Polar Codes

Henry Scheible, Warren Shepard

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Introduction

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The Basic Idea

For a given channel W, channel coding is trivial when the capacity of W, denoted $C(W) \in \{0, 1\}$

1. If we can transform any channel W into a collection of extreme channels, it could be easier to communicate.
2. If the ratio of "good" channels with capacity ~ 1 to the total number of channels is $C(W)$, we can communicate with the same capacity as W.

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Note: when referring to an arbitrary "channel" W we mean an arbitrary symmetric binary DMC.

Agenda

Outline

Simplest Case: 2 BEC's

Recursive Construction

Channel Polarization Theorem

Code Construction

Proof of Channel Polarization Theorem

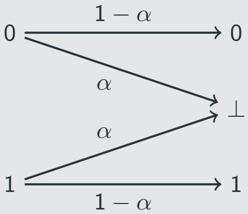
Further Results

Sending a message over two BECs

Recall:

Binary Erasure Channel (BEC)

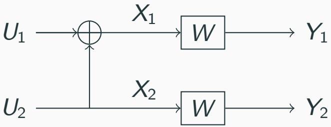
Let BEC_α be the channel with $\mathscr{X} = \{0,1\}$, $\mathscr{Y} = \{0,\perp,1\}$ and transition probabilities



for some $\alpha < 1$.

Sending a message over two BECs

For now, let $W = BEC_\alpha$ for some fixed α. Consider the compound channel W_2 sending a message in $\mathscr{X}^2$ to $\mathscr{Y}^2$ by sending each bit through W. Denote the input by the random variables X_1, X_2 and output by Y_1, Y_2.



When sending a string $u = U_1 U_2 \in \{0, 1\}$ through W_2, are we more likely to successfully decode U_1 or U_2?

Decoding & Virtual Channels

Assume that $U_1, U_2 \sim \text{Bern}(1/2)$. We will first attempt to decode U_1 knowing both Y_1 and Y_2, then if successful decode U_2 knowing Y_1, Y_2 and U_1. Call these "virtual" channels $W_2^{(1)}$ and $W_2^{(2)}$:

- $W_2^{(1)} : U_1 \mapsto Y_1 Y_2$
- $W_2^{(2)} : U_2 \mapsto Y_1 Y_2 U_1$

Conservation of Capacity

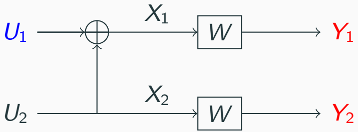
- $W_2^{(1)} : U_1 \mapsto Y_1 Y_2$
- $W_2^{(2)} : U_2 \mapsto Y_1 Y_2 U_1$

Using the Chain Rule for Mutual Information and the independence of U_1 and U_2, see that

$$\begin{aligned} C(W_2) &= I(U_1 U_2 : Y_1 Y_2) \\ &= I(U_1 : Y_1 Y_2) + I(U_2 : Y_1 Y_2 | U_1) \\ &= I(U_1 : Y_1 Y_2) + I(U_2 : Y_1 Y_2 U_1) \\ &= C\left(W_2^{(1)}\right) + C\left(W_2^{(2)}\right) \end{aligned}$$

Capacity of $W_2^{(1)}$

$W_2^{(1)} : U_1 \mapsto Y_1 Y_2$



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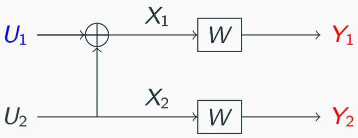

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C(W_2^{(1)}) &= I(U_1 : Y_1 Y_2) \\
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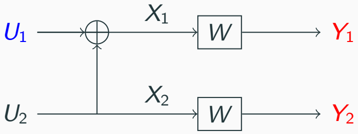

$$C(W_2^{(1)}) = I(U_1 : Y_1 Y_2)$$

$$= 1 - \sum_{(y_1, y_2) \in \{0,1,\perp\}^2} H(U_1 | Y_1 Y_2 = y_1 y_2) \mathbb{P}\{Y_1 Y_2 = y_1 y_2\}$$

