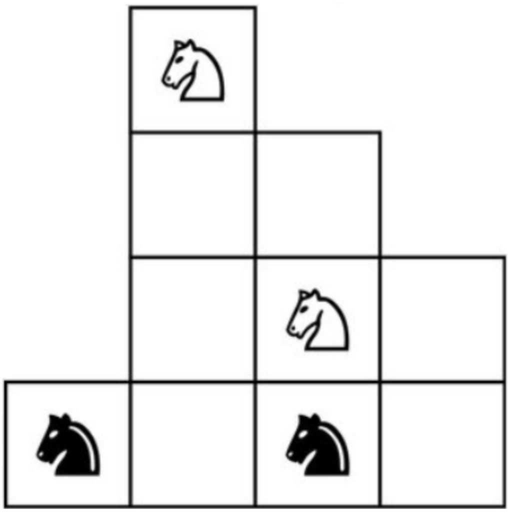
Pebble Motion Problems and NP Reductions

Introduction

Given an n x m board containing pairs of white and black knights with some squares removed, can you find a sequence of moves so that the white knights and black knights swapped positions? Two knights cannot be on the same square at any given time.



Goal: *Exchange the positions of the black and white knights*

June Huh shared this puzzle in an interview. It's a classic puzzle called the Knights Swap. I was immediately drawn to the strange allure of the puzzle, and it consumed a significant part of my summer after high school. After solving the specific case, I wanted to see whether there's an algorithm to find an optimal solution to the general case: an arbitrary number of pairs of knights on a general graph. But finding such an algorithm eluded me for the summer. It just seemed like the problem was "impossibly difficult", but I could not formalize how at the time.

Having taken relevant courses in college, I wanted to revisit the problem again, this time with greater rigor. I was also excited to learn that there are several theoretical graph problems that are similar to the Knights Exchange. More detail can be found at the end. One direction I wanted to take is to formalize why the Knights Exchange Problem is "so hard". To show a problem is "hard" in theoretical computer science, a classic approach is to reduce a known "hard" problem to your current problem. To show problem B is harder than problem A, if you have a black-boxed algorithm for problem B, then you can use that algorithm as a subroutine in problem A in a way so that everything else is "fast enough". The usual metric of "fast enough" is polynomial time. Then problem B must be at least as hard as problem A. For what follows, we can think of the knights as colored pebbles on a graph. The term "pebbles" is used in the literature, so I will adopt that terminology.

The Reduction

The Hamiltonian Cycle Problem (HCP): Given a graph $G = (V, E)$, is there a cycle that contains all the vertices?

The Hamiltonian Cycle on Bipartite Graphs Problem (HCBP): Given a bipartite graph $G = (V, E)$, with bipartition $V = A \cup B$ with $|A| = |B|$ is there a cycle that contains all the vertices?

Definitions:

- A **configuration** is an ordered pair of the form $\mathcal{C} = (G, f)$ where $G = (V, E)$ is an undirected graph, and $f : V \to \{-1, 0, 1\}$ can be thought of as the pebble assignment function where vertices mapped to -1 are occupied by a black pebble, vertices mapped to 1 are occupied by a white pebble, and vertices mapped to 0 are empty.
- A **slide** is the ordered pair of configurations $(\mathcal{C}_1, \mathcal{C}_2)$ where $\mathcal{C}_1 = (G, f_1)$, $\mathcal{C}_2 = (G, f_2)$, for $G = (V, E)$ such that there exists $uv \in E$ such that $f_1 \equiv f_2$ on $V \setminus uv$, and $f_1(v) = f_2(u) = 0$ and $f_1(u) = f_2(v) \neq 0$. For brevity, we will denote a slide as $u \to v$, where the pebble on u slides to v.

Closest Pebble Distance Problem (CPDP): Given a configuration $\mathcal{C} = (G, f)$ and a natural number k, is there a sequence of at most k slides that transforms $\mathcal{C} = (G, f)$ into $\mathcal{C}' = (G, f')$ so that $f \equiv -f'$ (the white and black pebbles have swapped places).

