MATH2621 — Higher Complex Analysis. VI
Differentiability

This lecture?

In this lecture, we investigate the differentiability of a function of a complex variable.

Many calculations are similar to the real-variable case; however, some functions that we might expect to be differentiable are not.

Definition

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Suppose that $S \subseteq \mathbb{C}$ and that $f : S \to \mathbb{C}$ is a complex function. Then we say that f is *differentiable at the point* z_0 in S if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \quad \text{or equivalently} \quad \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

exists. If it exists, it is called the *derivative* of f at z_0, and written $f'(z_0)$ or $\dfrac{df(z_0)}{dz}$.

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We say that f is differentiable in S if it is differentiable at all points of S.

Exercise 1

Suppose that $f_1(z) = z^2 + iz + 2$. Is f_1 differentiable at z_0 in $\mathbb{C}$? If so, find $f_1'(z_0)$?

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Answer. If $z \neq z_0$, then

$$\begin{aligned}\frac{f_1(z) - f_1(z_0)}{z - z_0} &= \frac{z^2 + iz + 2 - z_0^2 - iz_0 - 2}{z - z_0} \\ &= \frac{(z - z_0)(z + z_0) + i(z - z_0)}{z - z_0} \\ &= z + z_0 + i.\end{aligned}$$

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From the algebra of limits and standard limits,

$$\lim_{z \to z_0} \frac{f_1(z) - f_1(z_0)}{z - z_0} = \lim_{z \to z_0} z + z_0 + i = 2z_0 + i.$$

Hence f_1 is differentiable at z_0 and $f_1'(z_0) = 2z_0 + i$. This holds for all $z_0 \in \mathbb{C}$, so f_1 is differentiable in $\mathbb{C}$, and $f_1'(z) = 2z + i$.

Complex derivatives versus real derivatives

The computation above is almost identical to that to find the derivative of the real function $x^2 + x + 2$. Indeed, many formulae from the real case also hold in the complex case when x is replaced by z. So do many theorems.

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The following examples show that there is a twist to the story.

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Suppose that $f_2(z) = \overline{z}$. Is f_2 differentiable at z_0 in $\mathbb{C}$? If so, find $f_2'(z_0)$?

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Answer. If $w \neq 0$, then

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Now, if $\theta \in (-\pi, \pi]$, then

$$\lim_{\substack{w \to 0 \\ \operatorname{Arg}(w) = \theta}} \overline{w}/w = e^{-2i\theta},$$

which depends on θ.

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Now, if $\theta \in (-\pi, \pi]$, then

$$\lim_{\substack{w \to 0 \\ \mathrm{Arg}(w) = \theta}} \overline{w}/w = e^{-2i\theta},$$

which depends on θ. By the uniqueness of limits, $\lim_{w \to 0} \overline{w}/w$ does not exist, and so neither does

$$\lim_{w \to 0} \frac{f_2(z_0 + w) - f_2(z_0)}{w}\,.$$

As z_0 is arbitrary, f_2 is not differentiable anywhere in $\mathbb{C}$. $\triangle$

Exercise 3

Suppose that $f_3(z) = |z|^2$. Is f_3 differentiable at z_0 in $\mathbb{C}$? If so, find $f_3'(z_0)$?

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Answer. Fix z_0, and recall that $|z|^2 = z\overline{z}$. If $w \neq 0$, then

$$
\begin{aligned}
\frac{f_3(z_0 + w) - f_3(z_0)}{w} &= \frac{(z_0 + w)(\overline{z_0 + w}) - z_0\overline{z_0}}{w} \\
&= \frac{z_0\overline{w} + w\overline{z_0} + w\overline{w}}{w} \\
&= \overline{z_0} + \overline{w} + z_0\,\frac{\overline{w}}{w}.
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$$\begin{aligned}\frac{f_3(z_0+w) - f_3(z_0)}{w} &= \frac{(z_0+w)(\overline{z_0+w}) - z_0\overline{z_0}}{w}\\ &= \frac{z_0\overline{w} + w\overline{z_0} + w\overline{w}}{w}\\ &= \overline{z_0} + \overline{w} + z_0\,\frac{\overline{w}}{w}.\end{aligned}$$

If $z_0 = 0$, then the right hand side is equal to $\overline{w}$, and $\lim_{w\to 0}\overline{w} = 0$. Hence f_3 is differentiable at 0 with derivative 0.

