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Shifted-Lognormal LIBOR Market Model

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Abstract

This document provides a client overview of the Bloomberg implementation of the shifted-lognormal LIBOR market model using several volatility and correlation models.

Keywords. LIBOR Market Model, Monte Carlo simulation, shifted lognormal, correlation model, swaption calibration, CMS-spread calibration.

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1 Introduction

This document provides an overview of pricing interest rate derivatives using the Shifted-Lognormal LIBOR Market Model, with particular emphasis on the client's interaction with the calibration and Monte Carlo simulation component interfaces. We will also discuss the different flavors of volatility, correlation, and shift models. Calibration examples using the Caplet, Swaption and CMS-spread option markets are illustrated in terms of their quality-of-fit. Pricing examples using vanillas, exotics, and callable deals will also be given.

In what follows, the client's interaction with the LMM Pricing Function will be described in terms of the following steps:

1. specification of deal characterizing interest rate derivative being priced
2. specification of LIBOR Market (schedule of reset dates, accrual dates, and accrual coverages)
3. specification of LMM Model Parameters (shifts, volatility and correlation models)
4. specification of calibration market data (discount curve, swaption volatilities, and/or cap and cms spread option prices)
5. specification of calibration model data (initial values and bounds of calibrated model parameters, fixed values of non-calibrated model parameters)
6. specification of calibration configuration settings (price vs. Black volatility vs. Normal volatility)
7. specification of Monte Carlo configuration settings (number of simulation paths)

2 LIBOR Market Specification

The starting point for all pricing is the specification of a deal, *i.e.* translating a deal's term-sheet into parameters that can be specified to the DLIB APIs. Although the details of specifying a deal are not discussed in this document, certain elements of the deal's structure are necessary to pricing with the LIBOR Market Model. To the extent that specific underlying indexes comprising a deal are related to the LIBOR rates of some tenor and in some currency (*e.g.* US0003M or EUR006M), for example CMS rates or the LIBOR rates themselves, special attention must be given to the schedules implicit in the deal structure. In particular, if the deal requires LIBOR rates $\{F_1, \ldots, F_n\}$ which are fixed at fixing dates[1] $\{T_1^-, \ldots, T_n^-\}$ and are applied to accrual periods $\{T_1, \ldots, T_n, T_{n+1}\}$, then the following LIBOR Market would be diagrammatically specified as follows:

$$\text{today} = T_0 \longrightarrow T_1 \xrightarrow[\tau_1]{F_1} T_2 \xrightarrow[\tau_2]{F_2} \cdots \xrightarrow[\tau_{n-1}]{F_{n-1}} T_n \xrightarrow[\tau_n]{F_n} T_{n+1}$$

The time T_0 indicates the evaluation[2] date of the deal, and the coverages $\{\tau_1, \ldots, \tau_n\}$ are the accrual times given in units of year-fractions, whose exact values are deduced from the day-count

[1]Fixing dates and forward rates are often referred to as reset dates and reset rates, respectively.

[2]The evaluation date is sometimes referred to as the settlement date, the pricing date, or the as-of date.

conventions specified by the deal. For example, the following data are associated to the US0003M index, using the ACT/360 day-count convention and an effective date of December 11, 2012:

k	Reset Date T_k^-	Accrual Start T_k	Accrual End T_{k+1}	Days -	Coverage τ_k	Reset Rate F_k
1	03/11/2013	03/13/2013	06/13/2013	92	0.2555	0.2801
2	06/11/2013	06/13/2013	09/13/2013	92	0.2555	0.2865
3	09/11/2013	09/13/2013	12/13/2013	91	0.2527	0.3113

Table 2.1: US0003M (ACT/360) on 12/10/2012 (First Three LIBOR Periods)

In general, every deal will determine a Libor schedule identical to that of a swap which starts on the settlement date (normally today's date, unless it is an aged deal), and which extends to include the deal's horizon date. Note that if calibration instruments are later selected (see §4) whose maturities necessitate the inclusion of additional Libors, then the Libor schedule's end date is correspondingly extended beyond the horizon date.

3 Model Specification and Model Parameters

The LiborMarketSpec described in §2 only partially characterizes the LIBOR Market Model; in fact, it is only the "LIBOR Market" part of the "LIBOR Market Model". Specifying the "Model" part of LMM requires the specification of the main model (the shifted-lognormal model in the present case) and its three additional sub-models (shifts, volatility and correlation).

The shifted-lognormal LMM supports four flavors of correlation sub-model (Rebonato two-parameter full-factor, Rebonato two-parameter reduced-factor, and of course a fully specified correlation matrix), three flavors of the volatility sub-model (constant, piecewise-constant, linear-exponential), and two flavors of shift model (alpha-shift and beta-shift).

While we provide technical details in the appendices, for purposes of this section we merely state that the shifted-lognormal LMM models the dynamics of the LIBOR rates $F_k(t)$ using the following coupled system of stochastic differential equations[3]:

$$\boxed{dF_k(t) \quad = \quad \mu_k^Q(t)dt + \sigma_k(t)(F_k(t) + \alpha_k)dW_k^Q(t), \qquad (3.1)}$$

where the various Brownian motions $dW_k^Q(t)$ have the "forward-forward" correlation structure given by ρ:

$$dW^Q(t)' \cdot dW^Q(t) = \begin{bmatrix} dW_1^Q \\ \vdots \\ dW_n^Q \end{bmatrix} \cdot [dW_1^Q, \ldots, dW_n^Q] = \Big(\{\rho_{i,j}dt\}\Big) =: \rho dt. \qquad (3.2)$$

[3]The superscript Q here indicates the Q-measure associated with a choice of Numeraire.

These SDE's contain both diffusion terms $\sigma_k(t)F_k(t)dW_k^Q(t)$ and drift terms $\mu_k^Q(t)dt$ which are described in Appendix C. What concerns us in this section is identifying the model parameters $\sigma_k(t), \alpha_k(t)$ and $\rho_{i,j}(t)$, and explaining how they are specified by the client user.

Strictly speaking, the model parameters are the functions $\sigma_k(t), \alpha_k(t)$ and $\rho_{i,j}(t)$. However, inasmuch as we mandate parametric forms corresponding to a "volatility-model" and a "correlation-model" for these functions, the model parameters are identified with the associated functional parameters. For example, when using the piecewise-constant volatility-model we have $\sigma_i(t) = \sigma_{i,j}$ for $t \in [T_j, T_{j+1})$, and so $n(n+1)/2$ model parameters must be specified. Similarly, when using the Rebonato (two-parameter, full-factor) correlation-model[4] we have:

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)e^{-\rho_d|T_i - T_j|},$$

and so only the two parameters ρ_d and ρ_∞ must be specified.

Volatilities	Constant	Piecewise Constant	Linear Exponential
$\sigma_i(t)$	σ_i	$\sigma_{i,j}, \quad t \in [T_{j-1}, T_j]$	$\Phi_i\left[(a(T_i - t) + d)e^{-b(T_i-t)} + c\right]$
Parameters	σ_i	$\sigma_{i,j}, \quad j \le i$	a, b, c, d, Φ_i
Number Parameters	n	$n(n+1)/2$	$n+4$

Table 3.1: Volatility-Model Parameters

Shifts	Constant
$\alpha_i(t)$	α_i
Parameters	α_i (= 0 for lognormal)
Number Parameters	n

Table 3.2: Shift-Model Parameters

We summarize in tables Table 3.1, Table 3.3, and Table 3.2 the model parameters associated with the different flavors of volatility structure and forward-forward correlation structure, as well as the shift parameters specific to the shifted-lognormal base model:

Correlations	Rebonato Full-Factor	Rebonato Reduced-Factor	$A \cdot A'$ $\lvert A_{i,}\rvert = 1$
$\rho_{i,j}$	$\rho_\infty + (1-\rho_\infty)e^{-\rho_d\lvert T_i - T_j\rvert}$	f-factor PCA	$\sum_k a_{i,k}a_{j,k}$
Parameters	ρ_∞, ρ_d	ρ_∞, ρ_d, f	$\rho_{i,j} \quad {\scriptstyle j>i}$
Number Parameters	2	2	$n(n-1)/2$

Table 3.3: Correlation-Model Parameters (†3-parameter not currently exposed in DLIB.)

The starting point for constructing a LIBOR market model will therefore consist of specifying the

[4]None of the correlation models considered here are time-dependent; $\rho_{ij}(t)$ depends only on the indices (i, j).

Figure 3.1: Volatility and Shift model parameters

Figure 3.2: Correlation Parameters

choice of each sub-model, as well as the initial values (or default values if uncalibrated) for all of the corresponding model parameters.

To illustrate how a LIBOR market model can be constructed in practice, let's assume that in a three-period LIBOR market the lognormal model is chosen with a volatility model of constant type $(\sigma_1, \sigma_2, \sigma_3)$, and a correlation model of Rebonato full-factor type (ρ_d, ρ_∞). The selection of a lognormal model implies $\{\alpha_1 = \alpha_2 = \alpha_3 = 0\}$, and we arbitrarily assign $\{\sigma_1 = 0.11, \sigma_2 = 0.12, \sigma_3 = 0.13\}$ to the constant vols associated with the forwards F_1, F_2, F_3, and set $\{\rho_\infty = 0.35, \rho_d = 0.15\}$ in the Rebonato full-factor correlation parameters.

Note that the collection of $\sigma_{i,j}$, each of which is the volatility of the i'th LIBOR $F_i(t)$ during the time interval when $T_j < t < T_{j+1}$, has been flattened accordingly:

$$\begin{bmatrix} \sigma_{1,1} & 0 & 0 \\ \sigma_{2,1} & \sigma_{2,2} & 0 \\ \sigma_{3,1} & \sigma_{3,2} & \sigma_{3,3} \end{bmatrix} \longmapsto [\sigma_{1,1}, \sigma_{2,1}, \sigma_{2,2}, \sigma_{3,1}, \sigma_{3,2}, \sigma_{3,3}];$$
$$\begin{bmatrix} 0.11 & 0 & 0 \\ 0.21 & 0.22 & 0 \\ 0.31 & 0.32 & 0.33 \end{bmatrix} \longmapsto [0.11, 0.12, 0.21, 0.22, 0.23, 0.31, 0.32, 0.33].$$

Observe that $\sigma_{i,j} = 0$ when $j > i$ because $F_i(t > T_i)$ has no volatility after its fixing date, and is therefore omitted from the parametrization.

