# Higher Complex Analysis. XXIX
# The Fourier transformation

# Today?

In this lecture, we introduce the Fourier transform $\widehat{f}$ of a suitable kind of function $f$ on $\mathbb{R}$; we then compute some examples.
[P]
We will need to be able to exchange limits and integrals.

# Locally integrable and integrable functions on $\mathbb{R}$

We say that a function $f : \mathbb{R} \to \mathbb{C}$ is locally integrable if $f$ is Riemann integrable on all finite intervals $[-R, R]$. For example, continuous functions are locally integrable.

[P]

We write $L^1(\mathbb{R})$ for the collection of all locally integrable functions $f : \mathbb{R} \to \mathbb{C}$ such that the improper integral

$$\int_{-\infty}^{\infty} |f(x)| \, dx = \lim_{R\to\infty} \int_{[-R,R]} |f(x)| \, dx$$

converges (and is finite).

# Example

Define the Gaussian $\varphi : \mathbb{R} \to \mathbb{C}$ by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad \forall x \in \mathbb{R}.$$

Then $\varphi \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} \varphi(x)\, dx = 1$.

## The Fourier transform

### Definition

Suppose that $f \in L^1(\mathbb{R})$. The Fourier transform of $f$ is the function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ given by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx = \lim_{R\to\infty} \int_{[-R,R]} f(x)\, e^{-ix\xi}\, dx.$$

$$\wedge : L^1(\mathbb{R}) \longrightarrow \{\text{functions}\}$$

# Exercise 1

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-|x|}$. Show that

$$\widehat{f}(\xi) = \frac{2}{1+\xi^2} \quad \forall \xi \in \mathbb{R}.$$

Answer. [P] First of all, it is easy to check that $f \in L^1(\mathbb{R})$.
By definition,

$$\left| e^{-R(1-i\xi)} \right| = e^{-}$$

$$\widehat{f}(\xi) = \lim_{R\to\infty} \int_{[-R,R]} e^{-|x|}\, e^{-ix\xi}\, dx.$$

$$e^{|x|} = \begin{cases} e^{x} & x \geq 0 \\ e^{-x} & x < 0 \end{cases}$$

[P] Now

$$\int_{[-R,R]} e^{-|x|}\, e^{-ix\xi}\, dx = \int_{[-R,0]} e^{x}\, e^{-ix\xi}\, dx + \int_{[0,R]} e^{-x}\, e^{-ix\xi}\, dx$$

$$= \int_{-R}^{0} e^{x(1-i\xi)}\, dx + \int_{0}^{R} e^{-x(1+i\xi)}\, dx$$

# Answer to Exercise 1

$$\begin{aligned}\int_{[-R,R]} e^{-|x|}\, e^{-ix\xi}\, dx &= \left[\frac{e^{x(1-i\xi)}}{(1-i\xi)}\right]_{-R}^{0} + \left[\frac{e^{-x(1+i\xi)}}{-(1+i\xi)}\right]_{0}^{R} \\ &= \frac{1}{1-i\xi} - \frac{e^{-R(1-i\xi)}}{1-i\xi} - \frac{e^{-R(1+i\xi)}}{1+i\xi} + \frac{1}{1+i\xi},\end{aligned}$$

and since $|\, e^{iR\xi}\,| = 1$, it follows that

$$\left|\frac{e^{-R(1-i\xi)}}{1-i\xi}\right| = \left|\frac{e^{-R(1+i\xi)}}{1+i\xi}\right| = \frac{e^{-R}}{\sqrt{1+\xi^2}} \leq e^{-R} \to 0$$

as $R \to \infty$, so

$$\widehat{f}(\xi) = \frac{1}{1-i\xi} + \frac{1}{1+i\xi} = \frac{2}{1+\xi^2},$$

as required. $\triangle$

# Exercise 2

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(1 + x^2)$. Show that

$$\widehat{f}(\xi) = \pi e^{-|\xi|} \quad \forall \xi \in \mathbb{R}.$$

Answer. [P] We saw in Lecture 24 that

$$\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2 + 1}\, dx = \pi e^{-\xi}$$

when $\xi \geq 0$.

