# Computer Simulation

## Module 10: Comparing Systems

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## Introduction

# Module Overview

Last Module: Discussed simulation output analysis techniques, primarily batch means and independent replications for a single system.

This Module: Not all systems are created equal. How can we compare systems to determine which is the best?

# Module Overview

1. Introduction ← This lesson
2. Confidence interval for the mean
3. CIs for difference in two means
4. Paired CI for diff in two means
5. CIs for mean diffs in simulations

Variance Reduction Techniques

6. Common random numbers
7. Antithetic random numbers
8. Control variates

# Module Overview

9. Ranking and Selection Methods
10. Normal means selection problem
11. Single-stage procedure
12. Normal extensions
13. Bernoulli probability selection
14. Bernoulli extensions
15. Multinomial cell selection
16. Single-stage procedure + extensions

# Introduction

- Statistics / simulation experiments are typically performed to analyze or compare a "small" number of systems, say < 200.

- Method depends on the type of comparison and properties of the data.

- One system: could use traditional confidence intervals (CIs) based on the normal or *t*-distributions from baby stats.

- Two systems: could again use CIs from baby stats — maybe even clever ones based on paired observations.

- For > 2 systems, we can use *ranking and selection* techniques.

# Confidence Intervals

Lots of possible confidence intervals:

- means
- variances
- quantiles
- one-sample
- two-sample cases (e.g., differences in means)
- > 2 samples
- “classical” baby stats environment
- simulation environment (observations aren’t i.i.d. normal)

# Summary

This Lesson: Gave an overview on this module and issues involving the comparison of competing systems.

Next Time: We'll step back and review the easiest case – a CI for a single mean from a normal distribution.

The lesson will serve as a good starting point for what follows.

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## Confidence Interval for the Mean

# Lesson Overview

Last Lesson: Gave an intro to the problem of comparing simulated systems – which is the best?

This Lesson: We'll step back and review / derive the classical confidence interval for the mean of i.i.d. normal data.

After this little lesson, we'll graduate to the comparison of two systems.

## Confidence Intervals

*One-Sample Case:*

- Interested in obtaining a two-sided $100(1-\alpha)\%$ CI for the unknown mean $\mu$ of a normal distribution.
- Suppose we have independent and identically distributed (i.i.d.) normal data $X_1, X_2, \ldots, X_n$.
- Assume unknown variance $\sigma^2$.
- Use the well-known $t$-distribution based CI, which I'll derive for your viewing pleasure.

First of all, recall that

- The sample mean $\bar{X}_n \equiv \sum_{i=1}^{n} X_i/n \sim \text{Nor}(\mu, \sigma^2/n)$.
- The sample variance $S_X^2 \equiv \sum_{i=1}^{n}(X_i - \bar{X}_n)^2/(n-1) \sim \sigma^2\chi^2(n-1)/(n-1)$.
- $\bar{X}_n$ and $S_X^2$ are independent.

With these facts in mind, we have

$$T = \frac{\bar{X}_n - \mu}{\sqrt{S_X^2/n}} = \frac{\frac{\bar{X}_n - \mu}{\sqrt{\sigma^2/n}}}{\sqrt{S_X^2/\sigma^2}} \sim \frac{\text{Nor}(0,1)}{\sqrt{\frac{\chi^2(n-1)}{n-1}}} \sim t(n-1).$$

Letting the notation $t_{\gamma,\nu}$ denotes the $1-\gamma$ quantile of a $t$-distribution with $\nu$ degrees of freedom, we have

$$\begin{aligned}
1-\alpha &= P(-t_{\alpha/2,n-1} \le T \le t_{\alpha/2,n-1}) \\
&= P\left(-t_{\alpha/2,n-1} \le \frac{\bar{X}_n - \mu}{\sqrt{S_X^2/n}} \le t_{\alpha/2,n-1}\right) \\
&= P\left(\bar{X}_n - t_{\alpha/2,n-1} S_X/\sqrt{n} \le \mu \le \bar{X}_n + t_{\alpha/2,n-1} S_X/\sqrt{n}\right).
\end{aligned}$$

So we have the following $100(1-\alpha)\%$ CI for $\mu$,

$$\mu \in \bar{X}_n \pm t_{\alpha/2,n-1} S_X/\sqrt{n}.$$

# Summary

This Time: Derived the classical confidence interval for the mean of a single normal distribution having unknown variance.