4 Calibration Instruments

Calibration refers to the method of finding values of model-parameters which produce observable values (e.g. swaption volatilities or cap premiums or cms-spread correlations) of (vanilla) market data. In general, it may not be possible to find such values of the model-parameters, and so one constructs a nonnegative-valued "objective function" of the model-parameters which will be zero precisely when the market data is perfectly reproduced, and whose minimization is taken as a "best-fit" of the model-parameters to market data.

4.1 Objective Function

Writing the vector of model-parameters as $\mathbf{x}$, we can describe the objective function

$$\mathbf{F} : \mathbb{R}^{dim(\mathbf{x})} \to \mathbb{R}^1$$

as follows: Assume there are M calibration targets, each with a market value (whether a swaption volatility, undiscounted swaption price, cap premium, etc.) represented by C_k $(1 \leq k \leq M)$. Assume further that the model predicts (based upon its vector of model-parameters $\mathbf{x}$) a value of $\hat{C}_k(\mathbf{x})$ for each of the M calibration instruments. Then the objective function can be described as taking the length of the vector $\mathbf{e}(\mathbf{x})$ of errors:

$$\mathbf{x} \in \mathbb{R}^{dim(\mathbf{x})} \longrightarrow \|\mathbf{e}(\mathbf{x})\| = \sqrt{\sum_{1 \leq k \leq M} (\hat{C}_k(\mathbf{x}) - C_k(\mathbf{x}))^2}.$$

Note that the k'th component of $\mathbf{e}(\mathbf{x})$ is the signed-error in the k'th calibration instrument, and so the vector $\mathbf{e}(\mathbf{x})$ records all errors on an equal footing. If a weighting of the individual errors is desired (e.g. by the inverse of a bid-ask spread), one may replace the Euclidean norm with a weighted norm

$$\mathbf{F}(\mathbf{x}) = \sqrt{\sum_{1\le k\le M} \omega_k(\hat{C}_k(\mathbf{x}) - C_k(\mathbf{x}))^2} \tag{4.1}$$

with weights $\omega_k > 0$.

Note further that market instruments quoted in different units will create an objective function that combines terms of incompatible units. From this point of view, the weights may also be employed as unit conversions, for example multiplying terms quoted in Black volatility by a vega factor, effectively converting to a common unit of price which can be meaningfully combined with other terms quoted in price. The use of the ω_k to accommodate different calibration target units is depicted in Table 4.1, and described more fully in §5.3.

Target Type	Instrument Weight
Price	1.0
Black Volatility	Inverse Black Vega
Normal Volatility	Inverse Normal Vega

Table 4.1: Target types and their associated ω_k

4.2 Least Squares

The least-squares method addresses the minimization of the scalar objective function $\mathbf{F}(\mathbf{x})$ by applying specialized linear algebra algorithms to the geometry of the vector of signed errors $\mathbf{e}(\mathbf{x})$. The collection of all $\mathbf{e}(\mathbf{x})$ forms a hypersurface in Euclidean space of m dimensions and we seek an $\mathbf{x}$ for which $\mathbf{e}(\mathbf{x})$ is closest to the origin. The two basic algorithms used in this context are Steepest-Descent, which is stable but converges slowly, and Gauss-Newton, which is fast but need not converge. The Levenberg-Marquadt algorithm is a hybrid of Steepest-Descent and Gauss-Newton, and can be proved to converge in the presence of constraints. Note that unconstrained optimization is not applicable to our calibrations, since, for example, volatility and shift model parameters must be non-negative.

Unlike many other optimization approaches which employ a *bootstrapping* methodology, the LMM optimizer calibrates all model parameters in one step. An exception to this which relates to skew calibration is described in §5.5.

4.3 Instrument Selection

Calibration will require having market prices (or volatilities, or correlations, etc.) of a set of calibration instruments (*i.e.* the quantities $C_k(\mathbf{x})$ appearing in (4.1)). In general, the choice of these instruments will be tuned to the term-sheet being priced. The following instrument types are supported which can be quoted as Black Volatility or Normal Volatility or Spot Premium for ATM strike or Absolute Strike (OTM):

- Swaptions (Payer or Receiver if quoted in Premium)
- Caps (Floors)
- Caplets (Floorlets)

Additionally, the following CMS-Spread instrument types are supported which can be quoted as Normal Volatility or Spot Premium or Forward Premium for ATM strike or Absolute Strike (OTM):

- CMS-Spread Options (Straddles, Calls, Puts)

Finally, the following CMS-Spread instrument types is supported which is quoted a Spread Correlation for ATM strike or Absolute Strike (OTM):

- CMS-Spread Correlations

Market quotes for each of these instrument types are retrievable by invoking the appropriate Terminal function, or by using Bloomberg's `blph()` function in Excel with the appropriate ticker symbol. DLIB retrieves quotes from the same source as the VCUB function. Some examples are provided in Table 4.2.

Instrument	Function	Ticker example
Discount Factors	SWDF EU	'S0045D 18M BLC2 Curncy'
ATM Swaption Volatility	GVSP 295	'EUSV0A1 BVOL Curncy'
OTM Swaption Volatility	IAEP 2 1	'EUPA0C02 BVOL Curncy'
ATM Cap Premium	ICAE 17	'EUCPAM CMPN Curncy'
OTM Cap Premium		'EUCF201 CMPN Curncy'
CMS Premium over 3M LIBOR	ICAE 20	'EUCM025 CMPN Curncy'
CMS 10/2 Spread Options	GDCO 3449 3	'EUSO2023 Curncy'

Table 4.2: Example Instrument Tickers

Note that when calibrating to CMS spread options, it is important to include among the calibration instruments the swaption instruments which correspond to the reference swaps of each leg of the CMS spread. For example, calibrating to a CMS spread option 10/2 with expiry 3 years should include the 3x10 and the 3x2 swaptions. Also note that the ATM of the CMS spread straddle is a (shifted-lognormal) model-dependent convexity-corrected strike. Unless an absolute strike for the spread is supplied, the ATM strike is computed internally. One expedient pricing formula for the CMS spread premium uses a Gaussian bi-variate model whose ingredients are the normal volatilities (possibly converted from quoted ATM Black volatilities) of the reference swaps comprising the legs

Figure 4.1: ATM Swaption Matrix

of the spread, and also their terminal correlation (which may be historically determined). The well known formula for the resulting normal volatility of the spread

$$\sigma_{spread} = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \tag{4.2}$$

can be used (if not supplied directly) in a Bachelier pricing formula to price the straddle.

When internally calibrating to the CMS spread option price, whether (see Table 4.3 below) provided directly as a Spot Premium, or Forward Premium, or indirectly as the implied normal volatility σ_{spread} or implied terminal correlation of the spread, the calibrator uses an approximation formula for the price as a function of model parameters, specifically the volatility and shift parameters of the constituent Libors belonging to the legs. Clearly, without calibration instruments which influence the model parameters associated with the CMS legs, such as swaptions on their underlying reference swaps, the leg volatilities will be indeterminate.

Quotation Type	Quotation Meaning	Conversion to Forward Premium
Forward Premium	Forward Premium	None
Spot Premium	Discounted Forward Premium	Un-Discount to Forward Premium
SpreadVol	Normal Volatility σ_{spread} of Spread	Apply Bachelier Pricing Formula
IndexCorr	Correlation ρ between CMS legs	Apply (4.2) and Bachelier Pricing Formula

Table 4.3: Quotation alternatives for CMS Spread Options

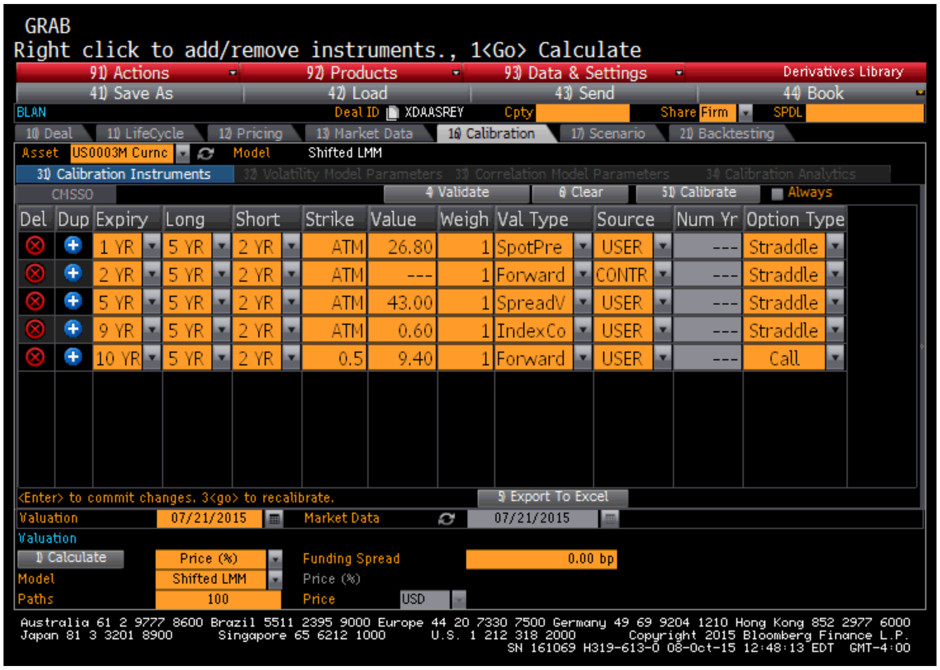


Figure 4.2: CMS Spread Options

5 Calibration

Having described the specification of calibration instruments, we proceed to describe the details of performing a calibration with the shifted-lognormal LMM.

Recalling that the calibration process is a matter of tuning model parameters through an optimization algorithm, we must consider the question of the "initial values" of the model parameters, and also whether certain model parameters will be "non-calibrated".

