)^{1/2} \\ &= \begin{cases} |z^2-1|^{1/2} e^{i\,\mathrm{Arg}(z-1)/2 + i\,\mathrm{Arg}(z+1)/2} & \text{when } z \neq \pm 1 \\ 0 & \text{when } z = \pm 1. \end{cases} \end{aligned}$$

Let us now determine whether these possible definitions coincide and where they are continuous.

## $|z^2-1|^{1/2}e^{i\,\mathrm{Arg}(z^2-1)/2}$

We consider where $|z^2-1|^{1/2}e^{i\,\mathrm{Arg}(z^2-1)/2}$ is continuous. Since $|z^2-1|^{1/2}$ is continuous, the question is linked to where $e^{i\,\mathrm{Arg}(z^2-1)/2}$ is continuous.

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The possible discontinuities are when $\mathrm{Arg}(z^2-1)$ is discontinuous, that is, when $z^2-1\in(-\infty,0]$, and thus when $0\leq z^2\leq 1$, that is, $-1\leq z\leq 1$, or when $z^2\leq 0$, that is, $z\in i\mathbb{R}$.

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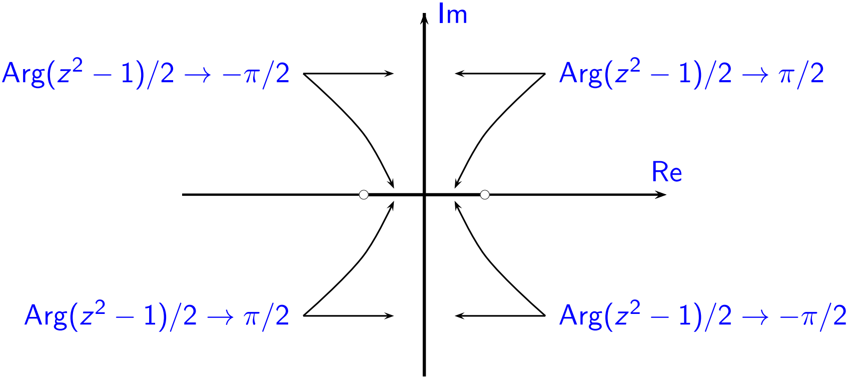
The possible discontinuities are when $\mathrm{Arg}(z^2 - 1)$ is discontinuous, that is, when $z^2 - 1 \in (-\infty, 0]$, and thus when $0 \leq z^2 \leq 1$, that is, $-1 \leq z \leq 1$, or when $z^2 \leq 0$, that is, $z \in i\mathbb{R}$.

Clearly

$$\lim_{z \to \pm 1} |z^2 - 1|^{1/2} e^{i\,\mathrm{Arg}(z^2-1)/2} = 0,$$

so the function is continuous at $\pm 1$. There are many possibilities to consider, best represented in a diagram.

# An argument diagram




This definition leads to the definition of a square root that is continuous in $\mathbb{C} \setminus (i\mathbb{R} \cup (-1,1))$. It is also differentiable there.

## $|z^2-1|^{1/2}e^{i(\mathrm{Arg}(z-1)+\mathrm{Arg}(z+1))/2}$

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The possible problems are when $\mathrm{Arg}(z-1)$ or $\mathrm{Arg}(z+1)$ are discontinuous, that is, when $z$ is real and $z \leq 1$. Clearly

$$\lim_{z\to\pm 1} |z^2-1|^{1/2}e^{i\,\mathrm{Arg}(z-1)/2+i\,\mathrm{Arg}(z+1)/2} = 0,$$

so the function is continuous at $\pm 1$.

## $|z^2 - 1|^{1/2} e^{i(\mathrm{Arg}(z-1)+\mathrm{Arg}(z+1))/2}$ (continued)

Observe that, on the one hand, if $-1 < x_0 < 1$, then

$$\lim_{\substack{z \to x_0 \\ \mathrm{Im}(z) > 0}} \mathrm{Arg}(z-1)/2 + \mathrm{Arg}(z+1)/2 = \pi/2$$

$$\lim_{\substack{z \to x_0 \\ \mathrm{Im}(z) < 0}} \mathrm{Arg}(z-1)/2 + \mathrm{Arg}(z+1)/2 = -\pi/2,$$

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and it follows that

$$\lim_{\substack{z \to x_0 \\ \mathrm{Im}(z) > 0}} e^{i\,\mathrm{Arg}(z-1)/2 + i\,\mathrm{Arg}(z+1)/2} = i$$

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$$\lim_{\substack{z\to x_0\\ \mathrm{Im}(z)>0}} e^{i\,\mathrm{Arg}(z-1)/2+i\,\mathrm{Arg}(z+1)/2}=i$$

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so that there is a discontinuity at $x_0$.

