# Regression Analysis

## Simple Linear Regression

**Nicoleta Serban, Ph.D.**
*Professor*
School of Industrial and Systems Engineering

Regression Concepts: Estimation

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# About This Lesson

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# Simple Linear Regression: Model

Our goal is to find the best line that describes a linear relationship; that is, find **$(\beta_0, \beta_1)$** where

$$Y = \beta_0 + \beta_1 x + \varepsilon$$

**Equivalently, estimating:**

1. $\beta_0$ *Intercept*
2. $\beta_1$ *Slope*

**ε** is the deviance of the data from the linear model

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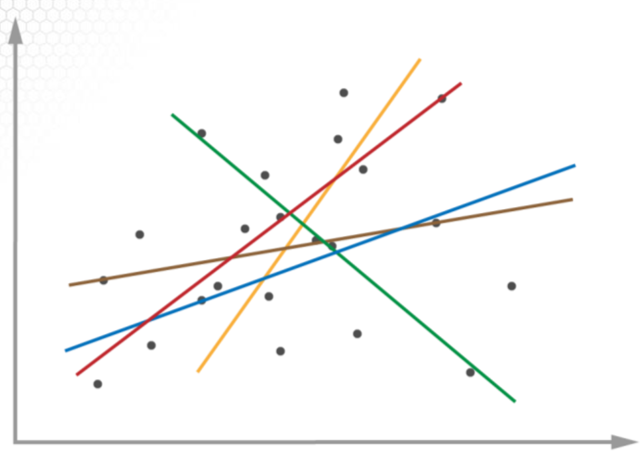
# Simple Linear Regression: Model



How to find the best line?

Our goal is to find the line that describes a linear relationship; that is, find **$(\beta_0, \beta_1)$** where

$$Y = \beta_0 + \beta_1 x + \varepsilon$$

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# Simple Linear Regression: Model

**Data**: $\{(x_1,y_1),\ldots,(x_n,y_n)\}$
**Model**: $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \quad i=1,\ldots,n$

**Assumptions**:

- *Linearity/Mean Zero Assumption:* $E(\varepsilon_i) = 0$
- *Constant Variance Assumption*: $Var(\varepsilon_i) = \sigma^2$
- *Independence Assumption* $\{\varepsilon_1,\ldots,\varepsilon_n\}$ *are independent random variables*
- (*Later we assume* $\varepsilon_i \sim$ *Normal)*

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# Simple Linear Regression: Model

**Data**: $\{(x_1,y_1),\ldots,(x_n,y_n)\}$
**Model**: $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \quad i=1,\ldots,n$

***The model parameters are:*** $\boldsymbol{\beta_0}$, $\boldsymbol{\beta_1}$, $\boldsymbol{\sigma^2}$
- ***Unknown regardless how much data are observed***
- ***Estimated given the model assumptions***
- ***Estimated based on data***

**Assumptions**:

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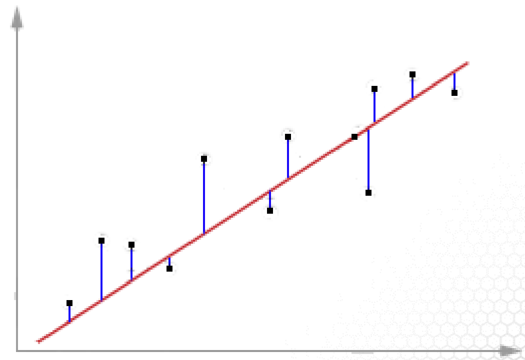
# Model Estimation: Approach

To estimate ($\beta_0$, $\beta_1$) , we find values that minimize sum of squared errors:

$$\sum_{i=1}^{n}(y_i-(\beta_0+\beta_1 x_i))^2$$



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# Model Estimation: Approach

To estimate ($\beta_0$, $\beta_1$), we find values that minimize squared error:

$$\sum_{i=1}^{n}(y_i-(\beta_0+\beta_1 x_i))^2$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$$

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \equiv \frac{\sum_{i=1}^{n} y_i(xi-\bar{x})}{\sum_{i=1}^{n}(xi-\bar{x})^2}$$

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# Model Estimation: Approach

Begin with the minimization problem: $$\min_{\beta_0,\beta_1} \sum_{i=1}^{n} \left(y_i - (\beta_0 + \beta_1 x_i)\right)^2$$

To solve, take the first order derivatives of the function to be minimized and equate to 0:

$$\frac{\partial}{\partial \beta_0} \sum_{i=1}^{n} \left(y_i - (\beta_0 + \beta_1 x_i)\right)^2 = 0$$

$$\frac{\partial}{\partial \beta_1} \sum_{i=1}^{n} \left(y_i - (\beta_0 + \beta_1 x_i)\right)^2 = 0$$

- Result into a system of linear equation in $\beta_0$ and $\beta_1$
- Solve using linear algebra
- Solutions to the system are $\hat{\beta}_0$ and $\hat{\beta}_1$

$$\boxed{\begin{aligned}\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\ \hat{\beta}_1 &= \frac{S_{xy}}{S_{xx}} = \frac{\sum_{i=1}^{n} y_i (x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}\end{aligned}}$$