5.1 Initial and Default Parameter Values

If a model parameter is a *calibrated parameter*, then its initial value plays the role of the "initial guess" used by the optimizer. If one views optimization as the process of incrementally improving the fit to market data by tuning model parameters, then the initial guess is simply the starting point for this process. In this context, the user need not (and typically would not) be burdened with specifying an initial guess, as default values are internally provided. Of course, users can coerce their own choice of an initial guess by overriding the internal defaults[5]

[5]If a user specified value is outside of the range of acceptable values for a given model parameter, for example a negative Black volatility, then an error will be thrown.

5.2 Fixed (non-calibrated) Parameter Values

If a model parameter is a *non-calibrated parameter*, then its initial value will be held fixed during the optimization process (and will thereby imply a limitation on how good of a fit the optimizer can achieve), and consequently this initial value will also be its final value.

More generally, users may provide their own choice of parameter values by explicitly overriding the internal defaults. Non-calibrated parameters typically arise when a user has already performed a calibration (in which all shifts, volatilities, and correlations are calibrated parameters as in §5.1), and subsequently wants to perform a quick recalibration of *only the volatilities* to a subset of market data. To explain this more fully, consider the following loosely accurate identifications:

- calibrate to ATM swaptions $\longleftrightarrow$ calibration of volatilities
- calibrate to OTM swaptions $\longleftrightarrow$ calibration of shifts
- calibrate to CMS-spread options $\longleftrightarrow$ calibration of correlations

A concrete example would be to perform a calibration to the ATM and OTM swaption markets and also the CMS-spread option market, which will result in all model parameters being calibrated. Alternatively, one could choose to calibrate the volatility parameters to the ATM swaption market only, while specifying the shifts and correlations as fixed non-calibrated parameters whose initial values would be specified under DLIB's Correlation tab.

5.3 Internal Optimizer Settings

There are several calibration settings which are in effect when performing a calibration. The most important setting is the specification of the *target type* of the calibration, which is to say the *units* used by the optimizer. Calibrating to the volatilities would be a natural choice in the case of a swaption-only market, whereas calibrating to the price is suggested when caps or cms-spread option premiums are quoted. The target type used in LMM is chosen to be "price using vega-weighting", which will respect the instrument quotation, whether price, Black volatility, or Normal volatility. Calibrating to quoted volatility will effectively change the units of price to units of volatility[6] by applying an appropriate vega-weighting to the instrument's price. This vega-weight adjustment can be thought of as a conversion factor from units of price to units of volatility based on the market sensitivity. Note that it is desirable to avoid a mix of Black and Normal volatilities by ensuring that all instruments have consistent *vol-type* settings. For example, calibrating to a mix of swaptions and cms-spread options should use Normal Volatilities exclusively, since the cms-spread options do not support Black Volatility.

Another calibration setting is the specification of the *error type* of the calibration, which in the LMM implementation is always Absolute Error, as given in (4.1).

[6] Whether the volatility is Black or Normal is determined by the *vol-type* associated with the instrument.

Smoothing of the shift and volatility model parameters through the use of a penalty function appended to the objection function is another feature of LMM calibration. See Appendix F for a description of how this feature is implemented.

The remaining internal configuration settings relate to the optimizer, and have been tuned to enhance performance.

5.4 Performing the Calibration

We have already described the minimal specification of the LMM model parameters, which will be supplemented with the selection of calibration instruments, LiborMarketSpec, and YieldCurve[7]. Performing a calibration will result in a *calibrated model* whose model parameters have been *fitted* to the market instruments.

Some remarks about the methodolgy for calibrating correlation model parameters. In the full-factor calibration, the Rebonato two-parameter full-factor correlation matrix

$$\boldsymbol{\rho}^N := (\rho_{i,j}) = \left(\rho_\infty + (1-\rho_\infty)e^{-\rho_d |T_i - T_j|}\right)$$

is generated as the (ρ_∞, ρ_d) are adjusted in each optimizer iteration. In the case of the reduced-factor correlation model of F factors, a rank F approximation $\boldsymbol{\rho}^F$ is obtained from $\boldsymbol{\rho}^N$ by applying a PCA (Principal Component Analysis) algorithm each optimizer iteration. It may happen that $\boldsymbol{\rho}^F$ exhibits features not present in $\boldsymbol{\rho}^N$, such as positivity or monotonicity in the columns (for example, `DLIB RACL` using 3-factor with initial values $\rho_\infty = 0.0, \rho_d = 0.6$). On the other hand, while producing *instantaneous* correlations which could possibly possess unexpected characteristics (such as non-monotonicity or non-positivity), the *terminal* correlations between the forward Libors, which are ultimately the correlations of interest, do not inherit any of these non-intuitive features.

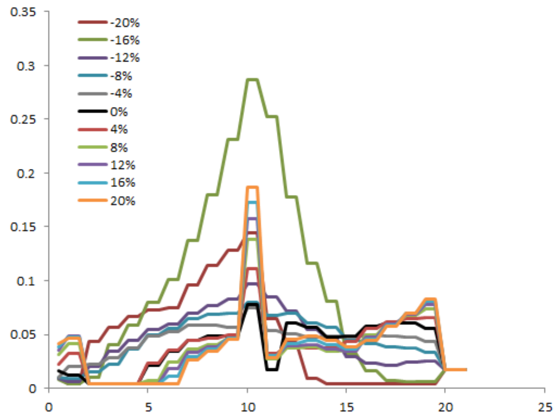
5.5 Decoupled-Shift Calibration

The "Decoupled-Shift" calibration is a methodology which was developed to address the regularity and robustness of the calibration of the shift parameters α_k. Although there are techniques that are available to impose regularity on the model parameters (see Appendix F), they are inadequate for correcting the erratic behavior visible in DLIB's Vega Scenario Analysis[8].

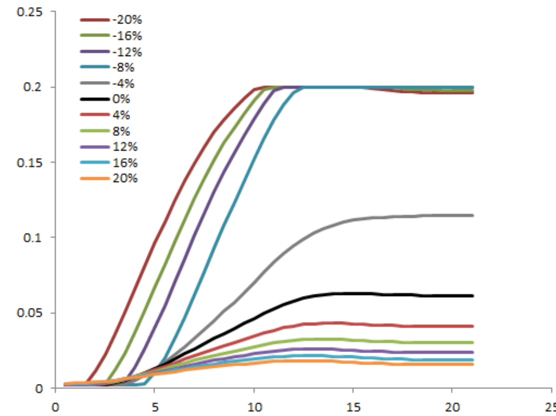
Results when calibrating the shifts in a Vega Scenario Analysis on a collared floater, which have been calibrated to 10Y swaptions (ATM & OTM) with 6M-10Y expiries, are shown in Figure 5.1(a) and Figure 5.1(b). The smooth structure of the shifts in Figure 5.1(b) is necessary to reproduce the market-implied shifts obtained in Figure 5.1(c), which shows implied shifts from each pair of swaption quotes at ATM & OTM strike, relative to systematic bumps to the market volatilities (VCUB).

[7] The YieldCurve data structure is described elsewhere.

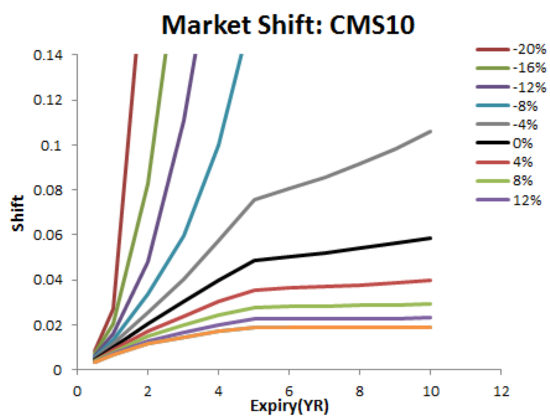
[8] The Vega Scenario Analysis in DLIB involves multiple calibrations to swaption skews by parallel shifting the market volatility data.



(a) Calibrated LMM shifts without DSC



(b) Calibrated LMM shifts with DSC



(c) Market implied shifts

Figure 5.1: Typical market-implied shifts directly obtained from swaption skew data and the calibration results.

The Decoupled-Shift calibration method achieves the desired stability and regularity shown in Figure 5.1(b) by separating the calibration of the shifts from the calibration of the volatilities. The key idea is to notice that the shift of the swap rate is, to a good approximation, a linear combination of LMM shifts per (C.16) and the swap rate shift, which can be directly computed by referring to the market swaption quotes at different strikes. Essentially, one collects all the equations (C.16) from the pairs of swaption quotes at different strikes, and then analytically solves the system of equations by using a constrained least-squares method. After mapping the market skew into LMM shifts by solving this system of equations, the standard calibration process is performed by fixing the shifts and calibrating the volatility and correlation parameters.

It should be noted that the Decoupled-Shift calibration cannot be applied to the cap instruments because of the numerical complexity. Therefore, a second phase of calibration needs to be applied with the result of the first phase as the initial guess.

The Decoupled-Shift approach yields a very good result as shown in Figure 5.1(b) and Figure 5.2, *i.e.* a systematic trend of very smooth shift profiles per varying bump levels as shown in Figure 5.1(c) and showing excellent fitting quality.

