The Homotopy Type of Projective Manifolds

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June 6, 2025

This writeup gives an overview of topological distinctions between Kähler manifolds and projective manifolds, focusing on a result by Voisin which establishes a distinction at the level of homotopy type in any dimension ≥ 4.

This question is especially important because (due to a theorem of Chow) projective complex manifolds are equivalent to algebraic varieties. Thus, the question is also asking whether there is a topological means of distinguishing which Kähler manifolds admit the structure of an algebraic variety.

1 Introduction

In this section, we present a historical overview of the problem of distinguishing Kähler and projective manifolds. In 1954, Kodaira answered the problem up to isomorphism of complex manifolds with the following theorem:

Theorem 1.1 (Kodaira Embedding Theorem, 1954, [Kodb]). *A compact complex manifold X is projective if and only if it admits a Kähler form whose cohomology class is rational.*

This theorem gives a complete criterion for which Kähler manifolds are projective, but does not consider whether they can be distinguished topologically. In 1960, Kodaira determined that Kähler surfaces and Kähler manifolds could not be topologically distinguished.

Theorem 1.2 ([Koda]). *A compact Kähler surface is deformation equivalent to a complex projective surface.*

Before continuing, we review a chain of equivalence relations on complex manifolds:

(1) Isomorphic/Biholomorphic
(2) Deformation Equivalent
(3) Diffeomorphic
(4) Homeomorphic
(5) Homotopy Equivalent

In the above chain, each relation implies the one following it. Most of these implications are standard, but (2) $\implies$ (3) is due to a theorem of Ehresmann, see [Voia, Theorem 9.3].

The natural generalization of Theorem 1.2 is whether the same result holds in higher dimensions, a question which came to be known as the **Kodaira problem**:

For each n, are all compact Kähler manifolds deformation equivalent to a complex projective manifold?

If the answer to the Kodaira question was yes, that would mean (by the chain of equivalence relations discussed above that there is no topological means of distinguishing Kähler manifolds from projective ones. Theorem 1.2 gives an answer of yes in dimension 2, and at that time all higher dimensions were still an open question.

The next significant development came in 2004 with a negative answer from Voisin for all $n \geq 4$.

Theorem 1.3 ([Voib, Theorem 2]). *In any dimension ≥ 4, there exist compact Kähler manifolds which do not have the homotopy type of a complex projective manifold.*

The majority of the rest of this writeup will be spent sketching the proof of Theorem 1.3, but Voisin left the $n = 3$ case open, so we discuss it briefly here.

A new development came in the $n = 3$ case in 2024 with the following theorem of Lin.

Theorem 1.4 (Lin, 2024). *Every compact Kähler manifold of dimension 3 has an algebraic approximation.*

Under our definition of the Kodaira question, however, this is not strictly a positive answer for dimension 3. This writeup is not focused on algebraic approximations so we will not carefully compare the definitions, but at a high level an algebraic approximation allows for the algebraic variety approximating our manifold to be singular (i.e. not necessarily a projective complex manifold), so we do not necessarily have that every compact Kähler manifold of dimension 3 is deformation equivalent to a projective manifold.

2 Construction of non-projective Kähler manifolds

To find a compact complex manifold which is not Kähler, one need not look further than complex tori. There are multiple ways to introduce reasons why complex tori cannot be projective (i.e. cannot be abelian varieties), but one is to require that they admit a certain type of endomorphism.

Let T be a complex torus with an endomorphism ϕ. This can be equivalently characterized as an endomorphism of the $\mathbb{Z}$-module Γ, the lattice of T, or as an endomorphism on $H_1(X, \mathbb{Z}) \cong \Gamma$.

Let f be the characteristic polynomial of ϕ. We have the following condition on T:

Lemma 2.1 ([Voib, Lemma 1]). *If $n \geq 2$ and the Galois group of the splitting field of f acts as the symmetric group on the roots of f, then T is not an abelian variety.*

It is easy to show that such pairs (T, ϕ) for any $n \geq 2$, but it turns out that there is no easy way to show that these tori are not *homotopy equivalent* to a projective manifold (and in fact that statement is not true in many cases.

