FAMILY NAME: ....
OTHER NAME(S): ....
STUDENT NUMBER: ....
SIGNATURE: ....

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Term 3 2022

MATH2621
Higher Complex Analysis

(1) TIME ALLOWED – 2 hours

(2) READING TIME – 10 Minutes

(3) TOTAL NUMBER OF QUESTIONS TO BE ANSWERED – 4

(4) ANSWER ALL QUESTIONS

(5) THE QUESTIONS ARE OF EQUAL VALUE

(6) ANSWER EACH QUESTION IN A SEPARATE BOOKLET

(7) THIS PAPER MAY **NOT** BE RETAINED BY THE CANDIDATE

(8) CANDIDATES MAY BRING TO THE EXAMINATION 2 A4 PAGES OF NOTES. UNSW APPROVED CALCULATOR

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked "Q1"

1. i) Let A, B and C be the following three regions in the complex plane:

$$A = \{z \in \mathbb{C} : |z - 2| \geq 2\},$$
$$B = \{z \in \mathbb{C} : \pi/4 < \operatorname{Arg}(z - 1) < \pi/2\},$$
$$C = \{z \in \mathbb{C} : 0 < \operatorname{Re}(z^2) < 1\}.$$

a) Sketch each region in a separate diagram.

b) State, with reasons, whether or not each region is a domain.

c) For those regions which are domains, state, with reasons, whether or not they are simply connected.

ii) Find all complex numbers z such that

$$\cosh(z) = 3i.$$

iii) Let

$$f(z) = \left(\frac{x^3}{3} - y^3 - y^2 x\right) + i\left(yx^2 - \frac{y^3}{3}\right),$$

where $z = x + iy$, with $x, y \in \mathbb{R}$.

a) Determine the set of points where f is differentiable.

b) Where is f holomorphic? Give a reason for your answer.

c) Find $f'(z)$ in terms of x and y, where it exists.

iv) Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $u(x, y) = e^y \cos(x) + 2xy + x$.

a) Show that u is harmonic on $\mathbb{R}^2$.

b) Find a harmonic conjugate v for u.

c) Find a known holomorphic function f in the variable z such that $f(z) = u(x, y) + iv(x, y)$ when $z = x + iy$.

Use a separate book clearly marked "Q2"

2. i) Suppose that
$$f(z) = \frac{5}{(z-1)} - \frac{3}{(z+4)^2}\,.$$
Find the Laurent series for f in powers of $(z+2)$ that converges when $z = -i$.

ii) State the definition of an essential singularity $z_0 \in \mathbb{C}$ of a function g.

iii) Suppose that
$$h(z) = \frac{\sin(z)}{z(z+2i)^2}\,.$$

a) Find all the singular points of h, classify them, and find the residue of h at each of them.

b) Hence calculate the integral
$$\int_\Gamma h(z)\,dz,$$
where Γ is the circle with centre -1 and radius 3, traversed in the anti-clockwise direction.

iv) Suppose that a holomorphic function k may be represented by the power series
$$\sum_{n=0}^{\infty} n(n-1)\,(z+2)^n.$$
in a particular region.

a) Find the centre and radius of convergence of the power series.

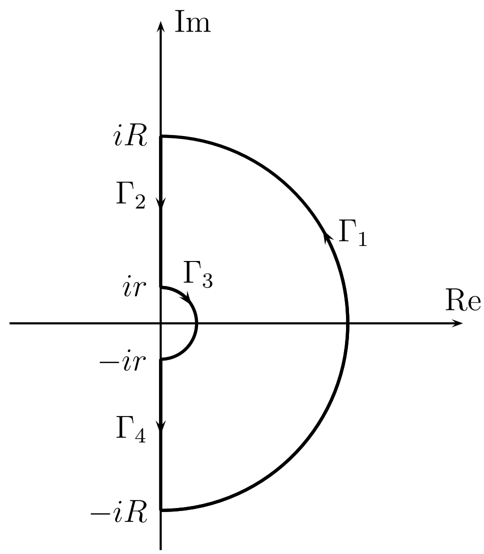
b) Find an expression for k as an elementary function.

Use a separate book clearly marked "Q3"

3. Consider the function f given by

$$f(z) = \frac{\mathrm{Log}(z)}{z^2 - a^2},$$

where a is a positive real number; here Log denotes the principal branch of log, where the argument is taken to lie between $-\pi$ and π. Consider also the simple closed contour Γ that is composed of the contours Γ_1, Γ_2, Γ_3, and Γ_4 shown in the figure.



The semicircular arcs Γ_1 and Γ_3 have centre 0 and radii R and r respectively; Γ_2 and Γ_4 are straight line segments. Further, $r < 1 < R$ and $r < a < R$.

i) Find the singular points of f inside Γ and the residues of f at these singular points.

ii) Using L'Hôpital's rule or otherwise, show that

$$\lim_{R\to+\infty} \frac{\ln(R)}{R} = 0 \quad \text{and} \quad \lim_{r\to 0+} r\ln(r) = 0.$$

iii) Explain why $\displaystyle\lim_{r\to 0+} \int_{\Gamma_3} f(z)\,dz = 0$.

iv) Find $\displaystyle\lim_{R\to+\infty} \int_{\Gamma_1} f(z)\,dz$.

v) Hence show that $\displaystyle\int_0^{+\infty} \frac{\ln(x)}{x^2 + a^2}\,dx = \frac{\pi\ln(a)}{2a}$.

vi) Hence find $\displaystyle\int_0^{+\infty} \frac{\ln(x)}{(x^2 + e^2)^2}\,dx$.

Use a separate book clearly marked "Q4"

4. i) a) State the Cauchy-Goursat Theorem for a complex function f defined on a polygonally simply connected domain Ω.

b) Hence, assuming that the assumptions and the conclusion of the Cauchy-Goursat Theorem hold, prove that there exists a function F on Ω such that

$$\int_\Gamma f(z)\,dz = F(q) - F(p),$$

for every simple contour Γ contained in Ω from p to q, with $p, q \in \Omega$.

c) Compute

$$I = \int_\Gamma \frac{1}{z-1}\,dz$$

where Γ is any simple contour from $-i$ to i that does not intersect the positive real axis.

ii) Consider the function g on $\mathbb{R}$ defined by

$$g(x) = e^{-2|x|} \qquad \forall x \in \mathbb{R}.$$

a) Compute the Fourier transform $\hat{g}$ of the function g.

b) Hence or otherwise, compute $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{4+\xi^2}\,d\xi$.

iii) Let p and q be nonzero polynomials with no common factors and with $p(-1) \neq 0$, of degrees m and n respectively, and let h be the rational function $\dfrac{z+1}{z^2-1}\dfrac{p}{q}$.

a) Find the total number of singularities of h (counting multiplicities).

b) Discuss in which cases there exist removable singularities and find them.

END OF EXAMINATION