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An Evaluation of Several Differencing Methods for Inviscid Fluid Flow Problems

ASHLEY F. EMERY

Sandia Corporation, Livermore, California 94550

and

University of Washington, Seattle, Washington 98105

ABSTRACT

An evaluation was made of five algorithms for ideal inviscid flow: Lax's, Rusanov's, Landshoff's, the Lax–Wendroff, and Richtmyer's. These algorithms were compared on the basis of ease of coding, spatial and temporal resolution, and execution time. The comparisons were drawn through studies of a one-dimensional reflected shock wave, a flat-faced step in supersonic flow, and passage of a normal shock wave over a stationary cone. Lax's method was found to be the easiest to implement and of good resolution; Rusanov's to be the most versatile and of better resolution; and a modified version of the Lax–Wendroff to yield the best resolution but to be the most difficult and time-consuming to use.

I. INTRODUCTION

Now that high-speed computers have become generally available and numerical techniques more sophisticated, many fluid dynamic problems are solved by direct numerical integration of the governing differential equations. Unfortunately, although there seem to be almost as many numerical methods as there are investigators, no comparison of the methods and their computational times is available.

This study was performed to compare the characteristics and usefulness of several differencing methods: Lax's, Landshoff's, Rusanov's, Lax and Wendroff's, and Richtmyer's. All of these methods are explicit, and no computations were made to determine the relative advantages of the explicit, implicit, or alternating-direction techniques. In the following sections a brief discussion of the methods is given, their solutions to several problems are compared, and finally some improvements are suggested.

Basic Equations

The governing equations of mass, momentum, and energy for inviscid, plane, two-dimensional flows are as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho uv}{\partial y} = -\frac{\partial P}{\partial x}, \tag{2}$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho uv}{\partial x} + \frac{\partial \rho v^2}{\partial y} = -\frac{\partial P}{\partial y}, \tag{3}$$

$$\frac{\partial \rho e}{\partial t} + \frac{\partial}{\partial x}\{\rho u[e + P/\rho + 1/2(u^2 + v^2)]\} + \frac{\partial}{\partial y}\{\rho v[e + P/\rho + 1/2(u^2 + v^2)]\} = 0. \tag{4}$$

There is no conceptual difficulty in treating cylindrical or spherical coordinate systems [1], [2], but a study of plane, rectangular coordinate system is sufficient for the purposes of comparison.

Equations (1)–(4) are written in "divergence free" or "control volume" form, since the advantages of this approach have already been well documented [3], [4]. By defining the vectors

$$\tilde{w}^T = \{\rho, \rho u, \rho v, E\}, \tag{5}$$

$$\tilde{f}^T = \rho u, -\frac{\gamma - 3}{2}\rho u^2 + (\gamma - 1)\left(E - \frac{\rho v^2}{2}\right), \rho uv, \frac{1 - \gamma}{2}\rho u(u^2 + v^2) + \gamma u E, \tag{6}$$

$$\tilde{g}^T = \rho v, \rho uv, -\frac{\gamma - 3}{2}\rho v^2 + (\gamma - 1)\left(E - \frac{\rho u^2}{2}\right), \frac{1 - \gamma}{2}\rho v(u^2 + v^2) + \gamma v E, \tag{7}$$

where

$$E = \rho[e + 1/2(u^2 + v^2)],$$

Eqs. (1)–(4) may be expressed as

$$\operatorname{div}(\tilde{w}, \tilde{f}, \tilde{g}) = \frac{\partial \tilde{w}}{\partial t} + \frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{g}}{\partial y} = 0. \tag{8}$$

All of the methods used to solve these equations may be shown to be consistent and to be stable according to either the Courant–Friederich–Lewey (CFL) criterion or some modification thereof.

II. Description of Methods

Lax's Method

Perhaps the simplest and most easily programmed method is that due to Lax [5]. In this method, the differencing is carried out in a straightforward manner—normally utilizing centered differences in space and forward differences in time. The system of equations may be shown to be conditionally stable according to

$$\Delta t = \min\left(\frac{\Delta x}{u + c}, \frac{\Delta y}{v + c}\right) \tag{9}$$

if the forward time difference for a specified component, w, is replaced by

$$\frac{\partial w}{\partial t} = \frac{w(t + \Delta t, x, y) - \bar{w}(t, x, y)}{\Delta t}, \tag{10}$$

where

$$\bar{w} = 1/4\{w(t, x + \Delta x, y) + w(t, x, y + \Delta y) + w(t, x - \Delta x, y) + w(t, x, y - \Delta y)\} \tag{11}$$

represents a spatial average of w. Lax's method, in effect, introduces an artificial viscosity (in the sense of second derivative terms) whose coefficients are

$$\frac{\Delta x^2}{4\Delta t} \text{ and } \frac{\Delta y^2}{4\Delta t}.$$

Thus, to yield the maximum resolution of the flow field, Δt must be as large as possible within the limits of stability and Δx must be as small as possible within the storage capabilities of the computer.

Landshoff's Method

Following Richtmyer and von Neumann's suggestion [6], Landshoff [7] modified the pressure by including an additional term, q, called the artificial pressure and given for one dimension by

$$q = -\alpha\rho c(\partial u/\partial x)\,\Delta x, \tag{12}$$

where α is an adjustable constant. Normally a cut-off procedure [8] is used to prevent the smearing of rarefaction waves, but when the nature of the wave and its direction are unknown, no cut-off is possible. This method was tested only in its one-dimensional form.

Rusanov's Method

One of the most useful differencing formulations is due to Rusanov [9]. This improvement of Lax's method maintains the minimum artificial viscosity at each nodal point and also weights the importance of the neighboring mesh points. The generic equation

$$\frac{\partial \tilde{w}}{\partial t} + \frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{g}}{\partial y} = 0$$

is represented in Lax's method by

$$\tilde{w}(t + \Delta t, x, y) = \bar{\tilde{w}}(t, x, y) - \Delta t \frac{\partial \tilde{f}}{\partial x} - \Delta t \frac{\partial \tilde{g}}{\partial y}.$$

Rusanov's method consists in weighting the points so that (*1*) the method can be reduced to Lax's method; (*2*) if all neighboring points are of equal value, the average reduces to that value; and (*3*) the closest points are weighted most heavily. These requirements can be most easily met by defining (for any component)

$$\begin{aligned} \bar{w} = {} & \frac{1}{2}[w(t, x + \Delta x, y) + w(t, x - \Delta x, y)] \frac{\Delta y^2}{\Delta^2} \\ & + \frac{1}{2}[w(t, x, y + \Delta y) + w(t, x, y - \Delta y)] \frac{\Delta x^2}{\Delta^2}, \end{aligned} \tag{13}$$

where $\Delta^2 = \Delta x^2 + \Delta y^2$. By adding and subtracting $w(t, x, y)$ we may write

$$\bar{w} = w(t, x, y) + \frac{\Delta x^2}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)\frac{\Delta y^2}{\Delta^2} + \frac{\Delta y^2}{2}\left(\frac{\partial^2 w}{\partial y^2}\right)\frac{\Delta x^2}{\Delta^2}. \tag{14}$$

Now to further decrease the damping, Rusanov multiplied Eq. (14) by a variable constant and the Courant number $\bar{\sigma}$ so that

$$\bar{w}(t, x, y) = w(t, x, y) + \frac{\Delta x^2}{2}\frac{\partial}{\partial x}\left(\alpha \frac{\partial w}{\partial x}\right) + \frac{\Delta y^2}{2}\frac{\partial}{\partial y}\left(\beta \frac{\partial w}{y}\right), \tag{15}$$

where

$$\begin{aligned} \alpha &= \bar{\sigma}\omega \frac{V + c}{(V + c)_{\max}} \frac{\Delta y^2}{\Delta^2}, \\ \beta &= \bar{\sigma}\omega \frac{V + c}{(V + c)_{\max}} \frac{\Delta x^2}{\Delta^2}, \\ V^2 &= u^2 + v^2, \end{aligned} \tag{16}$$

and where ω is a pure number of variable magnitude. If $\Delta x = \Delta y$, $\omega = 1$, and $V + c = (V + c)_{\max}$, then Eq. (15) reduces to Lax's equation. For other conditions, however, the presence of the term $(V + c)/(V + c)_{\max}$ tends to reduce the damping for points for which the sound pulse travels slowly and to maximize the damping only for those points for which $V + c \sim (V + c)_{\max}$. The result for most flow problems is a reduced damping at the majority of nodal points in the field and a larger damping only where it is really needed (e.g., in the vicinity of shocks).