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# Fitted Values and Residuals

Given the estimates of $b_0$ and $b_1$, we define:

- *Fitted values*: $\boxed{\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i}$
- *Residuals*: $\boxed{r_i = \hat{\varepsilon}_i = y_i - \hat{y}_i}$
- *Mean squared error*: Estimator for $\sigma^2$

$$\boxed{\mathrm{MSE} = \frac{\sum_{i=1}^{n} r_i^2}{n-2} = \frac{\mathrm{SSE}}{n-2}}$$

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# Variance Sampling Distribution

$$\hat{\sigma}^2 = \frac{\sum \hat{\epsilon}_i^2}{n-2} \sim \chi^2_{n-2}$$

(chi-squared distribution with n-2 degrees of freedom)

Assuming $\hat{\epsilon}_i \sim \epsilon_i \sim \mathrm{N}(0, \sigma^2)$

$\uparrow$

Estimating $\sigma^2$ $\longleftarrow$ Sample variance

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# Variance Sampling Distribution (cont'd)

**What is the sample variance estimation?**

**Basic statistic concept:**

**Consider** $Z_1,\ldots,Z_n \sim N(\mu, \sigma^2)$ with $\mu$ and $\sigma^2$ unknown

The sample variance estimator: $S^2 = \frac{\sum(Z_i - \bar{Z})^2}{n-1} \rightarrow \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

**Why n-1?**

We lose a degree of freedom because we replace $\mu \leftarrow \bar{Z}$

Now, going back to $\hat{\sigma}^2 = \frac{\sum \hat{\epsilon}_i^2}{n-2} \sim \chi^2_{n-2}$

This looks like the sample variance estimates except we use n-2 degrees of freedom.

**Why?**

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# Variance Sampling Distribution (cont'd)

Recall that $\epsilon_i = \left(y_i - (\beta_0 + \beta_1 x_i)\right)$

Replaced by $\hat{\epsilon}_i = \left(y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)\right)$

We lose two degrees of freedom because

$\beta_0 \leftarrow \hat{\beta}_0$

$\beta_1 \leftarrow \hat{\beta}_1$

Thus, assuming that $\epsilon_i \sim N\left(0, \sigma^2\right)$

$$\hat{\sigma}^2 = \mathrm{MSE} \sim \chi^2_{n-2}$$

**(This is called the sampling distribution of $\hat{\sigma}^2$)**

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# Model Parameter Interpretation

Commonly interested in the behavior of $\beta_1$

- A positive value of $\beta_1$ is consistent with a direct relationship between x and y; **e.g.**, higher values of height are associated with higher values of weight, or lower values of revenue are associated with lower values of profit;
- A negative value of $\beta_1$ is consistent with an inverse relationship between x and y**; e.g.**, higher price of a product is associated with lower demand, or a lower inflation rate is associated with a higher savings rate;
- A close-to-zero value of $\beta_1$ means that there is not a significant association between x and y.

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# Model Estimate Interpretation

The Least Squares estimated coefficients have specific interpretations:

- $\hat{\beta}_1$ is the estimated expected change in the response variable associated with one unit of change in the predicting variable;
- $\hat{\beta}_0$ is the estimated expected value of the response variable when the predicting variable equals zero.

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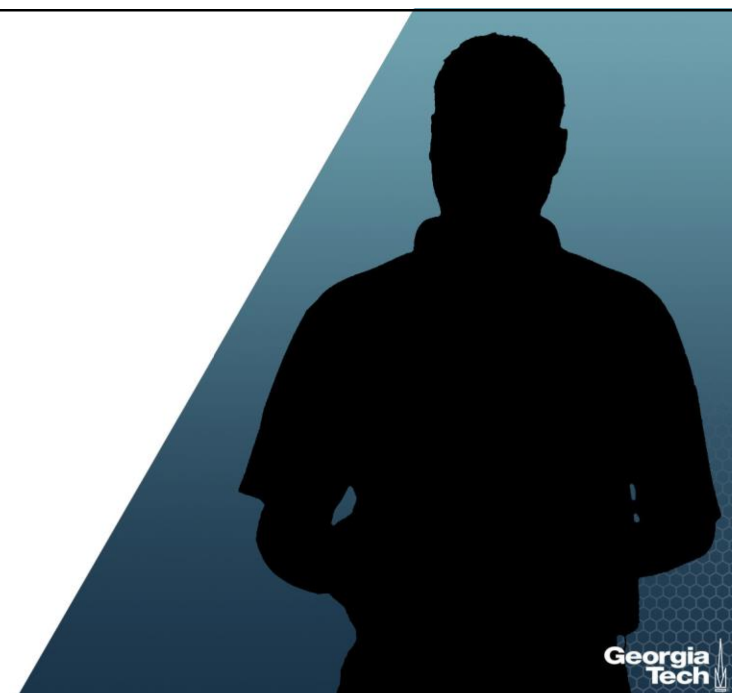
# Summary



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