Data Analytics in Business

Linear Regression

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Steps in Regression Analysis

Lessons in this Module

A. Steps in Regression Analysis
B. A Real Estate example
C. Notation
D. R^2, Adjusted R^2
E. Simple Regression (One Predictor Variable) Using R
F. Multiple Regression
G. R^2, Adjusted R^2 from Multiple Regression
H. Common Problems and Fixes in Linear Regression

Steps in Regression Analysis

1. Statement of the problem
2. Using regression for:
   - Diagnostic,
   - Predictive, or
   - Prescriptive analytics?
3. Selection of potentially relevant response and explanatory variables
4. Data collection
   - Internal data external data, purchased data, experiments, etc.

Adapted from Chatterjee, S., & Hadi, A. S. (2013). *Regression Analysis by Example* (5th ed.). Somerset: Wiley.

Steps in Regression Analysis (cont'd)

5. Choice of fitting method:
   - Ordinary least squares (OLS),
   - Generalized least squares,
   - Maximum likelihood,
   - Etc.
6. Model fitting
7. Model validation (diagnostics)
8. Refine the model & iterate from step 3
9. Use of the model

Adapted from Chatterjee, S., & Hadi, A. S. (2013). *Regression Analysis by Example* (5th ed.). Somerset: Wiley.

Business Examples

Y - Dependent Variable	X - Independent Variable(s)
1. Used car price	odometer reading, age of car, condition
2. Sales	advertisement spending
3. Time taken to repair a product	experience of technician in years
4. Product added to shopping cart?	ratings, price
5. Starting salary of new employee	work experience, years of education
6. Sale price of house	square feet, # of bedrooms, location
7. Will customer default?	credit balance, income, age
8. Will customer churn?	length of contract, age of customer

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Quiz (True/False)

- Could a variable, say price, be either a dependent or an independent variable?

  Answer: **TRUE**. Depends on the purpose of your model; see where **price** appears in examples #1 and #4 in the previous slide.

- A variable that takes binary values (pass/fail or true/false) cannot be a dependent variable.

  Answer: **FALSE**. We do use 0/1 dependent variables in logistic regression models; #7 in the previous slide is one example.

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Real Estate Pricing Example

Linear Regression: A Sample Problem

- Assume that you need to sell your house
- You want to predict the listing price based on how other houses are listed in the market
- How would you approach this task?
- A typical approach is to ask realtors:
  - Realtors often will use “comparables” (i.e., recent sales of houses in your neighborhood) and somehow come up with a suggested sale price
- However, you want to be more analytical in your approach
  - You have access to recent actual home sales in your city
  - You’d like to know what are the impacts of factors such as lotsize, # of bedrooms, # of bathrooms, etc., on the price
  - Could you use linear regression to help you get a “better” estimate of the listing price?

Use Housing Dataframe in Ecdat Package in R

- This data set is a sample of the real estate transactions in one city
- It is a cross-section of 546 home prices (from 1987) in the city of Windsor in Canada
- Alternatively, you could collect house prices from websites or scrape them from the web

str(Housing)

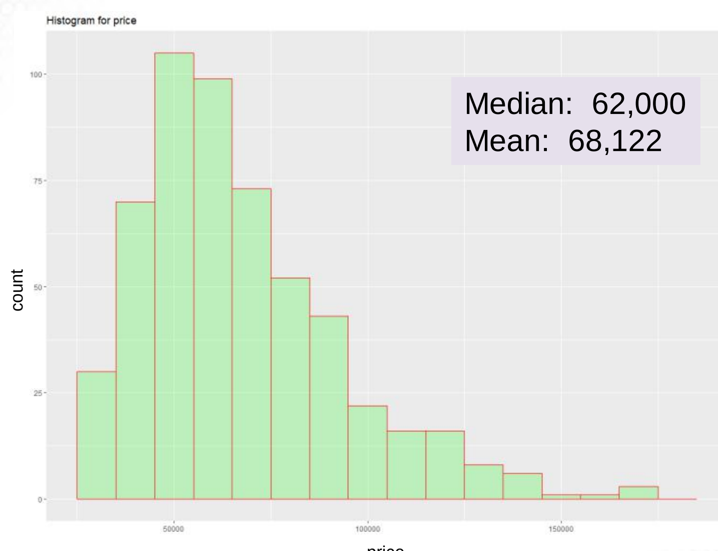
- 'data.frame': 546 obs. of 12 variables:
- $ price: num 42000 38500 49500 60500 61000 66000 66000 69000 83800 88500 ...
- $ lotsize: num 5850 4000 3060 6650 6360 4160 3880 4160 4800 5500 ...
- $ bedrooms: num 3 2 3 3 2 3 3 3 3 3 ...
- $ bathrms: num 1 1 1 1 1 1 2 1 1 2 ...
- $ stories: num 2 1 1 2 1 1 2 3 1 4 ...
- $ driveway: Factor w/ 2 levels "no","yes": 2 2 2 2 2 2 2 2 2 2 ...
- $ recroom: Factor w/ 2 levels "no","yes": 1 1 1 2 1 2 1 1 2 2 ...
- $ fullbase: Factor w/ 2 levels "no","yes": 2 1 1 1 1 2 2 1 2 1 ...
- $ gashw: Factor w/ 2 levels "no","yes": 1 1 1 1 1 1 1 1 1 1 ...
- $ airco: Factor w/ 2 levels "no","yes": 1 1 1 1 1 2 1 1 1 2 ...
- $ garagepl: num 1 0 0 0 0 0 2 0 0 1 ...
- $ prefarea: Factor w/ 2 levels "no","yes": 1 1 1 1 1 1 1 1 1 1 ...

The First 10 Records in Housing

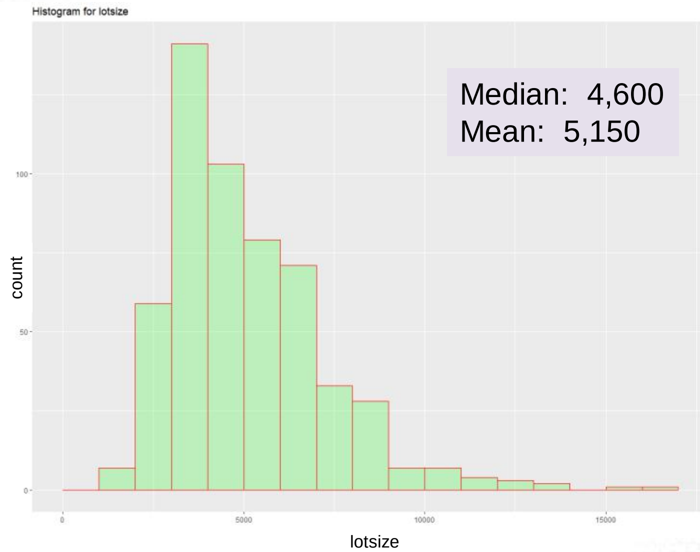
Housing Dataset in the *Ecdat* package in R

price	lotsize	bedrooms	bathrms	stories	driveway	recroom	fullbase	gashw	airco	garagepl	prefarea
42000	5850	3	1	2	yes	no	yes	no	no	1	no
38500	4000	2	1	1	yes	no	no	no	no	0	no
49500	3060	3	1	1	yes	no	no	no	no	0	no
60500	6650	3	1	2	yes	yes	no	no	no	0	no
61000	6360	2	1	1	yes	no	no	no	no	0	no
66000	4160	3	1	1	yes	yes	yes	no	yes	0	no
66000	3880	3	2	2	yes	no	yes	no	no	2	no
69000	4160	3	1	3	yes	no	no	no	no	0	no
83800	4800	3	1	1	yes	yes	yes	no	no	0	no
88500	5500	3	2	4	yes	yes	no	no	yes	1	no