The necessary stability criterion for Eq. (15) is

$$\bar{\sigma} < \omega < \frac{1}{\bar{\sigma}}, \qquad \bar{\sigma} = \frac{\Delta t \Delta}{\Delta x \Delta y} (V + c)_{\max} \leqslant 1. \tag{16a}$$

This criterion cannot be rigorously proven, but experience has demonstrated its validity.

Lax-Wendroff Method

The full development of the two-dimensional Lax-Wendroff difference formulation has been given by Burstein [10] and Emery [11], and only a brief description will be included here. The formulation consists in approximating the time derivative so that second-order terms are retained:

$$\tilde{w}(t + \Delta t, x, y) = \tilde{w}(t, x, y) + \frac{\partial \tilde{w}}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 \tilde{w}}{\partial t^2} \Delta t^2 \tag{17}$$

or

$$\begin{aligned} &\tilde{w}(t + \Delta t, x, y) \\ &\quad = \tilde{w}(t, x, y) - \Delta t \left(\frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{g}}{\partial y}\right) + \frac{\Delta t^2}{2} \frac{\partial}{\partial x} [A] \left(\frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{g}}{\partial y}\right) + \frac{\Delta t^2}{2} \frac{\partial}{\partial y} [B] \left(\frac{\partial \tilde{f}}{\partial x} + \frac{\partial \tilde{g}}{\partial y}\right), \end{aligned} \tag{18}$$

where $[A]$ and $[B]$ are matrices given by

$$\frac{\partial \tilde{f}}{\partial x} = [A] \frac{\partial \tilde{w}}{\partial x}. \tag{19}$$

The inclusion of the $\tilde{w}_{,tt}$ term produces a conditionally stable formulation capable of great spatial resolution—producing shocks which are about one-half the width predicted by other methods—but requires considerably more computational time. In addition, the method frequently becomes unstable. To avoid instabilities—caused by the disappearance of the $\tilde{w}_{tt}$ term at unusual points in the flow (e.g., sonic points and stagnation points in essentially one-dimensional portions of the flow [10], [11])—it is necessary to incorporate an additional artificial viscosity term. Following Lax and Wendroff's original paper [12], one writes for one dimension

$$\tilde{w}_{,t} = -\tilde{f}_{,x} + 1/2[Q(\tilde{a}, \tilde{b}) \cdot (\tilde{b} - \tilde{a})]_{,x} \tag{20}$$

where $\tilde{a}$ and $\tilde{b}$ are values of $\tilde{w}$ at x and $x + \Delta x$; and Q is the matrix

$$Q = d_0 I + d_1 A + d_2 A^2 + \cdots d_{n-1} A^{n-1}, \tag{21}$$

where n is the number of independent eigenvalues of $[A]$. In addition, Q must satisfy the following rule: $Q(a, b)$ is a matrix which should commute with $A(\frac{a+b}{2})$ and whose eigenvalues should be equal to the absolute values of the differences of the corresponding eigenvalues of $A(a)$ and $A(b)$ multiplied by the dimensionless factors B_1 through B_n of order unity. For a one-dimensional Eulerian system,

$$\begin{aligned} d_0 = \frac{K_x}{2} \Bigg\{ & a \left(\frac{\overline{u^2}}{\overline{c^2}} - 2 \frac{\bar{u}}{\bar{c}} \frac{\overline{uc}}{\overline{c^2}} + 1 \right) + b \left(\frac{\bar{u}}{\bar{c}} \frac{\overline{uc}}{\overline{c^2}} - \frac{\overline{u^2}}{2\overline{c^2}} - \frac{\bar{u}}{2\bar{c}} \right) \\ & + e \left(\frac{\bar{u}}{\bar{c}} \frac{\overline{uc}}{\overline{c^2}} + \frac{\bar{u}}{2\bar{c}} - \frac{\overline{u^2}}{2\overline{c^2}} \right) \Bigg\}, \end{aligned} \tag{22}$$

$$d_1 = \frac{K_x}{2\bar{c}} \left\{ 2a \frac{\overline{uc}}{\overline{c^2}} + \frac{b}{2} \left(1 - 2 \frac{\overline{uc}}{\overline{c^2}} \right) - \frac{e}{2} \left(1 + 2 \frac{\overline{uc}}{\overline{c^2}} \right) \right\}, \tag{23}$$

$$d_2 = \frac{K_x}{4\overline{c^2}} (-2a + b + e),^1 \tag{24}$$

where $a = |\Delta u|$, $b = |\Delta(u + c)|$, $e = |\Delta(u - c)|$, $\bar{u} = \frac{1}{2}\{u(x) + u(x + \Delta x)\}$.

The stability requirement now includes K_x and is given by

$$\Delta t \leqslant \frac{\Delta x}{|u + c|} \left\{ \left(1 + \frac{K_x^2}{16} \right)^{1/2} - \frac{K_x}{1} \right\}, \tag{25}$$

which, for $K_x = 1$, reduces to

$$\Delta t \leqslant 0.78(\Delta x / | u + c |)$$

This requirement is just slightly more stringent than the usual CFL requirement.

Modification of Lax–Wendroff Method for Two-Dimensional Problems

It is not possible to extend the Lax-Wendroff method to two dimensions by using a generalization of the above modification while satisfying the conditions

[1] The constant K_x need not be the same for d_0, d_1, and d_2.

of commutation of Q with A and B and of invariance of the artificial viscosity term with respect to rotation of the coordinate axis. It is possible, however, simply to add in the y direction an artificial term, K_y, with a separate coefficient, which is completely analogous to the term in the x direction.[2] Numerical tests have indicated that this term provides sufficient damping of oscillations of short wave length to prevent instabilities. For such a formulation, the stability criterion is

$$\sigma < \frac{1}{2 + \frac{1}{2}K} \frac{(1 + K)^{1/2}}{2} < \frac{1}{\sqrt{6}} \qquad (\text{for } K_x = K_y), \tag{26}$$

and reduces to Lax and Wendroff's value of $1/\sqrt{8}$ for $K = 0$. This value appears to have been substantiated by Burstein, who found that σ must be less than 0.355 for long-time stability.

Successful calculations have been carried out, however, with values of $K_x = 8$ and $\sigma = 0.7$ for one-dimensional flows and of $K = 4$ and $\sigma = 0.7$ for two-dimensional flows. This Courant number is far in excess of the 0.39 predicted by theory and the 0.355 shown by Burstein, a fact which the author cannot explain. The best that can be done at the present time is to present a rationale for the use of higher values of σ.

First, let us assume that one can derive the stability analysis for an Eulerian system of coordinates from that for a Lagrangian system. In Lagrangian coordinates the amplification matrix is

$$G = I + i\lambda A \sin\alpha + \tfrac{1}{2}(\lambda^2 A^2 + \lambda Q)(\cos\alpha - 1) \tag{27}$$

where $\alpha = k\Delta x$, and k is any real integer. Since

$$Q = \frac{B}{2} \frac{|\, c(a) - c(b) \,|}{\frac{1}{2}[c(a) + c(b)]} A^2, \tag{28}$$

the eigenvalues of G are simply

$$\mu_G = 1 + i\lambda\mu_A \sin\alpha + \tfrac{1}{2}(\lambda^2 \mu_A{}^2 + \lambda\mu_Q)(\cos\alpha - 1). \tag{29}$$

Stability is ensured if

$$\lambda^2 \mu_A{}^2 + \lambda\mu_Q \leqslant 1 \tag{30}$$

and, since $\mu_A = \pm c$,

$$\mu_Q = \tfrac{1}{2}B\,|\, c(a) - c(b)|. \tag{31}$$

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[2] This extension was derived independently and nearly simultaneously by Burstein [10] and the author [11].