Next Time: We'll review a couple of CIs for the difference in the means of two normal distributions – our next step towards comparing two simulated systems.

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## Module 10: Comparing Systems

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## Confidence Intervals for the Difference in Two Means

# Lesson Overview

Last Time: Reviewed the baby stats CI for the mean of a single normal distribution having unknown variance.

This Time: Compare two systems via two-sample CIs for the difference in two normal means.

Almost to the point where we can apply this stuff in simulations.

*Two-Sample Case:* Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. $\text{Nor}(\mu_X, \sigma_X^2)$ and $Y_1, Y_2, \ldots, Y_m$ are i.i.d. $\text{Nor}(\mu_Y, \sigma_Y^2)$.

A CI for the difference between $\mu_X - \mu_Y$ can be carried out by any of the following methods, all of which are from baby stats.

- pooled CI (use when $\sigma_X^2$ and $\sigma_Y^2$ are *equal but unknown*)
- approximate CI (use when $\sigma_X^2$ and $\sigma_Y^2$ are *unequal and unknown*)
- paired CI (use when $\text{Cov}(X_i, Y_i) > 0$)

In what follows, $\bar{X}_n$, $\bar{Y}_m$, $S_X^2$, and $S_Y^2$ are the obvious sample means and variances of the $X$'s and $Y$'s.

*Pooled CI:* If the $X$'s and $Y$'s are independent but with *common, unknown variance*, then the usual CI for the difference in means is

$$\mu_X - \mu_Y \;\in\; \bar{X}_n - \bar{Y}_m \pm t_{\alpha/2,n+m-2}\, S_P \sqrt{\frac{1}{n} + \frac{1}{m}},$$

where

$$S_P^2 \;\equiv\; \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}$$

is the pooled variance estimator for $\sigma^2$.

*Approximate CI:* If the $X$'s and $Y$'s are independent but with arbitrary unknown variances, then the usual CI for the difference in means is

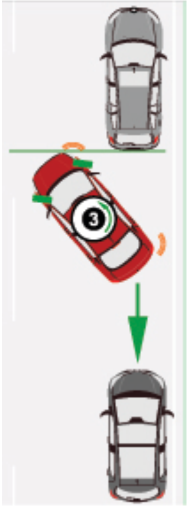
$$\mu_X - \mu_Y \ \in \ \bar{X} - \bar{Y} \pm t_{\alpha/2,\nu} \sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}.$$

This CI is not quite exact, since it uses an *approximate* degrees of freedom,

$$\nu \ \equiv \ \frac{\left(\frac{S_X^2}{n} + \frac{S_Y^2}{m}\right)^2}{\frac{(S_X^2/n)^2}{n+1} + \frac{(S_Y^2/m)^2}{m+1}} - 2.$$

Example: Times for people to parallel park two cars (assume normal).

| A guy parks Honda $X_i$ | Different (indep) guy parks Caddy $Y_i$ |
|---|---|
| 10 | 30 |
| 25 | 15 |
| 5 | 40 |
| 20 | 10 |
| 15 | 25 |



After a little algebra, we have

$$\bar{X} = 15, \quad \bar{Y} = 24, \quad S_X^2 = 62.5, \quad S_Y^2 = 142.5.$$

More algebra gives

$$\nu = \frac{6\left(62.5 + 142.5\right)^2}{(62.5)^2 + (142.5)^2} - 2 = 8.4 \approx 8 \quad \text{(round down).}$$

This yields the following 90% CI,

$$\mu_X - \mu_Y \in \bar{X} - \bar{Y} \pm t_{0.05,8}\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{n}} = -9 \pm 11.91,$$

which contains 0 and so is *inconclusive* about which of $\mu_X$ and $\mu_Y$ is bigger. $\square$

# Summary

This Time: Looked at a couple of CIs for the difference in two normal means. We assumed that the two samples were completely independent.

Next Time: What happens if the two samples happen to be correlated with each other? We'll review the classical paired CI.