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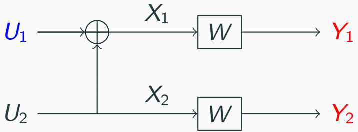

Note that $U_1 U_2 \sim \text{Bern}(1/2)^{\otimes 2}$ implies that $X_1 X_2 \sim \text{Bern}(1/2)^{\otimes 2}$, so $Y_1, Y_2 \sim p$, where

- $p(0) = 1/2(1-\alpha)$
- $p(\perp) = \alpha$
- $p(1) = 1/2(1-\alpha)$

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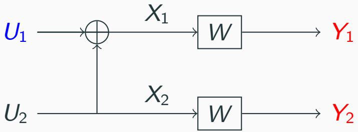

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If $Y_1, Y_2 \neq \perp$ (w.p. $(1-\alpha)^2$), then

$$H(U_1) = H_2(0) = 0$$

If $Y_1 = \perp$ or $Y_2 = \perp$ (w.p. $1-(1-\alpha)^2$), then

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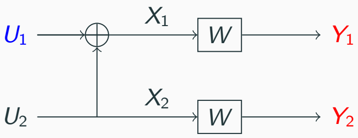

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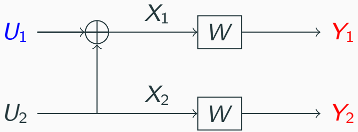

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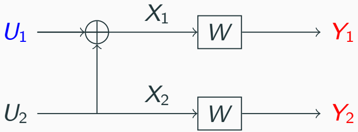

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Remark

$W_2^{(1)}$ is effectively $BEC_{1-(1-\alpha)^2}$

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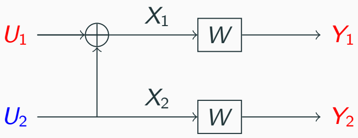

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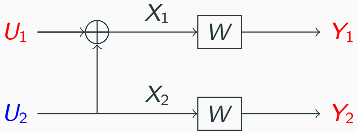

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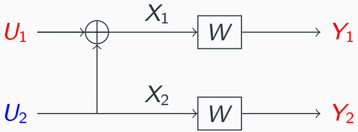

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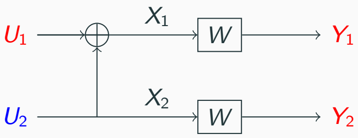

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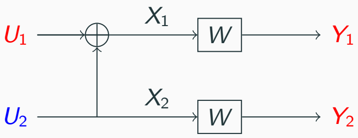

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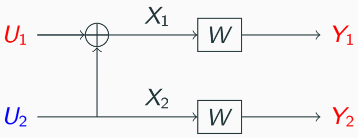

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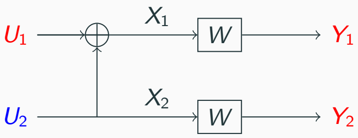

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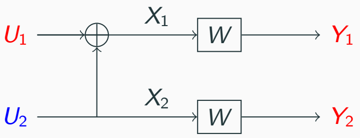

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Remark

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Summary of $W_2^{(1)}$ and $W_2^{(2)}$