At this point Lax and Wendroff replaced the absolute quantity by $\mu_A = c$. Since $| c(a) - c(b)| \leqslant c$, we retain the difference term and write:

$$\lambda^2 \mu_A{}^2 + \lambda \tfrac{1}{2} B \,| c(a) - c(b)| \leqslant 1. \tag{32}$$

Solving this equation and replacing the speed of sound term, c, in μ_A by $u + c$ to accomplish the change from the Lagrangian to the Eulerian system, we obtain

$$\sigma < \frac{(u + c)_{\max}}{u + c} \left\{ \left(1 + \frac{B^2 V}{16}\right)^{1/2} - \frac{BV}{4} \right\}, \tag{33}$$

where

$$V = \frac{|(u + c)_a - (u + c)_b\,|}{(u + c)}\,. \tag{34}$$

For the numerical tests reported below, the appropriate values of $u + c$ across a Mach-3 shock showed a value of $\sigma \leqslant 0.5$, while the value derived from the original analysis was $\sigma \leqslant 0.2$. Successful calculations were in fact made with a value of $\sigma = 0.7$, and since stability analyses are not particularly tight, such a value is not unreasonable.

Now in two dimensions this procedure cannot be used, since the matrices $\| \partial f/\partial w \|$ and $\| \partial g/\partial w \|$ do not possess the same eigen values. One may, however, use the requirement

$$\langle G\tilde{q}, \tilde{q} \rangle \leqslant 1, \tag{35}$$

where q is an arbitrary unit vector. Following Burstein [10], the result shown by Eq. (26) is obtained. G does not act, however, upon the arbitrary unit vector $\tilde{q}$, but rather on the field vector $\tilde{w}$ whose components are interrelated. If Eq. (35) is replaced by

$$\langle G\tilde{w}, \tilde{w} \rangle / \langle \tilde{w}, \tilde{w} \rangle \leqslant 1, \tag{35a}$$

which implies that for equal damping in the x and y directions

$$\frac{\| A\tilde{w} \|}{\| \tilde{w} \|} \leqslant \left(2 + \frac{K}{2}\right)^{-2} \frac{K + 1}{2} \leqslant \frac{1}{6} \tag{35b}$$

(and a similar equation for B), then this inequality can be investigated for some special values of $\tilde{w}$, albeit because of its complexity not for general values. For conditions near the stagnation point and the sonic line (where the instabilities tend to arise) one may find the following approximate results for $\gamma = 1.4$

(1) $u = v = 0,$ $\qquad\qquad \sigma \leqslant 2.5/\sqrt{6}$

$$
\begin{array}{llll}
(2)\ u = v = c, & \sigma \leqslant 2.2/\sqrt{6} & c \gg c_0 \\
 & \leqslant 2/\sqrt{6} & c \ll c_0 \\
 & \leqslant 1.7/\sqrt{6} & c = c_0 \\
(3)\ u = c \quad v = 0, & \sigma \leqslant 2.5/\sqrt{6} & c \gg c_0 \\
 & \leqslant 2/\sqrt{6} & c = c_0 \\
 & \leqslant 2/\sqrt{6} & c \ll c_0 .
\end{array}
$$

With these results it is plausible to find calculations successfully completed when $\sigma \sim 0.7$. Burstein's inability to utilize a value of σ greater than 0.35, however, remains inexplicable, though it is possible that difference methods which are too sophisticated are very susceptible to small changes in the order of the computations.

Richtmyer's Method

Richtmyer's method [13] is a two-step approximation to the Lax-Wendroff formulation. The appropriate difference forms in one-dimension are

$$\tilde{w}(t + \Delta t, x) = \frac{1}{2}\,[\tilde{w}(t, x + \Delta x) + \tilde{w}(t, x - \Delta x)] - \frac{\Delta t}{2}\,\frac{\partial \tilde{f}(t, x)}{\partial x}, \qquad (36a)$$

$$\tilde{w}(t + 2\Delta t, x) = \tilde{w}(t, x) - \Delta t\,\frac{\partial \tilde{f}(t + \Delta t, x)}{\partial x}. \qquad (36b)$$

The necessary stability requirement is

$$\sigma \leqslant 1/\sqrt{2}$$

Numerical tests with this method showed that it was susceptible to the same instabilities as the original Lax–Wendroff method and that the allowable computational time increments (Δt) were not much larger.

III. Problems and Results

One-Dimensional Shock Wave

Probably the simplest problem which can be numerically solved and which yields worthwhile information is the passage of a one-dimensional shock wave through a perfect gas and its reflection from a solid wall. For this problem, initial conditions were set at unity, and then at the left edge of the field the conditions appropriate to a Mach-3 shock were maintained.

Figure 1 indicates the results obtained using Lax's method. It is apparent that $\sigma \leqslant 0.7$ for acceptable overshoot of the pressure, although stable computations were carried out with $\sigma = 0.95$. It should be noted that the width of the shock and the overshoot are inversely related and depend directly on σ. [*Note added in proof.* A stability analysis for two-dimensional algorithms with $\Delta x = \Delta y$ reveals that $\Delta t \leqslant [\Delta x/(V + c)](1/\sqrt{2})$ or $\sigma < 0.707$ in agreement with the results shown on figure 1. P. D. Lax [16] has indicated that the correct stability criteria is given by Eq. (16a) where $\Delta x \Delta y/(\Delta x^2 + \Delta y^2)^{1/2}$ is the radius of the characteristic cone which is tangent to the chords connecting the nodal points used in the computation of $\tilde{w}$ at a given point.]

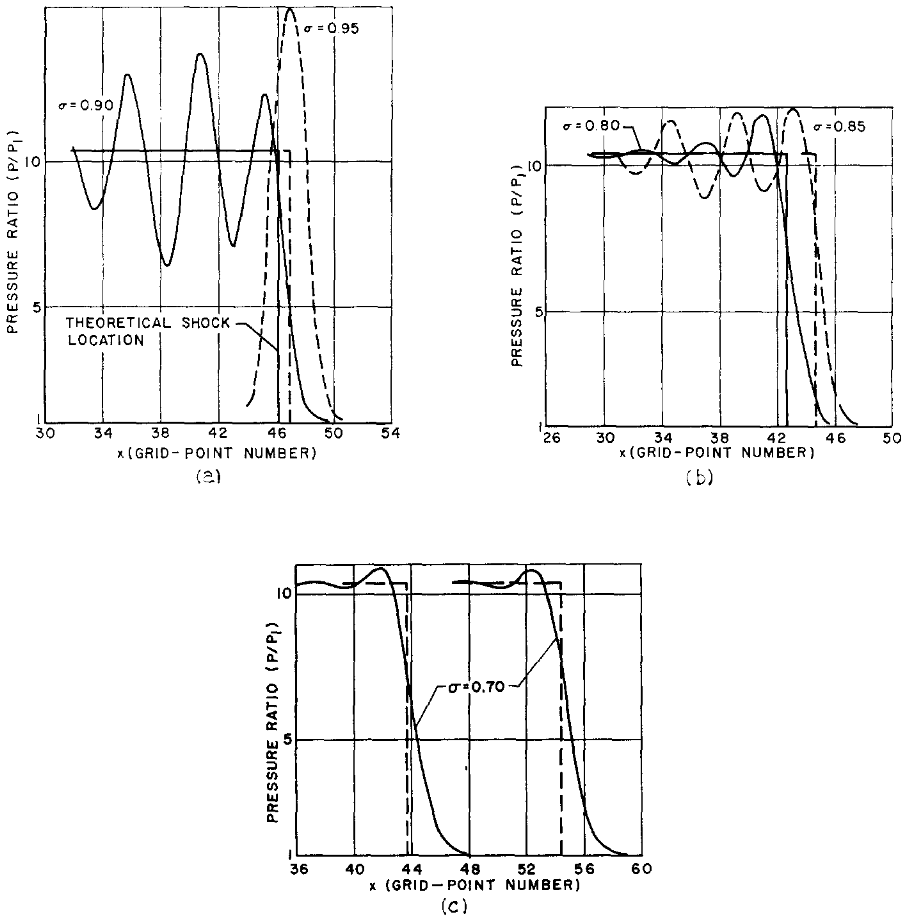


FIG. 1. Pressure profiles for one-dimensional passage as given by Lax's method. (a) $\sigma = 0.95$, 0.90; (b) $\sigma = 0.85$, 0.80; (c) $\sigma = 0.70$.