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Answer. Fix z_0, and recall that $|z|^2 = z\overline{z}$. If $w \neq 0$, then

$$\begin{aligned}\frac{f_3(z_0+w) - f_3(z_0)}{w} &= \frac{(z_0+w)(\overline{z_0+w}) - z_0\overline{z_0}}{w} \\ &= \frac{z_0\overline{w} + w\overline{z_0} + w\overline{w}}{w} \\ &= \overline{z_0} + \overline{w} + z_0\,\frac{\overline{w}}{w}.\end{aligned}$$

If $z_0 = 0$, then the right hand side is equal to $\overline{w}$, and $\lim_{w\to 0}\overline{w} = 0$. Hence f_3 is differentiable at 0 with derivative 0. If $z_0 \neq 0$, then $\lim_{w\to 0}\overline{z_0} + \overline{w} = \overline{z_0}$, but $\lim_{w\to 0}\overline{w}/w$ does not exist. Hence f_3 is not differentiable at z_0 if $z_0 \neq 0$. $\triangle$

The Cauchy–Riemann equations

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We will now investigate differentiability using theoretical tools. If $\lim_{w \to 0} q(w)$ exists, then

$$\lim_{\substack{w \to 0 \\ w \in \mathbb{R}}} q(w) = \lim_{\substack{w \to 0 \\ w \in i\mathbb{R}}} q(w) = \lim_{w \to 0} q(w),$$

in the sense that the first two limits also exist, and are equal to the third. This allows us to relate the complex derivative to partial derivatives.

Very important theorem

Theorem

Suppose that Ω is an open subset of $\mathbb{C}$, that f is a complex function defined in Ω, that $f(x+iy) = u(x,y) + iv(x,y)$, where u and v are real-valued functions of two real variables, and that f is differentiable at $z_0 \in \Omega$. Then the partial derivatives

$$\frac{\partial u}{\partial x}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0), \quad \frac{\partial v}{\partial x}(x_0, y_0) \quad \textit{and} \quad \frac{\partial v}{\partial y}(x_0, y_0)$$

all exist, and

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \qquad \textit{and} \qquad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

Further,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Terminology

The pair of equations that relate the partial derivatives of u and v are known as the Cauchy–Riemann equations.

Step 1 of the proof

Proof. If $f'(z_0)$ exists, then

$$\begin{aligned}
&f'(z_0)\\
&= \lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{f(z_0+w)-f(z_0)}{w}\\
&= \lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{u(x_0+w,y_0)+iv(x_0+w,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{w}\\
&= \lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{u(x_0+w,y_0)-u(x_0,y_0)}{w} + i\lim_{\substack{w\to 0\\ w\in\mathbb{R}}} \frac{v(x_0+w,y_0)-v(x_0,y_0)}{w}\\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0),
\end{aligned}$$

because a limit exists if and only if its real and imaginary parts do.

End of Step 1

Thus

$$\frac{\partial u}{\partial x}(x_0, y_0) = \mathrm{Re}(f'(z_0)) \qquad \text{and} \qquad \frac{\partial v}{\partial x}(x_0, y_0) = \mathrm{Im}(f'(z_0)).$$

Step 2 of the proof

Similarly, if $f'(z_0)$ exists, then

$$
\begin{aligned}
&f'(z_0) \\
&= \lim_{\substack{w\to 0\\ w\in i\mathbb{R}}} \frac{f(z_0+w)-f(z_0)}{w} \\
&= \lim_{\substack{h\to 0\\ h\in \mathbb{R}}} \frac{u(x_0,y_0+h)+iv(x_0,y_0+h)-u(x_0,y_0)-iv(x_0,y_0)}{ih} \\
&= \lim_{\substack{h\to 0\\ h\in \mathbb{R}}} \frac{v(x_0,y_0+h)-v(x_0,y_0)}{h} - i\lim_{\substack{h\to 0\\ h\in \mathbb{R}}} \frac{u(x_0,y_0+h)-u(x_0,y_0)}{h} \\
&= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0).
\end{aligned}
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Thus

$$\frac{\partial v}{\partial y}(x_0, y_0) = \mathrm{Re}(f'(z_0)) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x_0, y_0) = -\,\mathrm{Im}(f'(z_0)).$$

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Thus

$$\frac{\partial v}{\partial y}(x_0, y_0) = \mathrm{Re}(f'(z_0)) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x_0, y_0) = -\,\mathrm{Im}(f'(z_0)).$$

The Cauchy–Riemann equations follow by equating the expressions for the real part of $f'(z_0)$ and those for the imaginary part of $f'(z_0)$. The rest of the theorem also follows. $\square$

Significance of the Cauchy–Riemann equations

One consequence of the previous theorem is that if f is differentiable at every point of an open set Ω in $\mathbb{C}$, then the Cauchy–Riemann equations hold at every point of Ω. Later on, we will see that in addition the four partial derivatives $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$ and $\partial v/\partial y$ are all continuous. For open sets Ω, the converse is true.