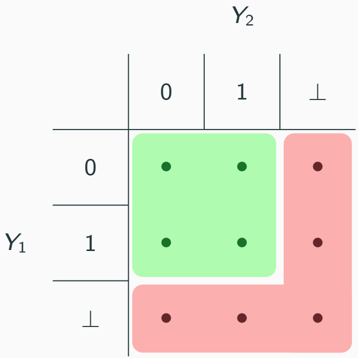


Figure 1: $W_2^{(1)}$

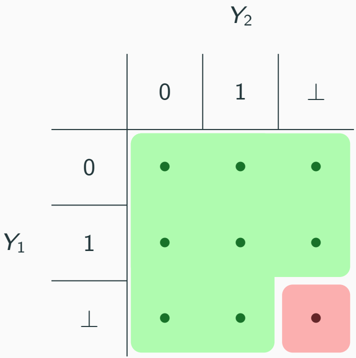


Figure 2: $W_2^{(2)}$

BEC_{α^2}
Erasure occurs if *either* channel erases

$BEC_{1-(1-\alpha)^2}$
Erasure occurs if *both* channels erase

Outline

Simplest Case: 2 BEC's

Recursive Construction

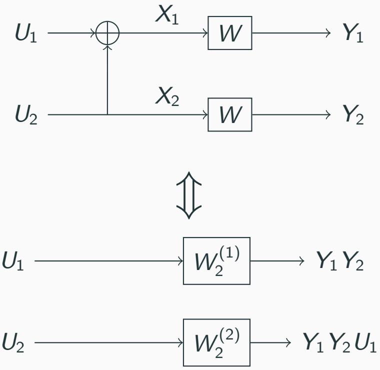
Channel Polarization Theorem

Code Construction

Proof of Channel Polarization Theorem

Further Results

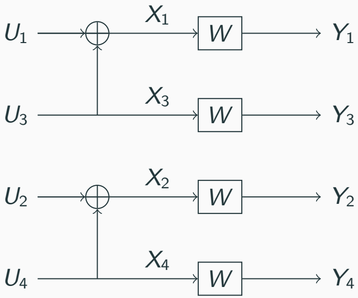
Channel Splitting



Recursive Construction of Polarizing Channels

Idea

We can use $W_2^{(1)}$ and $W_2^{(2)}$ to recursively construct more polarizing channels the same way we constructed $W_2^{(1)}$ and $W_2^{(2)}$ from W.



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$$U_1 \longrightarrow \boxed{W_2^{(1)}} \longrightarrow Y_1 Y_3$$

$$U_3 \longrightarrow \boxed{W_2^{(2)}} \longrightarrow Y_1 Y_3 U_1$$

$$U_2 \longrightarrow \boxed{W_2^{(1)}} \longrightarrow Y_2 Y_4$$

$$U_4 \longrightarrow \boxed{W_2^{(2)}} \longrightarrow Y_2 Y_4 U_2$$

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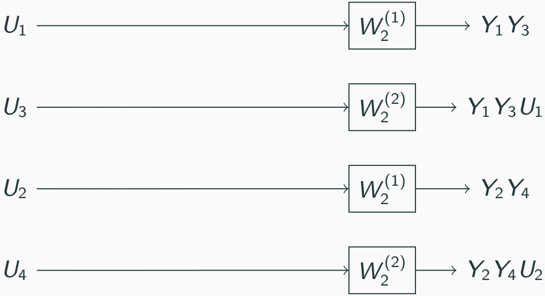

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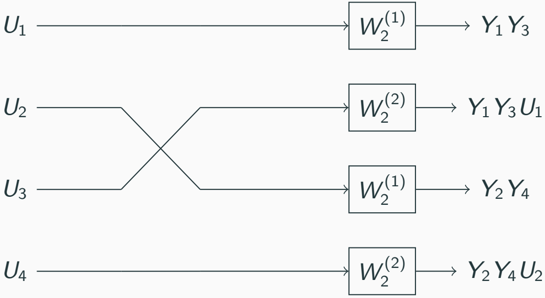

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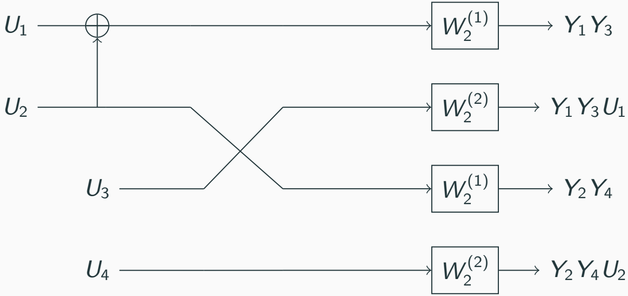

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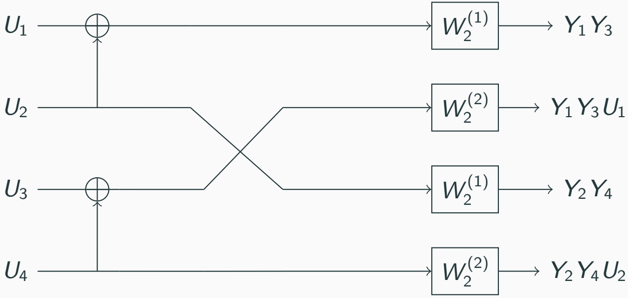