# Computer Simulation

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## Paired CI for the Difference in Two Means

# Lesson Overview

Last Time: Went over a couple of CIs for the difference in two means assuming completely independent samples.

This Time: What if the samples aren't indep of each other?

Not necessarily bad!
Paired CI for diff in means

*Paired CI*: Again consider two competing normal pop'ns with unknown means $\mu_X$ and $\mu_Y$. Suppose we collect observations from the two pop'ns in *pairs*.

Different pairs are *independent*, but the two obs'ns within the *same* pair may *not* be indep.

$$\text{indep}\begin{cases} \text{Pair } 1: & (X_1, Y_1) \\ \text{Pair } 2: & (X_2, Y_2) \\ \quad\vdots & \quad\vdots \\ \text{Pair } n: & \underbrace{(X_n, Y_n)}_{\text{not indep}} \end{cases}$$

# Paired *t* Set-up

Example: Think sets of twins. One twin takes a new drug, the other takes a placebo.

Idea: By setting up such experiments, we hope to be able to capture the difference between the two normal populations more precisely, since we're using the pairs to eliminate extraneous noise.

This will be the trick we use later on in this module when we use the simulation technique of *common random numbers*.

Here's the set-up. Take $n$ pairs of observations:

$$\begin{aligned} X_1, X_2, \ldots, X_n &\overset{\text{iid}}{\sim} \text{Nor}(\mu_X, \sigma_X^2) \\ Y_1, Y_2, \ldots, Y_n &\overset{\text{iid}}{\sim} \text{Nor}(\mu_Y, \sigma_Y^2). \end{aligned}$$

(Technical assumption: All $X_i$'s and $Y_j$'s are jointly normal.)

We assume that the variances $\sigma_X^2$ and $\sigma_Y^2$ are *unknown* and possibly *unequal*.

Further, pair $i$ is indep of pair $j$ (between pairs), but $X_i$ may not be indep of $Y_i$ (within a pair).

Define the *pair-wise differences*, $D_i \equiv X_i - Y_i$, $i = 1, 2, \ldots, n$.

Then $D_1, D_2, \ldots, D_n \overset{\text{iid}}{\sim} \text{Nor}(\mu_D, \sigma_D^2)$, where $\mu_D \equiv \mu_X - \mu_Y$ (which is what we want the CI for), and

$$\sigma_D^2 \;\equiv\; \sigma_X^2 + \sigma_Y^2 - 2\text{Cov}(X_i, Y_i).$$

Idea: We hope that $\text{Cov}(X_i, Y_i)$ is pretty positive, which will result in lower $\sigma_D^2$ — low variance is a good thing!

Now the problem reduces to the old $\text{Nor}(\mu, \sigma^2)$ case with unknown $\mu$ and $\sigma^2$. So let's calculate the sample mean and variance as before.

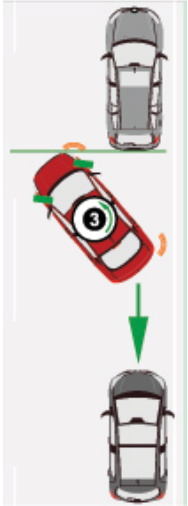
$$\begin{aligned}
\bar{D} &\equiv \frac{1}{n}\sum_{i=1}^{n} D_i \sim \text{Nor}(\mu_D, \sigma_D^2/n) \\
S_D^2 &\equiv \frac{1}{n-1}\sum_{i=1}^{n}(D_i - \bar{D})^2 \sim \frac{\sigma_D^2 \chi^2(n-1)}{n-1}.
\end{aligned}$$

Just like before, get the CI

$$\mu_D \in \bar{D} \pm t_{\alpha/2,n-1}\sqrt{S_D^2/n}.$$

Example: Times for *the same person* to parallel park two cars.

| Person | Park Honda | Park Cadillac | Difference |
|---|---|---|---|
| 1 | 10 | 20 | $-10$ |
| 2 | 25 | 40 | $-15$ |
| 3 | 5 | 5 | 0 |
| 4 | 20 | 35 | $-15$ |
| 5 | 15 | 20 | $-5$ |



The individual people are indep, but the times for the same individual to park the two cars are not indep.