3 Proof that X is not projective

Going forward, X will represent the manifold we wish to show is not homotopy equivalent to any projective manifold.

Now, we construct X as a blowup of $T \times T$, where T is one of the tori defined in the previous section. Thus, we can construct a version of X for any dimenison ≥ 4 as desired. Because T has an endomorphism ϕ, we have 4 submanifolds $0 \times T$, $T \times 0$, T_{diag}, and T_{graph}. It turns out that having these 4 submanifolds interacting in a particular way is enough to recover the structure of X as $T \times T$, so we need force these 4 submanifolds to be preserved under deformations.

A theorem of Kodaira claims that exceptional divisors resulting from blowups are rigid under deformations, giving the intuition that we should blow up these 4 submanifolds to attempt to preserve them.

These intersect at finitely many points (x_n, y_n), so blow up each of these points. The manifold is still Kähler. Now, we can blow up each of the 4 subspaces listed above, getting X which is still Kähler. The original blowup of points was necessary to ensure that 4 manifolds were disjoint before blowing them up (a necessary hypothesis for the manifold remaining Kähler).

It turns out that the following theorem suffices because the Picard variety of a projective manifold is always projective, so we just need to show that $\mathrm{Pic}^0(X)$ is not projective.

Proposition 3.1. *$T \times T$ is isomorphic to a subtorus of $\mathrm{Pic}^0(X)$.*

Proof. First, note that $H^1(T \times T, \mathbb{Z}) \cong H^1(X, \mathbb{Z})$ because the type of blowups considered do not affect first cohomology. Further, by the Kunneth Formula we have that $H^1(T \times T, \mathbb{Z}) = H^1(T, \mathbb{Z}) \oplus H^1(T, \mathbb{Z})$. Using this decomposition, we can define the natural projections pr_1 and pr_2.

Next, note that $H^1(T \times T, \mathbb{Z})$ has 4 subspaces:

(1) $L_1 = \ker pr_1$

(2) $L_2 = \ker pr_2$

(3) $L_3 = \ker pr_1 + pr_2$

(4) $L_4 = \ker pr_1 + \phi^* pr_2$

It turns out that these subspaces are in fact sub-Hodge structures because pr_1 and pr_2 are morphisms of Hodge structures, but we defer the proof of this to the next section.

Because $\mathrm{Pic}^0(X) \cong H^1(X, \mathbb{C}) / \left(H^{1,0}(X) \oplus H^1(X, \mathbb{Z})\right)$, each sub-Hodge structure of $H^1(X, \mathbb{Z})$ actually gives a subtorus of Pic^0.

The subtori corresponding to each of the above Hodge structures have the following properties

(1) $T_{L_1}, T_{L_2} \cong T_{L_3}$ and $T_{L_1} \oplus T_{L_2} = \mathrm{Pic}^0(X)$, so in fact $\mathrm{Pic}^0(X) \cong T' \times T'$ for some torus T'.

(2) T_{L_4} guarantees the existence of an endomorphism T.

$\square$

4 Proof that X' is not projective

To show that X is not homotopy equivalent to a projective manifold, we introduce a new manifold X' which shares X's integral cohomology ring, a homotopy invariant, and show that X' cannot be projective, which is encapsulated in the following theorem:

Theorem 4.1. *Let X' be a compact kahler manifold such that there exists a graded ring isomorphism $\gamma : H^*(X', \mathbb{Z}) \to H^*(X, \mathbb{Z})$. Then X' is not projective.*

Note that we immediately have 4 subspaces of $H^1(X', \mathbb{Z})$ with the same properties as $L_1, \ldots, L_4$ by considering the preimages of $L_1, \ldots, L_4$ under γ, but it is not clear that these subspaces remain sub-Hodge structures (which is necessary for the rest of the proof of non-projectivity).

To show that they are sub-Hodge structures, we need to view them also as the kernel of a endomorphism of the Hodge structure $H^1(X', \mathbb{Z})$.