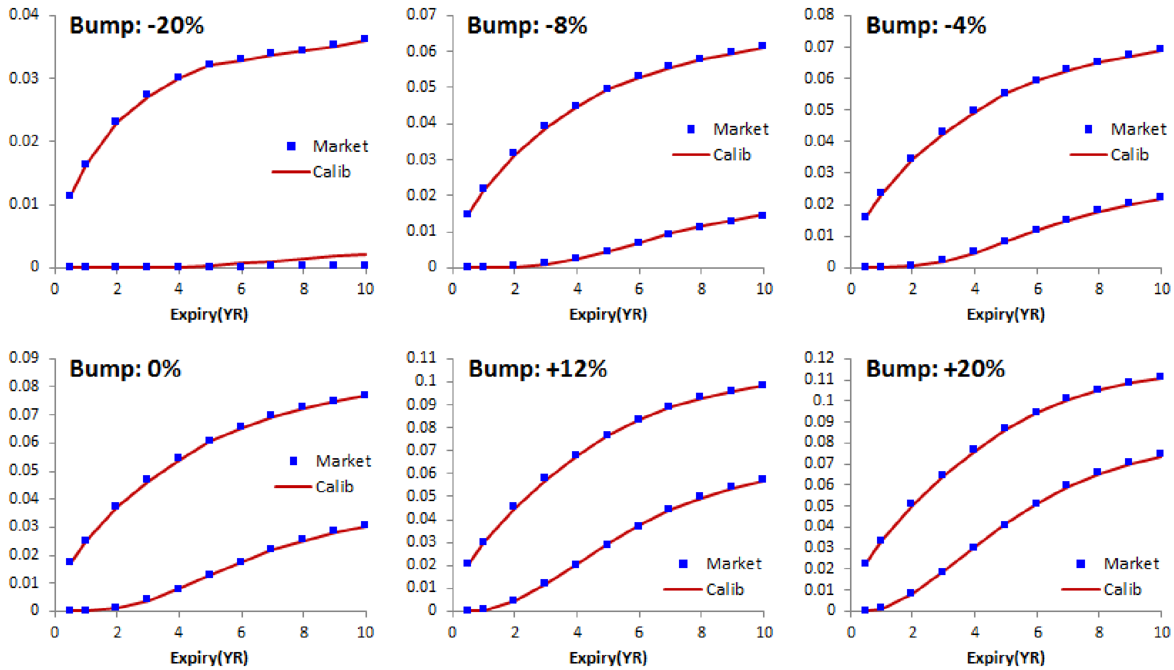


Figure 5.2: Calibrated LMM fitting quality of the Decoupled-Shift calibration. The fitting quality is excellent.

An *vega scenario analysis* shows a striking difference when compared with the Hull-White model. Since the swap rate of the Hull-White model essentially follows a normal distribution, an expected downward vega trend is observed. On the other hand, the shifted lognormal LMM will display an opposite effect with a positive vega trend, while achieving an improved fitting quality.

Another validation test was performed using calibration instruments which included the skews of caplets, swaptions, caps, and cms spread options. The results displayed in Figure 5.3 are consistent with the previous tests showing a much smoother shift profile and superior fitting quality.

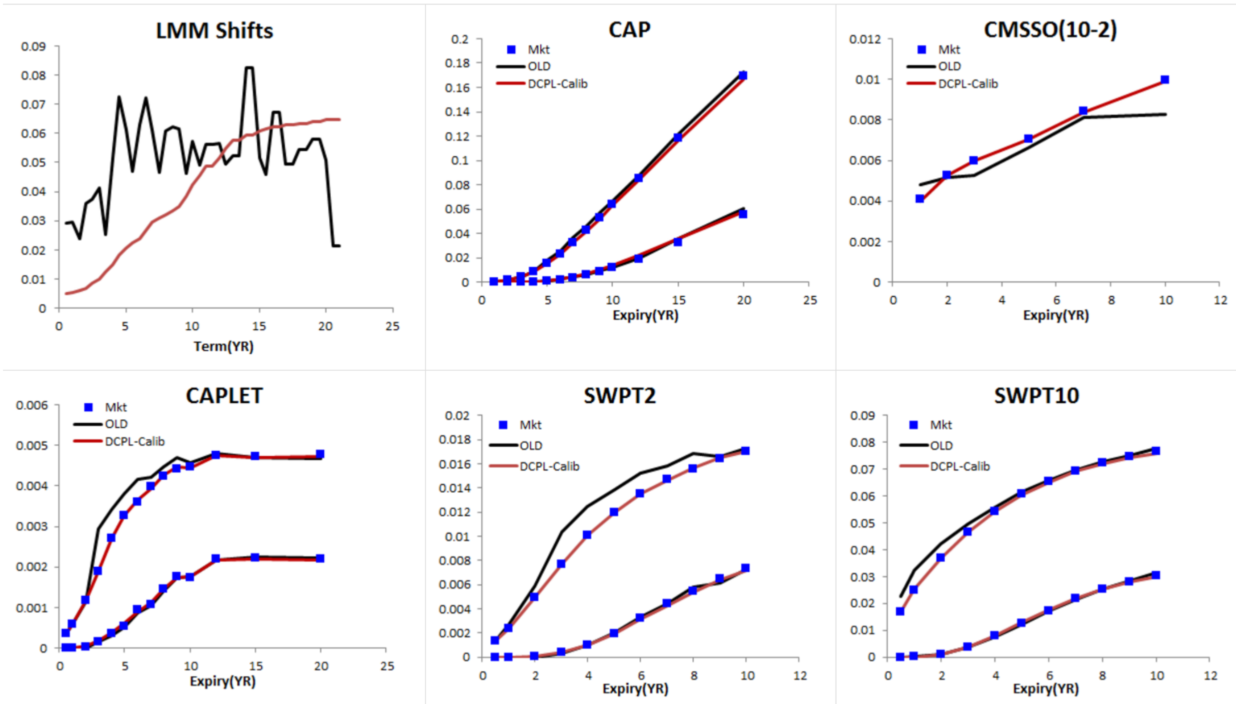


Figure 5.3: Stress test against the calibration instruments consisting of skews of caplets, (10Y/2Y) swaptions, caps and (10Y-2Y) cms spread options. Comparison with older methodology is indicated in plots with black curves.

5.6 Quality of Fit

After performing a calibration, it is of interest to determine the *quality of fit*, which is to say how well the optimizer was able to achieve fitting the model parameters to the set of calibration instruments. Moreover, in addition to determining the fit of the "in-sample" instruments to which the model parameters were calibrated, it may be desirable to determine the prices of "out-of-sample" instruments predicted by the calibrated model parameters (either by analytic formula prices in the case of caps and caplets, or approximation pricing formulas in the case of swaptions and cms-spread options).

6 Monte Carlo Pricing

Once model parameters have been determined by the calibration phase, the Monte-Carlo simulation will be invoked by the Pricer. Specifically, the Pricer will require an MCSimulator to generate simulated LIBOR rates which will be used in determining the price of the deal. The specifications required when constructing the MCSimulator, albeit only a subset of which are user-configurable, are summarized in Table 6.1 below:

Feature	Setting	Default
Number of Paths	$< 100,000$	$20,000$
Numeraire	Internal	Spot Libor
Variance Reduction	Internal	Moment Matching
Seed	Internal	N/A
Sampling Interval	Internal	N/A

Table 6.1: Simulator Configuration Settings

6.1 Numeraires

Recall that a numeraire is a choice of asset price used to normalize other derivative asset prices. The numeraires described below derive from purchasing bonds at time T_0 of certain maturity T_m, and then, at some intermediate time $t \leq T_m$ (typically $t = T_m$), cashing them out and using the proceeds to purchase new bonds of maturity $T_{m'}$, and continuing in this fashion for future times $T_{m''}, T_{m'''}, \ldots$.

$$\begin{aligned} \text{Spot-LIBOR} &: \quad m = 0,\ m' = 1,\ m'' = 2, \ldots \\ \text{Terminal} &: \quad m = N \\ T_k\text{-Forward} &: \quad m = k,\ m' = m + 1,\ m'' = m' + 1, \ldots \end{aligned}$$

In general, a choice of numeraire must be made before any Monte-Carlo simulation is performed, as different choices imply that different algorithms will be applied. However, in the present case of the Shifted-Lognormal Libor Market Model, only the Spot-Libor numeraire is supported.

Although Monte Carlo simulations using the Spot-Libor numeraire may exhibit blow-up by virtue of its drift being comprised of many positive terms, the small number of offending paths can be controlled by capping the increments to the state variables. In all other respects, the Spot Libor numeraire is found to be superior to alternative numeraires. For example, the Terminal numeraire, whose evaluation of discount bonds may even fail monotonicity with respect to maturity, may also exhibit blow-down from a large negative drift. Additionally, the Spot Libor numeraire has the important feature that unbiasing (see below) the discount bonds will automatically unbias the forward Libor rates, thereby enforcing two important no-arbitrage properties.

6.2 Internal Simulator Settings

In addition to the specification of the number of paths, there are several internal configuration settings applied to the LMM simulator. These include the integer *seed* used for random number generation, and the *sampling interval*, which indicates the time step used by the simulator during SDE simulation. In practice, a smaller amount of time might be used between two evolution dates if either of those dates correspond to a LIBOR fixing time or payment time or observation time. In no case, however, will the time between evolution dates exceed the sampling interval, as it provides a maximum granularity to the discretization of the time grid.

Variance reduction is an often used technique in Monte Carlo simulations, which is employed to achieve the same simulation variance using a small number of paths that would otherwise be obtainable only by using a large number of paths. The LMM implementation does not employ any variance reduction algorithms, including antithetic random number generation.

6.3 Pricing in a Dual-Curve Environment

Note that when pricing in a *dual-curve* setting, the simulation is performed by evolving OIS Forward rates only, and does not directly simulate market Libor rates. Any Libor rate required by the pricer is obtained from the simulated OIS Forward rate by adding a *basis*, which is obtained by retrieving the time-zero difference between OIS Forward rates and Libor Forward rates. This methodology is similarly applied to other Libor rates; the 6 month Libor rates are obtained from simulated 3 month OIS Forward rates by the addition of a basis obtained from the time-zero OIS and Forward curves. See Appendix E where more details are provided.

APPENDICES

A Spot-LIBOR Numeraire

In this section we describe the numeraire used in the LMM pricing. Notationally, $P_T(t)$ denotes the time t value of a zero-coupon bond of maturity T (necessarily a pathwise quantity for $t > 0$), so $P_T := P_T(0)$ is today's value of the bond maturing at time T which is determined from the initial discount curve; and $P_n := P_{T_n}(0)$ is today's value of the bond maturing at the Libor date T_n.

The Spot-Libor numeraire, also called the "discretely rebalanced bank account" numeraire, is a discrete version of the risk-neutral numeraire associated with the continuously compounded money market account. Specifically, its value at time T_0 is \$1. ; at time T_1 has increased in value to P_1^{-1} at which time the LIBOR from T_1 to T_2 has been fixed at $F_1(T_1)$; at time T_2 the numeraire has increased in value to $P_1^{-1}(1 + \tau_1 F_1(T_1))$; and so on, until Libor date T_k at which time its value becomes

$$\begin{aligned}\mathcal{N}(T_k) \quad &:= \quad P_1^{-1} \cdot \prod_{j=1}^{k-1}(1 + \tau F_j(T_k)) \\ &= \quad P_1^{-1} \cdot \prod_{j=1}^{k-1}(1 + \tau_j F_j(T_j)) \qquad \text{(since } F_j \text{ freezes at } T_j\text{)}.\end{aligned}$$

In other words, starting with a \$1. purchase of bonds paying P_1^{-1} at maturity T_1, one cashes out at each LIBOR date T_k and uses the proceeds to purchase new bonds paying a return of $(1+\tau_k F_k(T_k))$ at maturity T_{k+1}.