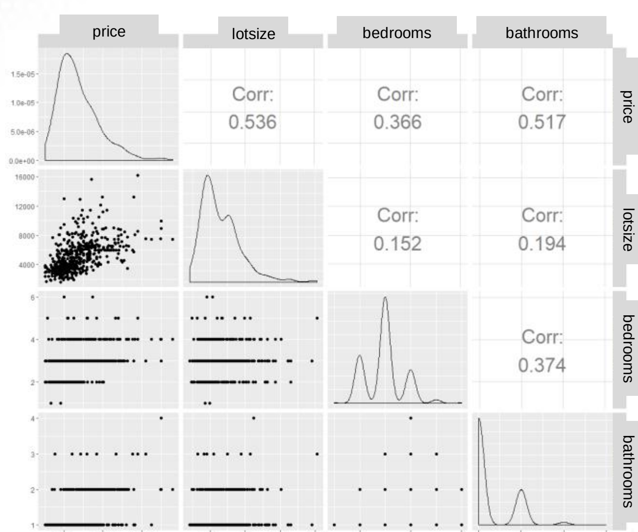
Histogram of House Prices



Histogram of lotsize



Correlation Matrix



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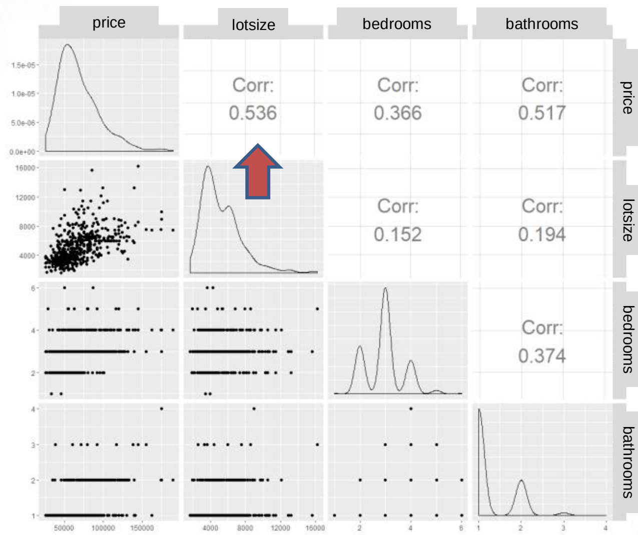
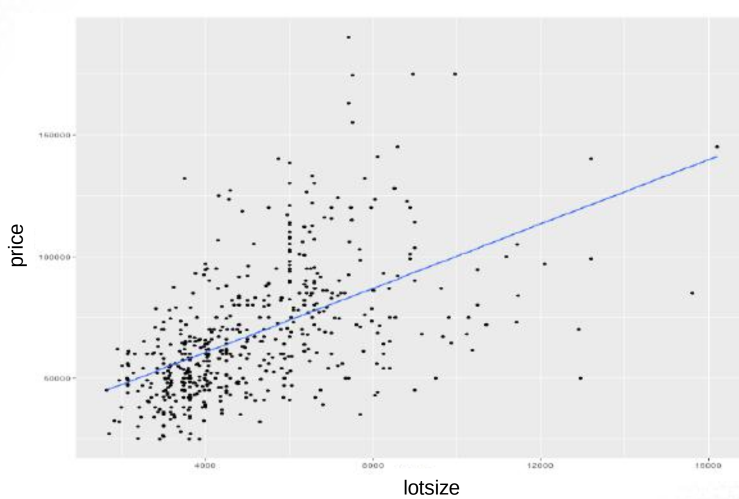


Scatter Plot

Scatter Plot of price (*y*) against lotsize (*x*), including the linear regression line



Quiz (True/False)

- The mean of a variable that has a right-skewed distribution is smaller than the median.

  Answer: **FALSE**.

- The correlation coefficient can capture the strength of both linear and non-linear relationships.

  Answer: **FALSE**.

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Linear Regression

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Notation

Linear Regression: Notation

Notation	Meaning
$i = 1,2,\ldots,n$	*i* refers to the *i*th observation or record in a data set of records (typically a sample of the population)
$(x_{11}, x_{21}, \ldots, x_{p1})$, $(x_{12}, x_{22}, \ldots, x_{p2})$, …, $(x_{1n}, x_{2n}, \ldots, x_{pn})$	*n observations of the p explanatory variables*
$y_1, y_2, \ldots, y_n$	*n* observations of the dependent variable
$\bar{y}$	Mean value of the dependent (*y*) variable
$\bar{x}_k$	Mean value of the x_kth explanatory (independent) variable

Linear Regression: Notation (cont'd)

Notation	Meaning
$\beta_0, \beta_1, \ldots, \beta_p$	Parameters of the regression line for the entire population
$b_0, b_1 \ldots, b_p$	Estimates of the β parameters obtained by fitting the regression to the sample data
ε_i	Error term for the *i*th observation in the population
e_i	Error term for the *i*th observation in the sample
$\hat{y}_i$	Estimated value of *y* for the *i*th observation in a sample. This is obtained by evaluating the regression function at x_i

Simple Linear Regression

- We observe the data in the Housing dataset (which is a sample)
- We want to build a model for the **population**:

  $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$ (which is the valid relation)
- ε_i are independent and identically distributed (i.i.d.) random variables, which are normally distributed with mean 0 and standard deviation σ
- However, we do not know β_0, β_1, or σ; so we need to estimate them based on the **sample** in the Housing dataset
- Using this sample, we are going to build a model

  $Y_i = b_0 + b_1 X_i + e_i$

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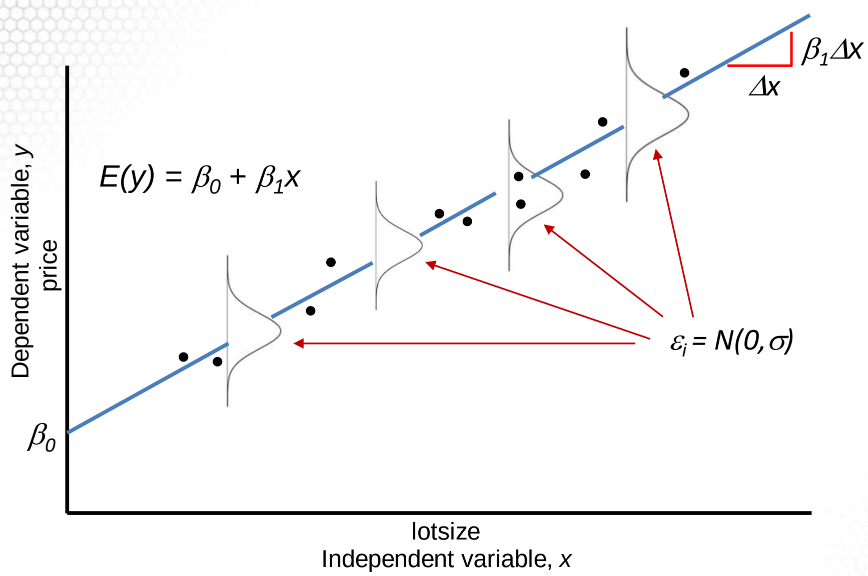
Population model, $Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$



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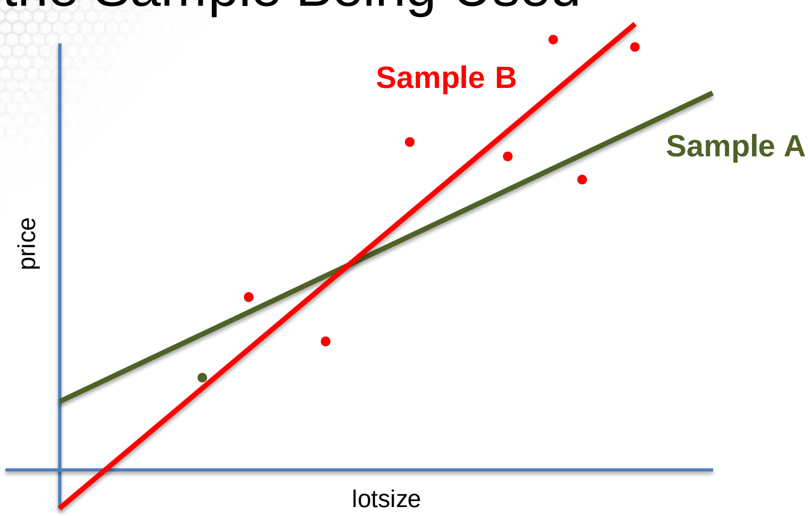
Estimates of Slope and Intercept Depend on the Sample Being Used