# Answer to Exercise 2

$$\int_{-\infty}^{\infty} \frac{\cos \xi x}{x^2+1} - i\int_{-\infty}^{\infty} \frac{\sin \xi x}{x^2+1}$$

odd function $= 0$

Further, regardless of the sign of $\xi$,

$$\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{x^2+1}\,dx = \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx.$$

[P]
Hence

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{x^2+1}\,dx &= \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2+1}\,dx \\ &= \int_{-\infty}^{\infty} \frac{e^{i|\xi| x}}{x^2+1}\,dx \\ &= \pi e^{-|\xi|},\end{aligned}$$

as required. $\triangle$

# Exercise 3

The Gaussian $\varphi$ is defined by $\varphi(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$. Show that

$$\widehat{\varphi}(\xi) = e^{-\xi^2/2}\,.$$

Answer. [P] First, we know that

$$\int_{-\infty}^{\infty} \varphi(x)\,dx = 1. \tag{1}$$

[P]

$-\frac{x^2}{2} - x\xi i$

By completing the square, we see that

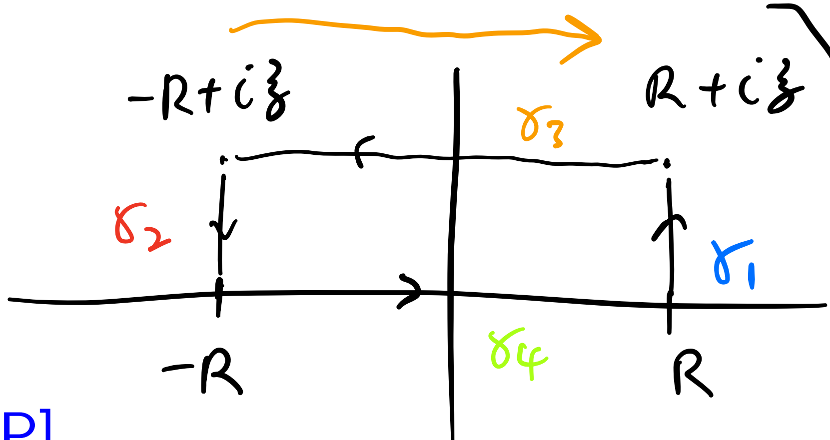
$$\widehat{\varphi}(\xi) = \lim_{R\to\infty} \int_{[-R,R]} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, e^{-ix\xi}\;dx$$

$$= \lim_{R\to\infty} e^{-\xi^2/2}\,\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-(x+i\xi)^2/2}\;dx.$$

# Answer to Exercise 3

The Cauchy–Goursat Theorem, applied to the rectangular contour with vertices at $-R$, $R$, $R+i\xi$ and $-R+i\xi$, implies that

$$\frac{1}{\sqrt{2\pi}}\int_{-R}^{R} e^{-(x+i\xi)^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{R}^{R+i\xi} e^{-z^2/2}\,dz - \frac{1}{\sqrt{2\pi}}\int_{-R}^{-R+i\xi} e^{-z^2/2}\,dz + \frac{1}{\sqrt{2\pi}}\int_{-R}^{R} e^{-x^2/2}\,dx.$$




$$\iff \frac{1}{\sqrt{2\pi}}\int_{R+i\xi}^{-R+i\xi} e^{-\frac{z^2}{2}}\,dz$$

[P]

As $R \to \infty$, the last term on the right hand side tends to 1. So we will get the desired result if we can show that the first two terms on the right hand side to go to 0.

