HW 9 Problem 5

a) $P(1 \text{ bucket empty}) = P(\text{Every bucket has an element except } 1)$

$m = n =$ # of buckets / keys therefore $m-1$ will have elements

$\left(\frac{m-1}{m}\right)$ is the probabity that one element lands in one of the buckets that leave 1 specific bucket/key empty

so $\left(\frac{m-1}{m}\right)^m$ will be the $P(1 \text{ bucket empty})$

$$P(1 \text{ bucket empty}) = \left(\frac{m-1}{m}\right)^m$$

$E[m] = \sum m_i p$ $\quad p =$ probability of (1 bucket empty)

$$E[m] = \sum_{i=1}^{m} m_i \left(\frac{m-1}{m}\right)^m$$

$$\sum_{i=1}^{m} m_i \left(\frac{m}{m} - \frac{1}{m}\right)^m$$

$$\boxed{\sum_{i=1}^{m} m_i \left(1 - \frac{1}{m}\right)^m}$$

b) Fullest a bucket will become means multinomial distribution as it could have been binomial disputan as well where $\binom{n}{k} p^k (1-p)^{n-k}$

In this case multinomal,

$$f(m) = \frac{n!}{m_1! \, m_2! \cdots m_k!} \, p_1^{m_1} \cdots p_k^{m_k}$$

$$f(m) = \frac{n!}{\prod_{i=1}^{k} m_i!} \prod_{i=1}^{k} p_i^{m_i} \qquad p = \left(1 - \frac{1}{m}\right)$$

$$\boxed{f(m) = \frac{n!}{\prod_{i=1}^{k} m_i!} \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{m_i}\right)^{m_i}}$$