Estimates of Slope and Intercept Depend on the Sample Being Used



Use Ordinary Least Squares (OLS) to Fit the Line $\hat{y} = b_0 + b_1$x

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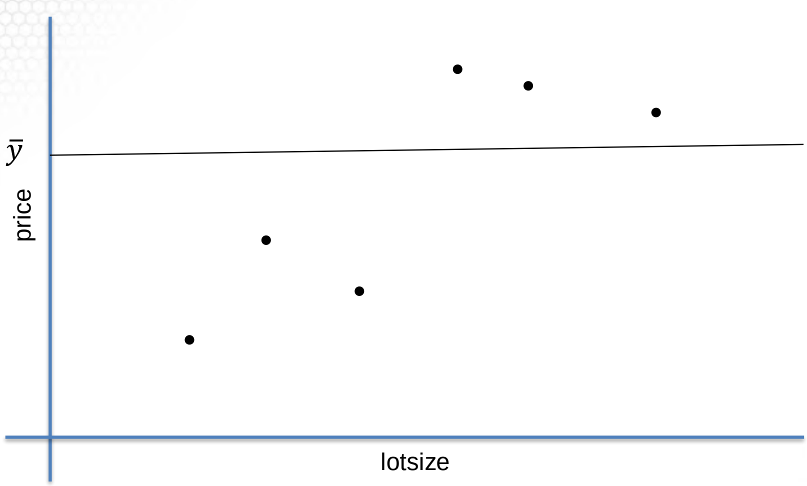

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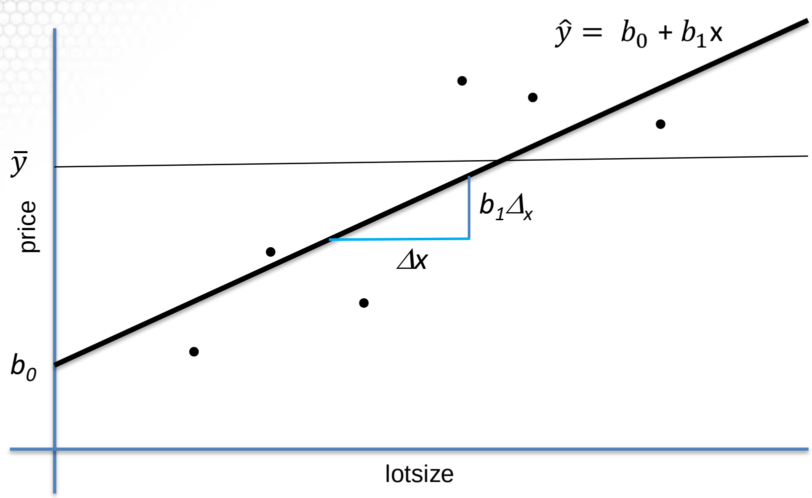
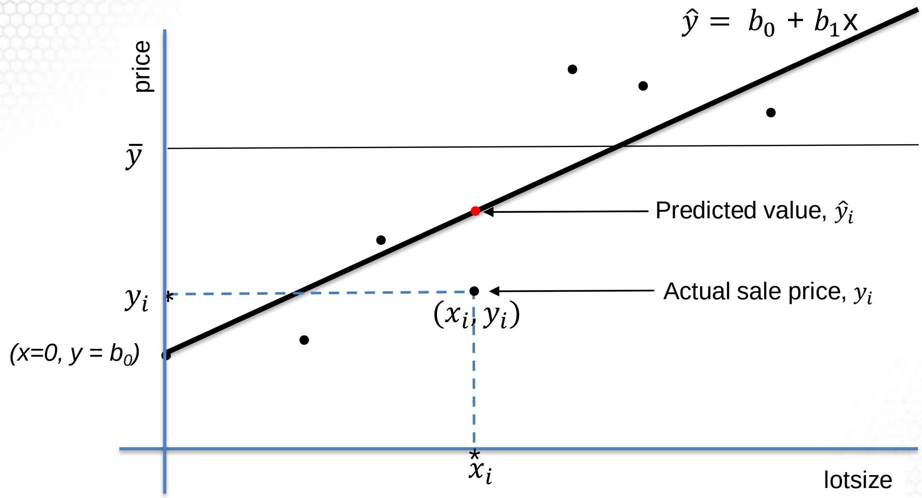


Total Deviation = Explained Deviation + Unexplained Deviation



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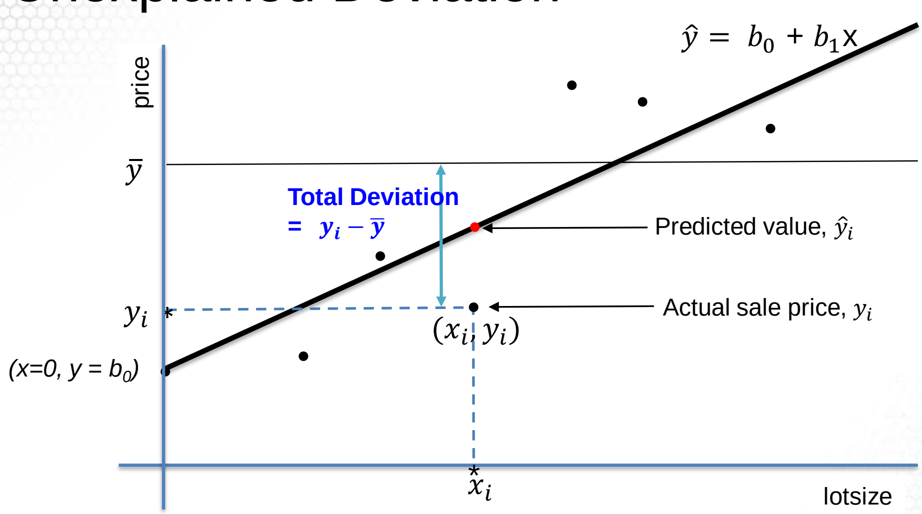
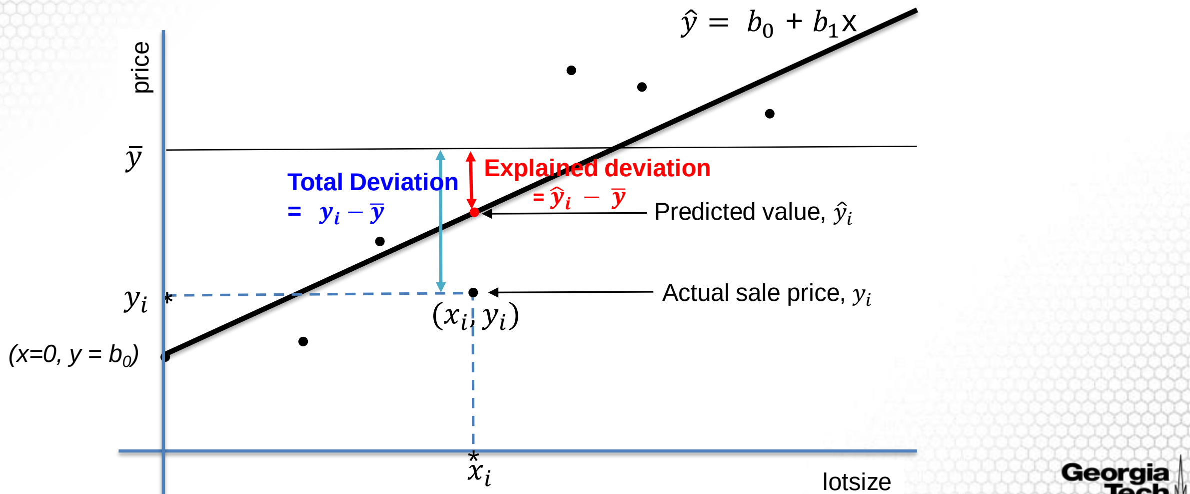
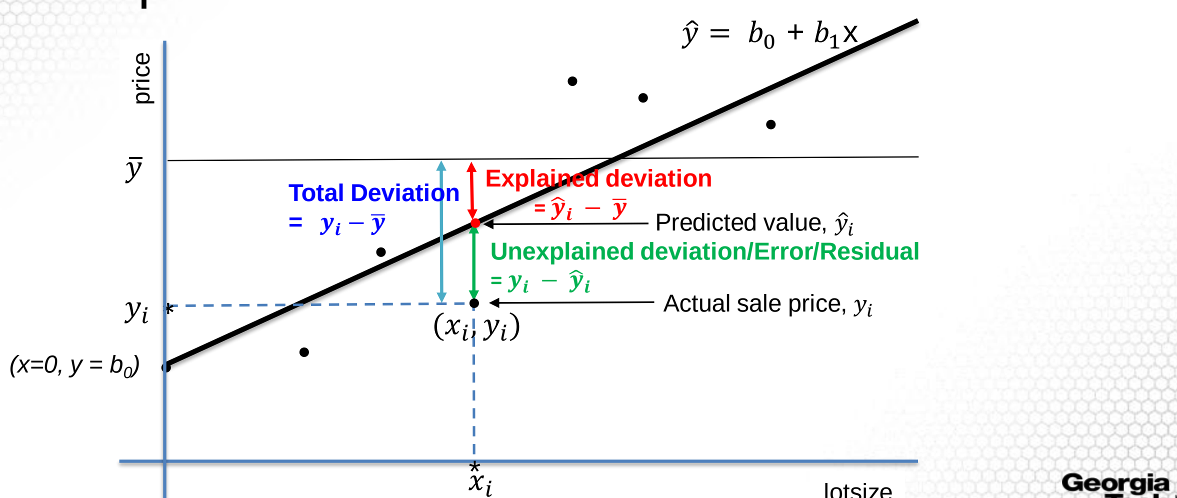