I will start with the assumption that the Hamiltonian Cycle Problem is NP-Hard. Then I will reduce it to the Hamiltonian Cycle Problem for bipartite graphs. The absence of odd cycles that we get for bipartite graphs is precisely what we need to tackle the closet pebble distance problem and show that it is NP-hard. What I find interesting is that the Hamiltonian Cycle Problem is still NP-hard on bipartite graphs. One possible interpretation is that the additional structure on bipartite graphs isn't enough to reduce the complexity of the problem of finding a Hamiltonian cycle.

Lemma 0.9. ***HCP*** $\leq_k$ ***HCBP***

Proof. Given as input $G = (V, E)$, construct vertices v^+ and v^- for each $v \in V$, and construct edges u^+v^- and u^-v^+ for each $uv \in E$. Also add edges v^+v^- for every $v \in V$. Call the resulting graph after the transformation G'. Then G' is a bipartite graph with one vertex set containing all v^+ and the other vertex set containing all v^-.

Polynomial Time Reduction

There are $O(n^2)$ edges to construct, and $O(n)$ vertices to construct, where n is the number of vertices in V.

Instance of HCP $\implies$ Instance of HCBP

Suppose $v_1 \cdots v_n v_1$ is a Hamiltonian Cycle in G. Then $v_1^- v_1^+ v_2^- v_2^+ v_3^- \cdots v_n^- v_n^+ v_1^-$ is a Hamiltonian Cycle in G'.

Instance of HCBP $\implies$ Instance of HCP

For some $\pi : [n] \to [n]$ suppose that $v_1^- v_{\pi(1)}^+ v_2^- v_{\pi(2)}^+ \cdots v_n^- v_{\pi(n)}^+ v_1^-$ is a Hamiltonian Cycle in G'. Any Hamiltonian Cycle in G' takes this form. The Hamiltonian Cycle in G will take the form $v_1 P v_1$ where P is a path of $n-1$ distinct vertices. Inductively, we will define $P(i)$, the ith vertex in the sequence. Define $P(1) = v_{\pi(1)}$ if $\pi(1) \neq 1$. Otherwise, set $P(1) = v_2$. Now suppose that $P(i)$ is defined to be the kth vertex in the Hamiltonian cycle. Define $P(i+1)$ to be the earliest vertex after the kth vertex in the Hamiltonian Cycle in G' that originates from a different vertex in G. Symbolically, define the function $H : [2n] \to V$ as follows: Suppose that the kth vertex in the Hamiltonian Cycle in G' is $v_j^\pm$ (the plus or minus doesn't matter, just that it originated from v_j in G). Then set $H(k) := v_j$.

Set $\tau := \min\{t > k : H(t) \neq H(k)\}$ and $P(i+1) := H(\tau)$. I claim that $v_1 P v_1$ is a Hamiltonian Cycle in G. Note that each vertex occurs once in $v_1 P$, otherwise there will be two of the same vertices in the Hamiltonian Cycle in G'. Also, the length of P is $n-1$ since the Hamiltonian Cycle in G' must include every vertex. in G. Therefore, $v_1 P v_1$ is a Hamiltonian Cycle in G.

□

Theorem 0.10. ***HCBP*** $\leq_k$ ***CPDP***

Proof. We will construct the graph $G' = (V', E')$ as follows: Set the new vertex set to include all the original vertices with two extra vertices: $V' = V \cup \{x, y\}$, where $x, y \notin V$. Construct the new edge set E' to contain all the edges in E with two extra edges: Fix an arbitrary $v_0 \in V^+$ and add edges $v_0 x$ and $v_0 y$. Call the resulting graph $G' = (V', E')$.

Let V^-, V^+ be the two bipartite sets of V. Place black pebbles on all the vertices in V^- and white pebbles on all the vertices in V^+. In other words, set

$$f(v) = \begin{cases} 1 \text{ if } v \in V^+ \\ -1 \text{ if } v \in V^- \\ 0 \text{ if } v \in \{x, y\} \end{cases}$$

Set the initial configuration to be $\mathcal{C} := (G' f)$. Set $k := n + 4$. Then our transformation is $G \to (\mathcal{C}, k)$.