The 90% two-sided CI is therefore

$$\begin{aligned} \mu_D &\in \bar{D} \pm t_{0.025,4} \sqrt{S_D^2/n} \\ &= -9 \pm 2.13 \sqrt{42.5/5} \;=\; -9 \pm 6.21 \end{aligned}$$

This interval is entirely to the left of 0, indicating $\mu_D < 0$, i.e., Caddys take longer to park, on average. $\square$

This CI is quite a bit shorter (more informative) than the previous "approximate" two-sample CI, $-9 \pm 11.91$, because the paired-$t$ takes advantage of the correlation within observation pairs.

Moral: Use paired-*t* when you can.

# Summary

This Time: Went over the paired-*t* CI for the difference in two normal means.  It works really well when the two samples are correlated.

Next Time: How can we use all of this CI stuff to compare simulated systems  (where the observations aren't i.i.d. normal)?

Not quite as tough as it seems.

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## CIs for Mean Differences in Simulations

# Lesson Overview

Last Lesson: Discussed the paired-$t$ CI for the difference in two normal means. Use when the two samples are correlated.

This Time: How to apply this stuff in simulations, where observations aren't i.i.d. normal.

Idea: Use replicate sample means as i.i.d. normals.

# Comparison of Simulated Systems

One of the most important uses of simulation output analysis is the comparison of competing systems or alternative system configurations.

Example: Evaluate two different “re-start” strategies that an airline can evoke following a disrupting snowstorm in the Northeast — which policy minimizes a certain cost function associated with the re-start?

Simulation is uniquely equipped to help with these type of comparisons. Many techniques:

- classical CI’s adopted to simulations.
- variance reduction methods, and
- ranking and selection procedures.

## Confidence Intervals for Mean Differences

With our airline example in mind, let $Z_{i,j}$ be the cost from the $j$th simulation replication of strategy $i$, $i = 1, 2$, $j = 1, 2, \ldots, b_i$.

Assume that $Z_{i,1}, Z_{i,2}, \ldots, Z_{i,b_i}$ are i.i.d. normal with unknown mean $\mu_i$ and unknown variance, $i = 1, 2$. Justification?…

(a) Get *independent* data by controlling the random numbers between replications.

(b) Get *identically distributed* costs between reps by performing the reps under identical conditions.

(c) Get approximately *normal* data by adding up (or averaging) many sub-costs to get overall costs for both strategies.

Goal: Obtain a $100(1-\alpha)\%$ CI for the difference in means, $\mu_1 - \mu_2$.

Suppose that the $Z_{1,j}$'s are independent of the $Z_{2,j}$'s and define

$$\bar{Z}_{i,b_i} \equiv \frac{1}{b_i}\sum_{j=1}^{b_i} Z_{i,j}, \quad i = 1, 2,$$

and

$$S_i^2 \equiv \frac{1}{b_i - 1}\sum_{j=1}^{b_i}(Z_{i,j} - \bar{Z}_{i,b_i})^2, \quad i = 1, 2.$$

An approximate $100(1-\alpha)\%$ CI is

$$\mu_1 - \mu_2 \ \in \ \bar{Z}_{1,b_1} - \bar{Z}_{2,b_2} \pm t_{\alpha/2,\nu}\sqrt{\frac{S_1^2}{b_1} + \frac{S_2^2}{b_2}}$$

where the (approx.) d.f. $\nu$ is given earlier in this module.

Suppose (as in airline example) that small cost is good.

- If the interval lies entirely to the left [right] of zero, then system 1 [2] is better.
- If the interval contains zero, then the two systems are, statistically, about the same.

Alternative strategy: Use a CI analogous to a paired-$t$ test.

Take $b$ replications from *both* strategies and set the difference $D_j \equiv Z_{1,j} - Z_{2,j}$ for $j = 1, 2, \ldots, b$.

$$\bar{D}_b \equiv \frac{1}{b}\sum_{j=1}^{b} D_j \text{ and } S_D^2 \equiv \frac{1}{b-1}\sum_{j=1}^{b}(D_j - \bar{D}_b)^2.$$

The $100(1-\alpha)\%$ paired-$t$ CI is very efficient if $\mathrm{Corr}(Z_{1,j}, Z_{2,j}) > 0$.

$$\mu_1 - \mu_2 \in \bar{D}_b \pm t_{\alpha/2,b-1}\sqrt{S_D^2/b}.$$

# Summary

This Time: The CIs for the mean difference of two simulated systems look like the approx. and paired-*t* CIs from the last lesson!

