Lecture5

Data Analysis: Propagation of Uncertainties

Textbook: Chapter3

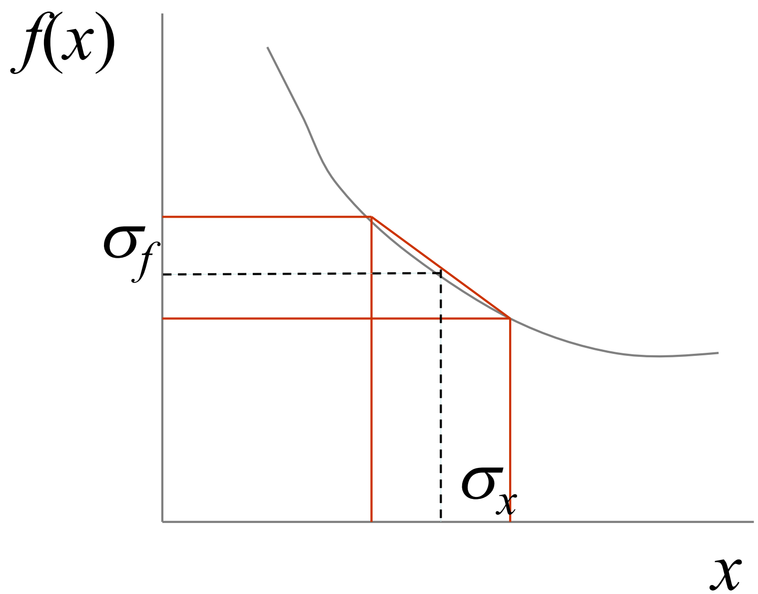
You need to know: differentiation rules (calculus class)

Office hours (based on doodle): Tuesdays 9.30-11am
Office #: C109 (C floor, Physics building)

Uncertainty propagation (one dimension/variable)

Knowing x uncertainty (σ_x), and dependence of f(x), we can evaluate f uncertainty (σ_f).

Graphical interpretation of evaluating uncertainties:



$$\sigma_f = \left|\frac{df}{dx}\right| \sigma_x$$

df/dx can be a number or a function of x

σ_f must be positive

REMINDER: Differentiation rules (calculus class)

1) Constant $f(x) = c$ $$\frac{d}{dx}(c) = 0$$

2) Power $f(x) = x^n$ $n \neq 0$ $$\frac{d}{dx}(x^n) = nx^{n-1}$$

3) Constant multiplication: $$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$$

4) Sum and difference: $$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \qquad \frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

5) Product: $$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

6) Ratio:
(f and g must be differentiable functions) $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

7) Chain (composition of functions):

$(f \ o \ g)(x) = f(g(x))$ or $y = f(u), u=g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

REMINDER: Differentiation rules (calculus class)

Exponential and Logarithmic functions, derivatives

$f(x) = e^x \qquad \frac{df}{dx} = e^x$

$f(x) = a^x \qquad \frac{df}{dx} = a^x \, ln(a)$

$f(x) = ln(x) \qquad \frac{df}{dx} = \frac{1}{x}$

$f(x) = log_a(x) \qquad \frac{df}{dx} = \frac{1}{xln(a)}$

REMINDER: Differentiation rules (calculus class)

Trigonometric functions, derivatives

$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x) \qquad \frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

$$\frac{d}{dx}\left(\tan(x)\right) = \sec^2(x) \qquad \frac{d}{dx}\left(\cot(x)\right) = -\csc^2(x)$$

$$\frac{d}{dx}\left(\sec(x)\right) = \sec(x)\tan(x) \qquad \frac{d}{dx}\left(\csc(x)\right) = -\csc(x)\cot(x)$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x} \qquad \csc x = \frac{1}{\sin x}$$

L5Ex1: Differentiate the following – single variable – functions

a. f(x) = πx

b. g(y)=5y^3

c. y (x) = - $\frac{8}{x^2}$

d. v(t)=$\sqrt{32}$ t$^{1/2}$

e. s(d) = 0.5$d^{\sqrt{2}}$

f. f(x)= 2x^8 + 4x^6 - 10x^5 + x^3 + 25x^2 - 11

g. g(x)=($x^3 + 2x)(5x^2 + 7x + 9)$

---

h. v(t) = $\frac{3t^3+7t}{t^2-8t+1}$

i. f(x) = $\sqrt{2x^2 - 8}$

j. g(x) = $\sin\sqrt{\alpha x^3 + \beta x^2 - 6x + \gamma}$ where α, β, γ are constants

Multivariate calculus, functions of several variables, partial derivatives

Let f depends on n variables: f= f(x1, x2, x3, …., xn)

Partial derivative of f with respect to xi: a derivative of f with respect to xi while other variables are treated as constants

First order partial derivatives:

$$\frac{\partial f(x_1, x_2, x_3, \ldots, xn)}{\partial x_i} \equiv \partial_{x_i} f(x_1, x_2, x_3, \ldots, xn) \quad i = 1,2,\ldots,n$$

e.g. $$\frac{\partial f(x_1, x_2, x_3, \ldots, xn)}{\partial x_3} \equiv \partial_{x_3} f(x_1, x_2, x_3, \ldots, xn)$$

