MATH2621 — Higher Complex Analysis. XXIII
Residues and Cauchy's Residue Theorem

This lecture?

In this lecture, we

- define residues,
- prove Cauchy's residue theorem, and
- see examples of finding residues.

[P]
This will enable us to find integrals over closed contours more efficiently.

Residues

Definition

Suppose that the function f has an isolated singularity at z_0. Then f is holomorphic in some punctured ball $B^\circ(z_0, r)$, and has a Laurent series $\sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$ there. The *residue of f at z_0*, written $\operatorname{Res}(f, z_0)$ or $\operatorname{Res}(f(z), z = z_0)$, is defined to be c_{-1}, the coefficient of $(z - z_0)^{-1}$ in this series.

[P] $\frac{c_{-100}}{(z-z_0)^{100}} \cdots + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0)$

From Laurent's theorem, we see that if f is holomorphic in $B^\circ(z_0, r)$, and Γ is a simple closed contour in $B^\circ(z_0, r)$ around z_0, then

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \int_\Gamma f(z)\, dz. \tag{1}$$

Observe that $\operatorname{Res}(g, z_0) = 0$ if g is holomorphic in $B(z_0, r)$; this follows from the definition, or from the Cauchy–Goursat theorem.

Cauchy's residue theorem

Theorem

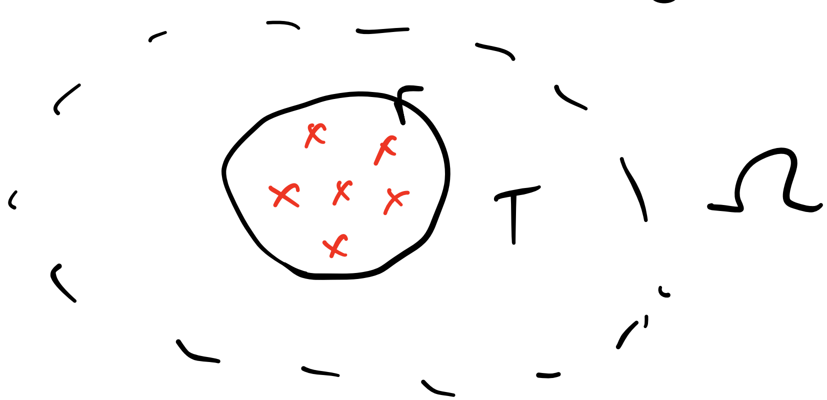
Suppose that Γ is a simple closed contour with the standard orientation in a domain Ω, that $f \in H(\Omega)$, and that

$$\mathrm{Int}(\Gamma) \cap \Omega = \mathrm{Int}(\Gamma) \setminus \{z_1, z_2, \ldots, z_K\}.$$

Then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{K} \mathrm{Res}(f, z_k).$$

We prove this theorem later. This theorem enables us to calculate integrals over closed curves as long as we can calculate residues.



Exercise 1

Suppose that

$$f(z) = \frac{\alpha_1}{z-a} + \frac{\beta_1}{z-b} + \frac{\beta_2}{(z-b)^2}.$$

Find the residues of f at a and b, and hence find $\int_\Gamma f(z)\,dz$, where Γ is a simple closed contour surrounding a and b.

Answer. [P] Both the singularities a and b lie inside Γ, and both may contribute to the sum of residues.
[P]
To find $\mathrm{Res}(f, a)$, we observe that the second and third terms are holomorphic inside $B(a, \varepsilon)$ when $\varepsilon < |a - b|$, so only contribute terms in $(z - a)^n$ where $n \geq 0$ to the Laurent series for f in $B^\circ(a, \varepsilon)$, and the residue is α_1.