Polynomial Time Reduction

There are $O(n^2)$ edges to construct, and $O(n)$ vertices to construct in G', where n is the number of vertices in V.

Instance of HCBP $\implies$ Instance of CPDP

Suppose $v_0 v_1 v_2 \cdots v_{n-1} v_0$ is a Hamiltonian cycle (I counted up to $n-1$ because there are n vertices and we're starting at 0). Recall that v_0 is the "special" vertex that has edges $v_0 x$ and $v_0 y$. Here is the sequence of slides:

- Initially:
  $\langle f(x), f(y), f(v_0), f(v_1), \cdots f(v_{n-1})\rangle = \langle 0, 0, 1, -1, 1, \cdots, -1\rangle$
- After slide 1 ($v_0 \to x$):
  $\langle f(x), f(y), f(v_0), f(v_1), \cdots f(v_{n-1})\rangle = \langle 1, 0, 0, -1, 1, \cdots, -1\rangle$
- After slides 2 through n ($v_1 \to v_0, v_2 \to v_1, v_3 \to v_2, \cdots v_{n-1} \to v_{n-2}$):
  $\langle f(x), f(y), f(v_0), f(v_1), \cdots f(v_{n-1})\rangle = \langle 1, 0, -1, 1, -1, \cdots, 0\rangle$
- After slide n+1 ($v_0 \to y$):
  $\langle f(x), f(y), f(v_0), f(v_1), \cdots f(v_{n-1})\rangle = \langle 1, -1, 0, 1, -1, \cdots, 0\rangle$
- After slide n+2 ($x \to v_0$):
  $\langle f(x), f(y), f(v_0), f(v_1), \cdots f(v_{n-1})\rangle = \langle 0, -1, 1, 1, -1, \cdots, 0\rangle$
- After slide n+3 ($v_0 \to v_{n-1}$):
  $\langle f(x), f(y), f(v_0), f(v_1), \cdots f(v_{n-1})\rangle = \langle 0, -1, 0, 1, -1, \ldots, 1\rangle$
- After slide n+4 ($y \to v_0$):
  $\langle f(x), f(y), f(v_0), f(v_1), \cdots f(v_{n-1})\rangle = \langle 0, 0, -1, 1, -1, \cdots, 1\rangle$

Instance of CPDP $\implies$ Instance of HCBP

For any valid sequence of slides, we can track the hole trajectories that start from x and y. We define the sequences by x_i (y_i) := position of the hole starting from x (y) after it has moved i times as a result of slides.

Hole trajectory starting from x: $x_0, x_1, \cdots, x_k$

Hole trajectory starting from y: $y_0, y_1, \cdots, y_j$

Observations:

1. The union of trajectories x and y visits every vertex in V: $v_0, v_1, \cdots v_{n-1}$
2. $x_0, x_k, y_0, y_j \in \{x, y\}$.
3. $j, k > 0$.

Proof of Observations:

Recall $V = \{v_0, v_1, \cdots v_{n-1}\} = V' \setminus \{x, y\}$ and $v_0 \in V^+$.

1. For each $v \in V$, a different pebble starts and ends there. Thus, there must have been a hole there at some point, so a pebble could move there.
2. Every $v \in V$ starts with a pebble, so the holes must start and return to x and y.
3. Without loss of generality, suppose a sequence of slides didn't use y. Then the pebble at v_0 can only shift between x and v_0. Both of these vertices aren't in V^-, which is where it needs to be after a valid sequence of slides.

Observations (2) and (3) imply that $j, k \geq 2$. Then observation (1) in combination implies that the total number of slides $(j + k)$ is at least $n + 4$. If there was a valid transformation with exactly $n + 4$ slides, trajectory x (without loss of generality) must be of the form x, v_0, P, v_0, x where P is some path of $n - 1$ vertices. P includes $v_1, \cdots, v_{n-1}$, each of which appears exactly once. Thus, we have a Hamiltonian Cycle, assuming $n \geq 4$.