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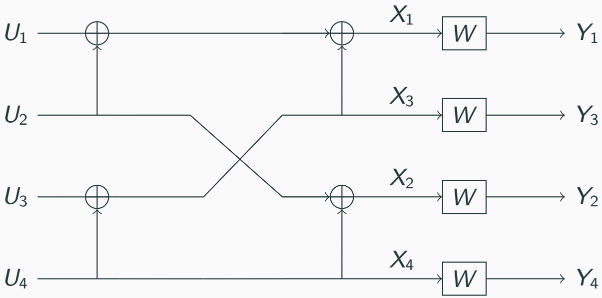

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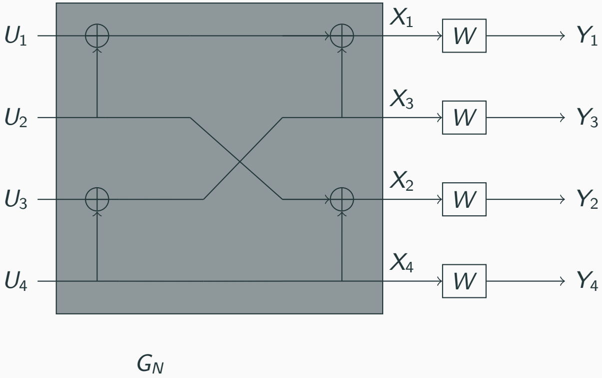

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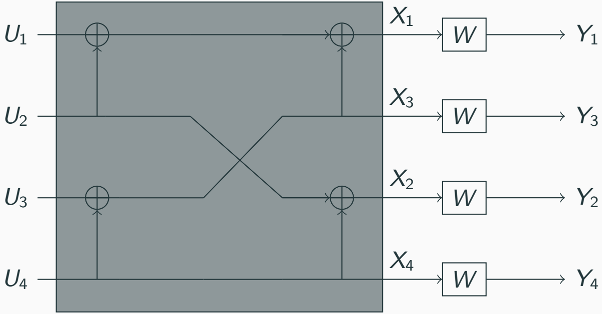


$$G_N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{\otimes 2}$$

Final Virtual Channels

From this process, we get four virtual channels from our four initial channels W:

- $W_4^{(1)} : U_1 \to Y_1 \cdots Y_4$
- $W_4^{(2)} : U_1 \to Y_1 \cdots Y_4 U_1$
- $W_4^{(3)} : U_1 \to Y_1 \cdots Y_4 U_1 U_2$
- $W_4^{(i)} : U_1 \to Y_1 \cdots Y_4 U_1 U_2 U_3$

Final Virtual Channels (General Case)

From this process, we get N virtual channels from our N initial channels W:

- $W_N^{(1)} : U_1 \to Y_1 \cdots Y_N$
- $W_N^{(2)} : U_1 \to Y_1 \cdots Y_N U_1$
- $\cdots$
- $W_N^{(i)} : U_1 \to Y_1 \cdots Y_N U_1 \cdots U_{i-1}$

Outline

Channel Polarization Theorem

Formalizing channel polarization:

Theorem: For any channel W and $\delta \in [0,1]$, consider the channels $\left\{W_N^{(i)}\right\}$ where $N = 2^n$ is the number of channels for an integer n. As $n \to \infty$,

1. the fraction of indices $i \in [N]$ such that $C(W_N^{(i)}) \in (1-\delta, 1]$ approaches $C(W)$
2. the fraction of indices i such that $C(W_N^{(i)}) \in [0, \delta)$ approaches $1 - C(W)$