$$\int_\Gamma f(z) = 2\pi i\,(\mathrm{Res}(f, a) + \mathrm{Res}(f, b))$$

Answer to Exercise 1

To find $\mathrm{Res}(f, b)$, we observe that the first term is holomorphic inside a very small ball $B(b, \varepsilon)$ centred at b, so only contributes terms in $(z-b)^n$ where $n \geq 0$ to the Laurent series for f in $B^\circ(b, \varepsilon)$, and the residue is β_1.
[P]
Consequently,

$$\int_\Gamma f(z)\,dz = 2\pi i \big(\mathrm{Res}(f, a) + \mathrm{Res}(f, b)\big) = 2\pi i \big(\alpha_1 + \beta_1\big). \qquad \triangle$$

[P]
This exercise shows that, for rational functions, residues are coefficients of certain terms in the partial fraction expansions, and that not all the coefficients are needed to compute contour integrals (in the example above, β_2 does not matter at all). If we can compute residues efficiently, then we will not need to find partial fraction expansions.

Computing residues

Suppose that f has an isolated singularity at z_0. There are three ways to compute $\mathrm{Res}(f, z_0)$, according to the type of singularity,
[P]
First, if the singularity is removable, that is, if f is bounded near z_0, the residue is 0.
[P]
Next, if f has an essential singularity at z_0, then different approaches may be needed. We will not be very interested in this problem.

Computing residues at poles

Third, if f has a pole of order N at z_0, then $f(z)$ is equal to

$$\begin{aligned} &c_{-N}(z-z_0)^{-N} + c_{1-N}(z-z_0)^{1-N} + \dots \\ &\qquad + c_{-1}(z-z_0)^{-1} + c_0(z-z_0)^0 + \dots \end{aligned}$$

so $(z-z_0)^N f(z)$ is equal to

$$\begin{aligned} &c_{-N}(z-z_0)^{0} + c_{1-N}(z-z_0)^{1} + \dots \\ &\qquad + c_{-1}(z-z_0)^{N-1} + c_0(z-z_0)^N + \dots \end{aligned}$$

and $\dfrac{d^{N-1}}{dz^{N-1}}\Big((z-z_0)^N f(z)\Big)$ is equal to

$$\begin{aligned} &\frac{d^{N-1}}{dz^{N-1}}\Big(c_{-N}(z-z_0)^{0} + c_{1-N}(z-z_0)^{1} + \dots \\ &\qquad + c_{-1}(z-z_0)^{N-1} + c_0(z-z_0)^N + \dots\Big) \\ &= (N-1)!\, c_{-1}(z-z_0)^0 + N!\, c_0(z-z_0)^1 + \dots, \end{aligned}$$

whence ...

Computing residues at poles. 2

$$\lim_{z \to z_0} \frac{d^{N-1}}{dz^{N-1}} (z - z_0)^N f(z) = (N-1)!\, c_{-1}.$$

Thus

$$\operatorname{Res}(f, z_0) = \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{d^{N-1}}{dz^{N-1}} (z - z_0)^N f(z).$$

[P]

This formula is not the definition of the residue; that definition is at the start of this lecture. To use this formula, we have to know the order of the pole.

Computing residues in general

If f has a singularity at z_0, we could proceed by looking at

$$\lim_{z \to z_0} (z - z_0)^n f(z)$$

for increasing values of n, starting at 0, until such time as we find a finite number; this is then the order of the pole, N. Once we know N, we use the formula on the previous slide. This would be very inefficient if the order of the pole is large, and would not give a result at all for an essential singularity! *It is therefore important to know how to find the order of the pole.*

[P]

The key information about the orders of zeros and poles is in the lecture on singularities.

Exercise 2

Suppose that $f(z) = \dfrac{z - \pi/2}{1 - \sin(z)}$. Find $\mathrm{Res}(f, \pi/2)$.

Answer. [P] By l'Hôpital's rule,

$$\lim_{z \to \pi/2} f(z) = \lim_{z \to \pi/2} \frac{z - \pi/2}{1 - \sin(z)} = \lim_{z \to \pi/2} \frac{1}{-\cos(z)},$$

which does not exist, so the singularity at $\pi/2$ is not removable.