# Answer to Exercise 3

To treat the first term, we parametrise the line segment between $R$ and $R + i\xi$ by writing $z = R + iy$, where $0 \le y \le \xi$. For such $z$,

$$\left|e^{-z^2/2}\right| = \left|e^{-(R+iy)^2/2}\right| = \left|e^{-(R^2+2iRy-y^2)/2}\right| = e^{-(R^2-y^2)/2}$$
$$= e^{(y^2-R^2)/2} \le e^{(\xi^2-R^2)/2}.$$

By the $ML$ Lemma,

$$\left|\int_R^{R+i\xi} e^{-z^2/2}\, dz\right| \le e^{\xi^2-R^2}|\xi| \to 0 \qquad \text{as } R \to \infty.$$

[P]
The second term is similar.

We conclude that $\widehat{\varphi}(\xi) = e^{-\xi^2/2}$ for all $\xi \in \mathbb{R}$. $\triangle$

# Exercise 4

Let $f(x) = 1$ if $x \in [-1, 1]$ and $f(x) = 0$ otherwise. Show that $\widehat{f} \notin L^1(\mathbb{R})$.

Answer. [P] We compute that $\widehat{f} : \xi \mapsto 2\sin(\xi)/\xi \notin L^1(\mathbb{R})$.

$\triangle$

[P]
The unfortunate fact that the Fourier transform of an integrable function need not be integrable causes some problems. However, Fourier transforms of integrable functions do have some good properties.

# The Riemann-Lebesgue Lemma

**Lemma**
*If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is bounded and continuous, and vanishes at infinity.*

**Proof.** [P] For the boundedness, observe that

$$\begin{aligned}\left|\widehat{f}(\xi)\right| &= \left|\int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx\right| \\ &\leq \int_{-\infty}^{\infty} \left|f(x)\, e^{-ix\xi}\right|\, dx \\ &= \int_{-\infty}^{\infty} |f(x)|\, dx < \infty,\end{aligned}$$

since $f \in L^1(\mathbb{R})$.

# The Riemann-Lebesgue Lemma

Further, assuming that it is legitimate to exchange limits and integrals, we see that

$$\begin{aligned}
\lim_{\xi\to\xi_0} \widehat{f}(\xi) &= \lim_{\xi\to\xi_0} \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx \\
&= \int_{-\infty}^{\infty} \lim_{\xi\to\xi_0} f(x)\, e^{-ix\xi}\, dx \\
&= \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi_0}\, dx \\
&= \widehat{f}(\xi_0),
\end{aligned}$$

and so $f$ is continuous.
[P]
We will not consider the vanishing at infinity. □

# The inversion formula

Now we show how to recover the original function $f$ from $\widehat{f}$. In other words, we find the inverse of the linear transformation $\mathcal{F}$ given by $\mathcal{F}(f) = \widehat{f}$.

## Definition

We define $\mathcal{M}(\mathbb{R})$ to be the vector space of "moderately nice" functions on $\mathbb{R}$, that is, of continuous bounded functions whose absolute value is integrable.

# The inversion formula for the Fourier transform

**Theorem**

*If* $f \in \mathcal{M}(\mathbb{R})$ *and* $\widehat{f} \in \mathcal{M}(\mathbb{R})$, *then*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{ix\xi}\, d\xi.$$

*Proof.* We give the proof later. $\square$

[P]

A consequence of the inversion formula is that the Fourier transformation is a bijective map on the vector space of functions on $\mathcal{M}(\mathbb{R})$ whose Fourier transform lies in $\mathcal{M}(\mathbb{R})$.

# Remarks on the inversion formula

The next example shows that the inversion theorem does not hold without a continuity hypothesis. We have shown that $\widehat{\varphi} = \sqrt{2\pi}\varphi$, where $\varphi$ is the Gaussian. Now define

$$f(x) = \begin{cases} \varphi(x) & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0. \end{cases}$$

[P] As changing the value of a function at a point does not change the integral of the function, it follows that $\widehat{f} = \widehat{\varphi} = \sqrt{2\pi}\varphi$, so

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{i\xi x_0}\; d\xi = \int_{-\infty}^{\infty} \widehat{\varphi}(\xi)\, e^{i\xi x_0}\; d\xi = \varphi(x_0) \neq f(x_0),$$

and the inversion formula for $f$ fails at $x_0$.
[P]
However, this artificial type of counterexample is the only problem.