Next Time: This leads to the variance reduction technique of *common random numbers*, where we intentionally induce positive correlation between the systems!

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## Common Random Numbers

# Lesson Overview

Last Time: Implemented baby stats CIs for the differences in means for use in simulations.

This Time: Start discussion of variance reduction techniques.

Common random numbers (similar to paired-*t* CIs)
Antithetic random numbers
Control variates

## Common Random Numbers

Idea behind paired-$t$ CI: Use *common random numbers*, i.e., use the same pseudo-random numbers in exactly the same ways for corresponding runs of each of the competing systems.

Example: Use the same customer arrival and service times when simulating different proposed configurations of a job shop.

By subjecting the alternative systems to identical experimental conditions, we hope to make it easy to distinguish which systems are best even though the respective estimators have sampling error.

Consider the case in which we compare two queueing systems, $A$ and $B$, on the basis of their expected customer transit times, $\theta_A$ and $\theta_B$ — the smaller $\theta$-value corresponds to the better system.

Suppose we have estimators $\hat{\theta}_A$ and $\hat{\theta}_B$ for $\theta_A$ and $\theta_B$, respectively.

We'll declare $A$ as the better system if $\hat{\theta}_A < \hat{\theta}_B$. If $\hat{\theta}_A$ and $\hat{\theta}_B$ are simulated independently, then the variance of their difference,

$$\operatorname{Var}(\hat{\theta}_A - \hat{\theta}_B) \;=\; \operatorname{Var}(\hat{\theta}_A) + \operatorname{Var}(\hat{\theta}_B),$$

could be very large; then our declaration might lack conviction.

GTx

If we could reduce $\mathrm{Var}(\hat{\theta}_A - \hat{\theta}_B)$, then we could be much more confident about our declaration.

CRN sometimes induces a high positive correlation between the point estimators $\hat{\theta}_A$ and $\hat{\theta}_B$.

Then we have

$$\begin{aligned} \mathrm{Var}(\hat{\theta}_A - \hat{\theta}_B) &= \mathrm{Var}(\hat{\theta}_A) + \mathrm{Var}(\hat{\theta}_B) - 2\mathrm{Cov}(\hat{\theta}_A, \hat{\theta}_B) \\ &< \mathrm{Var}(\hat{\theta}_A) + \mathrm{Var}(\hat{\theta}_B), \end{aligned}$$

and we obtain a savings in variance.

Demo Time! Queueing analysis. Exponential interarrival and service times. Which strategy yields shorter cycle times?

A. One line feeding into two parallel servers, or

B. Customers making a 50-50 choice between two lines each feeding into a single server?

Simulate each alternative for 20 replications of 1000 minutes.

The usual *independent* simulations of strategies A and B reveals gives a CI of $\mu_{\mathrm{A}} - \mu_{\mathrm{B}} \in -16.19 \pm 9.26$.

The use of *CRN* with the same arrival and service times across strategies gives $\mu_{\mathrm{A}} - \mu_{\mathrm{B}} \in -15.05 \pm 3.37$. Much tighter CIs! ☺

GTx

# Summary

This Time: Discussed the variance reduction technique of common random numbers, which induces positive correlation between competitors. Gives tighter CIs.

Next Time: It's Opposite Day! We'll study antithetic random numbers, which induces negative correlation between competitors.

# Computer Simulation

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## Antithetic Random Numbers

# Lesson Overview

Last Time: Introduced the common random numbers variance reduction technique.

This Time: The antithetic random numbers method, which intentionally introduces negative correlation between estimators.

Very useful for estimating certain means.

## Antithetic Random Numbers

Opposite of CRN — Suppose that $\hat{\theta}_1$ and $\hat{\theta}_2$ are i.i.d. unbiased estimators for some parameter $\theta$.

