Rudin, Principles of Mathematical Analysis, Chapter 1 Answer Key

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

   Answer: Assume otherwise that $r+x$ is rational with sum α. We write $r+x = \alpha \implies x = \alpha - r$. This implies that x can be expressed as the difference of two rational rational numbers, $\alpha - r$, which is rational. Thus, $r + x$ must be irrational. $\rightarrow\leftarrow$

   We follow a similar argument for the second part. Assume otherwise that rx is rational with product $\frac{\beta}{\gamma}$. We write $rx = \frac{\beta}{\gamma} \implies x = \frac{\frac{\beta}{\gamma}}{r}$. This implies that x can be expressed as a (compound) fraction, which is violates the irrational property. Thus, rx must be irrational. $\rightarrow\leftarrow$

2. Prove that there is no rational number whose square is 12.

   Answer: Assume otherwise that there exists a rational $p = \frac{m}{n}$ for m, n not both even, such that:

   $$p^2 = 12$$

   We can re-arrange as follows:

   $$\left(\frac{m}{n}\right)^2 = 12$$

   $$m^2 = 12n^2$$

   Since 12 is even, and the product of two terms involving an even number is always even, the RHS must be even. So, by the equality, m^2 is also even, which implies that m is even and n is odd. We can write $m = 2k$ for some integer k and substitute:

   $$4k^2 = 12n^2$$

   $$k^2 = 3n^2$$

   We know that n is odd, hence $3n^2$ is odd, which implies k^2 and k are also odd. We can now express $k = 2j + 1$ and $n = 2l + 1$ for some integers j, l and substitute:

$$(2j+1)^2 = 3\left(2l+1\right)^2$$

$$4j^2+4j+1 = 3\left(4l^2+4l+1\right)$$

$$4j^2+4j+1 = 12l^2+12l+3$$

$$4j^2+4j-12l^2-12l = 2$$

$$4\left(j^2+j-l^2-l\right) = 2$$

This equality cannot hold because 4 is not a multiple of 2. Thus, p must be irrational. $\rightarrow\leftarrow$

3. Prove Proposition 1.15.

4. Let E be a nonempty subset of an ordered set; suppose x is a lower bound of E and β is an upper bound of E. Prove that $x \leq \beta$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$