Second order partial derivatives:

$$\frac{\partial^2 f(x_1, x_2, x_3, \ldots, xn)}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f(x_1, x_2, x_3, \ldots, xn)}{\partial x_j}\right)$$

$$\frac{\partial^2 f(x_1, x_2, x_3, \ldots, xn)}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j}\left(\frac{\partial f(x_1, x_2, x_3, \ldots, xn)}{\partial x_i}\right)$$

Multivariate calculus, functions of several variables, partial derivatives

Let f depends on n variables: f= f(x1, x2, x3, …., xn)
A total (full) derivative of a function f of n variables $x_1, x_2, x_3, \ldots, x_n$ with

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

If $y = f(x_1, x_2, x_3, \ldots, x_n)$ then the total (full) derivative of f with respect to y is:

$$\frac{df}{dy} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dy} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dy} + \ldots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dy}$$

L5Ex2: Calculate partial derivatives

Calculate: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

a) $f(x,y) = 2x^2y^{-4}\text{-}0.5x^{0.5}$

b) $f(x,y) = \frac{x-y}{x+y}$

c) $f(x,y) = \frac{\sin y}{x}$

d) $f(x,y,z) = ze^{(x+2y)}$

e) $f(x,y,z) = \cos(z+5)\ e^{(x^2+2/y)}$

f) $h(t) = \frac{1}{2}gt^2$

Calculate: $\frac{\partial h}{\partial t}$

g) $h(g,t) = \frac{1}{2}gt^2$

Calculate: $\frac{\partial h}{\partial t}, \frac{\partial h}{\partial g}$

h) $N(t) = B_0 + N_0 e^{\frac{-2t}{\tau}}$

Calculate: $\frac{\partial N}{\partial B_0}, \frac{\partial N}{\partial N_0}, \frac{\partial N}{\partial \tau}$

Error propagation

Suppose we measured a set of e.g. 3 variables: x,y,z with uncertainties σ_x, σ_y and σ_z Consider a function f=f(x,y,z). What is the variance of f i.e. $(\sigma_f)^2$ (f is determined from x, y and z; we want to know what's the uncertainty on f knowing uncertainties on x,y and z and assuming x,y and z are uncorrelated)

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial f}{\partial z}\right)^2 \sigma_z^2$$

If there are more then 3 variables which are measured, one should add more terms in above equations. If there are less than 3 variables (e.g. only x and y are measured, one should remove all terms with z variable in above equations).

TO REMEMBER:

$$\boxed{\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}^2 + \ldots\ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}^2}$$

L5Ex3:

A bird flies a distance d = 120 $\pm$ 3 m during a time t = 20.0 $\pm$ 1.2 s.
The average speed of the bird is v = d/t = 6 m/s. What is the uncertainty of v?

α) σ_v (d,t, σ_d, σ_t) functional form

β) σ_v = (numerical value)

Use error propagation formula: $$\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}^2 + \ldots\ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}^2$$

L5Ex4:

Textbook problem 3.14

A visitor to a castle measures the depth of a well by dropping a stone and timing its fall. She finds the time to fall is t=3.0 +/- 0.5 s and calculates depth $d=gt^2/2$. What is her conclusion (numerical value), if she takes

a) g=9.80 m/s^2 with negligible uncertainty

b) g=9.81 +/- 0.2 m/s^2

? Round the results.

L5Ex4:

Textbook example Sect. 3.9

Pendulum experiment.
Find g and its uncertainty (error propagation):

$$l = 92.95 \pm 0.15 \text{ [cm]}$$

$$T = 1.936 \pm 0.004 \, [s]$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Use error propagation formula: $$\sigma_f^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}^2 + \ldots\ldots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}^2$$

Combining Uncorrelated Errors

Let f = f(x, y) and variables x, y are uncorrrelated

- Linear case:

$$f = x \pm y$$

$$\boxed{\sigma_f^2 = \sigma_x^2 + \sigma_y^2}$$ ← absolute errors are relevant

- Products:

$$f = x^a y^b$$

$$\boxed{\left(\frac{\sigma_f}{f}\right)^2 = a^2\left(\frac{\sigma_x}{x}\right)^2 + b^2\left(\frac{\sigma_y}{y}\right)^2}$$

$$f = xy, \; f = x/y$$

$$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$$

Fractional errors are relevant and must be small !
(for larger errors, use a numerical method)

L5Ex7

Derive the formulae on σ_f below, starting from the most general formula

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2$$

Let f = f(x, y) and variables x, y are independent/uncorrelated

$$\sigma_x, \sigma_y - known$$

a) $f = x \pm y$

$$\sigma_f^2 = \sigma_x^2 + \sigma_y^2$$

b) $f = x^a y^b$

$$\left(\frac{\sigma_f}{f}\right)^2 = a^2\left(\frac{\sigma_x}{x}\right)^2 + b^2\left(\frac{\sigma_y}{y}\right)^2$$

c) $f = xy$

$$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$$

d) f=x/y

$$\left(\frac{\sigma_f}{f}\right)^2 = \left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2$$

In addition, for part d) : Check that you can get the same formula as in c) assuming a=1 and b=-1.

L5Ex5:

Calculate σ_f for $f = \tan(x)$ and $x = 88 \pm 1^o$

L5Ex6:

Calculate σ_f for

Textbook example Sect. 3.10

Acceleration of a cart down a slope.

Thank you!

Questions, comments:
e-mail: Joanna.Kiryluk@stonybrook.edu