If we can induce *negative* correlation between $\hat{\theta}_1$ and $\hat{\theta}_2$, then the average of the two is also unbiased and may have very low variance,

$$
\begin{aligned}
\mathrm{Var}\left(\frac{\hat{\theta}_1 + \hat{\theta}_2}{2}\right) &= \frac{1}{4}\left[\mathrm{Var}(\hat{\theta}_1) + \mathrm{Var}(\hat{\theta}_2) + 2\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2)\right] \\
&= \frac{1}{2}\left[\mathrm{Var}(\hat{\theta}_1) + \mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2)\right] \\
&< \frac{\mathrm{Var}(\hat{\theta}_1)}{2} \quad (\leftarrow \text{“usual” avg of two reps!}).
\end{aligned}
$$

Example: Let's do some Monte Carlo integration, using ARN to obtain a nice variance reduction.

Consider the integral $I = \int_1^2 (1/x)\, dx$. (Because I have natural logger rhythm, I happen to know that the true answer is $\ell\text{n}(2) \approx 0.693$.)

We'll use the following $n = 5$ $\text{Unif}(0,1)$ random numbers to come up with the usual estimator $\bar{I}_n$ for $I$:

$$0.85 \qquad 0.53 \qquad 0.98 \qquad 0.12 \qquad 0.45$$

Using the Monte Carlo integration notation from waaaay back in time with $g(x) = 1/x$,

$$\begin{aligned}\hat{\theta}_1 = \bar{I}_n &= \frac{b-a}{n}\sum_{i=1}^{n} g(a + (b-a)U_i) \\ &= \frac{1}{5}\sum_{i=1}^{5} g(1+U_i) \\ &= \frac{1}{5}\sum_{i=1}^{5} \frac{1}{1+U_i} \\ &= 0.6563 \quad \text{(not bad)}.\end{aligned}$$

Now we'll use the following *antithetic* random numbers (which are all the "opposite" of the above PRNs, i.e., $1 - U_i$):

$$0.15 \qquad 0.47 \qquad 0.02 \qquad 0.88 \qquad 0.55$$

Then the antithetic version of the estimator is

$$\hat{\theta}_2 = \frac{1}{5}\sum_{i=1}^{5}\frac{1}{1+(1-U_i)} = 0.7475 \quad \text{(also not bad).}$$

But lookee here when you take the average of the two answers,

$$\frac{\hat{\theta}_1 + \hat{\theta}_2}{2} = 0.6989.$$

Wow, this is great!!! ☺

Much closer to the right answer of about 0.693!

# Summary

This Time: Antithetic random numbers intentionally induces negative correlation between estimators so as to decrease the variance of the their average.

Next Time: Another variance reduction technique – control variates.  Related to regression methods.

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## Control Variates

# Lesson Overview

Last Time: Discussed the variance reduction technique known as antithetic variates.

This Time: One last VRT – control variates.

Closely related to regression from baby stats class.

## Control Variates

Another method to reduce estimator variance is related to regression.

Suppose that our goal is to estimate the mean $\mu$ of some steady-state simulation output process, $X_1, X_2, \ldots, X_n$. Suppose we somehow know the expected value of some other RV $Y$, and we also know that $\mathrm{Cov}(\bar{X}, Y) > 0$, where $\bar{X}$ is the sample mean.

Obviously, $\bar{X}$ is the "usual" estimator for $\mu$. But let's look at another estimator for $\mu$, namely, the *control-variate* estimator,

$$C \;=\; \bar{X} - \beta(Y - \mathrm{E}[Y]), \quad \text{for some constant } \beta.$$

Note that

$$\mathrm{E}[C] \;=\; \mathrm{E}[\bar{X}] - \beta(\mathrm{E}[Y] - \mathrm{E}[Y]) \;=\; \mathrm{E}[\bar{X}] \;=\; \mu.$$

$$\mathrm{Var}(C) \;=\; \mathrm{Var}(\bar{X}) + \beta^2\,\mathrm{Var}(Y) - 2\beta\,\mathrm{Cov}(\bar{X}, Y).$$

And then we can minimize $\mathrm{Var}(C)$ with respect to $\beta$. Differentiating,

$$\beta \;=\; \frac{\mathrm{Cov}(\bar{X}, Y)}{\mathrm{Var}(Y)}.$$

$$\mathrm{Var}(C) \;=\; \mathrm{Var}(\bar{X}) - \frac{\mathrm{Cov}^2(\bar{X}, Y)}{\mathrm{Var}(Y)} < \mathrm{Var}(\bar{X}). \quad \square$$

Examples: We might try to estimate a population's mean weight $\mu$ using observed weights $X_1, X_2, \ldots$ with corresponding heights $Y_1, Y_2, \ldots$ as controls (assuming that $\mathrm{E}[Y]$ is known).