Total Deviation = Explained Deviation + Unexplained Deviation



Total Deviation = Explained Deviation + Unexplained Deviation



Quiz (True/False)

- The total deviation at observation *(x_i , y_i)* is $\boldsymbol{y_i} - \overline{\boldsymbol{y}}$.
  Answer: **TRUE**

- In OLS, the estimates of slope and intercept do not depend on the sample being used.
  Answer: **FALSE**

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R^2, Adjusted R^2

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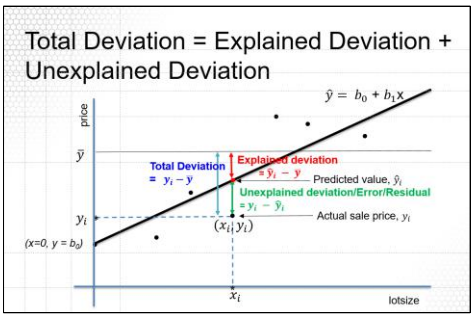
Regression (Ordinary Least Squares): Sum of Squared Errors (SSE)

Regression (OLS) determines the line that minimizes the Sum of Squared Errors

- i.e., b_0 and b_1 are determined such that they minimize:

$$SSE = \sum_i (y_i - \hat{y}_i)^2 = \sum_i (y_i - (b_0 + b_1 x_i))^2$$



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Summing the Deviations

$\sum_i (y_i - \bar{y})^2$	=	$\sum_i (y_i - \hat{y}_i)^2$	+	$\sum_i (\bar{y} - \hat{y}_i)^2$
SST	=	SSE	+	SSR
Total Sum of Squares	=	Sum of Squared Errors	+	Sum of Squares Regression

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Regression Output R^2 and Adjusted R^2

Coefficient of determination (R^2)

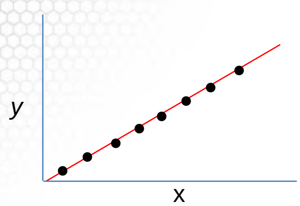
- A measure of the overall strength of the relationship between the dependent variable (Y) and independent variables (X)
- $R^2 = 1 - (SSE/SST) = SSR/SST$

  = Explained deviation (SSR)/Total Deviation (SST)
- $R^2 \rightarrow$ how much of the variation in Y (from the mean) has been explained

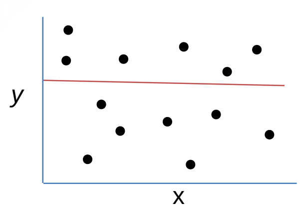
Adjusted R^2

- Adding a penalty for the number of independent variables (p)
- Adjusted $R^2 = 1 - \{SSE/(n - p - 1)\}/\{SST/(n - 1)\}$

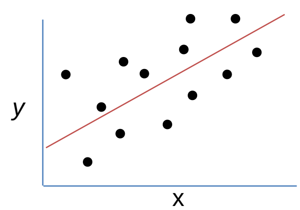
R^2



$R^2 = 1$,
X accounts for all *Y* variation



$R^2 = 0$,
X accounts for none of the *Y* variation



$R^2 = 0.75$,
X accounts for most of the *Y* variation

Quiz (True/False)

- $R^2 = 0$, implies that *X* values account for all of the variation in the *Y* values
  Answer: **FALSE**. $R^2 = 0$ implies that *X* values account for none of the variation in the *Y* values

- R^2 can take any value from – infinity to + infinity
  Answer: **FALSE**. It can take on values between 0 and 1

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Simple Regression (One Predictor Variable) Using R

Regression Output: Simple Linear Regression

```
lm(formula = price ~ lotsize, data = Housing)

Residuals:
   Min        1Q    Median    3Q      Max
-69551    -14626    -2858   9752.   106901

Coefficients:
              Estimate      Std. Error     t value     Pr(>|t|)
(Intercept)   3.414e+04   2.491e+03   13.7        <2e-16 ***
lotsize       6.599e+00   4.458e-01   14.8   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 22570 on 544 degrees of freedom
Multiple R-squared:  0.2871,      Adjusted R-squared:  0.2858
F-statistic: 219.1 on 1 and 544 DF,  p-value: < 2.2e-16
```

Unexplained deviation/Error/Residual = $y_i - \hat{y}_i$

Regression Output: Coefficients

b_0 and b_1 are estimates of the true parameters β_0 and β_1

H_0: the parameter is zero, H_1: The parameter is not zero

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```

Regression Output: t-values for Coefficients

p value: the probability of finding a t value of this size if the null hypothesis is true

H_0: the parameter is zero

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```

Interpreting Coefficients

	Estimate
(Intercept)	3.414e+04 ***
lotsize	6.599e+00 ***

b_0 = 34,140

Intercept of the regression line with the y-axis (when lotsize is zero). Not useful

b_1 = 6.599

An increase of 1,000 square feet is associated with an increase of the sale price of a house by $6,599, keeping all else constant (*ceteris paribus*)