[P]

By l'Hôpital's rule again,

$$\lim_{z \to \pi/2} (z - \pi/2) f(z) = \lim_{z \to \pi/2} \frac{(z - \pi/2)^2}{1 - \sin(z)} = \lim_{z \to \pi/2} \frac{2(z - \pi/2)}{-\cos(z)}$$

$$= \lim_{z \to \pi/2} \frac{2}{\sin(z)} = 2,$$

and so the pole is simple and $\mathrm{Res}(f, \pi/2) = 2$. $\triangle$

Comment

We could have worked out that the pole was simple by working out that the zero of the numerator is of order 1 and that of the denominator is of order 2, and then doing the second calculation only.

The p/q' formula

Proposition

Suppose that $f(z) = p(z)/q(z)$ in Ω, and that $p(z_0) \neq 0$ while $q(z_0) = 0$. If z_0 is a simple zero of q, that is, a zero of order 1, then

$$\mathrm{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}\,.$$

Proof.

[P] Since the pole of f is simple,

$$\begin{aligned}\mathrm{Res}(f, z_0) &= \lim_{z\to z_0}(z - z_0)\frac{p(z)}{q(z)} = \lim_{z\to z_0} p(z)\frac{(z - z_0)}{q(z)} \\ &= p(z_0)\lim_{z\to z_0}\frac{(z - z_0)}{q(z)} = p(z_0)\frac{1}{q'(z_0)} = \frac{p(z_0)}{q'(z_0)}\,,\end{aligned}$$

by l'Hôpital's rule. □

Exercise 3

Suppose that
$$f(z) = \tan(z).$$
Find the residue of f at $(2k+1)\pi/2$, where $k \in \mathbb{Z}$.

Answer. [P] This function has singularities at $(2k+1)\pi/2$, where $k \in \mathbb{Z}$, because $\tan = \sin / \cos$ and $\cos$ is zero at these points. The zeros of cos are of order 1. By the p/q' formula,

$$\operatorname{Res}(f, (2k+1)\pi/2) = \frac{\sin((2k+1)\pi/2)}{-\sin((2k+1)\pi/2)} = -1. \qquad \triangle$$

Of course, we could also have done this calculation directly, without using the formula.

Exercise 4

Suppose that
$$f(z) = 2z\sin(z^{-2}).$$
Find the residue of f at 0.

Answer. [P] By Taylor series,
$$\sin(w) = w - \frac{w^3}{3!} + \frac{w^5}{5!} - \dots$$
[P] for all $w \in \mathbb{C}$, and so, replacing w by z^{-2}, we see that
$$\begin{aligned} 2z\sin(z^{-2}) &= 2z\left(z^{-2} - \frac{z^{-6}}{3!} + \frac{z^{-10}}{5!} - \dots\right) \\ &= 2\left(z^{-1} - \frac{z^{-5}}{3!} + \frac{z^{-9}}{5!} - \dots\right) \end{aligned}$$
for all $z \in \mathbb{C} \setminus \{0\}$. [P] Hence $\mathrm{Res}(f, 0) = 2$. $\triangle$

Exercise 5

Suppose that
$$f(z) = \sin(z)\, e^{1/z}.$$
Find the residue of f at 0.

Answer. [P] Observe that
$$f(z) = \sin(z)\, e^{1/z} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\, z^{2m+1} \sum_{n=0}^{\infty} \frac{1}{n!}\, z^{-n}$$
$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\, \frac{1}{n!}\, z^{2m-n+1}\,.$$
[P] To get terms in z^{-1}, we require that $2m - n + 1 = -1$, that is, that $n = 2m + 2$. Thus
$$\mathrm{Res}(f, 0) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!\,(2m+2)!}\,.$$
△

Comments

The expression for the residue in the last example does not simplify but may be computed numerically to any desired degree of accuracy. This behaviour is common with essential singularities.
[P]
Essential singularities often appear because there is a "transcendental function" (such as exp or sinh or tan) which can be written as a power series, with something like $1/z$ in its argument. Residues involving transcendental functions can often be expressed as series which are derived from the corresponding power series.
[P]
We will not focus much on these singularities.