Figure 2 shows the results obtained with the modified Lax–Wendroff method. Note that (*1*) the shock width is significantly less than that achieved in Lax's method, and (*2*) the width is quite constant with varying K and σ.

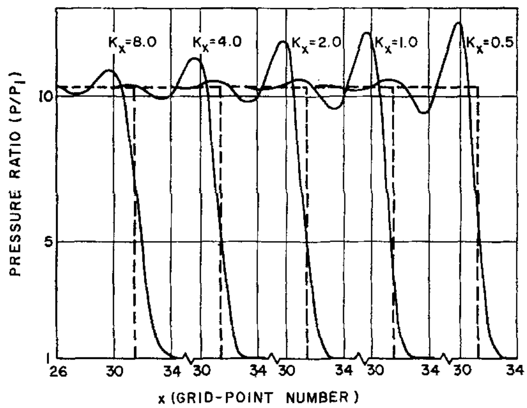


FIG. 2. Pressure profiles for one-dimensional passage as given by Lax–Wendroff method with nonlinear artificial viscosity term: $\sigma = 0.7$, $t = 10$ to $16\,\Delta x/c_1$.

The results of Rusanov's method are given in Fig. 3. It is quite apparent that ω must be greater than $\bar{\sigma}$. For $\omega \sim 1/\bar{\sigma}$, the shock is broad but with no pressure overshoot. This feature is of importance in transient external aerodynamics (see

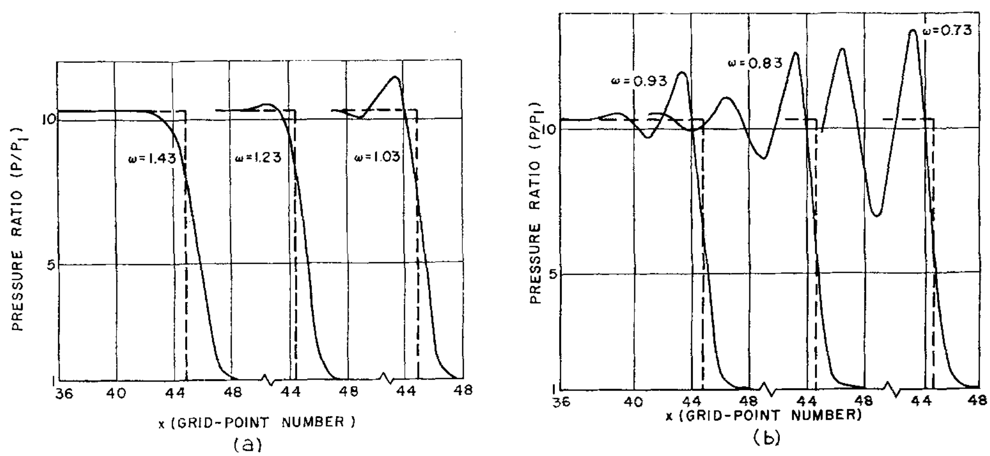


FIG. 3. Pressure profiles for one-dimensional passage as given by Rusanov's method. (a) $\bar{\sigma} = 0.7$, $\omega = 1.03, 1.23, 1.43$; (b) $\bar{\sigma} = 0.7$, $\omega = 0.73, 0.83, 0.93$.

Section IV below). This method appears to be better than Lax's, but poorer than the Lax–Wendroff; the ease of programming and the high computational speed, however, cannot be overlooked.

Figure 4 is given for completeness. Landshoff's method appears to yield satisfactory results upon examination of the pressure, but the density trace reveals incipient problems. This figure emphasizes the importance of examining all of the fluid mechanical variables when evaluating a method. None of the other methods showed such behavior.

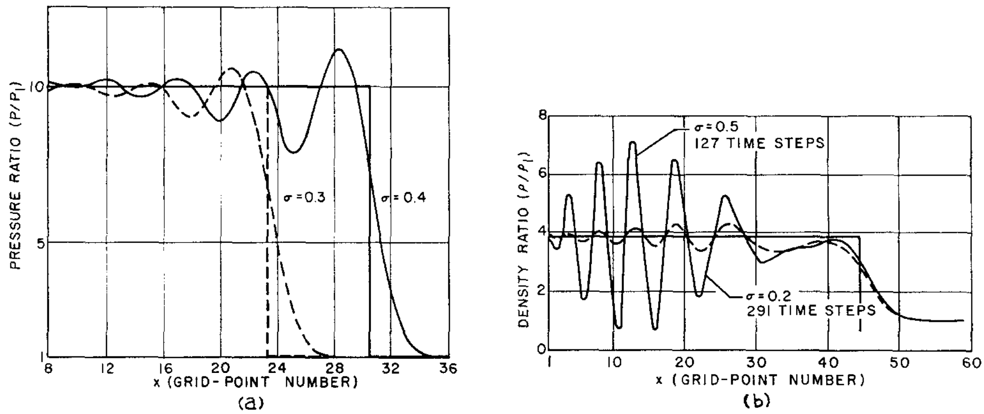


FIG. 4. Profiles for one-dimensional passage as given by Landshoff's method. (a) Pressure profile: $\alpha = 2.0$, $\sigma = 0.3, 0.4$; (b) density profile: $\alpha = 3.5$.

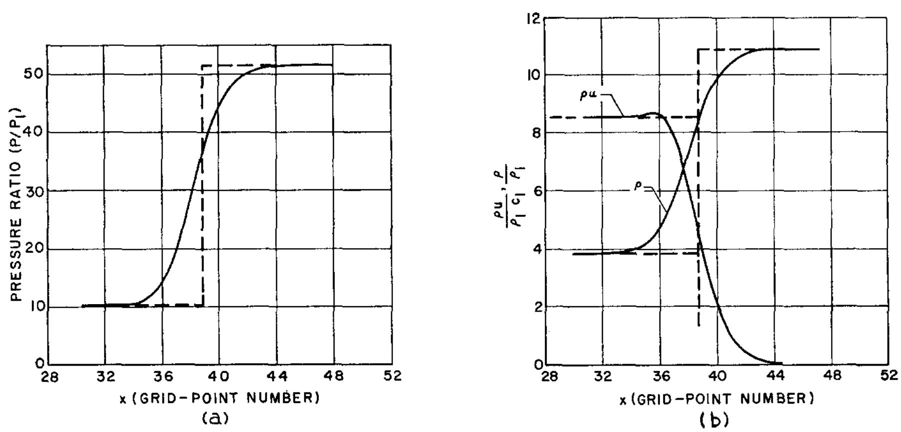


FIG. 5. Profiles for reflected shock as given by Lax's method: $M_s = 3.0$, $\sigma = 0.7$. (a) Pressure profile; (b) density profile.

Figures 5–8 are pressure-density traces for the reflected shock. Several important features are noticeable: (*1*) shock widths have increased very significantly; (*2*) shock widths for the Lax-Wendroff method are no longer independent of K or σ; and (*3*) with Rusanov's method it is now quite difficult to produce a very significant pressure overshoot. From Fig. 8 we can conclude that Landshoff's artificial viscosity method suffers from such curious problems that it should not be used in place of any of the other methods.

