Rudin, Principles of Mathematical Analysis, Chapter 1 Answer Key

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

   Answer: Assume otherwise that $r+x$ is rational with sum α. We write $r+x = \alpha \implies x = \alpha - r$. This implies that x can be expressed as the difference of two rational rational numbers, $\alpha - r$, which is rational. Thus, $r + x$ must be irrational. $\rightarrow\leftarrow$

   We follow a similar argument for the second part. Assume otherwise that rx is rational with product $\frac{\beta}{\gamma}$. We write $rx = \frac{\beta}{\gamma} \implies x = \frac{\frac{\beta}{\gamma}}{r}$. This implies that x can be expressed as a (compound) fraction, which is violates the irrational property. Thus, rx must be irrational. $\rightarrow\leftarrow$

2. Prove that there is no rational number whose square is 12.

   Answer: Assume otherwise that there exists a rational $p = \frac{m}{n}$ for m, n not both even, such that:

   $$p^2 = 12$$

   We can re-arrange as follows:

   $$\left(\frac{m}{n}\right)^2 = 12$$

   $$m^2 = 12n^2$$

   Since 12 is even, and the product of two terms involving an even number is always even, the RHS must be even. So, by the equality, m^2 is also even, which implies that m is even and n is odd. We can write $m = 2k$ for some integer k and substitute:

   $$4k^2 = 12n^2$$

   $$k^2 = 3n^2$$

   We know that n is odd, hence $3n^2$ is odd, which implies k^2 and k are also odd. We can now express $k = 2j + 1$ and $n = 2l + 1$ for some integers j, l and substitute:

$$(2j+1)^2 = 3\left(2l+1\right)^2$$

$$4j^2+4j+1 = 3\left(4l^2+4l+1\right)$$

$$4j^2+4j+1 = 12l^2+12l+3$$

$$4j^2+4j-12l^2-12l = 2$$

$$4\left(j^2+j-l^2-l\right) = 2$$

This equality cannot hold because 4 is not a multiple of 2. Thus, p must be irrational. $\rightarrow\leftarrow$

3. Prove Proposition 1.15.

   Answer: We want to prove the following statements that follow from the axioms of multiplication:

   - If $x \neq 0$ and $xy = xz$ then $y = z$.

$$xy = xz$$

$$x^{-1}(xy) = x^{-1}(xz) \quad \text{Existence of an inverse element}$$

$$\left(x^{-1}x\right)y = \left(x^{-1}x\right)z \quad \text{Associativity}$$

$$(1)y = (1)z \quad \text{Definition of inverse}$$

$$y = z \quad \text{Identity element}$$

   - If $x \neq 0$ and $xy = x$ then $y = 1$.

$$xy = x$$

$$x^{-1}(xy) = x^{-1}(x) \quad \text{Existence of an inverse element}$$

$$\left(x^{-1}x\right)y = \left(x^{-1}x\right) \quad \text{Associativity}$$

$$(1)y = 1 \quad \text{Definition of inverse}$$

$$y = 1 \quad \text{Identity element}$$

- If $x \neq 0$ and $xy = 1$ then $y = x^{-1}$.

$$xy = 1$$

$$x^{-1}(xy) = x^{-1}(1) \quad \text{Existence of an inverse element}$$

$$\left(x^{-1}x\right)y = x^{-1} \quad \text{Associativity, identity element}$$

$$(1)y = x^{-1} \quad \text{Definition of inverse}$$

$$y = x^{-1} \quad \text{Identity element}$$

- If $x \neq 0$ then $\frac{1}{x^{-1}} = x$.

$$\frac{1}{x^{-1}} = x$$

$$x^{-1}\left(\frac{1}{x^{-1}}\right) = \left(x^{-1}x\right) \quad \text{Multiply by } x^{-1}$$

$$1 = 1 \quad \text{Definition of inverse}$$

4. Let E be a nonempty subset of an ordered set; suppose x is a lower bound of E and β is an upper bound of E. Prove that $x \leq \beta$.

   Answer: By Definition 1.7, since β is an upper bound, then this must imply that $\alpha \leq \beta$ for all $\alpha \in E$. Simiarily, since x is a lower bound, then this also must imply that $x \leq \alpha$ for all $\alpha \in E$. Hence, $x \leq \alpha \leq \beta$ and $x \leq \beta$ as desired.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

   Answer: Denote $\alpha = \inf A$. By definition 1.8, α is a lower bound of A and no $x > \alpha$ is a lower bound of A. So, $\alpha \leq x, \forall x \in A$ implies $-\alpha \geq -x$ for $-x \in -A$, which means α is an upper-bound of of $-A$. In order to show $-\alpha$ is the *least* upper bound of $-A$, take some $\beta < -\alpha$, which implies $-\beta > \alpha$. Since $\alpha = \inf A$, there exists some $x \in A$ such that $\alpha < x < -\beta$ and $-x > \beta$, which implies β is not an upper bound of $-A$. Hence, $-\alpha = \sup(-A)$ and $\alpha = -\sup(-A)$.

6. Fix $b > 1$.

   (a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that
   $$(b^m)^{1/n} = (b^p)^{1/q}.$$
   Hence it makes sense to define $b^r = (b^m)^{1/n}$.

   (b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

   (c) If x is real, define $B(x)$ to be the set of all numbers b^r, where r is rational and $r \leq x$. Prove that
   $$b^x = \sup B(x)$$
   when r is rational. Hence it makes sense to define $b^x = \sup B(x)$ for every real x.

   (d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

7. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This is called the *logarithm of y to the base b*.)

   (a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

   (b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

   (c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

   (d) If w is such that $b^{-x} < y$, then $b^{1/n}x < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^x$.

   (e) If $b^r > y$, then $b^{r-(1/n)} > y$ for sufficiently large n.

   (f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

   (g) Prove that this x is unique.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

9. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

10. Suppose $z = a + bi$, $w = u + iv$, and
   $$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \quad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$
   Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.