STAT 150 - MIDTERM 1

SPRING 2023

Name (Last, First): ______________________________

Student ID: ______________________________

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

YOUR ANSWERS SHOULD BE CAREFULLY JUSTIFIED. ANSWERS WITHOUT PROPER JUSTIFICATION OR WORK, EVEN IF CORRECT, WILL NOT RECEIVE CREDIT.

THE EXAM CONSISTS OF 3 QUESTIONS. MAKE SURE TO READ THE QUESTIONS CAREFULLY. YOU ARE ALLOWED TO USE RESULTS FROM THE TEXTBOOK, HOMEWORK, AND LECTURE AS LONG AS THEY ARE CLEARLY REFERENCED.

1. (15 points) Let $(X_n)_{n=0}^{\infty}$ be an absorbing Markov chain on the finite state space $[N] = \{1, \dots, N\}$ with one-step transition probabilities $(p_{i,j})_{i,j \in [N]}$, transient states $\{1, \dots, r\}$, and absorbing states $\{r+1, \dots, N\}$. Suppose that the function $h : [N] \times [N] \to \mathbb{R}$ gives the cost of a transition between any two states: if $i, j \in [N]$, then a transition from the state i to the state j costs $h(i,j)$. Let $T = \min\{n \geq 0 : X_n \geq r+1\}$ be the time of absorption. Define

$$w_i = \mathbb{E}\left[\sum_{n=0}^{T-1} h(X_n, X_{n+1}) \middle| X_0 = i\right],$$

which is the expected total cost at the time of absorption assuming $X_0 = i$. For example, $w_i = 0$ if $i \in \{r+1, \dots, N\}$. So, we are interested in the values of w_i for $i \in \{1, \dots, r\}$.

(a) (7 points) Use first-step analysis to derive the system of equations relating the $(w_i)_{i=1}^{r}$.

(b) (8 points) Recall that for $i \in \{1, \ldots, r\}$ and $k \in \{r+1, \ldots, N\}$, we defined

$$u_{i,k} = \mathbb{P}(X_T = k | X_0 = i),$$

which is the probability of being absorbed into state k assuming we start at state i. Is there a cost function h such that $w_i = u_{i,k}$? If yes, explain why and find such a function. If no, explain why not.

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2. (15 points) Let $(X_n)_{n=0}^{\infty}$ be a branching process with common offspring distribution ξ, where $\mathbb{P}(\xi = k) = p_k$. Assume that ξ has probability generating function $\phi(s) = \sum_{k=0}^{\infty} p_k s^k$. Let $S = \min\{n \geq 0 : \xi_i^{(n)} = 0 \text{ for some } i \in [X_n]\}$, where $X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}$. In words, S is the earliest generation in which we observe an individual who does not have any children. For example, if $X_0 = 1$ and $X_1 = 0$, then $S = 0$.

(a) (6 points) Compute $\mathbb{P}(S \geq m+1 | S \geq m, X_m = k)$. For full credit, your answer should not include an infinite series.

(b) (9 points) Define $q_m = \mathbb{P}(S \geq m | X_0 = 1)$. Find a formula for q_{m+1} in terms of ϕ, q_m, and p_0. For full credit, your answer should not include an infinite series.

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3. (20 points) Let $(X_n)_{n=0}^{\infty}$ be a Markov chain on the state space $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ with one-step transition matrix $(p_{i,j})_{i,j\in\mathbb{N}_0}$ satisfying $p_{0,0} = \frac{1}{1+\alpha}$, $p_{0,1} = \frac{\alpha}{1+\alpha}$, and

$$p_{i,i-1} = \frac{1}{1+\alpha};$$
$$p_{i,i+1} = \frac{\alpha}{1+\alpha}.$$

if $i \geq 1$.

(a) (15 points) Assume that $\alpha \in (0, 1)$. Find a stationary distribution for this Markov chain.

(b) (5 points) Now assume that $\alpha \geq 1$. Does a stationary distribution exist? If so, find one. If not, explain why.

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