Regression Output: Sum of Squares

Analysis of Variance Table

	Df	Sum Sq
lotsize	1	1.1156e+11
Residuals	544	2.7704e+11

SSR = 1.1156e+11

SSE = 2.7704e+11

SST = SSR + SSE = 3.886e+11

Regression Output: R^2

```
lm(formula = price ~ lotsize, data = Housing)

Residuals:
   Min      1Q      Median     3Q     Max
-69551  -14626      -2858    9752   106901

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Regression Output R^2 and Adjusted R^2

SSR = 1.1156e+11

SSE = 2.7704e+11

SST = SSR + SSE = 3.886e+11

Multiple R-squared: 0.2871, Adjusted R-squared: 0.2858

$R^2 = 1 - (SSE/SST) = SSR/SST = 1.1156e+11/3.886e+11$

$= 0.2871$

Note: $\sqrt{R^2} = \sqrt{0.2871} = 0.536$ (which is the correlation coefficient between price and lotsize)

Adjusted $R^2 = 1 - \{SSE/(n - p - 1)\}/\{SST/(n - 1)\}$

$= 1 - \{(2.7704e+11/(546 - 1 - 1)\} / \{3.886e+11/(546 - 1)\}$

$= 0.2858$

F-test that the model is significant (H_0: $b_1 = 0$)

SSR = 1.1156e+11

SSE = 2.7704e+11

SST = SSR + SSE = 3.886e+11

If p is the number of independent variables, The F statistic

$= (SSR/p)/ (SSE/n - p - 1) = (R^2/p) / (1 - R^2)/(n - p - 1)$

The value of Prob(F) is the probability that H_0 is true (i.e., $b_1 = 0$).

For this model, p = 1,

$F = (0.2871/1) / (1 - 0.2871)/(546 - 1 - 1) = 219.1$

with (1,544) degrees of freedom.

F statistic: 219.1 with(1, 544) DF, p-value: < 2.2e-16. Hence H_0 is rejected.

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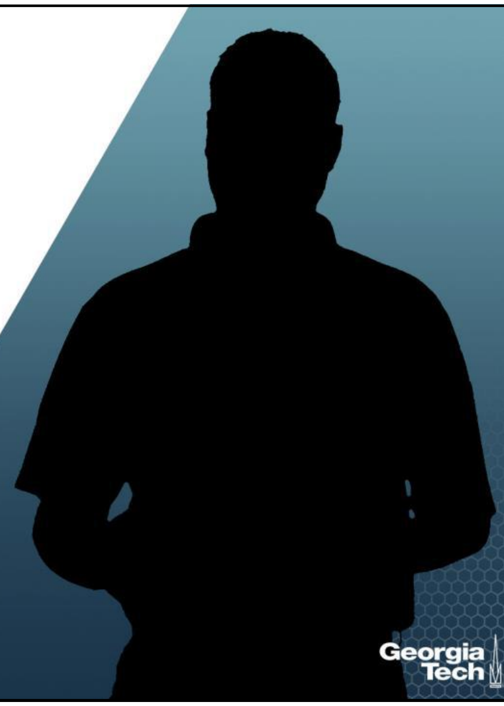

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Multiple Regression

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Multiple Linear Regression, with *p* Explanatory Variables

- **Regression coefficients**:
  $b_0, b_1, \ldots, b_p$ are estimates of $\beta_0, \beta_1, \ldots, \beta_p$
- **Prediction** for Y at x_i
  $\hat{y}_i = b_0 + b_1x_{1i} + b_2x_{2i} + \ldots + b_px_{pi}$
- **Residual**:
  $e_i = y_i - \hat{y}_i$

Goal: choose $b_0, b_1, \ldots, b_p$ to minimize the sum of squared errors

$$SSE = \sum_i (y_i - \hat{y}_i)^2 =$$

$$\sum_i (y_i - (b_0 + b_1x_{1i} + \ldots b_px_{pi}))^2$$

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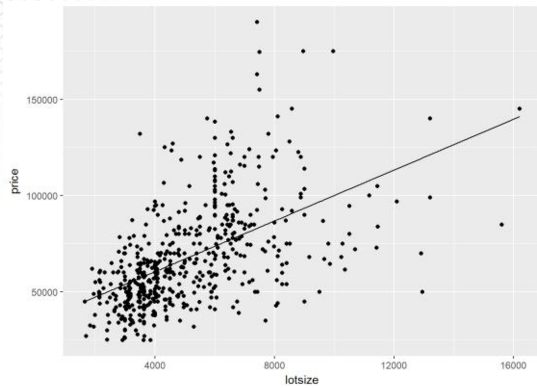
Using R to Estimate a Linear Model

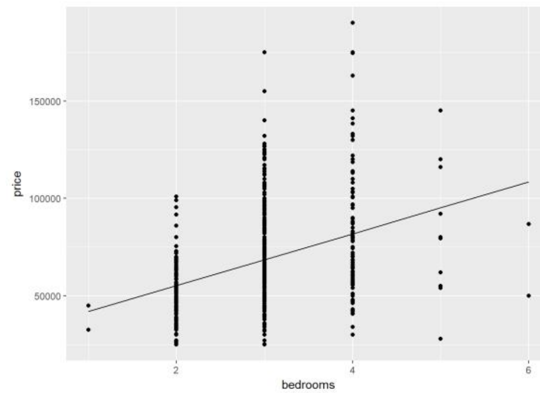
Using the *Housing* Dataset in the *Ecdat* package in R

Adding Bedrooms to the Analysis

price	lotsize	bedrooms
42,000	5,850	3
38,500	4,000	2
49,500	3,060	3
60,500	6,650	3
61,000	6,360	2
66,000	4,160	3
66,000	3,880	3
69,000	4,160	3
83,800	4,800	3
88,500	5,500	3
90,000	7,200	3
30,500	3,000	2
27,000	1,700	3
36,000	2,880	3
37,000	3,600	2

Visualize (Plots)





Do the slopes make sense?

Regression Output

lm(formula = price ~ lotsize + bedrooms, data = Housing)

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	5.613e+03	4.103e+03	1.368	0.172
lotsize	6.053e+00	4.243e-01	14.265	< 2e-16 ***
bedrooms	1.057e+04	1.248e+03	8.470	2.31e-16 ***

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 21230 on 543 degrees of freedom

Multiple R-squared: 0.3703, Adjusted R-squared: 0.3679

F-statistic: 159.6 on 2 and 543 DF, p-value: < 2.2e-16

Regression Output: Coefficients

$b_0, b_1, \ldots, b_p$ are estimates of the true parameters $\beta_0, \beta_1, \ldots, \beta_p$

H_0: the parameter is zero, H_1: The parameter is not zero

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lotsize	6.053e+00	4.243e-01	14.265	< 2e-16 ***
bedrooms	1.057e+04	1.248e+03	8.470	2.31e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 21230 on 543 degrees of freedom
Multiple R-squared: 0.3703, Adjusted R-squared: 0.3679
F-statistic: 159.6 on 2 and 543 DF, p-value: < 2.2e-16

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Regression Output: Standard Error of the Coefficients

Similar to Standard Deviation

lm(formula = price ~ lotsize + bedrooms, data = Housing)

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	5.613e+03	4.103e+03	1.368	0.172
lotsize	6.053e+00	4.243e-01	14.265	< 2e-16 ***
bedrooms	1.057e+04	1.248e+03	8.470	2.31e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 21230 on 543 degrees of freedom
Multiple R-squared: 0.3703, Adjusted R-squared: 0.3679
F-statistic: 159.6 on 2 and 543 DF, p-value: < 2.2e-16

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Regression Output: t-values for Coefficients

lm(formula = price ~ lotsize + bedrooms, data = Housing)

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	5.613e+03	4.103e+03	1.368	0.172
lotsize	6.053e+00	4.243e-01	14.265	< 2e-16 ***
bedrooms	1.057e+04	1.248e+03	8.470	2.31e-16 ***

Signif. codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 21230 on 543 degrees of freedom
Multiple R-squared: 0.3703, Adjusted R-squared: 0.3679
F-statistic: 159.6 on 2 and 543 DF, p-value: < 2.2e-16

Interpreting Coefficients

	Estimate
(Intercept)	5.613e+03
lotsize	6.053e+00
Bedrooms	1.057e+04

b_0 = 5613
Intercept of the regression line with the y-axis (when all x’s are zero). Not useful

b_1 = 6.053
An increase of 1,000 square feet is associated with an increase of the sale price of a house by $6,053, keeping all else constant

b_2 = 10570
An additional bedroom is associated with an increase of the sale price of a house by $10,570, keeping all else constant