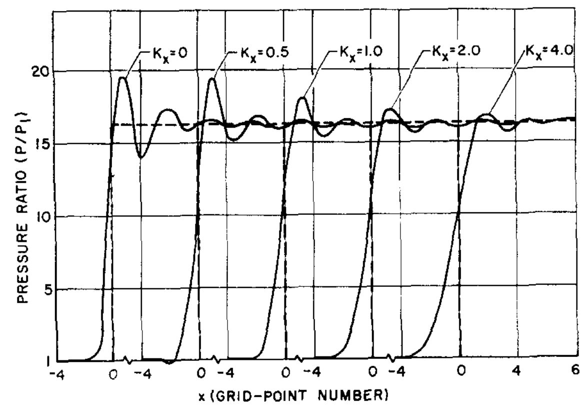


FIG. 6. Pressure profiles for reflected shock as given by Lax–Wendroff method with nonlinear artificial viscosity term: $\sigma = 0.7$, $t = 75\, \Delta x / c_1$.

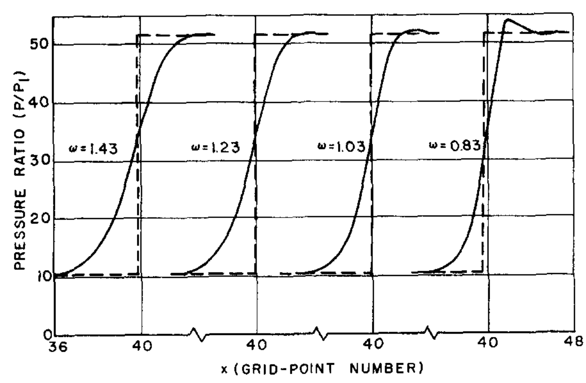


FIG. 7. Pressure profiles for reflected shock as given by Rusanov's method: $\sigma = 0.7$.

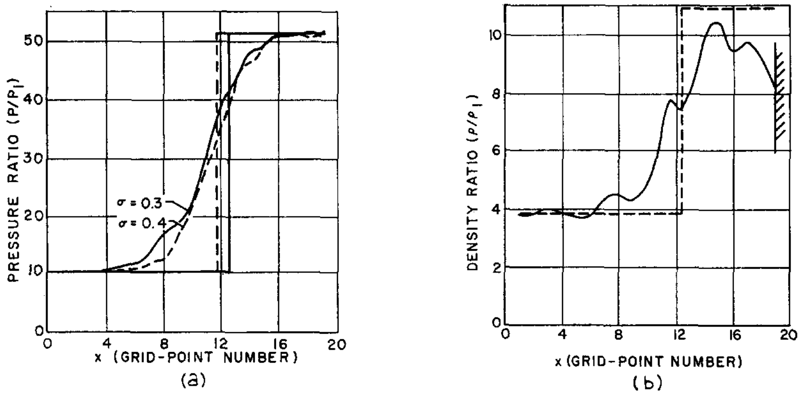


FIG. 8. Profiles for reflected shock as given by Landshoff's method. (a) Pressure profiles: $\alpha = 2.0$; (b) density profile: $\alpha = 0.2$, $\sigma = 0.3$.

Two-Dimensional Step

The second problem considered was a flat-faced step in two-dimensional flow. Figures 9–12 show the state of the flow at a time when it was nearly steady. Lax's results are in approximate agreement with physical intuition, although the spatial resolution is too poor to define the reflected shock at the upper surface or on the horizontal portion of the step. Rusanov's method (Fig. 10) quite adequately defines the shock structure and shows in good definition the expansion fans at the step corner. It also indicates the overexpansion of the flow at the corner and the lambda shock on the step surface. Using these results, however, only an experienced aerodynamicist would be able to determine what the true shape of the shock structure is. A measurement of the incident shock angle at the upper wall, for example, reveals that a regular reflection is not possible, although Fig. 11 gives the impression of a regular reflection. A close examination of the plot reveals the perpendicular nature of the $P/P_1 = 4$ isobar, and in fact the lambda shock on the upper wall is clearly seen in Fig. 12. On the other hand, the lower lambda shock is quite distorted (compare Fig. 11) on account of the nonsteady results. Because of the higher-order time term in the Lax–Wendroff method, a numerical steady state is more difficult to achieve than with the other methods.

Cone in a Mach-11 Flow

The third example tested was a cone of 11.5-degree half-angle in a Mach-11 flow. The calculations were performed for cylindrical coordinates using Rusanov's method (Fig. 13) and for spherical coordinates with Lax's method [14]. The conical shock flow was best represented by the spherical coordinates. Since shocks typically cover six mesh points, it is necessary to node the field very finely in the expected area of the shock.

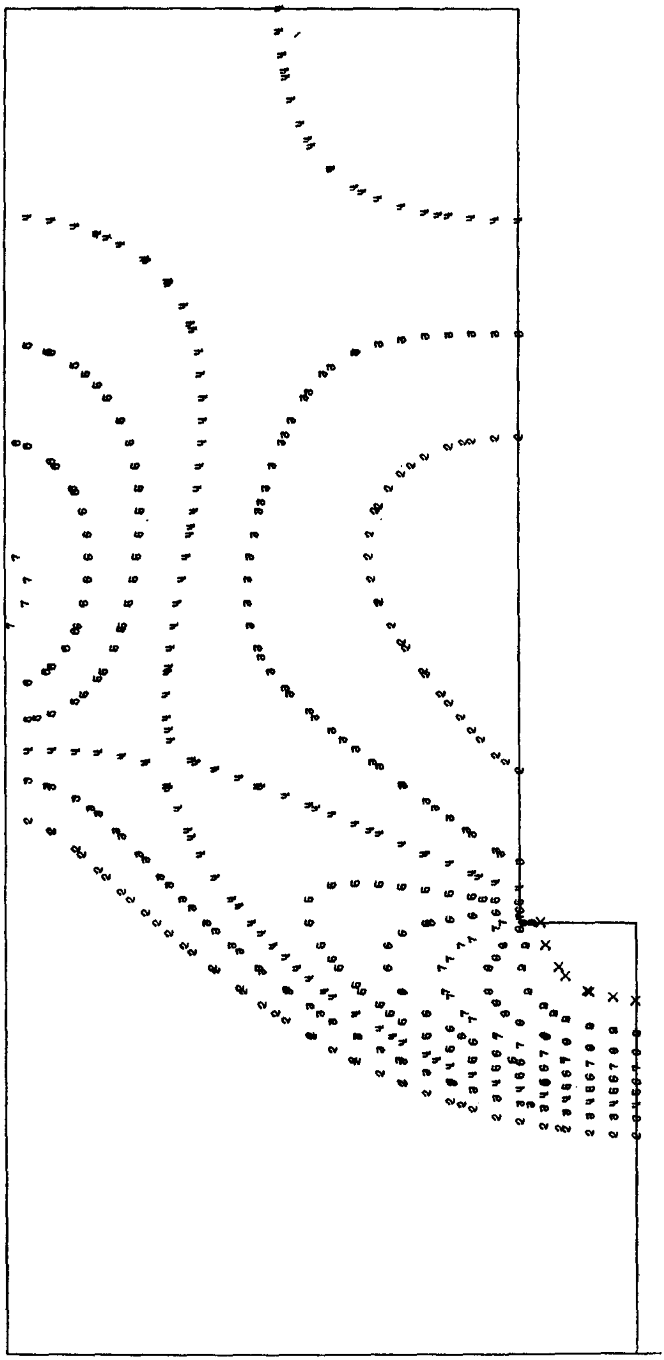


FIG. 9. Isobars (P/P_1) for two-dimensional step as given by Lax's method: $\sigma = 0.7$, $n = 850$, $n\Delta t = 147.5\ \Delta x/c_1$ ($x = 10$, $* = 11$, $\Delta = 12$).

