Regression Analysis

Poisson Regression

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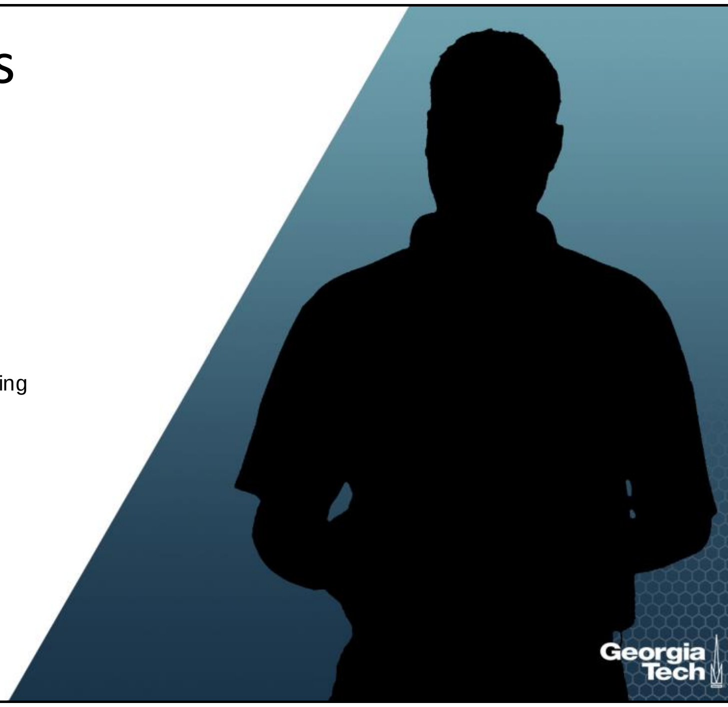
Introduction



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About This Lesson

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Other Distributions of the Response

➔ The response variable (e.g. rate) has a **Poisson distribution**

- What drives the rate of phone calls per day in a calling service center?
- What predicts the density per mile of trees in a forest?

➔ The response variable (e.g. wait time) has an **exponential distribution**

- What explains the wait time for a wellness visit at your physician offices?

➔ The response variable can have other distributions from the **exponential family of distributions**

Generalized Linear Model

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Standard Linear Regression

Model: $Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} + \varepsilon_i, \ i = 1, \ldots, n$

Assumptions:

- *Linearity/Mean Zero Assumption*: $E(\varepsilon_i) = 0$
- *Constant Variance Assumption*: $Var(\varepsilon_i) = \sigma^2$
- *Independence Assumption*: $\{\varepsilon_1, \ldots, \varepsilon_n\}$ are independent random variables
- ***Normality Assumption*: $\varepsilon_i \sim$ Normal**

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Generalized Linear Model

Data: $\{(x_{11},..,x_{1p}),Y_1\},....,\{(x_{n1},..,x_{np}),Y_n\}$ where $Y_1,..,Y_n$ response variable with **a distribution from the exponential family**

Model: Model the conditional expectation**:**

$$g(E(Y|x_1, \dots, x_p)) = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$$

OR

$$E(Y|x_1, \dots, x_p) = g^{-1}(\beta_0 + \beta_1 x_1 + \dots + \beta_p x_p)$$

where $g(\,)$ is a *link function* and $g^{-1}(\,)$ the *inverse link function* depending on the distribution of Y.

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Generalized Linear Model

Y~ *distribution in the exponential family* if its density function can be written as:

$$f(y;\theta) = h(y)e^{g(\theta)T(y)-B(\theta)}$$

where θ is the parameter of the distribution and $g(\theta)$ is the link function.

Distribution	Link	Regression Function
Normal	g(m) = m	m = $x^T\beta$
Poisson	g(m) = log(m)	m = $e^{x^T\beta}$
Bernoulli	g(m) = log(m/1-m)	m = $\frac{e^{x^T\beta}}{1+e^{x^T\beta}}$
Gamma	g(m) = 1/m	m = $\frac{1}{x^T\beta}$

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Poisson Regression

Data: $\{(x_{11},..,x_{1p}),Y_1\},....,\{(x_{n1},..,x_{np}),Y_n\}$ where $Y_1,..,Y_n$ response variable with **a Poisson distribution**

Model: Model the conditional expectation**:**

$$\log(E(Y|x_1,\ \dots,x_p)) = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$$

OR

$$E(Y|x_1,\ \dots,x_p) = \ e^{\beta_0+\beta_1 x_1+\dots+\beta_p x_p}$$

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Linear Regression versus Poisson Regression

Standard Linear Regression with log-transformation:

- $E(\log(Y)|x_1,\ \dots,x_p) = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$
- $V(\log(Y)|x_1,\ \dots,x_p)$ constant

Poisson Regression:

- $\log(E(Y|x_1,\ \dots,x_p)) = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$
- $V(Y|x_1,\ \dots,x_p) = e^{\beta_0+\beta_1 x_1+\dots+\beta_p x_p}$

OR

$$\log(V(Y|x_1,\ \dots,x_p)) = \beta_0 + \beta_1 x_1 + \dots + \beta_p x_p$$

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Linear Regression versus Poisson Regression

Standard Linear Regression with log-transformation:

- $E(\log(\mathrm{Y})|\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{p}) = \beta_0 + \beta_1 \mathrm{x}_1 + \ldots + \beta_p \mathrm{x}_\mathrm{p}$
- $\mathrm{V}(\log(\mathrm{Y})|\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{p})$ constant

Poisson Regression:

- $\log(\mathrm{E}(\mathrm{Y}|\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{p})) = \beta_0 + \beta_1 \mathrm{x}_1 + \ldots + \beta_p \mathrm{x}_\mathrm{p}$
- $\mathrm{V}(\mathrm{Y}|\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{p}) = e^{\beta_0 + \beta_1 \mathrm{x}_1 + \ldots + \beta_p \mathrm{x}_\mathrm{p}}$

OR

$\log(\mathrm{V}(\mathrm{Y}|\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{p})) = \beta_0 + \beta_1 \mathrm{x}_1 + \ldots + \beta_p \mathrm{x}_\mathrm{p}$

- Using Standard Linear Regression with log-transformation instead of Poisson Regression will result in violations of the assumption of constant variance.
- Alternatively, Standard Linear Regression could be use if the number of counts are large and with the variance stabilizing transformation $\sqrt{\mu + 3/8}$.

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Summary

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