Proof of Cauchy's Residue Theorem

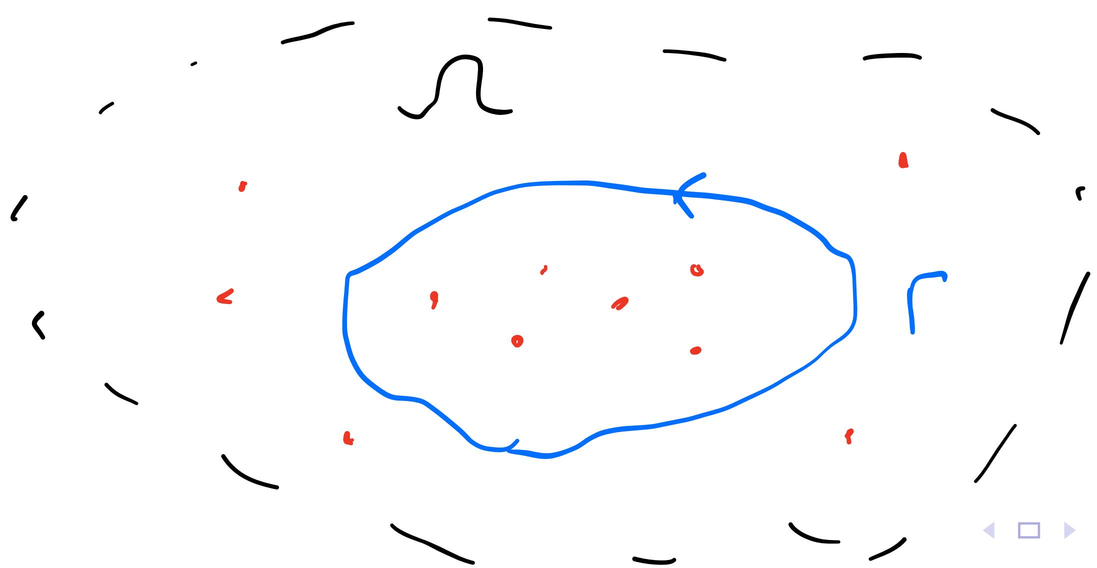
Theorem

Suppose that Γ *is a simple closed contour with the standard orientation in a domain* Ω*, that* $f \in H(\Omega)$*, and that*

$$\mathrm{Int}(\Gamma) \cap \Omega = \mathrm{Int}(\Gamma) \setminus \{z_1, z_2, \ldots, z_K\}.$$

Then

$$\int_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{K} \mathrm{Res}(f, z_k).$$