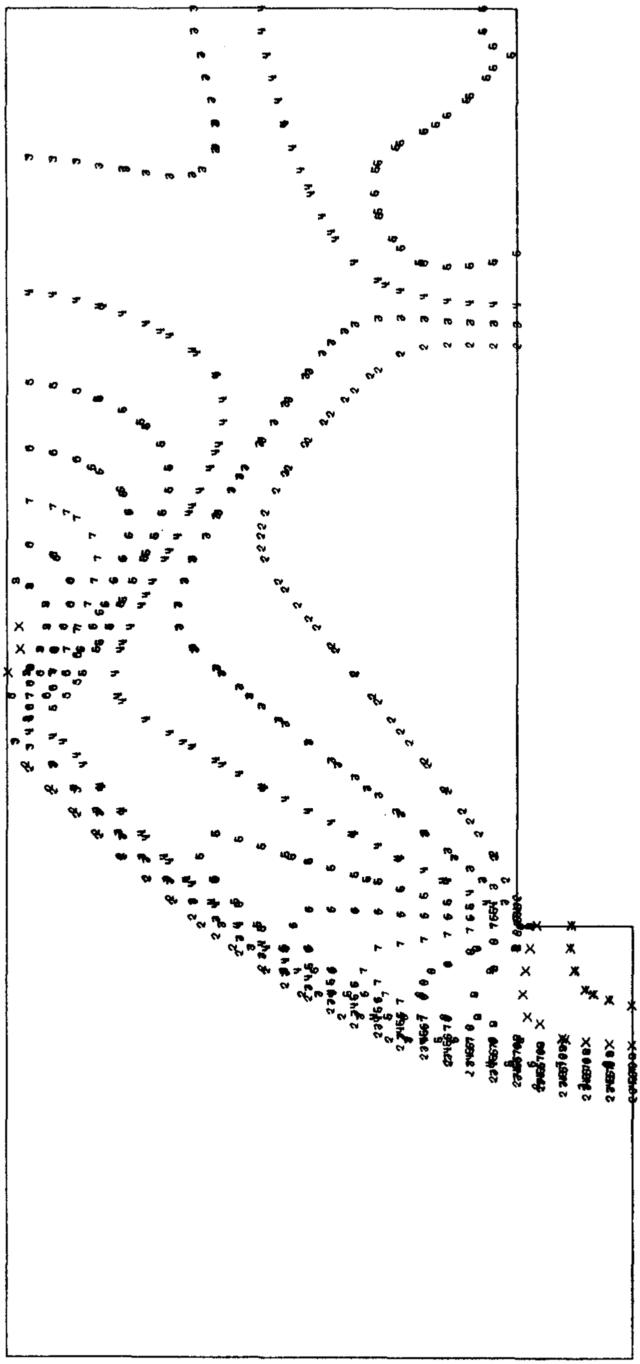


FIG. 10. Isobars (P/P_1) for two-dimensional step as given by Rusanov's method: $\sigma = 0.5$, $\omega = 0.4$, $n = 1000$, $n\Delta t = 122.3\ \Delta x/c_1$ ($x = 10$, $* = 11$, $\Delta = 12$).

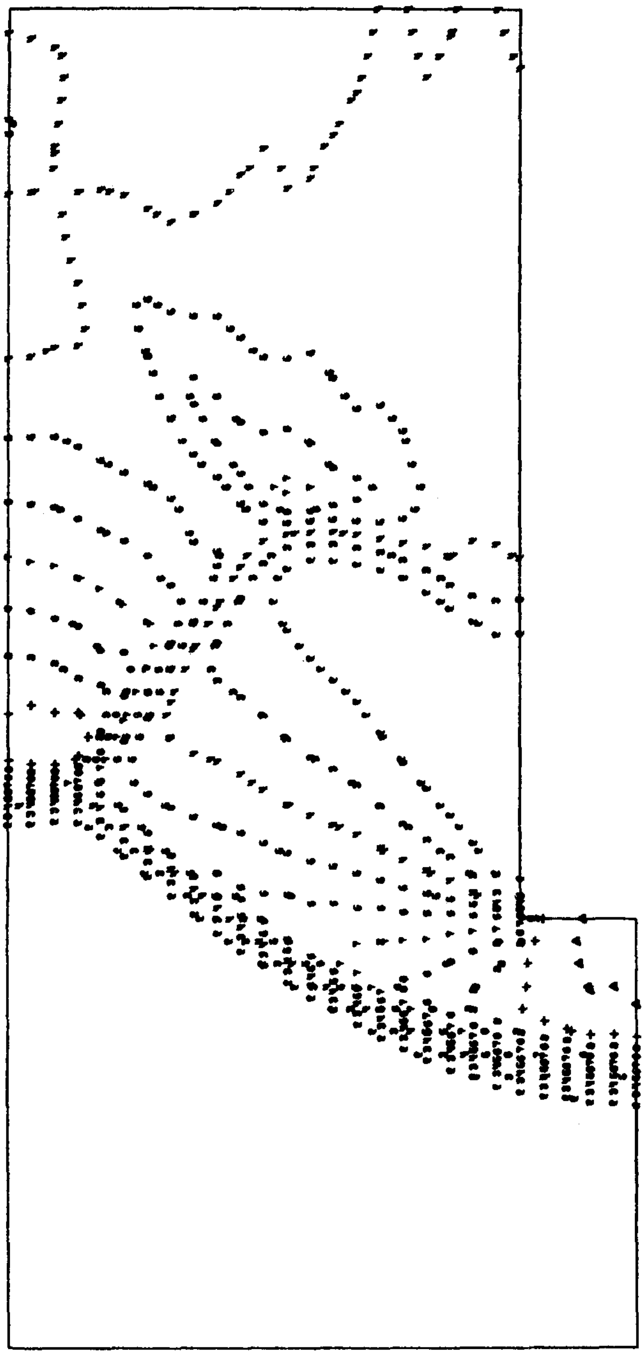


FIG. 11. Isobars (P/P_1) for two-dimensional step as given by Lax–Wendroff method: $\sigma = 0.7$, $n = 1397$, $n\Delta t = 238.8\ \Delta x/c_1$ ($+ = 10$, $\Delta = 11$) (K was changed from 2.0 to 4.0 at $n\Delta t = 150.9\ \Delta x/c_1$).