# $|z^2-1|^{1/2}e^{i(\mathrm{Arg}(z-1)+\mathrm{Arg}(z+1))/2}$ (continued)

On the other hand, if $x_0 < -1$, then

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so that there is no discontinuity at $x_0$.

# $|z^2-1|^{1/2}e^{i(\mathrm{Arg}(z-1)+\mathrm{Arg}(z+1))/2}$ (conclusion)

We could describe the choice $|z^2-1|^{1/2}e^{i(\mathrm{Arg}(z-1)+\mathrm{Arg}(z+1))/2}$ in words as the "branch of $(z^2-1)^{1/2}$ that is continuous in $\mathbb{C}\setminus(-1,1)$ and takes positive values on $(1,\infty)$".

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The second definition leads to a continuous inverse function with a larger domain, and is preferable. Suitable notation is

$$\mathrm{PV}(z-1)^{1/2}\,\mathrm{PV}(z+1)^{1/2}.$$

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We will be able to show later that this function is differentiable in $\mathbb{C}\setminus[-1,1]$.

# Exercise 1

Find an explicit formula for the "branch of $(1 - z^2)^{1/2}$ that is continuous on $\mathbb{C} \setminus ((-\infty, -1) \cup (1, \infty))$, and that takes the value 1 at 0".

Answer.

# Exercise 1

Find an explicit formula for the "branch of $(1 - z^2)^{1/2}$ that is continuous on $\mathbb{C} \setminus ((-\infty, -1) \cup (1, \infty))$, and that takes the value 1 at 0".

Answer. The allowed discontinuities for this function are the real intervals $(-\infty, -1)$ and $(1, \infty)$.
The function $\mathrm{PV}(1 - z)^{1/2}$ is discontinuous when $1 - z$ is a negative real number, that is, when $z \in (1, \infty)$. Similarly, the function $\mathrm{PV}(1 + z)^{1/2}$ is discontinuous when $1 + z$ is a negative real number, that is, when $z \in (-\infty, -1)$.

# Answer to Exercise 1

It follows that $\mathrm{PV}(1-z)^{1/2}\,\mathrm{PV}(1+z)^{1/2}$ is continuous except when $z \in (1, \infty)$ or $z \in (-\infty, -1)$. Further, when $z = 0$,

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So this is the solution. We may also write

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Alternatively, the function $\mathrm{PV}(1-z^2)^{1/2}$ is discontinuous when $1-z^2$ is a negative real number, that is, when $z^2 \in (1,\infty)$, that is, when $z \in (-\infty,-1)$ or $z \in (1,\infty)$.

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$$\left|1-z^2\right|^{1/2} e^{i\,\mathrm{Arg}(1-z^2)/2}.$$

△

# $(z(z^2-1))^{1/2}$

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The next step up in complication is the square root of a cubic polynomial. For instance, we might take the multi-valued function $w = (z(z^2-1))^{1/2}$. "Graphs" of functions like this are called *elliptic curves*. They are important in number theory—they are central to Andrew Wiles' proof of Fermat's last theorem—and they are important in cryptography—they give rise to "better" codes.

# Exercise 2

Show that there is a choice of definition of $(z(z^2 - 1))^{1/2}$ that is continuous except on the intervals $(-\infty, -1)$ and $(0, 1)$.

# More examples

The logarithm log is multi-valued because the argument arg is multi-valued. There are two common choices of argument, one between $-\pi$ and $\pi$, and the other between 0 and $2\pi$. Both of these give logarithms with the property that $\log(i) = \pi/2$, but the first has its branch cut along the negative real axis and the second has its branch cut along the positive real axis. If we want to deal with a logarithm that is continuous around 1, the first choice is better, but if we want to arrange continuity around $-1$, then the second is better.

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Soon we will need to look at expressions such as

$$\log(w + (w^2 - 1)^{1/2}).$$

We try to choose a branch of log and a branch of the square root to make the function continuous in as large a domain as possible.

# End notes

While the arg–Arg notation seems fairly standard, there does not seem to be any standard about whether $z^{1/n}$ or $\sqrt[n]{z}$ is uniquely defined. We use "multivalued" wherever appropriate to try to avoid any ambiguity.

Later we will be able to show that if a choice of a branch multi-valued function such as those considered above is continuous, then it is also differentiable.