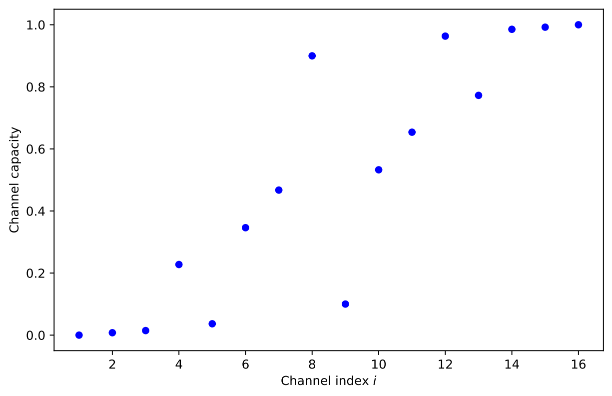
where $C(W)$ denotes the symmetric capacity of W (same as standard capacity for BEC).

Polarization Theorem Demonstration

Setup

Lets revisit our example of a Binary Erasure Channel with erasure probability $\alpha < 1$. For simplicity set $\alpha = \frac{1}{2}$. Now we want to (empirically) show that as *n* increases, the resulting channels polarize.

Polarization Theorem Demonstration



n=4, $\alpha = 0.5$

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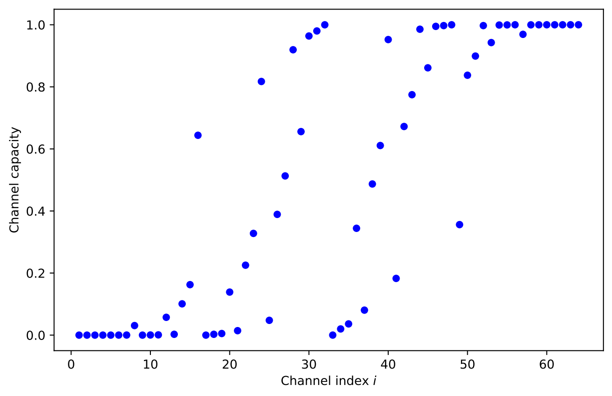

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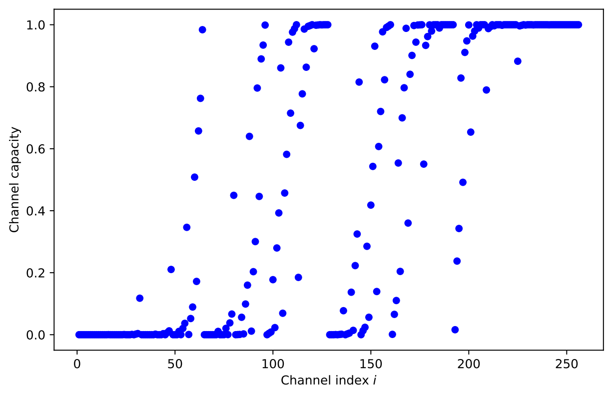
Polarization Theorem Demonstration



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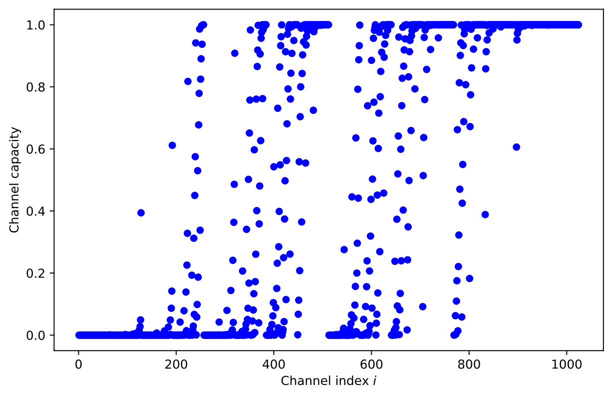

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Theorem 1 Demonstration

Setup

Lets revisit our example of a Binary Erasure Channel with erasure probability $\alpha < 1$. For simplicity set $\alpha = \frac{3}{10}$. Note $C = 1 - \alpha = \frac{7}{10}$, so we expect the fraction of "good" channels to approach $\frac{7}{10}$ for $\delta = 0.05$.