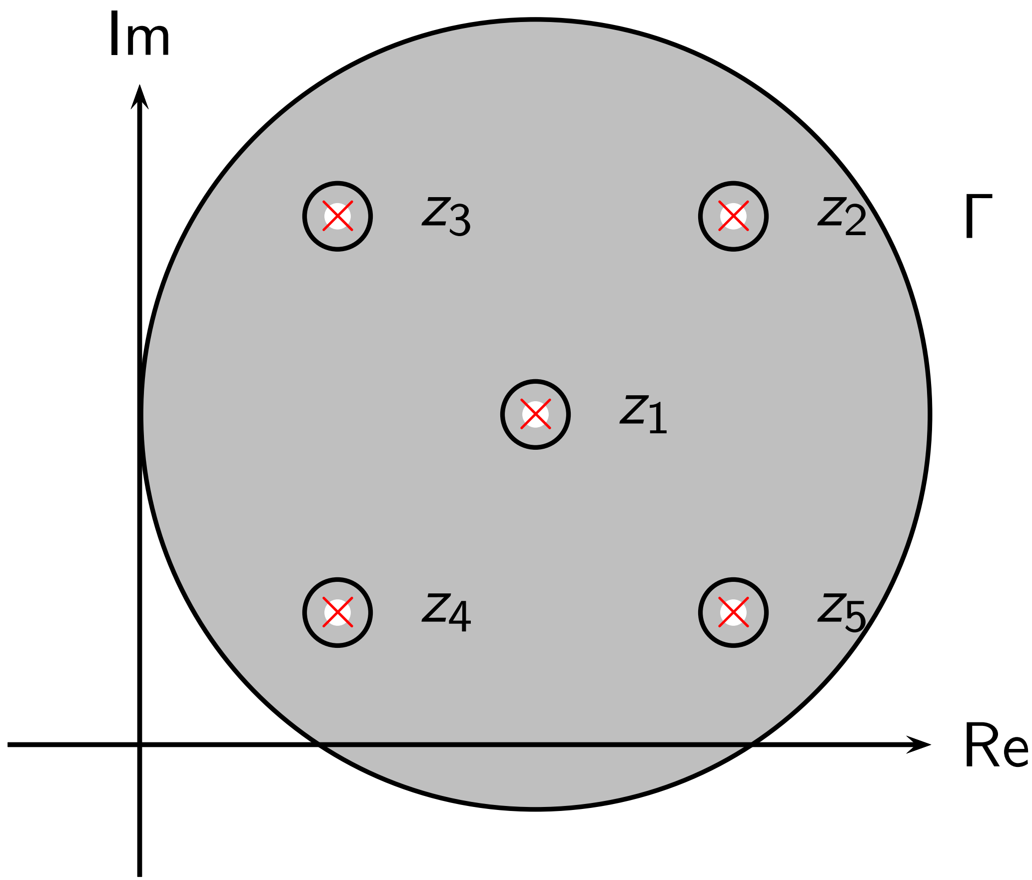
Proof of the theorem

Proof. We take balls $B(z_k, \varepsilon)$ centred at the singularities z_k, where the ε are chosen small enough that the closed balls $\overline{B}(z_i, \varepsilon)$ and $\overline{B}(z_j, \varepsilon)$ are disjoint if $i \neq j$, and such that each closed ball $\overline{B}(z_k, \varepsilon)$ is contained in $\mathrm{Int}(\Gamma)$.
[P]
Define

$$\Upsilon = \mathrm{Int}(\Gamma) \setminus \left(\bigcup_{k=1}^{K} \overline{B}(z_k, \varepsilon) \right).$$

Then f is holomorphic on an open set containing Υ and its boundary, and $\partial\Upsilon$ is made up of Γ, traversed anti-clockwise, together with the boundaries $\partial B(z_k, \varepsilon)^*$, that is, the circle with centre z_k and radius ε, traversed clockwise.

Proof of the theorem. 2



The Cauchy–Goursat theorem implies that

$$0 = \int_{\partial \Upsilon} f(z)\, dz = \int_{\Gamma} f(z)\, dz + \sum_{k=1}^{K} \int_{\partial B(z_k,\varepsilon)^*} f(z)\, dz,$$

that is, ...

Proof of the theorem. 3

$$\int_\Gamma f(z)\,dz = \sum_{k=1}^{K} \int_{\partial B(z_k,\varepsilon)} f(z)\,dz.$$

[P]
From (1),

$$\mathrm{Res}(f, z_k) = \frac{1}{2\pi i} \int_{\partial B(z_k,\varepsilon)} f(z)\,dz,$$

so

$$\int_{\partial B(z_k,\varepsilon)} f(z)\,dz = 2\pi i\,\mathrm{Res}(f, z_k),$$

[P] and hence

$$\int_\Gamma f(z)\,dz = \sum_{k=1}^{K} \int_{\partial B(z_k,\varepsilon)} f(z)\,dz = 2\pi i \sum_{k=1}^{K} \mathrm{Res}(f, z_k);$$

the theorem is proved. $\square$