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Theorem

If the four partial derivatives $\partial u/\partial x$, $\partial v/\partial x$, $\partial u/\partial y$ and $\partial v/\partial y$ are all continuous in an open set Ω, then f is complex differentiable at $z_0 \in \Omega$ if and only if the Cauchy–Riemann equations hold at z_0, and if so, then

$$f'(x_0 + iy_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Example 1

Suppose that $f_1(z) = z^2 + iz + 2$. Then

$$u(x, y) = x^2 - y^2 - y + 2 \quad \text{and} \quad v(x, y) = 2xy + x,$$

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so

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = -2y - 1, \qquad \frac{\partial v}{\partial x} = 2y + 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = 2x,$$

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and hence the Cauchy–Riemann equations hold for all (x, y). Since the partial derivatives are continuous and $\mathbb{C}$ is open, f_1 is differentiable in $\mathbb{C}$, and

$$f_1'(z) = 2x + i(2y + 1) = 2z + i.$$

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and hence the Cauchy–Riemann equations do not hold for any (x, y) in $\mathbb{R}^2$.

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and hence the Cauchy–Riemann equations do not hold for any (x,y) in $\mathbb{R}^2$.

Thus f_2 is not differentiable at any point in $\mathbb{C}$.

Example 3

Suppose that $f_3(z) = |z|^2$. Then

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and hence the Cauchy–Riemann equations hold if and only if $x = y = 0$. The partial derivatives are continuous in $\mathbb{C}$, which is open, and hence f_3 is differentiable at 0. Finally, f is not differentiable at any other point than 0, since the Cauchy–Riemann equations do not hold except at 0.

Example 4

Suppose that $f_4(z) = e^z$. Then

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and hence the Cauchy–Riemann equations hold for all (x, y) in $\mathbb{R}^2$. Since the partial derivatives are continuous and $\mathbb{C}$ is open, f_4 is differentiable in $\mathbb{C}$, and

$$f_4'(z) = e^x(\cos y + i \sin y) = e^z.$$

Trying to do this calculation using limits is rather messy!

Differentiability implies continuity

Theorem

Suppose that f is a complex function and that $z_0 \in \mathrm{Domain}(f)$. If f is differentiable at z_0, then f is continuous at z_0.

Algebra of derivatives

Theorem

Suppose that $z_0 \in \mathbb{C}$, that the complex functions f and g are differentiable at z_0, and that $c \in \mathbb{C}$. Then the functions cf, $f + g$ and $f\,g$ are differentiable at z_0, and

$$\begin{aligned} (cf)'(z_0) &= c\,f'(z_0), \\ (f+g)'(z_0) &= f'(z_0) + g'(z_0), \\ (f\,g)'(z_0) &= f'(z_0)\,g(z_0) + f(z_0)\,g'(z_0). \end{aligned}$$

Further, if $g(z_0) \neq 0$, then the function f/g is differentiable at z_0, and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)\,g(z_0) - f(z_0)\,g'(z_0)}{g(z_0)^2}\,.$$

Composed functions and l'Hôpital's rule

Theorem

Suppose that $z_0 \in \mathbb{C}$, that the complex function f is differentiable at $g(z_0)$, and that the complex function g is differentiable at z_0. Then the function $f \circ g$ is differentiable at z_0, and

$$(f \circ g)'(z_0) = f'(g(z_0))\, g'(z_0).$$

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Theorem

Suppose that $z_0 \in \mathbb{C} \cup \{\infty\}$ and that the complex functions f and g are differentiable at z_0. If $\lim_{z \to z_0} f(z)/g(z)$ is indeterminate, that is, of the form $0/0$ or ∞/∞, and if $\lim_{z \to z_0} f'(z)/g'(z)$ exists, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}\,.$$

End notes

We use d, ∂ and δ in this course and it is important to write these clearly and correctly. Marks may be deducted for incorrectly written letters!