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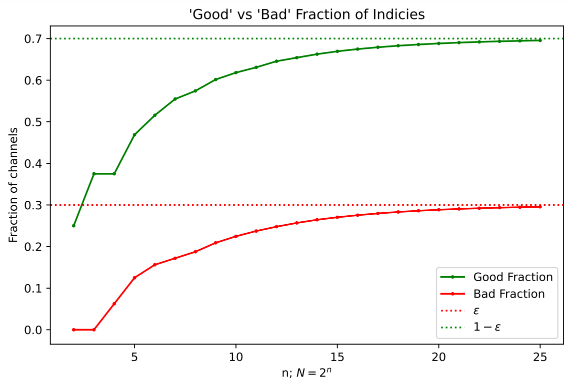

Outline

Constructing a Polar Code

Given a symmetric DMC W^N which is split into $W_n^{(1)}, ..., W_n^{(n)}$ polarized channels, we define a polar code with rate k/n by picking k channels with $I(W_n^{(i)})$ maximized.
Alice and Bob agree in advance on the values to be sent on the $n-k$ remaining channels, then Alice sends her message on those k channels.
Following the theorem, as $n \to \infty$ Alice will be able to pick a capacity-achieving rate with polarized channel capacity approaching 1.

Successive Cancellation Decoder

To *actually* split W^N into $W_N^{(1)}, ..., W_N^{(N)}$, we must use a particular decoding strategy: *successive cancellation decoding*:

For each $i \in 1, ..., N$:

- If i is in a meaningful channel: Decode $\hat{U}_i$ based on maximum likelihood from $Y_1, ..., Y_N, \hat{U}_1, ..., \hat{U_{i-1}}$
    - Compare $W_N^{(i)}(Y_1 \cdots Y_N U_{1\,i-1}|0)$ with $W_N^{(i)}(Y_1 \cdots Y_N U_{1\,i-1}|1)$
- If i is in a meaningless channel: set $\hat{U}_i = U_i$

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Simplest Case: 2 BEC's

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Proof of Main Theorem: Goal

Theorem: For any Binary DMC W and $\delta \in [0,1]$, consider the channels $\left\{W_N^{(i)}\right\}$ where $N = 2^n$ is the number of channels for an integer n. As $n \to \infty$,

1. the fraction of indices $i \in [N]$ such that $C(W_N^{(i)}) \in (1-\delta, 1]$ approaches $C(W)$
2. the fraction of indices i such that $C(W_N^{(i)}) \in [0, \delta)$ approaches $1 - C(W)$

where $C(W)$ denotes the symmetric capacity of W (same as standard capacity for BEC).

Proof of Main Theorem: Preliminaries

Let $Z(W)$ be the "reliability" of a channel, defined as

$$Z(W) = \sum_{y \in \mathcal{Y}} \sqrt{W(y \mid 0) W(y \mid 1)}.$$

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$$C(W) = \sum_{x \in \mathscr{X}} \sum_{y \in \mathscr{Y}} \frac{1}{2} W(y \mid x) \log \frac{W(y \mid x)}{\frac{1}{2} W(y \mid 0) + \frac{1}{2} W(y \mid 1)}.$$

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Consider the binary tree representation of a polar code construction. Let $\{K_n\} = \{K_n \mid n \geq 0\}$ be a random walk through the tree beginning at the root ($K_0 = W$) and ending at a leaf.

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$\mathscr{F}$ is the Borel Field generated by the cylinder sets $S(b_1, \dots b_n)$ where $b_1, \dots b_n$ is a binary sequence denoting the traversal of the binary tree representation.

Proof of Main Theorem: Lemma 1

Lemma 1

$\{C_n, F_n\}$ is a martingale (a sequence of RVs with the conditional expectation equal to the current value), i.e.*

1. $\mathbf{Exp}[|C_n|] < \infty$
2. $c_n = \mathbf{Exp}[C_{n+1} \mid C_n = c_n]$

Notably, $\{C_n\}$ converges almost always to C_∞ with $\mathbf{Exp}[C_\infty] = C_0$.