A more complete description would account for cashing out the bonds at an intermediate time $T_k \leq t < T_{k+1}$. Using the γ-notation to indicate, at an arbitrary time, the Libor index of the first rate "not yet frozen":

$$\gamma(t) = k + 1 \Longleftrightarrow T_k \leq t < T_{k+1}^{-1},$$

we write more generally (by discounting the overshoot from $(1 + F_k(T_k))$ by $P_{k+1}(t)$):

$$\begin{aligned}\mathcal{N}(t) &:= \mathcal{N}(T_{k+1}) \cdot P_{k+1}(t) \qquad (T_k \leq t < T_{k+1}) \\ &= P_1^{-1} \cdot \prod_{j=1}^{\gamma(t)-1} (1 + \tau_j F_j(T_j)) \cdot P_{\gamma(t)}(t). \end{aligned} \tag{A.1}$$

Strictly speaking $P_{\gamma(t)}(t)$, the time t value of the bond maturing at time T_{k+1} whose rate was reset at $F_k(T_k)$, is not mandated by the discrete rates $F_k(t)$ modeled in the Libor Market Model. A fuller discussion of these off-grid "stub discounts", and their related "stub rates"

$$F(t; t, T_{\gamma(t)}) := \frac{P_{\gamma(t)}(t)^{-1} - 1}{\tau_k}$$

is given in Appendix B.

Unbiasing Algorithm

Unbiasing, also called moment-matching, is a technique used to enforce agreement between simulated quantities, known to be martingales, with their initial values. For example, zero-coupon bonds are tradable assets, paying \$1 at maturity, and when discounted by (any) numeraire their expected values (with respect to the numeraire's associated measure) will, in theory, agree with their initial values. In practice, however, when using a reasonable number of Monte Carlo paths, one finds

$$P_T \neq \left\langle \mathcal{N}(T)^{-1} \right\rangle,$$

where we use the notation $\langle \cdot \rangle$ to indicate the sample mean over all paths.

The unbiasing methodology replaces $F_k(T) \to \hat{F}_k(T)$ so that

$$\hat{\mathcal{N}}(T) := P_1^{-1} \prod_{k=1}^{\gamma(T)-1} (1 + \tau_k \hat{F}_k) \cdot P_{\gamma(T)}(T), \tag{A.2a}$$

$$P_T = \left\langle \hat{\mathcal{N}}(T)^{-1} \right\rangle. \tag{A.2b}$$

Unbiasing formulas are developed so that all fixed-coupon bonds (with arbitrary coupon schedules) and on-grid float-coupon bonds are exactly priced, while the off-grid float-coupon bonds are priced very nearly exactly.

B Determining the LIBOR Grid $\{T_k\}$

There are certain scheduling dates that are naturally included when determining the Libor grid. One perspective is that T_0 is the deal's evaluation date, and T_n is the deal's horizon date. Another view is that T_0 is today's date, and T_n is the maturity of the furthest dated calibration instrument. These views are equally valid, but have different implications for calibration and simulation.

B.1 Market Aligned

The "market aligned" approach is more natural for the purist, whose view is that the unadulterated quotes of the liquid instruments in today's market are the most reliable inputs to which model parameters should be calibrated. In this view, the Libor grid should agree with that of the prevailing swap whose tenor encompasses the market instruments relevant to the deal. This choice will, for the most part, give consistency between the Libor grid and the calibration instruments, and hence will be straightforward for the calibration phase. On the other hand, if the deal dates, which will include fixing and accrual schedules of Libor and CMS underlyings relevant to the deal, is misaligned with today's market (for example, when pricing an aged deal), then some interpolation methodology at the pricing phase will have to account for the mismatch between those underlying Libors being priced and the internally simulated Libor states.

The interpolation scheme used in adapting a market-aligned Libor grid to the pricing of off-grid Libor rates amounts to the evaluation of "short-dated bonds" $P_{\gamma(t)}(t)$ and their associated "stub rates" $L(t;T,T_{\gamma(T)})$, namely the forward rate from $T \to T_{\gamma(T)}$ evaluated at time t. This approach suffers from some performance degradation during the pricing phase, but is completely flexible in that it does not impose any constraint on the set of pricing dates. It should be remarked that when

$$t < T_k < T < T_{k+1} \quad \gamma(T) = k+1,$$

the evaluation of the interpolated stub rate $L(t;T,T_{k+1})$ is determined by the arbitrage-free requirement to be a multiple (whose value depends only on the initial discount curve) of the simulated state $[L(t;T_k,T_{k+1}) + \alpha_k]$:

$$[L(t;T,T_{k+1}) + \alpha_k] \quad = \quad c \cdot [L(t;T_k,T_{k+1}) + \alpha_k]\,. \tag{B.1}$$

On the other hand, when $T_k \leq t \leq T < T_{k+1}$, which includes the special case when $t = T$ relevant to evaluating $P_{\gamma(T)}(T)$, the right hand side of (B.1) has been frozen at L_k's fixing time T_k^-, and so the stub rate will suffer from the *premature freezing* of $L(T_k;T_k,T_{k+1})$ which has no evolution between T_k and t. The method of [Wer] was adopted to overcome this premature freezing by allowing L_k to continue evolving for $T_k \leq t < T_{k+1}$, and was found to have many advantages over competing algorithms.

Finally, it should be noted that the evaluation of stub rates described here is critical to the time t evaluation of discount factors, in particular the numeraire $\mathcal{N}^S(t)$ discussed in Appendix A, when t is not a grid point T_k.

B.2 Deal Aligned

The "deal aligned" approach locks in the Libor grid to be compatible with the pricing dates relevant to the deal. Of course, this may not be possible, as in the case of a range accrual with daily observations and monthly coupons based on quarterly Libors, in which case the methods of §B.1 will be necessary. However, if the deal dates are a subset of a natural Libor schedule, but T_0 does not agree with today, then another approach is available. Since the calibration phase cannot handle calibrating model parameters associated with a given Libor grid to calibration instruments associated with a different Libor grid, a new set of calibration instruments whose expiries (but not tenors) must be translated to be relative to T_0 instead of today.

For this purpose, one can invoke the functionality of the terminal's `VCUB` function to generate quotes for *synthetic* caps and swaptions whose expiries are aligned with the deal's T_0 (hence the description "deal-aligned"), but misaligned with today's market. Other than any possible overhead arising from `VCUB` generating the synthetic quotes, the calibration and simulation phases are unaffected. As mentioned above, this option is appropriate only when all of the dates relevant to pricing belong to a single Libor schedule. Furthermore, one should note the following additional drawbacks to this approach. The interpolation algorithms employed by VCUB make certain model-dependent assumptions[9], and may not be consistent with the shifted-lognormal LMM model specification.

Furthermore, as described elsewhere, one may want to attach weights to individual strikes when calibrating to a swaption smile, one possible scheme using the reciprocal of the bid-ask spread as an approximate liquidity factor. This kind of strike weighting, and similar techniques that rely on authentic market quotations, are unavailable when using the VCUB transformation whose algorithms will lose all but the volatility data being interpolated.

C Model Dynamics

C.1 Conventions and Notations

With regard to mathematical notation, we will generally denote scalar quantities in lower-case, vector quantities in lower-case bold-face, and matrix quantities in upper-case. Array representations of vectors will use row-vector notation, while transposes are indicated using an apostrophe. Thus

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v}' &= \sum_k u_k \cdot v_k =: \langle \mathbf{u}, \mathbf{v}\rangle, \\
\mathbf{u}'\cdot\mathbf{v} &= \big(\{u_i v_j\}\big) =: \mathbf{u}\wedge\mathbf{v}, \\
\langle \mathbf{u}A, \mathbf{v}\rangle &= \sum_{i,j} u_i a_{i,j} v_j = \langle \mathbf{u}, \mathbf{v}A'\rangle.
\end{aligned}$$

We will also need to consider interest-rate swaps, forward starting at time T_a, and where, at each time T_j $(j = a+1, \ldots, b)$, a fixed rate K is exchanged for the Libor rate $F_{j-1}(T_j)$. The corresponding forward swap rate at time t is denoted by $S^{ab}(t)$. We have:

$$S^{ab}(t) \quad := \quad \frac{P_a(t) - P_b(t)}{\sum_{i=a}^{b-1}\tau_i P_{i+1}(t)} = \sum_{k=a}^{b-1}\omega_k^{ab}(t)F_k(t), \qquad \omega_k^{ab}(t) := \frac{\tau_k P_{k+1}(t)}{\sum_{i=a}^{b-1}\tau_i P_{i+1}(t)}. \tag{C.1}$$

Here, the denominator $\sum_{i=a}^{b-1}\tau_i P_{i+1}(t)$ is referred to as the "level" or "annuity", and in cases when the floating-leg and fixed-leg schedules are misaligned, is more properly written:

$$A^{ab}(t) := \sum_{i=\widetilde{a}}^{\widetilde{b-1}} \widetilde{\tau}_i P(t, \widetilde{T}_i). \tag{C.2}$$

[9] See the VCUB Model Parameters option (38).

For example, in the US swaption market the payment schedule for the floating leg is quarterly ($\tau_i \approx 0.25$), whereas the payment schedule for the fixed leg payments is semi-annual ($\widetilde{\tau_i} \approx 0.5$). In any case, the accrual start and end dates of the entire swap must agree, as must the overall coverage:

$$\begin{aligned} \widetilde{T_{\widetilde{a}}} &= T_a \\ \widetilde{T_{\widetilde{b}}} &= T_b \\ \sum_{i=\widetilde{a}}^{\widetilde{b-1}} \widetilde{\tau_i} &= \sum_{i=a}^{b-1} \tau_i. \end{aligned}$$

Again, for simplicity of presentation we may suppress this notational detail unless it is essential for a mathematical calculation.