Data Analytics in Business

Linear Regression

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R^2, Adjusted R^2 from Multiple Regression

Regression Output: Sum of Squares

Analysis of Variance Table

	Df	Sum Sq
lotsize	1	1.1156e+11
bedrooms	1	3.2329e+10
Residuals	543	2.4472e+11

SSR = (1.1156 + 0.32329) e+11
SSE = 2.4472e+11
SST = SSR + SSE = 3.88609e+11

Regression Output: R^2

lm(formula = price ~ lotsize + bedrooms, data = Housing)

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	5.613e+03	4.103e+03	1.368	0.172
lotsize	6.053e+00	4.243e-01	14.265	< 2e-16 ***
bedrooms	1.057e+04	1.248e+03	8.470	2.31e-16 ***

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 21230 on 543 degrees of freedom

Multiple R-squared: 0.3703, Adjusted R-squared: 0.3679

F-statistic: 159.6 on 2 and 543 DF, p-value: < 2.2e-16

Regression Output R^2 and Adjusted R^2

SSR = (1.1156 + 0.32329) e+11

SSE = 2.4472e+11

SST = SSR + SSE = 3.88609e+11

Multiple R-squared: 0.3703, Adjusted R-squared: 0.3679

R^2 = 1 – SSE/SST = SSR/SST = 1.43889e+11/3.88609e+11
= 0.3703

Adjusted R^2 = 1 – {SSE/(n – p – 1)}/{SST/(n – 1)}
= 1 – ((2.4472e+11/(546 – 2 – 1)} / {3.88609e+11/(546 – 1)}
= 0.3679

F-test of the Overall Significance of the Model (H_0: $b_1 = b_2 = 0$)

SSR = (1.1156 + 0.32329) e+11
SSE = 2.4472e+11
SST = SSR + SSE = 3.88609e+11
The F statistic
$= (SSR/p) / (SSE/(n - p - 1)) = (R^2/p) / ((1 - R^2)/(n - p - 1))$

The value of Prob(F) is the probability that H_0 is true.

For this example, F = (0.3703/2)/(1 – 0.3703)/(546 – 2 – 1) = 159.6
F-statistic: 159.6 on 2 and 543 DF, p-value: < 2.2e-16. Hence H_0 is rejected.

Simple vs. Multiple Regression

- For the Simple Regression we got:
  Multiple R-squared: 0.2871, Adjusted R-squared: 0.2858
- For the Multiple Regression we got:
  Multiple R-squared: 0.3703, Adjusted R-squared: 0.3679
- As you add variables, R-square will not decrease

Comparing the Two Models

n: number of observations

p: number of variables (do not count the intercept)

Bigger model 2 which has (more) p_2 variables

Smaller model 1 which has (fewer) p_1 variables

We want to determine whether model 2 gives a *significantly* better fit to the data. Then use the F statistic shown below

- $F(p_2-p_1, n - p_2 - 1)$
- F test statistic is calculated as

$$F = \frac{(R_2^2 - R_1^2)/(p_2 - p_1)}{(1 - R_2^2)/(n - p_2 - 1)}$$

Quiz (True/False)

- In general, adding more variables decreases the overall R-Square value of the multiple regression.

  Answer: **FALSE**.

- In the regression output shown below, a p-value of < 2e-16 *** means that there is not much evidence for the coefficient of lotsize to be different from zero.

  Answer: **FALSE**.

lm(formula = price ~ lotsize + bedrooms, data = Housing)

Coefficients:

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	5.613e+03	4.103e+03	1.368	0.172
lotsize	6.053e+00	4.243e-01	14.265	< 2e-16 ***
bedrooms	1.057e+04	1.248e+03	8.470	2.31e-16 ***

Data Analytics in Business

Linear Regression

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Common Problems and Fixes in Fitting Linear Regression, Part 1

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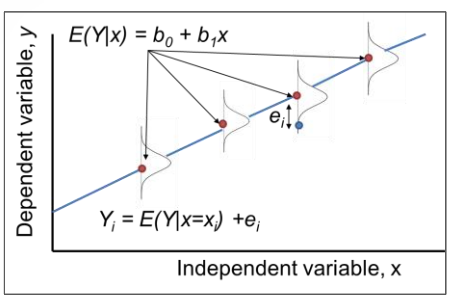
Assumptions of Linear Regression



- **Linearity assumption**: $E(y) = b_0 + b_1x$, i.e., the expected value of Y at each value of X approximates to a straight line
- **Assumption about errors**: The error terms e_i are independently and identically distributed (iid) normal random variables, each with mean zero and constant variance σ^2 (homoscedasticity)
- **Assumptions about predictors**: In multiple regression, the predictor variables are assumed to be linearly independent of one another

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Most Common Problems in Fitting Linear Regression

1. Non-linearity of the response-predictor relationships
2. Correlation of error terms
3. Non-constant variance of error terms
4. Outliers
5. High-leverage points
6. Collinearity

James, Gareth, et al. *An introduction to statistical learning: with applications in R* (Section 3.3.3). Springer, 2017.

1. Is the Relationship Nonlinear?

- Check the scatter plots of **Y** vs. each **X** variable. Linear?
- Another plot to use is the residuals plot vs. fitted values plot (especially useful in multiple regression)
  - We want to see no patterns

1. Is the Relationship Nonlinear?

- If there are concerns, then:
  - Can you model non-linear relationship with higher order terms (e.g., square)?
  - Use variance reducing transformation (such as log) that will give a better linear fit
  - Are there outliers or certain sections of the observations that seem to drive the non-linearity?
  - Is there any important variable that you left out from your model (e.g., age or gender)?
  - Or, maybe, was there systematic bias when collecting data, hence redesign data collection
- Checking residuals helps discover useful insights about your model and data

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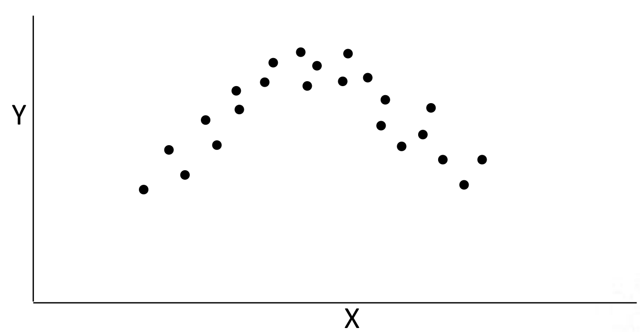
Scatterplot of the Response Variable Versus the Explanatory Variable

- This is useful to do before fitting a model
- You can plot Y vs. X to identify any patterns. For example, the scatter plot below suggests using X^2 rather than X as a predictor



