Noether–Lefschetz General Complete Intersection K3s

Asher Auel, Henry Scheible

Department of Mathematics, Dartmouth College

Abstract

We construct K3 surfaces of degree 6 and 8 defined over $\mathbb{Q}$ with geometric Picard rank 1. While the existence of such surfaces follows from results of Terasoma, no explicit examples were known. Previously, explicit examples of K3 surfaces over $\mathbb{Q}$ with geometric Picard rank 1 were constructed in degree 4 by van Luijk and in degree 2 by Elsenhans and Jahnel, and we further develop their techniques to handle the new cases.

Introduction

It is a standard fact that the generic (algebraic) K3 surface over $\mathbb{C}$ has Picard rank 1, equivalently, is Noether–Lefschetz general. However, these could *a priori* exclude all surfaces defined over $\overline{\mathbb{Q}}$. Ellenberg addressed this in 2002 by proving that there exists a K3 surface in each degree defined over a number field with geometric Picard rank 1.

For complete intersection surfaces $X_{a_1,\ldots,a_d} \subset \mathbb{P}^{2+d}$ of type $(a_1,\ldots,a_d)$, Terasoma gave the following existence result over $\mathbb{Q}$, answering a question of Shioda.

Theorem 1 (Terasoma [3]). *There exist Noether–Lefschetz general complete intersection surfaces $X_{a_1,\ldots,a_d} \subset \mathbb{P}^{2+d}$ defined over $\mathbb{Q}$, except in the Fano cases $X_2 \subset \mathbb{P}^3$, $X_3 \subset \mathbb{P}^3$ and $X_{2,2} \subset \mathbb{P}^4$.*

Terasoma's proof, however, is not constructive. The K3 surfaces that are complete intersections are $X_4 \subset \mathbb{P}^3$, $X_{2,3} \subset \mathbb{P}^4$, and $X_{2,2,2} \subset \mathbb{P}^5$. Van Luijk [4] gave the first examples of Noether–Lefschetz general degree 4 K3 surfaces, and we give the first examples of Noether–Lefschetz general degree 6 and 8 K3 surfaces. Let $\mathcal{K}_d$ be the moduli space of polarized K3 surfaces of degree d.

Theorem 2. *The set of polarized Noether–Lefschetz general K3 surfaces defined over $\mathbb{Q}$ is Zariski dense in $\mathcal{K}_d$ for $d = 4, 6, 8$.*

Van Luijk proved this in degree 4 by finding specific quartic K3 surfaces over $\mathbb{F}_{p_1}$ and $\mathbb{F}_{p_2}$ with geometric Picark rank 2 and incompatible Néron–Severi lattices. Later, Elsenhans–Jahnel [1] developed a technique for K3 surfaces of degree 2 that only required working modulo a single prime. We further develop this later technique for the cases of K3 surfaces X of degree 6 and 8 by reducing to a degree 2 model modulo a single prime.

- In degree 6, we consider X containing a line, then projection from the line yields a degree 2 model.
- In degree 8, we consider X Hodge-isogenous to a degree 2 K3 surface containing a tritangent line.

In both cases, we construct the K3 surface over $\mathbb{F}_p$ to have a specific rank 2 Néron–Severi lattice, and then find a lift to $\mathbb{Q}$ that is incompatible.

Problem. *For K3 surfaces with degree > 8, even the existence of a such a surface over $\mathbb{Q}$ is still open.*

The Bombieri–Lang conjecture suggests that Noether–Lefschetz general K3 surfaces of degree d defined over $\mathbb{Q}$ will not be Zariski dense in $\mathcal{K}_d$ for $d > 122$, when it is of general type by Gritsenko, Hulek, and Sankaran.

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Prerequisites

The **geometric Picard rank** of a variety X over k is the $\mathbb{Z}$-rank of the Néron–Severi group $\mathrm{NS}(\overline{X})$, where $\overline{X} = X \times_{\mathsf{Spec}(k)} \operatorname{Spec}\overline{k}$. We denote this by $\rho(X)$.

A **K3 surface** is a smooth, projective, geometrically integral surface X over a field k whose canonical sheaf ω_X on X is trivial and $H^1(X, \mathcal{O}_X) = 0$.

For a K3 surface X, $\operatorname{Pic}(\overline{X}) \cong \mathrm{NS}(\overline{X}) \cong \mathbb{Z}^{\oplus\rho(X)}$ where $1 \le \rho(X) \le 22$.

As a consequence of the Tate conjecture, for a K3 surface X over $\mathbb{F}_p$, we have that $\rho(X)$ is always even [2, Corollary 17.2.9]. In particular, $\rho(X) \geq 2$.