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Proof. (1) is true based on the fact that $0 \leq C_n \leq 1$. To show (2), consider a binary sequence $(b_1, \ldots, b_n) \in \{0,1\}^n$ and let $W_{b_1 \ldots b_n}$ be the channel obtained by traversing the tree representation based on the binary sequence.

$$\begin{aligned}\mathbf{Exp}[C_{n+1} \mid S(b_1, \ldots, b_n)] &= \frac{1}{2} C(W_{b_1 \ldots b_n 0}) + \frac{1}{2} C(W_{b_1 \ldots b_n 1}) \\ &= C(W_{b_1 \ldots b_n})\end{aligned}$$

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Since (C_n, F_n) is a martingale, it follows that $\{C_n\}$ converges almost always (explicitly, a set of points with measure zero can be excluded from convergence) to some random variable C_∞ such that $\mathbf{Exp}[C_\infty] = C_0$.

Proof of Main Theorem: Lemma 2

Lemma 2

$\{Z_n, F_n\}$ is a super-martingale (conditional expectation of next value is bounded by current), i.e.*

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Notably, $\{Z_n\}$ converges to Z_∞ which almost always takes value $\{0, 1\}$.

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Proof. (1) is satisfied because $Z(W)$ takes on values in $[0, 1]$. To prove (2), we follow a same structure as for I_n. Again, consider a binary $b_1, \ldots, b_n$. Then,

$$\begin{aligned}\mathbf{Exp}[Z_{n+1} \mid S(b_1, \ldots, b_n)] &= \frac{1}{2} Z(W_{b_1 \ldots b_n 0}) + \frac{1}{2} Z(W_{b_1 \ldots b_n 1}) \\ &\leq Z(W_{b_1 \ldots b_n})\end{aligned}$$

where $W_{b_1 \ldots b_n}$ is the value of Z_n on $b_1, \ldots b_n$

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Since $\{Z_n, F_n\}$ is a super-martingale, Z_n converges to a random variable Z_∞ such that $\mathbf{Exp}[|Z_n - Z_\infty|] \to 0$. Note $Z_{n+1} = Z_n^2$ with probability $1/2$, so

$$\mathbf{Exp}[|Z_{n+1} - Z_n|] \geq \frac{1}{2}\mathbf{Exp}[Z_n^2 - Z_n] = \frac{1}{2}\mathbf{Exp}[Z_n(1 - Z_n)] \geq 0.$$

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Therefore,

$$\mathbf{Exp}[Z_n(1 - Z_n)] \to 0 \implies \mathbf{Exp}[Z_\infty(1 - Z_\infty)] \to 0.$$

For the above to hold, Z_∞ must almost always be 0 or 1.

Proof of Main Theorem: Conclusion

We have that Z_∞ converges to 0 or 1 almost always. It is a fact [proved in appendix of paper] that:

$$C(W) \geq \log \frac{2}{1+Z(W)} \quad \text{and} \quad C(W) \leq \sqrt{1-Z(W)^2}.$$

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Theorem: For any channel W and $\delta \in [0,1]$, consider the channels $\left\{W_N^{(i)}\right\}$ where $N = 2^n$ is the number of channels for an integer n. As $n \to \infty$,

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Bound on the Rate of Polarization

Theorem: For any Binary DMC W with $C(W) > 0$ and any fixed $R < C(W)$, there exists a sequence of sets $\mathscr{A}_N \subset \{1, ..., N\}, N \in \{1, 2, ..., 2^n, ...\}$, such that $|\mathscr{A}_N| \geq NR$ and $Z(W_N^{(i)}) \leq O(N^{-5/4})$ for all $i \in \mathscr{A}_n s$

Recall that

$$C(W) \geq \log \frac{2}{1 + Z(W)}$$

Recent Result: List Decoding for Polar Codes (2015)

Our current channel polarization result relies on *successive cancellation decoding*, which is significantly weaker than *maximum likelihood decoding*

A recent advance proposes keeping L candidate strings at all times then picking the most likely one at the end of the decoding process, a decoding strategy which is between the other two options.

This makes Polar Codes actually useful for small to medium block lengths.

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