□

0.0.1 Remark:

After the proof, one might suspect that the decision problem: given a configuration of pebbles on a graph is there a **finite length** sequence of moves that swap the pebbles is a NP-hard problem. But that's not what this reduction shows. Here's a counterexample:

Let $V = \{1, 2, 3, 4, 5, 6\}$ with $V^- = \{1, 2, 3\}$ and $V^+ = \{4, 5, 6\}$. Let $E = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{3, 4\}\}$, and $x = 7$ and $y = 8$ be connected to vertex 1. This graph doesn't have a Hamiltonian Cycle, but there is a finite sequence of slides in the tranformed graph: 1

Relation to Problems in the Literature

The Knights Swap is an instance of a more general problem called the Multi-Color Pebble Motion on Graphs: https://link.springer.com/article/10.1007/s00453-009-9290-7 The authors show that deciding the feasibility for general graphs can be done in linear time (Not an optimal solution, but whether a solution exists).

Another variant is the pebble motion problem where every pebble can be thought of as having a distinct color: https://ieeexplore.ieee.org/document/715921. The authors present an algorithm to find a solution with $O(n^3)$ number of moves on general graphs, if there exists a solution. They also show that this bound is tight.

Step	Configuration $(1,2,3,4,5,6,7,8)$	Slide (source $\to$ dest)
0	(-1, -1, -1, 1, 1, 1, 0, 0)	– Start
1	(0, -1, -1, 1, 1, 1, -1, 0)	$1 \to 7$
2	(1, -1, -1, 0, 1, 1, -1, 0)	$4 \to 1$
3	(0, -1, -1, 0, 1, 1, -1, 1)	$1 \to 8$
4	(0, 0, -1, -1, 1, 1, -1, 1)	$2 \to 4$
5	(0, 1, -1, -1, 0, 1, -1, 1)	$5 \to 2$
6	(1, 1, -1, -1, 0, 0, -1, 1)	$6 \to 1$
7	(0, 1, -1, -1, 1, 0, -1, 1)	$1 \to 5$
8	(-1, 1, -1, 0, 1, 0, -1, 1)	$4 \to 1$
9	(0, 1, -1, 0, 1, -1, -1, 1)	$1 \to 6$
10	(0, 1, 0, -1, 1, -1, -1, 1)	$3 \to 4$
11	(-1, 1, 0, 0, 1, -1, -1, 1)	$4 \to 1$
12	(-1, 0, 0, 1, 1, -1, -1, 1)	$2 \to 4$
13	(-1, 0, 1, 0, 1, -1, -1, 1)	$4 \to 3$
14	(0, 0, 1, -1, 1, -1, -1, 1)	$1 \to 4$
15	(0, 1, 1, -1, 0, -1, -1, 1)	$5 \to 2$
16	(-1, 1, 1, -1, 0, -1, 0, 1)	$7 \to 1$
17	(0, 1, 1, -1, -1, -1, 0, 1)	$1 \to 5$
18	(1, 1, 1, -1, -1, -1, 0, 0)	$8 \to 1$

Table 1:
$V' = \{1,2,3,4,5,6,7,8\}$
$V^- = \{1,2,3\}$
$V^+ = \{4,5,6\}$
$E' = \{\{1,4\},\{1,5\},\{1,6\},\{2,4\},\{2,5\},\{3,4\},\{1,7\},\{1,8\}\}$
,
White pebbles are placed on vertices in V^+ and black pebbles are placed on vertices in V^-.

What about finding an optimal solution? The classic way to show that these optimality types of problems is NP-hard is first showing that finding an optimal solution to the N^2-1 puzzle is NP-hard (move tiles on a board with one square empty) because the pebble motion problem can be considered a generalization of it. But my approach was different, since I reduced to the problem directly. Please fact-check my reduction above. I can't find it anywhere in the literature.