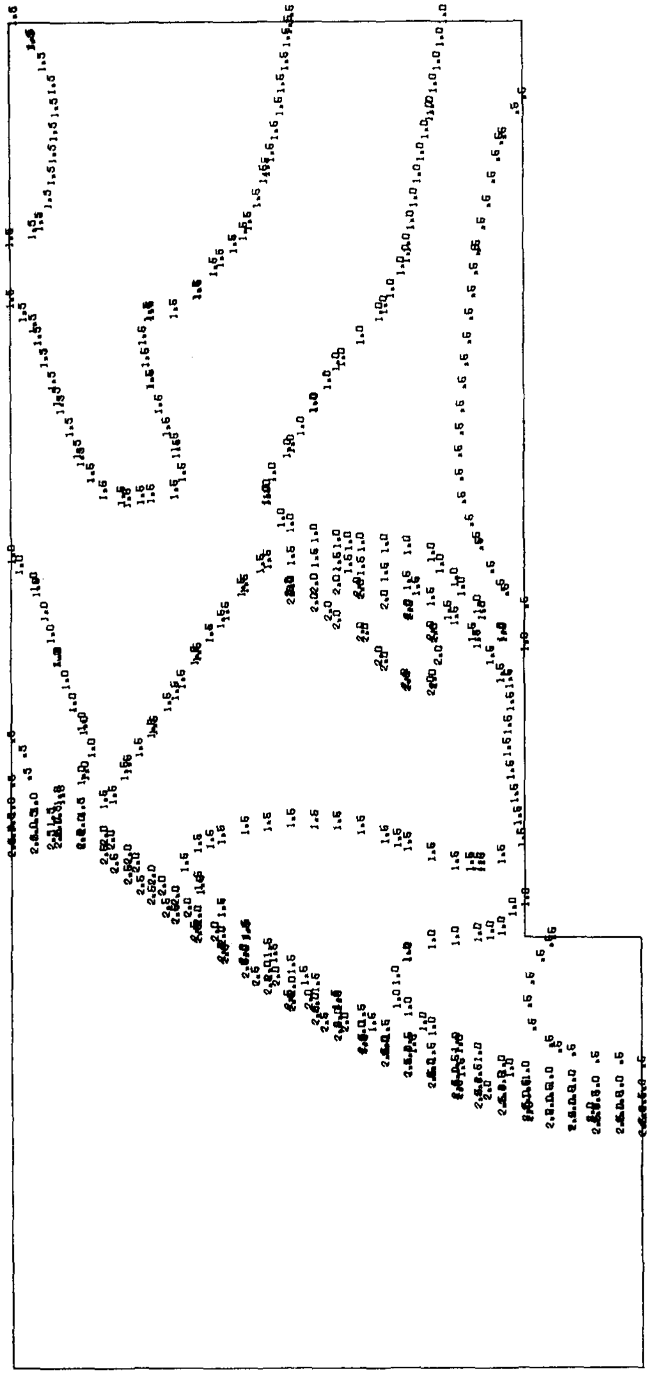


FIG. 12. Mach-number profiles for two-dimensional step as given by Lax–Wendroff method: $\sigma = 0.7$, $n = 1420$, $n\Delta t = 231.8\ \Delta x/c_1$ (K was changed from 2.0 to 4.0 at $n\Delta t = 150.9\ \Delta x/c_1$).

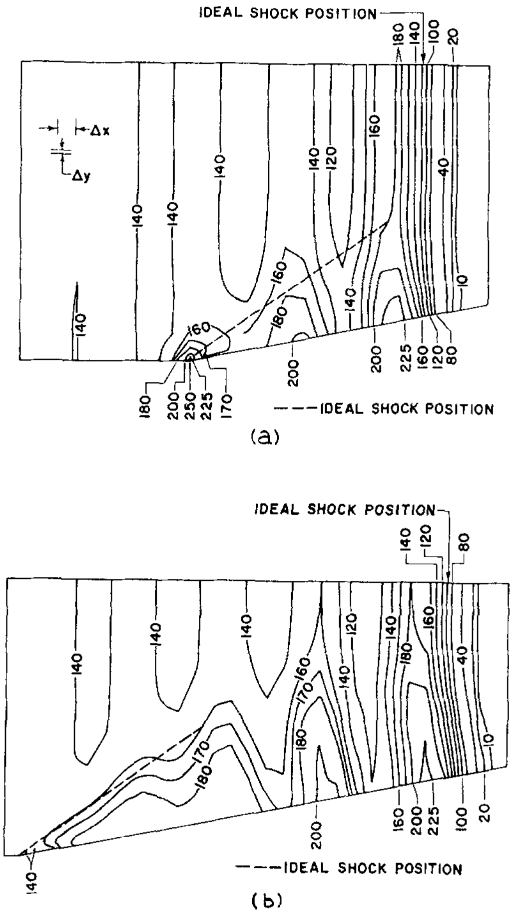


FIG. 13. Passage of a Mach-11 shock wave over an 11.5-degree half-angle cone at zero angle of attack. (a) Blunted tip; (b) sharp tip.

IV. LIMITATIONS AND IMPROVEMENTS

Transient Flow behind Shock Waves

It is apparent in the figures for the moving shock that large oscillations in density, pressure, and energy may be present in all but the most highly damped calculations. In cases where a shock passes over a body and forms a steady bow shock (e.g., passage over a cone or wedge), these oscillations may frequently be of the same magnitude as the changes the flow experiences in crossing the bow shock—

particularly if it is attached. They may therefore radically alter the flow variables on the vehicle's surface and produce anomalous transient shock shapes and pressure profiles. Figure 13 displays these effects and shows that the flow becomes smooth only after the shock is far downstream from the tip. Unless these conditions are recognized when they occur, very faulty interpretations of the results will be made.

When the flow properties behind a moving shock or on a body after a shock has passed over it are desired, extreme care must be taken not only in interpreting the results but also in making the calculations. Only if the oscillations are small compared to the expected changes can the numerical methods be used with moderate amounts of damping. An exception is provided by the Lax–Wendroff method, since its spatial resolution—even with moderate damping—is good, and the oscillations behind shocks are small. When using the other methods in cases where interference is expected, the damping (i.e., artificial viscosity) must be increased, although this increase may widen and smear the shocks and pressure waves so badly that the results may not be acceptable—particularly if the local transient pressure history on the body is desired to determine the dynamic movements of the body.

Corners and Singular Points

Probably the most difficult configurations to handle are those which involve corners and singular points. Through extensive numerical tests it has been found best to handle corner points by neglecting them and rounding the sharp edge slightly. Attempts to consider sharp-edged corners have shown that the flow field in the neighborhood of the corner is radically affected and frequently generates numerical instabilities.

Singular points are far more difficult. Consider the nodal point at the tip of a cone. It is apparent that, in an all-numerical approach to the flow problem, this nodal point must either be assigned to the conical surface or to the free stream. In the former case, the cone effectively has a cusp shape and in the latter is effectively blunted. The latter effect is permissible only if the cone's half-angle is large; otherwise the flow turning angle will be too large and the flow will tend to separate on passing to the conical surface. Figure 13(b) is a graphic example of this problem coupled with the transient shock problem. The fluctuation of pressure behind the shock interacting with the flow separation at the blunt nose produces apparent vortices which shed very nearly periodically from the nose. The singular point of the conical tip in spherical coordinates is a more difficult problem, for which one should see Reference [14].

Overlay

To reduce computational time it is frequently of advantage to alter the mesh spacing in the flow field. This operation can be most conveniently performed by changing the spacing by multiples. It is necessary, however, to remember that the artificial viscosity is dependent upon the mesh size and must be changed to maintain uniform shock thickness.

Almost all difference equations yield shock thicknesses which are proportional to the square root of the artificial viscosity and an overshoot which is inversely proportional to the artificial viscosity. To produce minimum shock thicknesses with an acceptable degree of overshoot, an optimum value of the artificial damping coefficient must be used. For Lax's method, the damping coefficient is

$$\Delta x^2/4\Delta t$$

and the optimum thickness—six mesh points—and acceptable overshoot are obtained when

$$\Delta t = 0.7(\Delta t)_{\max}$$

and

$$\Delta t_{\max} = \min\left\{\frac{\Delta x}{(u + c)_{\max}}, \frac{\Delta y}{(v + c)_{\max}}\right\}.$$

For the purposes of discussion we may simply require that the time increment be that associated with the space increment through the CFL stability requirement.

Now consider a flow region which has been divided into two subregions, one with Δx_1, the other with Δx_2, where

$$\Delta x_2 > \Delta x_1 .$$

For an optimum shock thickness in both regions, the damping coefficients in each region must be

$$d_1 \sim \frac{\Delta x_1^2}{\Delta t_1}, \qquad d_2 \sim \frac{\Delta x_2^2}{\Delta t_2}.$$

For ease of computation, however, we use the same time increment in both regions; and for stability this increment must be the smaller of the values Δt_1 and Δt_2. Thus in region 2 ($\Delta x = \Delta x_2$), the damping will be

$$d_2 \sim \frac{\Delta x_2^2}{\Delta t_1} = \frac{\Delta x_2^2}{\Delta t_2}\frac{\Delta t_2}{\Delta t_1} = \frac{\Delta x_1^2}{\Delta t_1}\frac{\Delta x_2}{(u + c)_1}\frac{(u + c)_2}{\Delta x_1}.$$

If $(u + c)_{\max}$ is approximately the same in both regions,

$$d_2 \sim \frac{\Delta x_2^2}{\Delta t_1} = \frac{\Delta x_2^2}{\Delta t_2}\frac{\Delta x_2}{\Delta x_1} = \frac{\Delta x_2}{\Delta x_1}\left(\frac{\Delta x^2}{\Delta t}\right)_{\text{optimum}}$$

Thus, in the use of the "overlay" technique, the coefficient in $\bar{\bar{w}}$ should be

$$\tfrac{1}{4}(\Delta x_1/\Delta x_2)$$

to yield the best shock shapes. Figure 14 a shows the result of not correcting the damping coefficient, while Fig. 14(b) shows the result when the correction is applied.