When considering CMS options, we will require use of the Annuity numeraire, also called the Forward Swap numeraire, which is defined as the following bond portfolio

$$\mathcal{N}^{ab}(t) \quad := \quad \sum_{k=a}^{b-1} \tau_k P(t, T_{k+1}), . \tag{C.3}$$

The Annuity numeraire is associated with the Forward Swap measure Q^{ab} in which the forward swap rate $S^{ab}(t)$ becomes a martingale. Indeed, denoting by $\mathbf{E}^{ab}$ the expectation corresponding to Q^{ab}, we have

$$S^{ab}(0) \quad := \quad \frac{P_a - P_b}{\sum_{k=a}^{b-1} \tau_k P_{k+1}} = \frac{P_a(0) - P_b(0)}{\mathcal{N}^{ab}(0)} = \mathbf{E}^{ab}\left(\frac{P_a(t) - P_b(t)}{\mathcal{N}^{ab}(t)}\right), \tag{C.4}$$

where the latter equality reflects the pricing of a difference of bonds in units of the Annuity numeraire.

C.2 Diffusion Terms

The rank of ρ is the number of independent Brownian drivers for the correlated Brownian motions dW_j^Q, and is called the number of "factors" of our model. Denoting the rank of ρ by r, we have $r \geq 1$ since $\rho_{0,0} = 1$, and when $r = N$ we say that our model is "full-factor". In the particular case when $\rho = \mathbf{I}$, the SDE system $\{dF_j(t)\}$ will decouple into N independent shifted-lognormal evolutions. When ρ is rank-deficient, we can nonetheless find (statistically) independent Brownian drivers $dZ_i^Q(t)$ for which

$$\mathbb{E}\left[dZ_i^Q(t) \cdot dZ_j^Q(t)\right] = \delta_{i,j} dt, \qquad 0 \leq i, j < r, \tag{C.5}$$

and for which

$$[dW_0^Q, \ldots, dW_{N-1}^Q] \quad = \quad [dZ_0^Q, \ldots, dZ_{r-1}^Q] \cdot \begin{pmatrix} \lambda_{0,0} & \ldots & \lambda_{0,N-1} \\ \vdots & \ddots & \vdots \\ \lambda_{r-1,0} & \ldots & \lambda_{r-1,N-1} \end{pmatrix} =: \; dZ^Q \cdot \Lambda. \tag{C.6}$$

Furthermore, combining (3.2), (C.5), and (C.6) gives

$$\begin{aligned}\rho dt &= \mathbb{E}\left[dW^{Q'} \cdot dW^{Q}\right] \\ &= \Lambda' \cdot \mathbb{E}\left[dZ^{Q'} \cdot dZ^{Q}\right] \cdot \Lambda \\ &= \Lambda' \cdot \mathbf{I} \cdot \Lambda dt,\end{aligned}$$

and so

$$\rho = \Lambda' \cdot \Lambda. \tag{C.7}$$

When $r = N$ and Λ is upper-triangular, (C.7) is referred to as the Cholesky factorization of ρ. When $r < N$, Λ is often referred to as the "pseudo square root" of ρ.

C.3 Drift Terms

By definition, in the shifted-lognormal LMM each forward rate F_j evolves under the corresponding forward measure $Q^{T_{j+1}}$ as a shifted geometric Brownian motion. Precisely, the instantaneous volatility of F_j is assumed to be given by

$$\sigma_j(t) \cdot (F_j(t) + \alpha_j),$$

where α_j's are real constants and σ_j's are deterministic functions of time, so that the dynamics of F_j under $Q^{T_{j+1}}$ is

$$dF_j(t) = \sigma_j(t) \cdot (F_j(t) + \alpha_j)\; dW_j^j(t), \tag{C.8}$$

where W_j^j is a standard Brownian motion under $Q^{T_{j+1}}$.

As (C.8) implies

$$d(F_j(t) + \alpha_j) = \sigma_j(t)(F_j(t) + \alpha_j)\; dW_j^j(t),$$

it follows that the forward rate F_j can be explicitly written as

$$\boxed{F_j(T) = -\alpha_j + (F_j(t) + \alpha_j) e^{-\frac{1}{2}\int_t^T \sigma_j^2(u)\,du + \int_t^T \sigma_j(u)\,dW_j^j(u)} \qquad t < T \le T_j.} \tag{C.9}$$

The distribution of $F_j(T)$, conditional on $F_j(t)$, $t < T \le T_{j-1}$, is then shifted-lognormal with density

$$p_{F_j(T)|F_j(t)}(x) = \frac{1}{(x + \alpha_j) U_j(t,T)\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(\frac{\ln \frac{x+\alpha_j}{F_j(t)+\alpha_j} + \frac{1}{2} U_j^2(t,T)}{U_j(t,T)} \right)^2 \right\}, \tag{C.10}$$

for $x > -\alpha_j$, where $U_j(t,T)$ is the cumulative volatility defined by

$$U_j(t,T) := \sqrt{\int_t^T \sigma_j^2(u) du}\;.$$

We give below the dynamics of the forwards $F_j(t)$ in the Spot LIBOR measure Q_S associated with the Spot LIBOR numeraire (A.1).

Recalling (3.1), we write

$$dX_j(t) = \mu_j^Q(t)dt + \sigma_j(t)X_j(t)dW_j^Q(t), \tag{C.11}$$
$$X_j(t) := F_j(t) + \alpha_j. \tag{C.12}$$

The drifts μ_j^Q of the forwards F_j are computed by requiring that the forwards be martingales in their own measure, *i.e.* by applying the change-of-numeraire technique to (C.8). Derivations of the explicit expression for μ_j^Q are provided in [AP, §15.2], [BM, §6.3], and give the following:

$$\mu_j^{Q_S}(t) = \sigma_j(t)X_j(t)\sum_{i=\gamma(t)}^{j}\frac{\rho_{i,j}\sigma_i(t)\tau_i X_i(t)}{1+\tau_i F_i(t)}. \tag{C.13}$$

C.4 Swap Rate Dynamics

It is not possible to model the dynamics of a swap rate exactly since the sum of shifted-lognormal processes is not itself a shifted-lognormal process. On the other hand, one may strive to model the swap rate S^{ab} *approximately* as a shifted-lognormal process in the forward swap measure Q^{ab} associated with the annuity numeraire A^{ab}. One seeks, therefore, to determine scalar parameters σ^{ab}, α^{ab}, and a single Q^{ab}-Brownian motion $W^{ab}(t)$, such that

$$X^{ab} := S^{ab} + \alpha^{ab},$$
$$dX^{ab} \approx \sigma^{ab}(t) \cdot X^{ab}(t) \cdot dW^{ab}(t).$$

Specifically, consider the following quantities from which the swap rate from T_a to T_b is derived:

$$X_k(t) := F_k(t) + \alpha_k, \tag{C.14a}$$
$$A^{ab}(t) := \sum_{k=a}^{b-1}\tau_k P_t(T_{k+1}), \qquad \text{(annuity)} \tag{C.14b}$$
$$\omega_k^{ab}(t) := \frac{\tau_k P_t(T_{k+1})}{A_{ab}(t)} \qquad k = a, \ldots, b-1, \tag{C.14c}$$
$$S^{ab}(t) := \sum_{k=a}^{b-1}\omega_k^{ab}(t)F_k(t). \qquad \text{(swap rate)} \tag{C.14d}$$

Following [BM, §6.15], one derives the following approximate shifted-lognormal dynamics of S^{ab}:

$$dS^{ab}(t) \approx \left(S^{ab}(t) + \alpha^{ab}\right) \cdot \langle \gamma_{ab}, dW_{ab}\rangle, \tag{C.15}$$

where α^{ab} is the “shift of the swaption”

$$\alpha^{ab} := \sum_{k=a}^{b-1}\omega_k^{ab}\alpha_k, \tag{C.16}$$

and dW_{ab} is the "row vector of correlated Brownian motions"

$$dW_{ab}(t) \quad := \quad \big[dW_a(t), \dots, dW_{b-1}(t)\big],$$

and γ_{ab} is the "vectorized volatility of the swaption"

$$\gamma_{ab}(t) \quad := \quad \frac{\big[\tau_a P_{a+1} X_a \sigma_a(t), \dots, \tau_{b-1} P_b X_{b-1} \sigma_{b-1}(t)\big]}{\sum_{k=a}^{b-1} \tau_k P_{k+1} X_k},$$

and the constants $X_k = X_k(0)$ and $P_k = P_0(T_k)$ are values frozen at time T_0. Furthermore, by giving the explicit expression for the variance of the one dimensional Brownian motion $\langle \gamma_{ab}, dW_{ab} \rangle$ in terms of a quadratic form in the model parameters:

$$\begin{aligned}
\rho_{ab} \quad &:= \quad \big(\rho_{i,j}\big)_{i=a,\dots,b-1; j=a,\dots,b-1} \\
\langle \gamma_{ab}, dW_{ab} \rangle^2 \quad &= \quad \langle \gamma_{ab}, dW_{ab} \rangle \cdot \langle dW_{ab}, \gamma_{ab}{}' \rangle = \gamma_{ab} \cdot dW'_{ab} \cdot dW_{ab} \cdot \gamma_{ab}{}', \\
\mathbb{E}\left[\langle \gamma_{ab}, dW_{ab} \rangle^2\right] \quad &\overset{(3.2)}{=} \quad \big(\gamma_{ab} \cdot \rho_{ab} \cdot \gamma_{ab}{}'\big)\, dt = \langle \gamma_{ab} \cdot \rho_{ab}, \gamma_{ab} \rangle dt,
\end{aligned}$$

we can simplify (C.15) by introducing the "volatility of the swaption"

$$\sigma^{ab}(t) \quad := \quad \langle \gamma_{ab} \cdot \rho_{ab}, \gamma_{ab} \rangle^{\frac{1}{2}}. \tag{C.17}$$