Given an integral model of a K3 surface defined over $\mathbb{Q}$, reduction modulo p induces a **specialization homomorphism** $\overline{\mathrm{sp}} : \operatorname{Pic}(\overline{X}) \to \operatorname{Pic}(\overline{X}_p)$ which is injective and compatible with the intersection products on $\overline{X}$ and $\overline{X}_p = X_p \times_{\operatorname{Spec}(\mathbb{F}_p)} \operatorname{Spec}(\overline{\mathbb{F}}_p)$ [2, Proposition 2.10].

Lemma 3 ([1, Corollary 3.7]). *Let $p \neq 2$ be a prime number and X be a scheme proper and flat over $\mathbb{Z}$. Suppose that the special fiber X_p is nonsingular. Then, the cokernel of the specialization homomorphism $\overline{\mathrm{sp}} : \operatorname{Pic}(\overline{X}) \to \operatorname{Pic}(\overline{X}_p)$ is torsion free.*

Bounding the Picard rank of X_p using the Weil Polynomial

Let f be the Frobenius morphism on X_p, and let φ^* be its pullback on $H^2_{\text{ét}}(\overline{X}_p, \mathbb{Q}_\ell)$. We call the characteristic polynomial of φ^* the **Weil polynomial** of X_p.

The cycle class map $c_1 : \operatorname{Pic}(\overline{X_p}) \to H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$, is Galois equivariant (in particular, respects the action of Frobenius), and is an injection when X is a K3 surface [4].

Since a basis of divisor classes over $\overline{\mathbb{F}}_p$ is defined in some finite extension of $\mathbb{F}_p$, some power of φ^* acts as the identity on the image of $\operatorname{Pic}(\overline{X}_p)$, all eigenvalues of Frobenius acting on the image of $\operatorname{Pic}(\overline{X}_p)$ are roots of unity. If we let $\varphi^*(1)$ be the induced automorphism on $H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)(1)$, we then know that $\operatorname{rk}\operatorname{Pic}(\overline{X}_p)$ is bounded above (and equal to, by the Tate conjecture) by the number of eigenvalues of $\varphi^*(1)$ that are roots of unity, counted with multiplicity. Eigenvalues of $\varphi^*(1)$ differ from eigenvalues of φ^* by a factor of p, so we have the following lemma:

Lemma 4. *The geometric Picard rank $\rho(X)$ is bounded above (and equal to, by the Tate conjecture) by the number of eigenvalues λ of φ^* for which λ/p is a root of unity, counted with multiplicity.*

The best we can hope for over a finite field is geometric Picard rank 2, which we can find by randomly generating K3 surfaces until the Weil polynomial has only two scaled root of unity roots. `Magma` has an existing algorithm for computing Weil polynomials of degree 2 K3s contributed by Elsenhans and Jahnel.

Degree 8 Case

General polarized K3 surfaces (X, H) of degree 8 are complete intersections of three quadrics in $\mathbb{P}^5$

The three quadrics span a net of quadrics parameterized by $\mathbb{P}^2$. Let $C \subseteq \mathbb{P}^2$ be the locus of degenerate quadrics, which is a sextic curve. Let $Y \to \mathbb{P}^2$ be the discriminant double cover of the net, which is branched over C. When C is smooth, Y is a polarized K3 surface of degree 2.

We can also view $Y = M_X(v)$ as a moduli space of stable sheaves on X with Mukai vector $v = (2, H, 2)$, i.e. sheaves on X with rank 2, first chern class H, and Euler characteristic $2 + 2 = 4$. In this case, Y is a **Fourier-Mukai partner** of X.

Theorem 5. *Let $X_0 = V(q_1, q_2, q_3) \subset \mathbb{P}^5_{\mathbb{F}_{47}}$, where*