Recall that to any complex manifold X we can associate a complex torus $\operatorname{Alb} X$ and a map $\operatorname{alb}_X : X \to \operatorname{Alb} X$ which is a holomorphism. This is the universal complex torus in the sense that for any other complex torus Y with a morphism $f : X \to Y$, f factors through a morphism of complex tori $\operatorname{Alb} X \to Y$.

The Albanese map is the natural blowdown map for our choice of X, so its kernel is generated by the exceptional divisors of the blown up subvarieties.

We will use a combination of the following three lemmas to complete the proof of our theorem.

Lemma 4.2. *Let $\tau : \tilde{X} \to X$ be a blowup of X along Y with exceptional divisor E. Let $j : Y \to X$ be the inclusion of Y. Then*

$$\ker(\cup E) \circ \tau^* = \ker j^*$$

Lemma 4.3. *The class of E (the exceptional divisor of Y) is a Hodge class for $Y = T \times 0, 0 \times T, T_{diag}, T_{graph}$. (In fact, this is true for any class in $\ker \operatorname{alb}_{X*}$ or $\ker \operatorname{alb}_{X'*}$)*

Proof. Because the Albanese maps are birational, hence of degree 1, we have that $\operatorname{alb}_{X*} \circ \operatorname{alb}_X^* = \operatorname{Id}$ and same for X'.

Moreover, $\operatorname{alb}_X^* : H^0(\operatorname{Alb} X, \Omega^2_{\operatorname{Alb} X}) \to H^0(X, \Omega^2_X)$ is an isomorphism so the morphism of hodge structures is an isomorphism on $H^{2,0}$. Thus, $\ker \operatorname{alb}_{X*}$ has no $(0,2)$-part, so it lies in $H^{1,1}$. Thus every class in $\ker \operatorname{alb}_{X*}$ is a Hodge class. $\square$

Lemma 4.4. *If $\alpha \in \ker \operatorname{alb}_{X*}$, then $\gamma^{-1}(\alpha) \in \ker \operatorname{alb}_{X'*}$.*

We omit the proof of Lemma 4.4 here, but it can be found on p. 334 of [Voib].

By Lemma 4.2, we know that the maps pr_1 and pr_2 on X are given by cupping with the cohomology class of the exceptional divisor E generated by blowing up $0 \times T$ and $T \times 0$.

If E is the class of the exceptional divisor generated by blowing up $0 \times T$, then $\ker \cup E = \ker pr_1 = L_1$, i.e. $\gamma^{-1}(\ker \cup E) = \gamma^{-1}(L_1)$. But γ is compatible with the cup product, so $\gamma^{-1}(L_1) = \ker \cup \gamma^{-1}(E)$. By Lemma 4.4, we know that $\gamma^{-1}(E) \in \ker \operatorname{alb}_{X'*}$, so by Lemma 4.3 $\gamma^{-1}(E)$ is a Hodge class. Thus $\cup \gamma^{-1}(E)$ is a morphism of Hodge structures, so its kernel, i.e. $\gamma^{-1}(L_1)$, is a sub-Hodge structure of $H^1(X', \mathbb{Z})$. The same argument applies for L_2, L_3, L_4.

Thus, we can continue the proof from Section 3 to show that X' is not projective (and note that we have completed our deferred proof from Section 3 by considering the case where $X' = X$ and $\gamma = \mathrm{Id}$).

5 Recent Results

Finall, note that this construction given relies on blowups, which leave open the possibility that there are bimeromorphically equivalent models of X that do have the homotopy type of a projective complex manifold. In particular, it turns out that tbe blowups used really were necessary, i.e. $T \times T$ on its own does not provide the necessary counterexample. In 2006, Voisin patched this potential question with the following result:

Theorem 5.1 ([Voic]). *In any even dimensin ≥ 10, there exist compact Kähler manifolds X, such that no smooth compact bimeromorphic model X' of X has the homotopy type of a projective complex manifold.*

References

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