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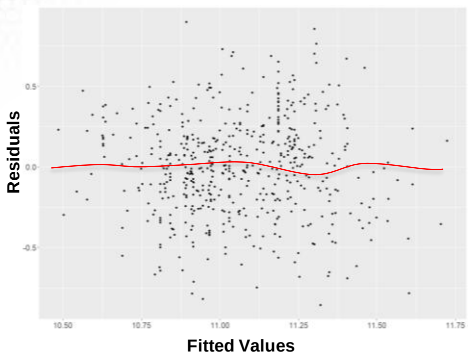
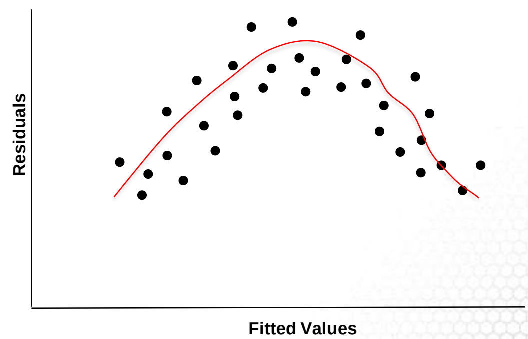
Residuals vs. Fitted Values Plot

Case 1



Case 2



This plot should be examined after a model is fitted

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2. Correlation of Error Terms

- An important assumption is that error terms e_1, e_2, …,e_n are uncorrelated. If they aren't, then we have autocorrelation
- So knowing the value of e_i should not have any influence on the magnitude or size of e_{i+1}
- This property is used to estimate the standard errors of the parameters of the model
- If there is correlation in the error terms:
  - The estimated standard errors will underestimate the true standard errors
  - Confidence and prediction intervals will be narrower than they should be and p values will be lower than they should be
  - We may have sense of confidence in the model that is not warranted
- The Durbin-Watson test is used to detect autocorrelations in a linear model

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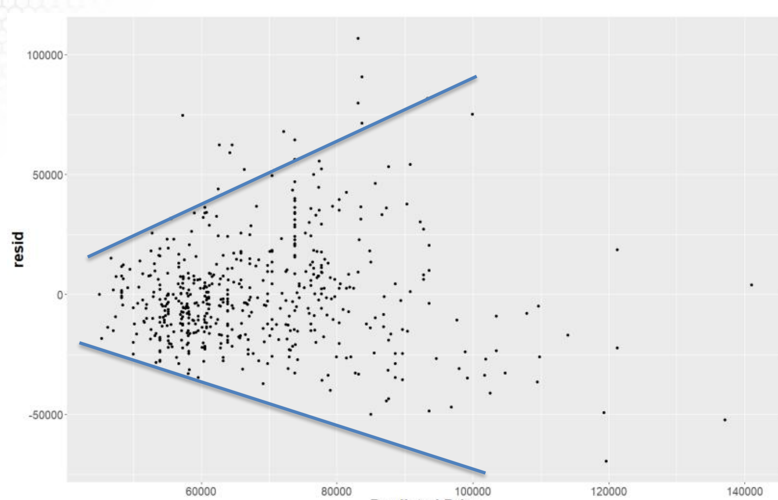
3. Heteroskedasticity (Non-constant Error Variance)

- The assumption is that the spread of the responses around the straight line is the same at all levels of the explanatory variable (i.e., we have constant variance or homoscedasticity)
- You may have non-constant error present (e.g., the errors increase in size with the fitted values). You can detect this with the residuals vs. fitted values plot
- If non-constant error is present, then Hypotheses tests and Confidence Intervals can be misleading
- If there is Heteroscedasticity, then transformation of the Y variable may be called for
  - Example: ln(Y), or 1/Y, etc.

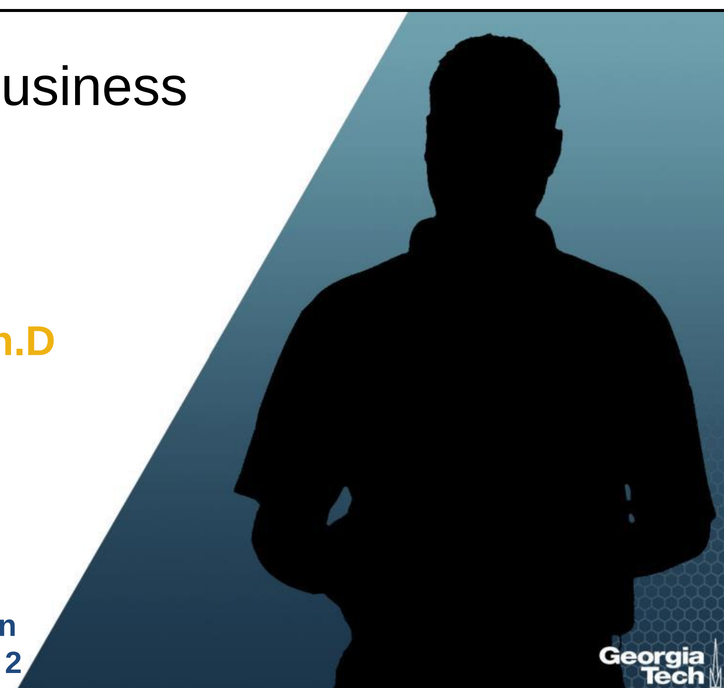


Quiz (True/False)

- If the scatterplot of Y vs. X shows a nonlinear pattern, then we should not change our linear regression model.
  Answer: **FALSE**
- Autocorrelation is the correlation between each of the e_i variables.
  Answer: **TRUE**
- Heteroskedasticity means having constant Error Variance.
  Answer: **FALSE**

Data Analytics in Business

Linear Regression



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Common Problems and Fixes in Fitting Linear Regression, Part 2

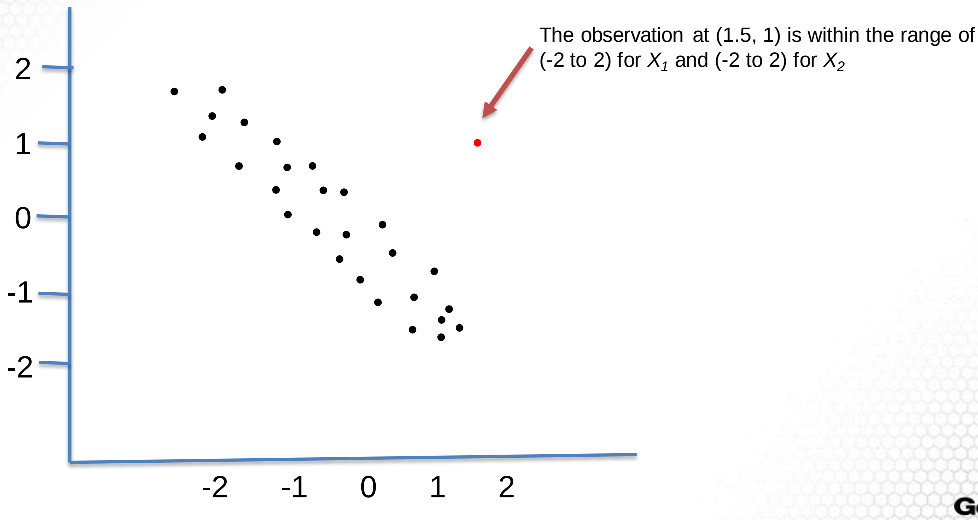
4. Outliers

- An outlier is a point that has a y_i value that is far from its predicted value, $\hat{y}_i$
- One way to visualize outliers is to plot residuals (or, better yet, standardized residuals) against predicted values of y
- Outliers could occur because of incorrect data recording or because the phenomenon could very well be non-linear
- Do not assume that an outlier observation should be removed. It may signal a model deficiency (for e.g., a missing predictor)

5. High Leverage Points

- In simple regression, look for observations that have a predictor value outside the normal range of observations
- A point has high leverage if its deletion (by itself or with 2 or 3 other points) causes noticeable changes in the model
- With many predictors in a model, one could have an observation that is within the range of each predictor's value but still be unusual

5. High Leverage Points - Example



Cook's Distance

- One statistic to identify influential points is the ***Cook's Distance C_i*** that measures the difference between the regression coefficients obtained
  - (a) from the full data and
  - (b) from deleting observation *i*
- A rule of thumb is to identify points with $C_i > 1$ as highly influential

Plot(model) in R

- The plot function in R, when applied to a linear model, provides four plots that are:
    - Residual vs. Fitted (check if residuals have non-linear patterns)
    - Normal Q-Q (check if residuals are normally distributed)
    - Scale-Location (check if √ (standardized residuals) are spread equally along the range of fitted values)
    - Residuals vs Leverage (to find influential points, if any, with $C_i > 1$)

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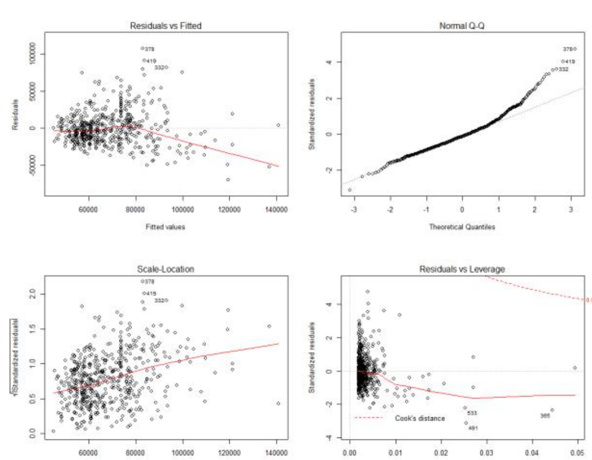
a.lm <- lm(formula = price ~ lotsize , data = Housing) plot(a.lm)



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Outliers and Influential Points

- **Objective**: In fitting a model to a given body of data, we would like to ensure that the fit is not overly determined by one or a few observations, aka **outliers**
- There are two types of outliers:
  1. **Y (response) outlier**
  2. **X (predictor) outlier, Leverage Point**
- An outlier has the potential to be identified as an **influential** data point if it unduly influences the regression analysis
- The next few slides will define the two types of outliers and how to classify the outlier as an influential point

Reference: Chatterjee S, Hadi AS (2012) *Regression Analysis by Example* 5 edition. (Wiley, Hoboken, New Jersey).

Outliers in the Response Y Variable

- An outlier is a point that has a y_i value that is far from its predicted value $\hat{y}_i$. It will have a large standardized residual
  - Since the standardized residuals are approximately normally distributed with mean = 0 and a standard deviation = 1, points with standardized residuals larger than 2 or 3 standard deviation away from the mean (zero) are called outliers