$$\begin{aligned}
q_1 &= 5x_0^2 + 6x_0x_1 + x_1^2 + 10x_0x_2 + 45x_1x_2 + x_2^2 + 6x_0x_3 \\
&\quad + 45x_1x_3 + 45x_2x_3 + 37x_0x_4 + 39x_1x_4 + 39x_2x_4 \\
&\quad + 2x_4^2 + 10x_0x_5 + 10x_1x_5 + 45x_2x_5 + 8x_4x_5 + 2x_5^2 \\
q_2 &= 43x_0^2 + 41x_0x_1 + 41x_0x_2 + 8x_1x_2 + 44x_2^2 + 2x_0x_3 \\
&\quad + 39x_1x_3 + 5x_3^2 + 45x_0x_4 + 2x_1x_4 + 45x_2x_4 + 10x_3x_4 + 3x_4^2 \\
&\quad + 43x_0x_5 + 6x_1x_5 + 2x_2x_5 + 2x_3x_5 + 10x_4x_5 + 3x_5^2 \\
q_3 &= 5x_0^2 + 45x_0x_1 + 46x_1^2 + 37x_0x_2 + 2x_1x_2 + 4x_2^2 + 2x_0x_3 \\
&\quad + 8x_1x_3 + 6x_2x_3 + 42x_3^2 + 8x_0x_4 + 39x_1x_4 + 43x_2x_4 + 4x_3x_4 \\
&\quad + 2x_1x_5 + 43x_2x_5 + 43x_3x_5 + 39x_4x_5 + 44x_5^2
\end{aligned}$$

If $X \subset \mathbb{P}^5_{\mathbb{Q}}$ is a degree 8 K3 surface reducing to X_0 modulo 47 whose degree 2 Fourier–Mukai partner Y has no tritangent lines over $\overline{\mathbb{Q}}$, then $\rho(X) = 1$.

The nonexistence of tritangent lines on the degree 2 Fourier–Mukai partner can be verified by a quick Groebner basis computation. We provide an explicit example.

Corollary 6. *Let $X = V(q_1, q_2, q_3) \subset \mathbb{P}^5_{\mathbb{Q}}$, where*

$$\begin{aligned}
q_1 &= -183x_0^2 + 382x_0x_1 - 234x_1^2 - 460x_0x_2 + 280x_1x_2 + 236x_2^2 + 6x_0x_3 \\
&\quad + 280x_1x_3 - 190x_2x_3 + 141x_3^2 + 178x_0x_4 - 384x_1x_4 + 86x_2x_4 - 94x_3x_4 \\
&\quad + 96x_4^2 + 386x_0x_5 + 292x_1x_5 - 378x_2x_5 - 470x_3x_5 + 102x_4x_5 - 233x_5^2 \\
q_2 &= 137x_0^2 - 100x_0x_1 + 47x_1^2 + 464x_0x_2 + 196x_1x_2 + 91x_2^2 - 374x_0x_3 \\
&\quad - 196x_1x_3 - 282x_2x_3 + 5x_3^2 + 186x_0x_4 - 468x_1x_4 + 186x_2x_4 + 292x_3x_4 \\
&\quad - 138x_4^2 + 466x_0x_5 + 100x_1x_5 - 280x_2x_5 - 468x_3x_5 + 480x_4x_5 - 44x_5^2 \\
q_3 &= 99x_0^2 + 280x_0x_1 + 187x_1^2 + 272x_0x_2 - 186x_1x_2 + 98x_2^2 + 378x_0x_3 \\
&\quad + 290x_1x_3 - 88x_2x_3 + 42x_3^2 - 86x_0x_4 - 102x_1x_4 + 184x_2x_4 + 98x_3x_4 \\
&\quad + 47x_4^2 - 376x_0x_5 + 472x_1x_5 + 560x_2x_5 - 286x_3x_5 - 290x_4x_5 - 97x_5^2
\end{aligned}$$

Then X is a degree 8 K3 surface with geometric Picard rank 1.

Degree 6 Case

Every polarized K3 surface (X, H) of degree 6 is a complete intersection of type $(2, 3)$ in $\mathbb{P}^4$.

Lemma 7. *If a sextic K3 surface (X, H) contains a line L in $\mathbb{P}^4$, then projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ from L restricts to a double cover $X \to \mathbb{P}^2$.*

Indeed, projection from L is induced by the linear system of $D = H - L$, and we check that $D^2 = 6 - 2 \cdot 1 - 2 = 2$, so D presents $X \to \mathbb{P}^2$ as a double cover branched over a sextic plane curve.

Over $\mathbb{F}_p$, we randomly generate sextic K3 surfaces X_p containing a fixed line L until we find one with $\rho(X) = 2$, which we can verify using the Elsenhans–Jahnel algorithm to compute the Weil polynomial.

Consider a lift X of X_p to $\mathbb{Q}$ such that $\overline{X}$ contains no lines. Because the reduction mod p respects the polarization and $\overline{\mathrm{sp}}$ preserves the intersection product, $\overline{\mathrm{sp}}$ automatically preserves degree as well.

We conclude with the following general result.