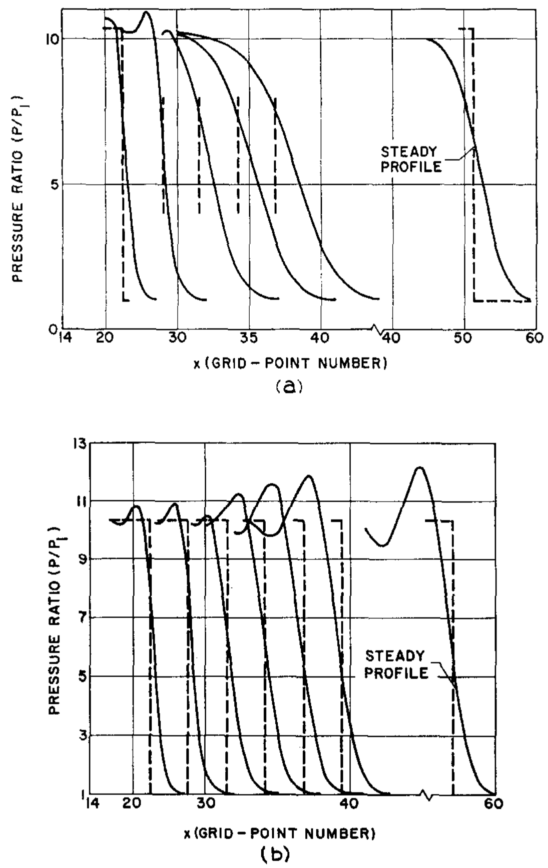


FIG. 14. Pressure profiles for one-dimensional passage as given by Lax's method with overlay for a grid change at $x = 30$. (a) Constant Damping Coefficient; (b) Different (Optimum) Damping Coefficients.

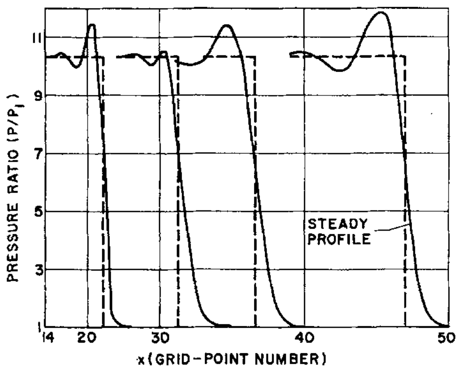
When using Rusanov's method, there is no necessity to incorporate any change in the damping coefficient. In this case the coefficient is

$$d \sim \omega\bar{\sigma}\,\frac{\Delta y^2}{\Delta^2}\,\frac{\Delta x^2}{\Delta t} = \omega(V + c)\,\Delta x\,\frac{\Delta y}{(\Delta x^2 + \Delta y^2)^{1/2}}$$

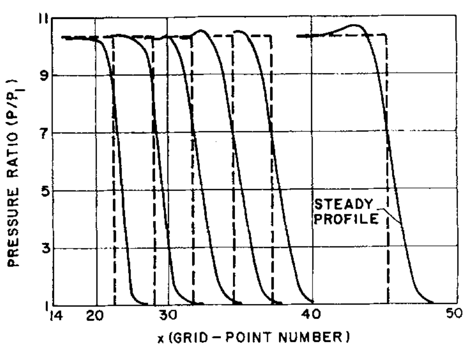
and we see that it is independent of the time increment used. Hence the same value of ω may be used in each region. Furthermore,

$$\bar{\sigma} < \omega < 1/\bar{\sigma}$$

and $\omega = 1$ represents the geometric center of the line extending from $\bar{\sigma}$ to $1/\bar{\sigma}$. Thus if $\omega \sim 1$ it will continue to represent the center, regardless of the value of $\bar{\sigma} \propto \Delta t/\Delta x$. Hence the shock shape should not be significantly different for the different sub-regions. Figures 15 a and 15 b show the results for Rusanov's method.



(a)



(b)

FIG. 15. Pressure profiles for one-dimensional passage as given by Rusanov's method with overlay for a grid change at $x = 30$. (a) $\bar{\sigma} = 0.7$, $\omega = 1.0$ (for entire field); (b) $\bar{\sigma} = 0.7$, $\omega = 1.4$ (for entire field).

One must remember that, although the shock width will remain constant at about six mesh points, the physical thickness doubles if $\Delta x_2 = 2\Delta x_1$. This increase can only be avoided by reducing the damping factor. Since the overshoot, however, is inversely proportional to the damping, the shock may either exhibit unacceptable overshoot or actually become unstable.

Since it is difficult to maintain an equal shock thickness in the two regions, it is best to change Δx across the entire y dimension of the flow field rather than only across a portion of it. Experiments performed with changes so that

$$\Delta x = \Delta x_1 \text{ for } y > y^*, \qquad \Delta x = \Delta x_2 \text{ for } y < y^*,$$

have shown that the computed flow field frequently becomes distorted enough to cause difficulty in interpreting the results.

Computational Times

Table I presents the computational times for each of the differencing techniques studied. When an equal number of nodal points are used, the Lax-Wendroff method is capable of spatial resolution approximately three times greater than that of Lax's method. Equal spatial resolution can be obtained by Lax's method by increasing the number of nodal points, at the cost of requiring total computational times nearly equal to those of the Lax–Wendroff method. One of the major features of the latter method, however, is the accuracy of its temporal resolution, the error being of the order of Δt^3, while that of the other methods is of the order of Δt^2. Thus, even though Lax's method can give equal spatial resolution, the Lax–Wendroff method is to be preferred for transient problems and problems in which overshoot must be at a minimum.

TABLE I

TIMES REQUIRED TO COMPUTE CONDITIONS AT AN INTERIOR POINT ON THE CDC 3600[a]

Method	Time (msec)	Ratio
Lax	1.7	1.0
Landshoff	2.0	1.2
Rusanov	2.3	1.35
Richtmyer	1.7	1.0
Lax–Wendroff (modified)	6.5	3.8

[a] An interior point is any point wholly within the flow—i.e., any point not a boundary point.

Recommendations

Of the three major methods—Lax's, Rusanov's, and the Lax–Wendroff—Rusanov's is the preferable. Lax's method is the simplest to program but offers the least flexibility. The Lax–Wendroff yields the best spatial and temporal resolution, but the difficulties in programming this method (particularly at corners and singular points) render it useful only to those interested in numerical methods for their own sake. Rusanov's method is easily programmed, offers an adjustable parameter, and yields very good resolution.

One point should be noted. Numerical tests have shown that if the number of nodal points is increased to require equal computational times, all methods are equivalent in accuracy. If temporal resolution is desired, however, then only the Lax–Wendroff method should be used. Figure 16 shows the development of the shock standoff distance for the step problem. A simple solution based upon Miles [15] is also given. It is apparent that the Lax–Wendroff method is much preferable to the other two methods.

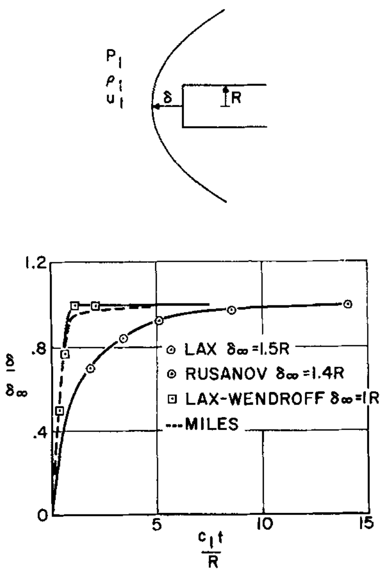


FIG. 16. Shock standoff distance on axis of symmetry.

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