We could estimate the price of an American stock option (which is tough) using the corresponding European option price (which is easy) as a control.

In any case, many simulation texts give advice on how to run the simulations of the competing systems so as to use CRN, ARN, and control variates.

# Summary

This Time: Completed our discussion on VRTs by talking about control variates.

Next Time: Start our material on ranking and selection methods – choosing the best system.

My favorite topic ever!

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## Ranking and Selection Methods

# Lesson Overview

Last Lesson: Finished off our material on variance reduction techniques.

This Time: Now on to ranking and selection methods!

Goal: Give procedures for finding the best system. Provide a statistical guarantee of being correct.

# Intro to Ranking and Selection

Ranking, selection, and multiple comparisons methods form another class of statistical techniques used to compare alternative systems.

Here, the experimenter is interested in selecting the best of a number ($\geq 2$) of competing processes.

Specify the desired probability of correctly selecting the best process, especially if the best process is significantly better than its competitors.

These methods are simple to use, fairly general, and intuitively appealing (see Bechhofer, Santner, and Goldsman 1995).

# Intro to Ranking and Selection

For > 2 systems, we could use methods such as simultaneous CIs and ANOVA. But those methods don't tell us much except that "at least one of the systems is different than the others", which is no surprise.

And what measures do you use to compare different systems?

- Which has the biggest mean?
- The smallest variance?
- The highest probability of yielding a success?
- The lowest risk?
- A combination of criteria?

# Intro to Ranking and Selection

Remainder of this module: We present ranking & selection procedures to find the best system with respect to one parameter.

Examples:

- Great Expectations: Which of 10 fertilizers produces the largest mean crop yield? (Normal)
- Great Expectorants: Find the pain reliever that has the highest probability of giving relief for a cough. (Binomial)
- Great Ex-Patriots: Who is the most-popular former New England football player? (Multinomial)

# Intro to Ranking and Selection

R&S selects the best system, or a subset of systems that includes the best.

- Guarantee a probability of a correct selection.
- Multiple Comparisons Procedures (MCPs) add in certain confidence intervals.

R&S is relevant in simulation:

- Normally distributed data by batching.
- Independence by controlling random numbers.
- Multiple-stage sampling by retaining the seeds.

# Summary

This Time: Introduced the topic of ranking and selection – choosing the best of a number of competing systems.

Next Time: Let's get more specific – choosing the normal distribution having the largest mean.

It's a classic!

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## Module 10: Comparing Systems

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## Normal Means Selection

# Lesson Overview

Last Time: Introduced the topic of ranking and selection – where the goal is to choose the best of a number of competing systems.

This Time: Look at the classic problem of choosing the normal distribution having the largest mean.

We'll guarantee a certain probability of correct selection.

# Find the Normal Distribution with the Largest Mean

We give procedures for selecting that one of $k$ *normal* distributions having the largest mean.

We use the *indifference-zone* approach.

Assumptions: Independent $Y_{i1}, Y_{i2}, \ldots$ $(1 \leq i \leq k)$ are taken from $k \geq 2$ normal populations $\Pi_1, \ldots, \Pi_k$. Here $\Pi_i$ has *unknown* mean $\mu_i$ and *known* or *unknown* variance $\sigma_i^2$.

Denote the vector of means by $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ and the vector of variances by $\boldsymbol{\sigma}^2 = (\sigma_1^2, \ldots, \sigma_k^2)$.

The ordered (but unknown) $\mu_i$'s are $\mu_{[1]} \leq \cdots \leq \mu_{[k]}$.

The system having the largest mean $\mu_{[k]}$ is the "best."

Goal: To select the population associated with mean $\mu_{[k]}$.