Theorem 8. *If $X \subset \mathbb{P}^n_{\mathbb{Q}}$ is a K3 surface containing no lines over $\overline{\mathbb{Q}}$ and the reduction X_p contains a line and $\rho(X_p) = 2$ for some prime p of good reduction, then $\rho(X_p) = 1$.*

Proof. Let $H \in \operatorname{Pic}(\overline{X})$ be the hyperplane section. Assume to get a contradiction that X has geometric Picard rank at least 2. If $\operatorname{rk}\operatorname{Pic}(\overline{X}) > 2$, we have a contradiction because $\overline{\mathrm{sp}}$ is an injective homomorphism from $\operatorname{Pic}(\overline{X})$ to $\operatorname{Pic}(\overline{X}_p)$, and $\operatorname{rk}\operatorname{Pic}(\overline{X}_p) = 2$. Thus, we can assume that $\operatorname{rk}\operatorname{Pic}(\overline{X}) = 2$. Because $\overline{\mathrm{sp}}$ is injective, the image of $\overline{\mathrm{sp}}$ has rank 2 as well, so the cokernel has rank 0.

However, by Lemma 3 we know that the cokernel is torsion free, so it must be 0, so $\overline{\mathrm{sp}}$ is surjective.

Let L be the line in $\overline{X}_p$. Because $\overline{\mathrm{sp}}$ is surjective, we have a divisor class $E \in \operatorname{Pic}(\overline{X})$ such that $\overline{\mathrm{sp}}(E) = L$, but by Lemma 3 we have that E is the class of a line, a contradiction. □

We performed these calculations to check the following explicit example:

Corollary 9. *Let $X = V(f_2, f_3) \subset \mathbb{P}^4_{\mathbb{Q}}$ where*

$$\begin{aligned}
f_2 &= x_0^2 - 3x_0x_1 + 3x_1^2 + 5x_0x_2 + 4x_1x_2 + 5x_2^2 - x_0x_3 - 2x_1x_3 - 3x_2x_3 - 5x_0x_4 + 5x_1x_4 \\
&\quad + 47x_3^2 + 47x_4^2 \\
f_3 &= 2x_0^3 + 3x_0^2x_1 + 3x_0x_1^2 + x_1^3 - x_0x_1x_2 - 3x_1^2x_2 + 4x_0x_2^2 - 4x_1x_2^2 + 5x_2^3 + 4x_0^2x_3 + x_0x_1x_3 \\
&\quad + 5x_1^2x_3 + 4x_0x_2x_3 + 4x_1x_2x_3 + -3x_2^2x_3 + 4x_1x_3^2 - x_2x_3^2 + 5x_0^2x_4 - 4x_0x_1x_4 + 2x_1^2x_4 \\
&\quad + x_0x_2x_4 + 4x_1x_2x_4 - 2x_2^2x_4 + 4x_0x_3x_4 - 3x_2x_3x_4 - x_0x_4^2 + -x_1x_4^2 + 5x_2x_4^2,
\end{aligned}$$

Then X is a sextic K3 surface with geometric Picard rank 1.

Proof. We computed that X_{47} contains the line $V(x_0, x_1, x_2)$, and projecting from this line, we computed that the Weil polynomial of X_{47} contains no cyclotomic roots besides $(t-47)^2$ coming from the hyperplane section and the line. Thus, by Lemma 4, X_p has geometric Picard rank 2.

Finally, we use a Groebner basis calculation to verify that X contains no lines over $\overline{\mathbb{Q}}$, so $\rho(X) = 1$ by Theorem 8. □

Proof of Theorem 2

In both the degree $d = 6, 8$ cases, we have effectively shown that there is a subset $R \subset \mathcal{K}_d$ consisting of K3 surfaces defined over $\mathbb{Q}$ having geometric Picard rank one, where R is of the form $T \cap U$, where T is the set of surfaces that reduce to a fixed surface mod p, and U is the complement of the Noether–Lefschetz divisor consisting of K3 surfaces containing a certain additional divisor (in the degree 6 case, a line, and in the degree 8 case, a tritangent line). Since U is open, so it suffices to show that T is Zariski dense in $\mathcal{K}_d$. In both cases, this holds by applying the following "weak approximation" type lemma to $\mathcal{K}_6$ and $\mathcal{K}_8$.

Lemma 10. *Let X be a unirational variety over $\mathbb{Z}$, and let x_0 be an $\mathbb{F}_p$-point of X. Assume that x_0 lifts to an $\mathbb{F}_p$-point under some unirational parameterization $\mathbb{P}^n \dashrightarrow X$. Then the set T of $\mathbb{Q}$-points of X that reduce to x_0 modulo p is Zariski dense in X.*

References

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