Using the above we can write the dynamics of $S^{ab}(t)$ in terms of (approximate) shifted-lognormal parameters α^{ab} and $\sigma^{ab}(t)$ and Q^{ab}-Brownian motion $dW^{ab}(t)$:

$$\begin{aligned}
X^{ab}(t) \quad &:= \quad S^{ab}(t) + \alpha^{ab}, \\
dW^{ab}(t) \quad &:= \quad \langle \gamma_{ab}, dW_{ab} \rangle / \sigma^{ab}, \\
dX^{ab}(t) \quad &\approx \quad \sigma^{ab}(t) \cdot X^{ab}(t) \cdot dW^{ab}(t).
\end{aligned} \tag{C.18}$$

D Pricing Formulas

D.1 Exact Caplet Pricing Formula

In this section we show that caplet and floorlet prices in the shifted-lognormal LMM can be calculated in closed form. Consider a caplet whose payoff, fixed at T_j and paid at T_{j+1}, is given by

$$\tau_j [F_j(T_j) - K]^+$$

where K is its strike price. Then, assuming unit notional, the price of the caplet can be calculated as follows:

$$\begin{aligned}
\mathbf{Cpl}(0, T_j, T_{j+1}, \tau_j, K) \quad &= \quad \tau_j P(0, T_{j+1}) E^{Q_{j+1}} \left[(F_j(T_j) - K)^+\right] \\
&= \quad \tau_j P_{j+1} E^{Q_{j+1}} \left[\big(F_j\,(T_j) + \alpha_j - (K + \alpha_j)\,\big)^+\right].
\end{aligned} \tag{D.1}$$

Since, according to (3.1), $X_j(T_j) = F_j(T_j) + \alpha_j$ is a lognormal random variable in the Q_j measure, the last expectation in (D.1) yields the adjusted Black caplet price that corresponds to a shifted geometric Brownian motion:

$$\boxed{\mathbf{Cpl}(0, T_j, T_{j+1}, \tau_j, K) \quad = \quad \tau_j P_{j+1} \mathrm{Bl}\big(F_j + \alpha_j, K + \alpha_j, V_j\big) \qquad \text{(D.2)}}$$

where the terminal volatility up to time T_j is given by

$$V_j \quad := \quad U_j(0, T_j) = \sqrt{\int_0^{T_j} \sigma_j^2(s)\, ds}\ , \tag{D.3}$$

and

$$\boxed{\mathrm{Bl}(F, K, v) := F \cdot \Phi\left(\frac{\ln(F/K) + v^2/2}{v}\right) - K \cdot \Phi\left(\frac{\ln(F/K) - v^2/2}{v}\right) \text{(D.4)}}$$

is the Black-Scholes formula with Φ denoting the standard normal distribution function.

Likewise, the shifted-lognormal LMM price of the floorlet paying out at time T_{j+1} the quantity

$$\tau_j [K - F_j(T_j)]^+$$

is given by

$$\boxed{\mathbf{Flt}(0, T_j, T_{j+1}, \tau_j, K) = \tau_j P_{j+1}\ \mathrm{Bl}\big(K + \alpha_j, F_j + \alpha_j, V_j\big). \qquad \text{(D.5)}}$$

D.2 Approximate Swaption Pricing Formula

A European payer (receiver) swaption is an option giving its owner the right to enter, at a given maturity T_a, an interest-rate swap where a fixed rate K is paid (received) on dates $T_a, \ldots, T_{b-1}$. The swaption payoff at time T_a is given by

$$\Big[\omega(S^{ab}(T_a) - K)\Big]^+ \sum_{i=a}^{b-1} \tau_i P(T_a, T_{i+1})$$

where $\omega = 1$ for a payer and $\omega = -1$ for a receiver, and $S^{ab}(t)$ denotes the forward swap rate (C.14) at time t for the set of times $T_a, \ldots, T_b$:

$$S^{ab}(t) \quad = \quad \frac{P(t, T_a) - P(t, T_b)}{\sum_{k=a}^{b-1} \tau_k P(t, T_{k+1})}.$$

Using the swap rate dynamics developed in §C.4, we can price swaptions by following the same procedure as in the caplet case discussed in §D.1. To this end, we first consider a European payer swaption with maturity T_α and strike K, whose underlying swap pays on times $T_{a+1}, \ldots, T_b$. Assuming unit notional, the swaption price at time zero can be calculated as follows:

$$\begin{aligned}\mathbf{PS}(0; a, b, K) &= \sum_{h=a}^{b-1} \tau_h P\left(0, T_{h+1}\right) \mathbf{E}^{ab}\left[\left(S^{ab}\left(T_a\right) - K\right)^+\right] \\ &= A^{ab}(0) \cdot \mathbf{E}^{ab}\left[\left(X^{ab}\left(T_\alpha\right) - \left(K + \alpha^{ab}\right)\right)^+\right].\end{aligned}$$

Since, according to (C.18), $X^{ab}(T_j) = S^{ab}(T_j) + \alpha^{ab}$ is (approximately) a lognormal random variable in the Q^{ab} measure, the last expectation in (D.1) yields the adjusted Black swaption price that corresponds to a shifted geometric Brownian motion. We thus obtain:

$$\boxed{\mathbf{PS}(0; a, b, K) \approx A^{ab}(0) \cdot \mathrm{Bl}\big(S^{ab}(0) + \alpha^{ab}, K + \alpha^{ab}, \Gamma^{ab}\big)} \tag{D.6}$$

where Γ^{ab} is the terminal volatility defined by

$$\Gamma^{ab} := \sqrt{\int_0^{T_\alpha} \left[\gamma^{ab}(s)\right]^2 ds} = \sqrt{\sum_{i,j=a}^{b-1} \rho_{i,j} \int_0^{T_a} \gamma_i(s) \gamma_j(s) ds}$$

and we have used expressions (C.16) and (C.17) derived in §C.4.

Likewise, the price of a receiver swaption is given by

$$\boxed{\mathbf{RS}(0; a, b, K) \approx A^{ab}(0) \cdot \mathrm{Bl}\big(K + \alpha^{ab}, S^{ab}(0) + \alpha^{ab}, \Gamma^{ab}\big)} \tag{D.7}$$

D.3 Approximate CMS Spread Option Pricing Formula

In this section we provide an approximation formula for the CMS spread option price, which gives straddle, call, and put prices as a function of the strike and the model parameters $\alpha_k, \sigma_k, \rho_{i,j}$.

The basic tool for pricing the difference of two instruments is the Margrabe spread option pricing formula [Mar1]. Now the Margrabe formula applies to the difference between two processes which are lognormal in a common measure. The first issue to confront is adapting the Margrabe formula to the difference between two shifted-lognormal processes, which is easily addressed. Less straightforward is the second issue, namely that the formulas for the CMS rates are naturally derived in their respective annuity measures Q^{ab} and Q^{ac}, and therefore these formulas must be adapted to a common measure which we choose to be that of the T_a-Forward. Thirdly, it will turn out that the CMS rates are no longer shifted-lognormal processes once adapted to this shared Q^{T_a} measure, and so we will need to invoke the technique of moment-matching to determine genuine shifted-lognormal process which best approximate (in the sense of agreement with their first two moments) the CMS rates.

Summarizing, in order to price CMS spread options we need to consider the two CMS rates $S^{ab}(t)$ and $S^{ac}(t)$, $a < b < c$, at time T_a under the same measure Q^{T_a}. Having derived the dynamics of the CMS rate processes in their natural annuity measures, the adapted formulas giving their moments in the common Q^{T_a} measure can be deduced by applying the change-of-measure technique. Next, one performs moment-matching to determine a new shifted-lognormal processes under the Q^{T_a} measure which possess the same first and second moments as the original processes. Finally, one invokes a variant of the Margrabe spread option pricing formula adapted to two shifted-lognormal underlying processes.

We mandate a new shifted-lognormal process under the Q^{T_a} measure of the following form:

$$\hat{S}^{ab}(T_a) = -\alpha^{ab} + (\hat{S}^{ab}(0) + \alpha^{ab})e^{-\frac{1}{2}T_a(\hat{\sigma}^{ab})^2 + \hat{\sigma}^{ab}W_{cms1}(T_a)}$$

Here, we use the same shift α^{ab} as that of $S^{ab}(T_a)$ in the Q^{ab} measure.

Equating the first two moments of $\hat{S}^{ab}(T_a)$ to those of $S^{ab}(T_a)$, and using well known moment formulas for a shifted lognormal process, gives:

$$\begin{aligned}
\mathbf{E}^{T_a}[S^{ab}(T_a)] &= \mathbf{E}^{T_a}[\hat{S}^{ab}(T_a)] = \hat{S}^{ab}(0); \\
\mathbf{E}^{T_a}[S^{ab}(T_a)^2] &= \mathbf{E}^{T_a}[(\hat{S}^{ab}(T_a))^2] \\
&= (\hat{S}^{ab}(0) + \alpha^{ab})^2 e^{T_a(\hat{\sigma}^{ab})^2} - \alpha^{ab}(2\hat{S}^{ab}(0) + \alpha^{ab}).
\end{aligned}$$

which may be solved for $\hat{S}^{ab}(0)$ and $\hat{\sigma}^{ab}$ in terms of the moments $\mathbf{E}^{T_a}[S^{ab}(T_a)^n]$ of S^{ab}:

$$\begin{aligned}
\hat{S}^{ab}(0) &= \mathbf{E}^{T_a}[S^{ab}(T_a)], \\
(\hat{\sigma}^{ab})^2 &= \frac{1}{T_a}\ln\left(\frac{\mathbf{E}^{T_a}[S^{ab}(T_a)^2] + \alpha^{ab}(2\hat{S}^{ab}(0) + \alpha^{ab})}{(\hat{S}^{ab}(0) + \alpha^{ab})^2}\right).
\end{aligned}$$

Since $\mathbf{E}^{T_a}[S^{ab}(T_a)]$ and $\mathbf{E}^{T_a}[S^{ab}(T_a)^2]$ can be derived from their moments $\mathbf{E}^{ab}[S^{ab}(T_a)^n]$ in the annuity measure Q^{ab} using the Linear Swap Model [Pel], we have just determined the parameters $\hat{S}^{ab}(0)$ and $\hat{\sigma}^{ab}$ in our moment matched process $\hat{S}^{ab}(T_a)$ in terms of $\alpha^{ab}, \sigma^{ab}, \rho$, which is to say in terms of the shifted-lognormal LMM model parameters. A similar calculation produces the moment matched process of the second CMS rate $\hat{S}^{ac}(T_a)$.