# Proof of the inversion formula

First observe that

$$
\begin{aligned}
&\int_{-\infty}^{\infty} g(x)\,\widehat{\psi}(x)\,dx \\
&= \int_{-\infty}^{\infty} g(x)\left(\int_{-\infty}^{\infty} \psi(\xi)\,e^{-i\xi x}\,\,d\xi\right)\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,\psi(\xi)\,e^{-i\xi x}\,\,d\xi\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,e^{-ix\xi}\,\psi(\xi)\,dx\,d\xi \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(x)\,e^{-ix\xi}\,\,dx\right)\psi(\xi)\,d\xi \\
&= \int_{-\infty}^{\infty} \widehat{g}(\xi)\,\psi(\xi)\,d\xi. \qquad (1)
\end{aligned}
$$

# Proof of the inversion formula. 2

Suppose that $\psi_\delta(x) = e^{-\delta^2 x^2/2}$, where $\delta \in \mathbb{R}^+$. Then by a variant of Exercise 3,

$$\widehat{\psi}_\delta(\xi) = \frac{\sqrt{2\pi}}{\delta}\, e^{-\xi^2/2\delta^2}\,.$$

We put this function into (1), and get

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)\frac{\sqrt{2\pi}}{\delta}\, e^{-x^2/2\delta^2}\; dx &= \int_{-\infty}^{\infty} g(x)\, \widehat{\psi}_\delta(x)\, dx \\ &= \int_{-\infty}^{\infty} \widehat{g}(\xi)\, \psi_\delta(\xi)\, d\xi \\ &= \int_{-\infty}^{\infty} \widehat{g}(\xi)\, e^{-\delta^2\xi^2/2}\; d\xi.\end{aligned}$$

[P]
Swapping the limit and integral, we see that as $\delta \to 0+$, the right hand side tends to

$$\int_{-\infty}^{\infty} \widehat{g}(\xi)\, d\xi.$$

# Proof of the inversion formula. 3

By the change of variables $x = \delta y$, the left hand side is

$$\int_{-\infty}^{\infty} g(x)\,\frac{\sqrt{2\pi}}{\delta}\,e^{-x^2/2\delta^2}\;dx = \sqrt{2\pi}\int_{-\infty}^{\infty} g(\delta y)\,e^{-y^2/2}\;dy \quad \Leftarrow \; \frac{dx}{\delta}$$

$$\to \sqrt{2\pi}\int_{-\infty}^{\infty} g(0)\,e^{-y^2/2}\;dy = g(0) \quad \Leftrightarrow \quad g(0)\int_{-\infty}^{\infty}\varphi(y)\,dy$$

as $\delta \to 0+$ (again, changing the order of limits and integrals).
[P]
Hence

$$g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g}(\xi)\,d\xi. \tag{2}$$

# Proof of the inversion formula. 4

Now suppose that $x \in \mathbb{R}$, $f \in \mathcal{M}(\mathbb{R})$ and $g(y) = f(y + x)$. Then, by the change of variable $u = y + x$, we see that

$$\begin{aligned} \widehat{g}(\xi) &= \int_{-\infty}^{\infty} g(y)\, e^{-iy\xi}\, dy = \int_{-\infty}^{\infty} f(y+x)\, e^{-iy\xi}\, dy \\ &= \int_{-\infty}^{\infty} f(u)\, e^{-i(u-x)\xi}\, du = e^{ix\xi} \int_{-\infty}^{\infty} f(u)\, e^{-iu\xi}\, du = e^{ix\xi}\, \widehat{f}(\xi) \end{aligned}$$

for all $\xi \in \mathbb{R}$, and by applying the previous formula to this $g$, we obtain

$$f(x) = g(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{g}(\xi)\, d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{ix\xi}\, d\xi,$$

completing the proof of the inversion theorem. $\square$