  - If removal of the outlier causes substantial change to the regression analysis, then it is an influential outlier (see influential point definition)
- **Detection**: One way to visualize/identify outliers is to plot residuals (or standardized residuals) against predicted values of y ($\hat{y}_i$)
- Why do outliers occur?
  - Outliers could occur because of incorrect data recording, because of real anomalous events recorded correctly, or because the phenomenon could very well be non-linear
  - Do not assume that an outlier observation should automatically be removed. It may signal a model deficiency (i.e., a missing predictor)

Outliers in the Predictor X Variable (Leverage Points)

- Extreme *x* values (*x* is the predictor variable) are high leverage points
  - The data point x_i will be unusually out of range of the other predictor X values
  - Does not have a large standardized residual
  - Can affect regression results
- **Detection**: Identify leverage points via index plot, dot plots, box plot, or Cook's Distance (next slide)
  - If the leverage point is flagged via Cook's distance, then it is also an influential point and thus has substantial influence on the fitted model (next slide)
- Why does the leverage point exist?
  - Requires a case-by-case data analysis
  - Often, it is best to analyze the leverage point by creating models with and without the data point to see the effect it has on the fitted line
  - This method of analysis applies to both *y* (response) outliers and influential points

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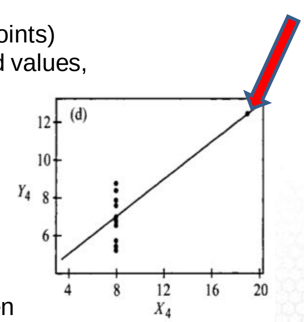
Influential Points

- An outlier is an *influential point* if its deletion (by itself or with 2 or 3 other points) causes substantial changes in the fitted model (estimated coefficients, fitted values, slope, t-tests, hypothesis tests, etc.)
  - Deletion must cause large changes, thus the data point has undue influence
  - See plot of (X,Y) least squares fitted line with an influential point



Detection:

- With several variables, we cannot detect influential points graphically
- Measure influential points via **Cook's Distance (C_i):** The difference between
  1. the regression coefficients obtained from the full data WITH '*i*th'data point
  2. and the regression coefficients obtained by DELETING the '*i*th' data point
     - Rule of thumb is to identify points with C_i > 1 as highly influential

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6. (Multi)Collinearity

- Run these two linear regression models in R and then compare their respective coefficient of cylinders:
  - Reg1 <- lm(formula = mpg ~ cylinders, data = Auto)
  - Reg2 <- lm(formula = mpg ~ cylinders + displacement + weight, data = Auto)
- Reg1

  cylinders -3.5581 0.1457 -24.43 <2e-16 ***
- Reg2

  cylinders -0.2678 0.4131 -0.648 0.517
- In Reg2, cylinder's coefficient is no longer statistically significant!

What Could Be the Reason?

- Such a change in a parameter estimate could indicate the presence of multicollinearity in Reg2
- Multicollinearity: two or more of the explanatory variables are more or less linearly related
- To detect multicollinearity, one approach is to use **Variance Inflation Factors (VIF)**
- Regress predictor variable X_j against all other predictor (X) variables. Name the resulting R^2 as R_j^2
- Define $VIF_j = 1/(1-R_j^2)$, $j = 1,2,\ldots, p$
- If X_j has a strong linear relationship to other X variables, then R_j^2 is close to 1, and VIF_j will be large.
- Values of VIF > 5 signify presence of multicollinearity (rule of thumb)

VIF

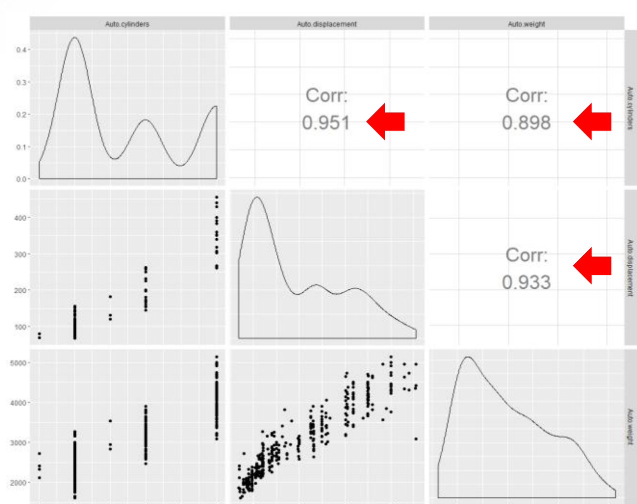
- We use the vif function on the predictors of the Reg2 model and obtain

  vif(Reg2)

cylinders	displacement	weight
10.515508	15.786455	7.788716

- These high VIF values indicate Multicollinearity problems

Correlation Matrix



Consequences of Multicollinearity

- Multicollinearity: the explanatory variables are highly correlated
  - If VIF$_j$ =1/(1-R_j^2) > 5, multicollinearity present…
- Consequences of multicollinearity:
  - OLS estimated parameters may have large variances and covariances, thus making precise estimation difficult
  - The confidence intervals of the estimated parameters tend to be bigger, hence we may not be able to reject H_0 (the null hypothesis, b_i = 0)
  - Regression coefficients have the wrong sign, or
  - Regression coefficients are not significantly different from 0 although R^2 is high
  - Adding an explanatory variable changes other variables' coefficients

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Consequences of Multicollinearity

- Solution?
  - Pick one variable if two measure the same "thing"
  - Use Principal Components Analysis or Factor Analysis to create more useful variable(s)

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Recap of this Module

A. Steps in Regression Analysis
B. A Real Estate example
C. Notation
D. R^2, Adjusted R^2
E. Simple Regression (One Predictor Variable) Using R
F. Multiple Regression
G. R^2, Adjusted R^2 from Multiple Regression
H. Common Problems and Fixes in Linear Regression