The final step before applying the Margrabe spread option formula is to establish the correlation between $W_{cms1}(T_a)$ and $W_{cms2}(T_a)$:

$$\begin{aligned}
\rho &:= Correl\left(\ln\left(\frac{\hat{S}^{ab}(T_a) + \alpha^{ab}}{\hat{S}^{ab}(0) + \alpha^{ab}}\right), \ln\left(\frac{\hat{S}^{ac}(T_a) + \alpha^{ac}}{\hat{S}^{ac}(0) + \alpha^{ac}}\right)\right) \\
&\approx \frac{\int_0^{T_a}\langle\gamma_{ab}, \gamma_{ac}\rho_{ac}^{ab}\rangle dt}{\sqrt{\int_0^{T_a}\langle\gamma_{ab}, \gamma_{ab}\rho_{ab}\rangle dt \ \int_0^{T_a}\langle\gamma_{ac}, \gamma_{ac}\rho_{ac}\rangle dt}}
\end{aligned} \tag{D.8}$$

where ρ_{ac}^{ab} is the rectangular sub-matrix of ρ given by

$$\begin{pmatrix} \rho_{a,a} & \cdots & \rho_{a,b} \\ \vdots & \rho_{i,j} & \vdots \\ \rho_{c,a} & \cdots & \rho_{c,b} \end{pmatrix}.$$

By making the additional definitions where the symbol x indicates an integration parameter:

$$\begin{aligned}\widetilde{K} &= K + \alpha^{ab} - \alpha^{ac},\\ X_1(x) &= \hat{S}^{ab}(0)\exp\left(\hat{\sigma}^{ab} x\sqrt{T_a} - \frac{1}{2}T_a(\hat{\sigma}^{ab})^2\right),\\ X_2(x) &= \hat{S}^{ac}(0)\exp\left(\hat{\sigma}^{ac} x\rho\sqrt{T_a} - \frac{1}{2}\rho^2 T_a(\hat{\sigma}^{ac})^2\right),\end{aligned}$$

we can express the price of a call option on a CMS spread of strike K using the Margrabe spread option formula:

$$\begin{aligned}\mathbf{CMSSC}(0;T_a,K,\hat{S}^{ab},\hat{\sigma}^{ab},\alpha^{ab},\hat{S}^{ac},\hat{\sigma}^{ac},\alpha^{ac},\rho) &= B_a(0)\cdot\mathbf{E}^{T_a}\left[\left(\hat{S}^{ac}(T_a) - \hat{S}^{ab}(T_a) - K\right)^+\right]\\ &\approx B_a(0)\cdot\int_{-\infty}^{\infty}\mathrm{Bl}(X_2(x), X_1(x)+\widetilde{K}, \sqrt{T_a(1-\rho^2)}\hat{\sigma}^{ac})\frac{e^{-x^2/2}dx}{\sqrt{2\pi}}. \qquad \text{(D.9)}\end{aligned}$$

The price of a put option on a CMS spread of strike K is derived similarly to (D.9):

$$\begin{aligned}\mathbf{CMSSP}(0;T_a,K,\hat{S}^{ab},\sigma^{ab},\alpha^{ab},\hat{S}^{ac},\sigma^{ac},\alpha^{ac},\rho) &= B_a(0)\cdot\mathbf{E}^{T_a}\left[\left(-\hat{S}^{ac}(T_a) + \hat{S}^{ab}(T_a) + K\right)^+\right]\\ &\approx B_a(0)\cdot\int_{-\infty}^{\infty}\mathrm{Bl}(X_1(x)+\widetilde{K}, X_2(x), \sqrt{T_a(1-\rho^2)}\hat{\sigma}^{ac})\frac{e^{-x^2/2}dx}{\sqrt{2\pi}}. \qquad \text{(D.10)}\end{aligned}$$

E Dual Curve Calculations

When calculating in a dual-curve setup, one makes the distinction between the OIS forward rates derived from the OIS discount curve $P_t(T)$

$$F_k(t) := \frac{1}{\tau_k}\left[\frac{P_t(T_k)}{P_t(T_{k+1})} - 1\right], \qquad \text{(E.1)}$$

and the LIBOR rates $L_k(t)$ derived from the quoted LIBOR index. In particular, the implied LIBOR spread over OIS, associated with each LIBOR rate $L_k(t)$, is given by

$$\beta_k(t) \quad := \quad L_k(t) - F_k(t). \qquad \text{(E.2)}$$

We make the simplifying assumption that $\beta_k(t)$ is time independent, hence $L_k(t) = F_k(t) + \beta_k$ where $\beta_k := L_k(0) - F_k(0)$. As only the OIS forward rates $F_k(t)$ are evolved in the Monte Carlo simulation, we need only explore the implications of the basis adjustment to calibration and deal pricing. Consider the following expression for the swaption price $\mathbf{PS}(t; a, b, K)$ on a payer swap from T_a to T_b struck at K:

$$\begin{aligned} S^{ab}(t) &:= \sum_{k=a}^{b-1} \omega_k^{ab}(t) L_k(t) \\ &\overset{(E.2)}{=} \sum_{k=a}^{b-1} \omega_k^{ab}(t)(F_k(t) + \beta_k), \qquad \text{(swap rate[10])} \\ \mathbf{PS}(t; a, b, K) &= N_{\mathbb{Q}}(t)\ \mathbb{E}_t^{\mathbb{Q}}\left[\frac{A^{ab}(T)\big(S^{ab}(T) - K\big)^+}{N_{\mathbb{Q}}(T)}\right], \end{aligned}$$

which, when priced with respect to the forward swap measure $\mathbb{Q} = Q^{ab}$ at $t = 0, T = T_a$, gives

$$\begin{aligned} \mathbf{PS}(0; a, b, K) &= A^{ab}(0)\ \mathbb{E}^{Q^{ab}}\left[\left(\sum_{k=a}^{b-1} \omega_k^{ab}(T_a)\big[F_k(T_a) - (K - \beta_k)\big]\right)^+\right] \\ &= A^{ab}(0)\ \mathbb{E}^{Q^{ab}}\left[\left(\sum_{k=a}^{b-1} \omega_k^{ab}(T_a)\big[(X_k(T_a) - (K + \alpha_k - \beta_k)\big]\right)^+\right] \\ &= A^{ab}(0)\ \mathbb{E}^{Q^{ab}}\left[\left(X^{ab}(T_a) - \sum_{k=a}^{b-1} \omega_k^{ab}(T_a)(K + \alpha_k - \beta_k)\right)^+\right]. \end{aligned} \tag{E.3}$$

Comparing (E.3) with the single-curve pricing of a swaption with respect to the shifted lognormal LMM as in (D.2), we obtain its analog in the dual-curve setup:

$$\begin{aligned} \mathbf{PS}(0; a, b, K) &\approx A^{ab}(0)\ \mathrm{Bl}\big(K + \alpha^{ab} - \beta^{ab}, X^{ab}(0), \Gamma^{ab}(T_a)\big) \\ &= A^{ab}(0)\ \mathrm{Bl}\big(K + \alpha^{ab} - \beta^{ab}, S^{ab}(0) + \alpha^{ab} - \beta^{ab}, \Gamma^{ab}(T_a)\big) \end{aligned} \tag{E.4}$$

where

$$\beta^{ab} := \sum_{k=a}^{b-1} \omega^{ab}(0)\beta_k(0).$$

Setting $b = a + 1$ in (E.4) gives the analog of (D.1) for dual-curve caplet pricing. In other words, the dual-curve calibrated shifts are essentially the single-curve shifts $(\alpha_k - \beta_k)$, calibrated to the dual-curve quoted swaptions using the dual-curve ATM, and then adjusted by the LIBOR spreads β_k. As explained earlier, the Monte Carlo simulation evolves the OIS forward rates $F_k(t)$, or more precisely the $\log(F_k(t) + \alpha_k)$. Therefore, with respect to pricing, when a pathwise LIBOR underlying is required for evaluating a payoff, the returned rate is obtained from the OIS forward by the addition of the basis adjustment $F_k \to F_k + \beta_k$.

[10] This expression differs from the single-curve formula by the quantity $\sum_{k=a}^{b-1} \omega_k^{ab} \beta_k$.

F Shift and Volatility Regularization

The regularization technique employed in the calibration process utilizes a penalty function which imposes smoothness by penalizing spikes in "adjacent" model parameter values. In the one dimensional case (shift parameters or constant volatility parameters), the penalty on variations with respect to the *forward index* takes the form:

$$\begin{aligned} p(\mathbf{x};\omega_1,\omega_2) &= \sqrt{\omega_1 f_1(\mathbf{x}) + \omega_2 f_2(\mathbf{x})}, \\ f_1(\mathbf{x}) &:= \sum_{i=1}^{N-1} (x_{i+1} - x_i)^2, \\ f_2(\mathbf{x}) &:= \sum_{i=2}^{N-1} (x_{i+1} - 2x_i + x_{i-1})^2, \end{aligned} \tag{F.1}$$

where $\mathbf{x}$ is the vector of model parameters and ω_1, ω_2 are weights.

In the two dimensional case (piecewise constant volatility parameters), we have a similar formulation obtained by taking the rms of the $p(\cdot)$ evaluated over all rows and columns of X:

$$\begin{aligned} q^2(X;\omega_1,\omega_2,\eta_1,\eta_2) &= \omega_1 \sum_{j=1}^{N-1} f_1(X_{\cdot,j}) + \omega_2 \sum_{j=1}^{N-2} f_2(X_{\cdot,j}) + \eta_1 \sum_{i=2}^{N} f_1(X_{i,\cdot}) + \eta_2 \sum_{i=3}^{N} f_2(X_{i,\cdot}) \\ &= \sum_k p^2(X_{\cdot,k};\omega_1,\omega_2) + \sum_k p^2(X_{k,\cdot};\eta_1,\eta_2), \end{aligned} \tag{F.2}$$

where ω_1, ω_2 are as above, and η_1, η_2 are independent weights applied to variations with respect to the time dimension.

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