A *correct selection* (CS) is made if the Goal is achieved.

GTx

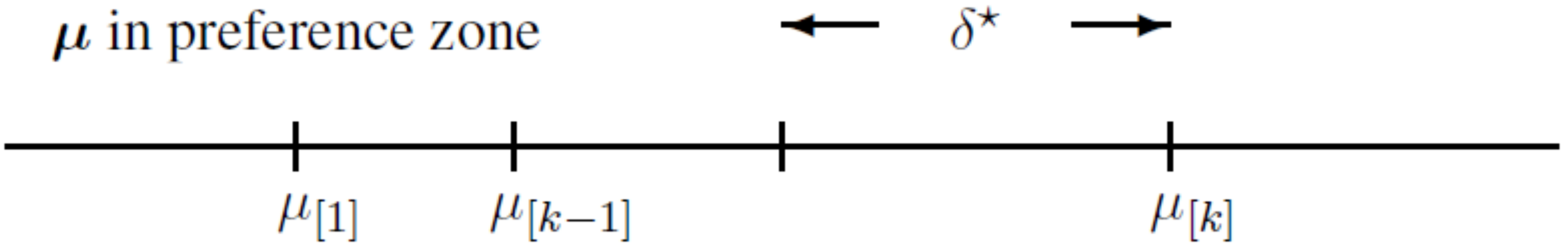
**Indifference-Zone Probability Requirement:** For specified constants $(P^\star, \delta^\star)$ with $\delta^\star > 0$ and $1/k < P^\star < 1$, we require

$$P(\mathrm{CS}) \;\geq\; P^\star \quad \text{whenever} \quad \mu_{[k]} - \mu_{[k-1]} \;\geq\; \delta^\star. \tag{1}$$

The constant $\delta^\star$ can be thought of as the "smallest difference worth detecting."

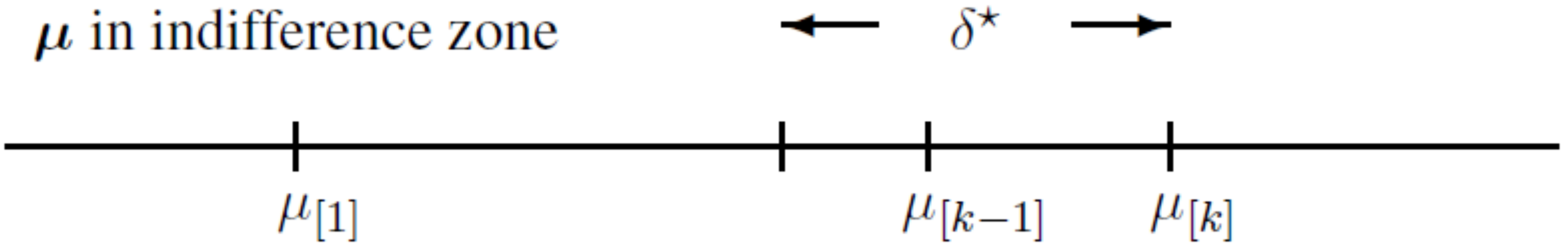
The probability in (1) depends on the differences $\mu_i - \mu_j$, the sample size $n$, and $\boldsymbol{\sigma}^2$.

Parameter configurations $\boldsymbol{\mu}$ satisfying $\mu_{[k]} - \mu_{[k-1]} \geq \delta^\star$ are in the *preference-zone* for a correct selection.




*You're traveling through another dimension, a dimension not only of sight and sound but of mind.*

If $\mu_{[k]} - \mu_{[k-1]} < \delta^\star$, then you're in the *indifference-zone*.




Any procedure that guarantees (1) is said to be employing the indifference-zone approach.

# So Many Procedures!

100’s of such procedures.

Highlights:

- Single-Stage (Bechhofer 1954)
- Two-Stage (Rinott 1979)
- Sequential (Kim and Nelson 2001)

We’ll go into some detail on the single-stage procedure.

- Just the basics
- Assumes common, known variance

# Summary

This Time: Gave background / notation for the R&S problem of finding the normal population with the largest mean.

Next Time: We'll look at a Bechhofer's famous single-stage procedure for the normal means problem.